Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 488 (2027) 114839

Journal of Differential Equations

www.elsevier.com/locate/jde

Full flexibility of the Monge-Ampère system in codimension $d_* - d + 1$ ☆

Wentao Cao [a], Jonas Hirsch [b], Dominik Inauen [c,*], Marta Lewicka [d]

[a] *Academy for Multidisciplinary Studies, Capital Normal University, West 3rd Ring North Road 105, Beijing, 100048, PR China*

[b] *Institut für Mathematik, Universität Leipzig, D-04109, Leipzig, Germany*

[c] *Université de Fribourg, CH-1700, Fribourg, Switzerland*

[d] *University of Pittsburgh, Department of Mathematics, 139 University Place, Pittsburgh, PA 15260, USA*

Received 13 April 2026; revised 24 September 2026; accepted 5 October 2026

Abstract

We prove that $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère system in dimension d and codimension $k = d_* - d + 1$, where d_* denotes the Janet dimension, are dense in the space of continuous functions, for every Hölder exponent $\alpha < 1$. Our result strengthens the statement in [24], obtained for $k = 2d_*$ and based on ideas from [22] in the context of the isometric immersion system. It also generalizes the result of [19], where full flexibility was established in dimension $d = 2$ and codimension $k = 2$. The same proof scheme further yields local full flexibility of isometric immersions of d-dimensional Riemannian metrics into Euclidean space of dimension $d_* + 1$, generalizing the result in [27] proved for $d = k = 2$. By using techniques of [12], the result can then be extended to compact manifolds, in codimension $(d + 1)d_* - d + 1$.
© 2026 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

MSC: 35J96; 53C42; 53A35

☆ W.C. was supported in part by the National Natural Science Foundation of China (No. 12471224). M.L. was partially supported by NSF grant DMS-2006439.

* Corresponding author.

E-mail addresses: cwtmath@cnu.edu.cn (W. Cao), jonas.hirsch@math.uni-leipzig.de (J. Hirsch), dominik.inauen@unifr.ch (D. Inauen), lewicka@pitt.edu (M. Lewicka).

https://doi.org/10.1016/j.jde.2026.114839
0022-0396/© 2026 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

In this paper, we prove a new *flexibility* result for the Monge-Ampère system, in arbitrary dimension d, and with the specific target codimension k:

$$k = \bar{d} \quad \text{where} \quad \bar{d} \doteq \frac{d^2 - d + 2}{2}.$$

We note the relation of $\bar{d}$ to the Janet dimension $d_* = d(d+1)/2$:

$$\bar{d} = d_* - d + 1.$$

In particular, $\bar{d}$ is significantly smaller than $2d_*$, that is the codimension in which *full flexibility*, in the sense of Theorem 1.1 below, was previously established. More precisely, we prove:

Theorem 1.1. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, d-dimensional set. For the target codimension:*

$$\bar{d} = d_* - d + 1 \quad \textit{where} \quad d_* = \frac{d(d+1)}{2}.$$

Given vector fields $v \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $w \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^d)$ and a matrix field $A \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, assume:

$$\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w < A \qquad \textit{in } \bar{\omega},$$

in the sense of matrix inequalities. Then, for every exponent α, satisfying:

$$\alpha < \min\left\{\frac{r+\beta}{2}, 1\right\}, \tag{1.1}$$

and for every $\epsilon > 0$, there exist $\tilde{v} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $\tilde{w} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^d)$ such that:

$$\begin{aligned} &\|\tilde{v} - v\|_0 \le \epsilon, \qquad \|\tilde{w} - w\|_0 \le \epsilon, \\ &\frac{1}{2}(\nabla \tilde{v})^T \nabla \tilde{v} + \mathrm{sym}\nabla \tilde{w} = A \qquad \textit{in } \bar{\omega}. \end{aligned}$$

Recall that the Monge-Ampère system, introduced in [24], seeks a k-dimensional solution v to:

$$\begin{cases} v : (\omega \subset \mathbb{R}^d) \to \mathbb{R}^k, \\ \mathfrak{Det}\, \nabla^2 v \doteq \left[\langle \partial_{is} v, \partial_{jt} v\rangle - \langle \partial_{it} v, \partial_{js} v\rangle\right]_{i,j,s,t:1\ldots d} = F \quad \text{in } \omega, \end{cases} \tag{1.2}$$

where $F : \omega \to \mathbb{R}^{d^4}$ is a given table field that satisfies symmetry conditions (1.6), necessary for the resolvability of (1.2). The first main reason why the system (1.2) is worthy of attention, is that it provides the multi-dimensional version of the Monge-Ampère equation:

$$\begin{cases} v : (\omega \subset \mathbb{R}^2) \to \mathbb{R}, \\ \det \nabla^2 v \doteq \partial_{11} v \partial_{22} v - (\partial_{12} v)^2 = f \quad \text{in } \omega, \end{cases} \tag{1.3}$$

arising from the prescribed curvature problem. Indeed, the family of Riemannian metrics on ω, generated by immersions $\{u^\epsilon = (id_d, \epsilon v)\}_{\epsilon \to 0}$, have their Riemann curvatures satisfy:

$$Riem\big((\nabla u^\epsilon)^T \nabla u^\epsilon\big) = Riem\big(\mathrm{Id}_d + \epsilon^2 (\nabla v)^T \nabla v\big) = \epsilon^2 \mathfrak{Det} \nabla^2 v + o(\epsilon^2).$$

When $d = 2, k = 1$ as in (1.3), the above is consistent with the formula for the Gaussian curvature of surfaces described as the graph of $\{\epsilon v\}_{\epsilon \to 0}$, namely:

$$\kappa(\epsilon v) = \frac{\epsilon^2 \det \nabla^2 v}{(1 + \epsilon^2 |\nabla v|^2)^2} = \epsilon^2 \det \nabla^2 v + o(\epsilon^2),$$

which yields (1.3) as the prescription of Gaussian curvature. The second reason to study (1.2), is related to its weak formulation, given in the form of the following Von Kàrmàn system:

$$\begin{cases} v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^d, \\ \dfrac{1}{2} (\nabla v)^T \nabla v + \mathrm{sym} \nabla w = A \quad \text{in } \omega. \end{cases} \tag{1.4}$$

Here, one seeks a solution pair, consisting of k-dimensional v and d-dimensional w, for a given matrix field $A : \omega \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ determined by F. The system (1.4) arises in the context of finding an isometric immersion u of the given Riemannian metric $g : \omega \to \mathbb{R}^{d \times d}_{\mathrm{sym},>}$, into $\mathbb{R}^{d+k}$:

$$\begin{cases} u : (\omega \subset \mathbb{R}^d) \to \mathbb{R}^{d+k}, \\ (\nabla u)^T \nabla u = g \quad \text{in } \omega. \end{cases} \tag{1.5}$$

Indeed, (1.5) yields (1.4) when equating the leading order terms in the family of Riemannian metrics $\{\mathrm{Id}_d + 2\epsilon^2 A\}_{\epsilon \to 0}$ and the metrics generated by the immersions $\{\bar{u}^\epsilon = (id_d + \epsilon^2 w, \epsilon v)\}_{\epsilon \to 0}$:

$$(\nabla \bar{u}^\epsilon)^T \nabla \bar{u}^\epsilon = \mathrm{Id}_d + \epsilon^2 \big((\nabla v)^T \nabla v + 2\,\mathrm{sym} \nabla w\big) + o(\epsilon^2).$$

When $d = 2, k = 1$, the above derivation is consistent with the left hand side of (1.4) appearing in the stretching energy of a thin film, subject to the out-of-plane displacement v and the in-plane displacement w. Hence, the three problems (1.2), (1.4) and (1.5) are intrinsically related.
We remark that the results as in Theorem 1.1 (as well as Corollary 1.2 below) remain valid in any target space $\mathbb{R}^k$ replacing $\mathbb{R}^{\bar{d}}$, provided that $k \geq \bar{d}$. Since our proofs rely only on corrugations in $\bar{d}$ distinct directions, the same construction of *stage* as in Theorem 1.3 applies. Another observation is that by extending our proofs - at the expense of additional technical constructions outlined in Section 5 - analogous full flexibility results hold in the setting of isometric immersions, that is for solutions of system (1.5), posed on $\omega \subset \mathbb{R}^d$ and with target dimension $d + \bar{d} = d_* + 1$, as stated in Theorem 5.1. In particular, this establishes the local existence of $\mathcal{C}^{1,1-}$-regular isometric immersions of d-dimensional $\mathcal{C}^2$ Riemannian metrics into the $(d_* + 1)$-dimensional Euclidean space. By using techniques of [12], the result can then be extended to compact manifolds, in

codimension $k \geq (d+1)d_* - d + 1$.
As a consequence and in light of [24, Theorem 1.8, Theorem 1.10], we further obtain:

Corollary 1.2. *Let $F = [F_{ij,st}]_{i,j,s,t=1\dots d} \in L^\infty(\omega, \mathbb{R}^{d^4})$ defined on an open, bounded, contractible domain $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, satisfy, for all $i, j, s, t, q = 1 \dots d$:*

$$\begin{aligned} &F_{ij,st} = -F_{ji,st} = -F_{ij,ts}, \qquad F_{ij,st} = F_{st,ij}, \\ &F_{ij,st} + F_{is,tj} + F_{it,js} = 0, \\ &\partial_q F_{ij,st} + \partial_s F_{ij,tq} + \partial_t F_{ij,qs} = 0 \quad \text{in the sense of distributions on } \omega. \end{aligned} \tag{1.6}$$

Then, for any exponent $\alpha \in (0,1)$, the set of $\mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{\bar{d}})$-regular weak solutions to: (1.2), is dense in $\mathcal{C}^0(\bar{\omega}, \mathbb{R}^{\bar{d}})$. Namely, every $v \in \mathcal{C}^0(\bar{\omega}, \mathbb{R}^{\bar{d}})$ is the uniform limit of some sequence $\{v_n \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{\bar{d}})\}_{n=1}^\infty$, such that: $\mathfrak{Det}\, \nabla^2 v = F$ on ω, for all $n \geq 1$.

1.1. Comparison with the literature

Our present result, in addition to addressing the well-posedness of the geometrical system (1.2), contributes to the growing body of work on the flexibility of $\mathcal{C}^{1,\alpha}$ isometric immersions and solutions to related PDEs.
The study of flexibility in the context of the system (1.5) originates from the classical Nash-Kuiper theorem [23,29] about $\mathcal{C}^1$ isometric immersions of a d-dimensional Riemannian manifold into $\mathbb{R}^{d+1}$ (in our terminology, this corresponds to codimension $k = 1$). The Hölder-regular isometric immersions were studied by Borisov [1,2], who conjectured that the Nash-Kuiper theorem extended to $\mathcal{C}^{1,\alpha}$-regularity for $\alpha < (1 + d^2 + d)^{-1}$ and $\alpha < 1/5$ for $d = 2$. He provided a proof for the exponent $\alpha < 1/7$, dimension $d = 2$ and the analytic metric g in [3]. Borisov's conjecture was confirmed by Conti, De Lellis and Székelyhidi in [12], whereas the case $\alpha < 1/5$, $d = 2$ was resolved by De Lellis, Inauen and Székelyhidi in [13]. These results were generalized to compact manifolds by Cao and Székelyhidi in [11]. Locally, it is very recently improved to $\alpha < (d^2 - d + 1)^{-1}$ by Cao, Hirsch and Inauen in [7] and $\alpha < (2d-1)^{-1}$ by Inauen in [17].
If the codimension k is larger than 1, one can construct more regular isometric immersions. In this direction, Nash first showed [30] that any manifold d-dimensional Riemannian manifold with metric $g \in \mathcal{C}^r$, $r \geq 3$, admits a $\mathcal{C}^r$-regular isometric immersion into $\mathbb{R}^{d+k}$ for sufficiently large k. The codimension bounds were later established and refined by Gromov [15] and Günther [16], while the case of $g \in \mathcal{C}^{r,\beta}$ with $r + \beta > 2$, was treated by Jacobowitz [21]. When the metric's regularity is lower, i.e. $r + \beta < 2$, Källén [22] constructed a $\mathcal{C}^{1,\alpha}$ immersion for any $\alpha < (r+\beta)/2$ when $k \geq 2d_*(d+1)$ is sufficiently large. See also [8,10,14] for further results about isometric immersions in high codimension. Recently, Lewicka proved in [27] that in dimension $d = 2$ and codimension $k \geq 2$, every subsolution to (1.5) with the metric $g \in \mathcal{C}^{r,\beta}$ admits an approximate sequence of isometric immersions with regularity $\mathcal{C}^{1,\alpha}$, for all $\alpha < \min\{(r+\beta)/2, 1\}$. This property is referred to as *full flexibility*. In the present paper, we extend this result to arbitrary dimension $d \geq 2$ and the corresponding codimension $\bar{d}$.
The parallel version of the Nash-Kuiper scheme as in the papers cited above, was first developed to handle flexibility of $\mathcal{C}^{1,\alpha}$ weak solutions to the Monge-Ampère equation (1.3) by Lewicka and Pakzad in [28], where the result was proved for $\alpha < 1/7$. This regularity exponent was later increased to $\alpha < 1/5$ and $\alpha < 1/3$ by Cao and Szekelyhidi [9], and by Cao, Hirsch and Inauen

[6], respectively. Closer to our present context, in [24] Lewicka introduced the Monge-Ampère system (1.2) and obtained flexibility of its solutions for any $\alpha < \min\{\beta/2, (1+2d_*/k)^{-1}\}$ given $A \in \mathcal{C}^{0,\beta}$, where the dimension d and k were arbitrary; and also for any $\alpha < \min\{\beta/2, 1\}$ provided that $k \geq 2d_*$. For the special case $d = 2$, weak solutions of progressively higher regularity, depending on the value of k, were obtained in the sequence of papers [18,25,26]. In [19], Inauen and Lewicka established the full flexibility of the Monge-Ampére system for $d = k = 2$, serving as a direct precursor to [24] (see also [31] for a related existence result) and to the present paper. We also note that the convex integration technique has been successfully applied to address Onsager's conjecture for the Euler equations, as well as to other fluid mechanics equations (see [4,5,20] and references therein).

1.2. Overview of the strategy of proofs

Our main technical contribution is the following *stage* construction, whose iteration via the Nash-Kuiper algorithm in Theorem 4.1 ultimately provides the proof of Theorem 1.1. To ease the notation, we define the *defect* relative to a matrix field A and vector fields v and w:

$$\mathcal{D}(A, v, w) \doteq A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big).$$

Theorem 1.3. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, d-dimensional set. Fix two integers $N, K \geq 1$. There exists $\sigma_0 \geq 1$ depending only on ω, N, K, such that the following holds. Given $v \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $w \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$, and $A \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $r \in \{0, 1\}$, $\beta \in (0, 1]$, satisfying:*

$$0 < \|\mathcal{D}\|_0 \leq 1 \quad \textit{where} \quad \mathcal{D} = \mathcal{D}(A, v, w), \tag{1.7}$$

and given two positive constants $\mathcal{M}, \sigma$ such that:

$$\mathcal{M} \geq \max\{\|v\|_2, \|w\|_2, 1\}, \quad \sigma \geq \sigma_0, \tag{1.8}$$

there exist $\tilde{v} \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $\tilde{w} \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$, such that the following bounds are valid:

$$\|\tilde{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \qquad \|\tilde{w} - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}\big(1 + \|\nabla v\|_0\big), \tag{1.9}_1$$

$$\|\nabla^2 \tilde{v}\|_0 \leq C\mathcal{M}\sigma^{dK+N}, \qquad \|\nabla^2 \tilde{w}\|_0 \leq C\mathcal{M}\sigma^{dK+N}\big(1 + \|\nabla v\|_0\big), \tag{1.9}_2$$

$$\|\mathcal{D}(A, \tilde{v}, \tilde{w})\|_0 \leq C\Big(\frac{\|A\|_{r,\beta}}{\mathcal{M}^{r+\beta}}\|\mathcal{D}\|_0^{(r+\beta)/2} + \frac{\|\mathcal{D}\|_0}{\sigma^{KN}}\Big). \tag{1.9}_3$$

Above, the constants C depend only on ω, r, β and N, K.

We emphasize that, compared to the existing literature, we are able to reduce the magnitude of the defect at the expense of a smaller blow-up rate for the second derivatives of the subsolutions in $(1.9)_2$–$(1.9)_3$. We now briefly outline how this is achieved.
In the course of the proof, we inductively define K intermediate subsolutions $(v_k, w_k)_{k=1}^K$. The initial pair (v_0, w_0) is constructed via a mollification of the given (v, w), while the final pair satisfies $(\tilde{v}, \tilde{w}) = (v_K, w_K)$. The construction of (v_{k+1}, w_{k+1}) from (v_k, w_k) is carried out by

successively adding *Kuiper's corrugations*, as introduced in Lemma 2.3, with the aim of canceling the d_* rank-one *primitive defects* of the form $a^2\eta\otimes\eta$. These primitive defects arise from the decomposition of $\mathcal{D}(A_0, v_k, w_k)$, see Lemma 2.2, where A is replaced by its mollified version A_0, yielding the first term on the right-hand side of the estimate $(1.9)_3$.
First, codimension $n=1$ is used to cancel the primitive defects corresponding to $\eta=\eta_{1j}$ for all $j=1\dots d$, through a higher tier of intermediate subsolution pairs $(V_{1j}, W_{1j})_{j=1}^{d}$. Here, Lemma 2.4 is applied N times to reduce the order of the resulting error terms. A prior application of *Källen's iteration*, see Lemma 2.5, allows one to additionally absorb into $\mathcal{D}$ those portions of the error that cannot be handled by Lemma 2.4. The subsequent codimensions $n=2\dots\bar{d}$ are used to consecutively cancel the remaining primitive defects, namely those corresponding to $\eta=\eta_{ij}$ with $2\le i\le j\le d$, thereby constructing the next intermediate subsolution pairs $(V_n, W_n)_{n=2}^{\bar{d}}$. The main observation is that the first error term on the right-hand side of (2.4), which involves $\nabla^2 V_{n-1}^n$, can be written as a sum of terms of the same structure, now involving only $\nabla^2 v_0$, together with an expansion of oscillatory contributions remaining from previous induction steps. All these terms oscillate in a fixed direction and can therefore be further decomposed and reduced by an argument similar to (2.5) in Lemma 2.4, but without the P_i^j components. This observation constitutes the main new ingredient of our analysis. We then conclude the induction step by setting $(v_{k+1}, w_{k+1}) = (V_{\bar{d}}, W_{\bar{d}})$.
With Theorem 1.3 at hand, Theorem 1.1 follows via a further iterative procedure, referred to as the *Nash-Kuiper algorithm*, see Theorem 4.1. We briefly recall its main idea, in which the solution $(\tilde{v}, \tilde{w})$ to (1.4) is obtained as the limit of the sequence $(\tilde{v}_n, \tilde{w}_n)_{n=1}^{\infty}$, generated through repeated applications of Theorem 1.3. Along this iteration, the decay rate of $\|\tilde{v}_n-\tilde{v}_{n-1}\|_1$ is roughly $\sigma^{-nKN/2}$, as follows from $(1.9)_1$ and $(1.9)_3$. Combining this with the blow-up rate $\sigma^{n(dK+N)}$ of $\|\nabla^2\tilde{v}_n\|_0$ in $(1.9)_2$, and applying an interpolation inequality in Hölder spaces, we obtain a bound on the corresponding $\mathcal{C}^{1,\alpha}$ norm:

$$\|\tilde{v}_n-\tilde{v}_{n-1}\|_{1,\alpha}\le C\sigma^{n(\alpha(dK+N+KN/2)-KN/2)}.$$

For the sequence $\{\tilde{v}_n\}_{n=1}^{\infty}$ to be Cauchy in $\mathcal{C}^{1,\alpha}$, the power of the relative frequencies σ along the iteration must be negative, which is equivalent to:

$$\alpha<\alpha_0\doteq\frac{KN/2}{dK+N+KN/2}=\frac{1}{1+2r_{K,N}}\qquad\text{where}\qquad r_{K,N}=\frac{dK+N}{KN}.$$

We observe that $r_{K,N}$ is the ratio of the blow-up rate of $\|\nabla^2\tilde{v}\|_0$ to the decay rate of $\|\mathcal{D}(A,\tilde{v},\tilde{w})\|_0$ (specifically, the portion corresponding to the second term on the right-hand side of $(1.9)_3$), and that it can be made arbitrarily small:

$$r_{K,N}\to 0\qquad\text{as } N, K\to\infty,$$

which corresponds to $\alpha_0=1$. Since the Hölder regularity obtained by iterating Theorem 1.3 is ultimately limited only by the regularity of A and by α_0, this yields the range stated in (1.1).

1.3. Organization of the paper and notation

In Section 2, we collect several preparatory statements used in the proof of Theorem 1.3. The proof, along with the stage construction, is carried out in Section 3. The Nash-Kuiper scheme

argument, which yields the proof of Theorem 1.1 is recalled in section 4. Finally, Section 5 outlines the additional arguments needed to extend Theorem 1.1 to the isometric immersion system (1.5).

By $\mathbb{R}^{d\times d}_{\text{sym}}$ we denote the space of symmetric $d \times d$ matrices, and $\mathbb{R}^{d\times d}_{\text{sym},>}$ stands for the set of such matrices which are positive definite. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ consists of restrictions of all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^k)$ to the closure of an open, bounded set $\omega \subset \mathbb{R}^d$. The $\mathcal{C}^m(\bar{\omega}, \mathbb{R}^k)$ norm of this restriction is denoted by $\|f\|_m$, while its Hölder norm in $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ is $\|f\|_{m,\alpha}$. By $\nabla^{(m)} f$ we denote the table-valued field of all partial derivatives of f of order m; in the introduction section, we kept the notation $\nabla^2 f$ for the matrix of second derivatives. If $d = k$, the symmetric part of ∇f is: $\text{sym}\nabla f = (\nabla f + (\nabla f)^T)/2$. By C we denote a universal constant that may change from line to line, but it depends only on the specified parameters.

2. The preparatory statements

In this section, we collect several technical ingredients that will be used in the proof of Theorem 1.3. The first lemma concerns the convolution and commutator estimates from [12].

Lemma 2.1. *Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1) \subset \mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \phi \, \mathrm{d}x = 1$. Denote:*

$$\phi_l(x) \doteq \frac{1}{l^d}\phi(\frac{x}{l}) \quad \text{for all } l \in (0,1],\ x \in \mathbb{R}^d.$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$, every $m \geq 0$ and $r \in \{0,1\}$, $\beta \in (0,1]$, there holds:

$$\|\nabla^{(m)}(f * \phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.1}_1$$

$$\|f - f * \phi_l\|_0 \leq l^{r+\beta}\|f\|_{r,\beta}, \tag{2.1}_2$$

$$\|\nabla^{(m)}\big((fg) * \phi_l - (f * \phi_l)(g * \phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.1}_3$$

with constants $C > 0$ depending only on d and m, where the right-hand sides of (2.1)$_2$ and (2.1)$_3$ are ∞ if f or g are not regular enough.

The next lemma provides a decomposition of symmetric matrices into linear combinations of rank-one *primitive matrices*. The result is elementary and its proof is left to the reader.

Lemma 2.2. *For all $i, j = 1 \dots d$, define the unit vectors:*

$$\eta_{ij} \doteq \frac{e_i + e_j}{|e_i + e_j|} \in \mathbb{R}^d.$$

Then $\{\eta_{ij} \otimes \eta_{ij}\}_{i,j=1\dots d,\, i\leq j}$ is a basis of the linear space $\mathbb{R}^{d\times d}_{\text{sym}}$, so that:

$$H = \sum_{i\leq j} \bar{a}_{ij}(H)\eta_{ij} \otimes \eta_{ij} \quad \text{for all } H \in \mathbb{R}^{d\times d}_{\text{sym}},$$

where $\{\bar{a}_{ij} : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}\}_{i,j=1\dots d,\, i\le j}$ *are the linear coordinate functions. Moreover, denoting:*

$$H_0 \doteq \sum_{i\le j} \eta_{ij} \otimes \eta_{ij}, \tag{2.2}$$

there exists $r_0 \in (0,1)$*, depending only on* d*, such that:*

$$|\bar{a}_{ij}(H) - 1| \le \frac{1}{2} \quad \text{for all } H \in B(H_0, r_0) \subset \mathbb{R}^{d\times d}_{\text{sym}} \quad \text{and all } i \le j.$$

As the next ingredient, we recall the *step* construction from [24, Lemma 2.1], in which a single codimension is used to cancel one rank-one defect of the form $a(x)^2 e_i \otimes e_i$.

Lemma 2.3. *Let* $v \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^{\bar{d}})$*,* $w \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$*,* $\lambda > 0$ *and* $a \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ *be given. Denote:*

$$\Gamma(t) \doteq 2\sin t, \quad \bar{\Gamma}(t) \doteq -\frac{1}{2}\sin(2t), \quad \bar{\bar{\Gamma}}(t) \doteq \frac{1}{2}\cos(2t), \quad \bar{\bar{\bar{\Gamma}}}(t) \doteq 1 - \frac{1}{2}\cos(2t).$$

Fix a coordinate $n = 1 \dots \bar{d}$ *and a unit vector* $\eta \in \mathbb{R}^d$*. Using the notation:*

$$t_\eta \doteq \langle x, \eta \rangle,$$

let $\tilde{v} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^{\bar{d}})$*,* $\tilde{w} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ *be given by:*

$$\begin{aligned} \tilde{v} &\doteq v + \frac{\Gamma(\lambda t_\eta)}{\lambda} a(x) e_n, \\ \tilde{w} &\doteq w - \frac{\Gamma(\lambda t_\eta)}{\lambda} a(x) \nabla v^n + \frac{\Gamma(\lambda t_\eta)}{\lambda} a(x)^2 \eta + \frac{\Gamma(\lambda t_\eta)}{\lambda^2} a(x) \nabla a(x). \end{aligned} \tag{2.3}$$

Then, the following identity is valid on $\mathbb{R}^d$*:*

$$\begin{aligned} &\Big(\frac{1}{2}(\nabla \tilde{v})^T \nabla \tilde{v} + \operatorname{sym} \nabla \tilde{w}\Big) - \Big(\frac{1}{2}(\nabla v)^T \nabla v + \operatorname{sym} \nabla w\Big) - a(x)^2 \eta \otimes \eta \\ &= -\frac{\Gamma(\lambda t_\eta)}{\lambda} a \nabla^2 v^n + \frac{\bar{\bar{\Gamma}}(\lambda t_\eta)}{\lambda^2} a \nabla^2 a + \frac{\bar{\bar{\bar{\Gamma}}}(\lambda t_\eta)}{\lambda^2} \nabla a \otimes \nabla a. \end{aligned} \tag{2.4}$$

The proof of (2.4) follows by direct inspection, in view of the identities:

$$\frac{1}{2}(\Gamma')^2 + \bar{\Gamma}' = 1, \qquad \Gamma'\Gamma + 2\bar{\Gamma} + \bar{\bar{\Gamma}}' = 0, \qquad \frac{1}{2}\Gamma^2 + \bar{\bar{\Gamma}} = \bar{\bar{\bar{\Gamma}}}.$$

Next, we present the *integration by parts* argument, put forward in [7]. We remark that if Γ_0 below has the general form $\alpha \sin(\beta t)$ or $\alpha \cos(\beta t)$, as is the case for the zero-mean periodic profiles Γ, $\bar{\Gamma}$ $\bar{\bar{\Gamma}}$ and $\bar{\bar{\bar{\Gamma}}} - 1$ in Lemma 2.3, then the primitives Γ_i defined in (2.6) are of the same form $\frac{\alpha}{\beta^i}\sin(\beta t)$ or $\frac{\alpha}{\beta^i}\cos(\beta t)$. In particular, all their derivatives are bounded, which ensures the uniformity of the estimates in the proof of Theorem 1.3.

Lemma 2.4. *Given $H \in \mathcal{C}^{k+1}(\mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\lambda > 0$, $j = 1 \dots d$ and $\Gamma_0 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, there holds:*

$$\begin{aligned}\frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} H &= (-1)^{k+1} \frac{\Gamma_{k+1}(\lambda t_{\eta_{1j}})}{\lambda^{k+2}} \mathrm{sym}\nabla L_k^j(H) \\ &\quad + \mathrm{sym}\nabla\Big(\sum_{i=0}^{k} (-1)^i \frac{\Gamma_{i+1}(\lambda t_{\eta_{1j}})}{\lambda^{i+2}} L_i^j(H)\Big) + \sum_{i=0}^{k} (-1)^i \frac{\Gamma_i(\lambda t_{\eta_{1j}})}{\lambda^{i+1}} P_i^j(H),\end{aligned} \tag{2.5}$$

where $t_{\eta_{1j}} = \langle x, \eta_{1j}\rangle$ with $\eta_{1j} = \frac{e_1+e_j}{|e_1+e_j|}$. The functions $\Gamma_i \in \mathcal{C}^i(\mathbb{R}, \mathbb{R})$ satisfy the recursive definition:

$$\Gamma'_{i+1} = \Gamma_i \quad \text{for all } i = 0 \dots k, \tag{2.6}$$

while $L_i^j(H) \in \mathcal{C}^{k+1-i}(\mathbb{R}^d, \mathbb{R}^d)$ and $P_i^j(H) \in \mathcal{C}^{k+1-i}(\mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ have the following properties:

(i) *the entries of $L_i^j(H)$ and $P_i^j(H)$ depend linearly on H and consist of the linear combinations of the entries of $\nabla^{(i)}H$,*
(ii) *$P_i^j(H) \in \mathrm{span}\big(\eta_{st} \otimes \eta_{st};\ 2 \le s \le t\big)$, so that, recalling the decomposition in Lemma 2.2, there holds: $\bar{a}_{1t}(P_i^j(H)) = 0$ for all $t = 1 \dots d$.*

Proof. We assume that the matrix field H is smooth and validate (2.5) inductively for any $k \ge 0$, constructing the fields $\{L_i^j = L_i^j(H)\}_{i\ge 0}$ and $\{P_i^j = P_i^j(H)\}_{i\ge 0}$ with the properties stated in (i), (ii). At $k = 0$, we may check directly that these are valid in:

$$\begin{aligned}\frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} H &= \frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} \mathrm{sym}\big(L_0^j \otimes \eta_{1j}\big) + \frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} P_0^j \\ &= -\frac{\Gamma_1(\lambda t_{\eta_{1j}})}{\lambda^2} \mathrm{sym}\nabla L_0^j + \mathrm{sym}\nabla\Big(\frac{\Gamma_1(\lambda t_{\eta_{1j}})}{\lambda^2} L_0^j\Big) + \frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} P_0^j,\end{aligned} \tag{2.7}$$

where: $P_0^j = H - \mathrm{sym}(L_0^j \otimes \eta_{1j})$, $\quad L_0^1 = 2He_1 - H_{11}e_1$,
$L_0^j = 2\sqrt{2}He_1 - \sqrt{2}H_{11}(e_1 + e_j)$ for $j = 2 \dots d$.

For the induction step, assume that $\{L_i^j,\ P_i^j\}_{i=0}^{k-1}$ are defined and (2.5) holds at some $k - 1 \ge 0$:

$$\begin{aligned}\frac{\Gamma_0(\lambda t_{\eta_{1j}})}{\lambda} H &= (-1)^k \frac{\Gamma_k(\lambda t_{\eta_{1j}})}{\lambda^{k+1}} \mathrm{sym}\nabla L_{k-1}^j \\ &\quad + \mathrm{sym}\nabla\Big(\sum_{i=0}^{k-1} (-1)^i \frac{\Gamma_{i+1}(\lambda t_{\eta_{1j}})}{\lambda^{i+2}} L_i^j\Big) + \sum_{i=0}^{k-1} (-1)^i \frac{\Gamma_i(\lambda t_{\eta_{1j}})}{\lambda^{i+1}} P_i^j.\end{aligned} \tag{2.8}$$

We now apply (2.7) to $H = \mathrm{sym}\nabla L_{k-1}^j$ and write:

$$\frac{\Gamma_k(\lambda t_{\eta_{1j}})}{\lambda}\mathrm{sym}\nabla L^j_{k-1} = -\frac{\Gamma_{k+1}(\lambda t_{\eta_{1j}})}{\lambda^2}\mathrm{sym}\nabla L^j_k + \mathrm{sym}\nabla\Big(\frac{\Gamma_{k+1}(\lambda t_{\eta_{1j}})}{\lambda^2}L^j_k\Big) + \frac{\Gamma_k(\lambda t_{\eta_{1j}})}{\lambda}P^j_k,$$

where: $L^1_k = 2(\mathrm{sym}\nabla L^j_{k-1})e_1 - \langle\partial_1 L^j_{k-1}, e_1\rangle e_1$,

$L^j_k = 2\sqrt{2}(\mathrm{sym}\nabla L^j_{k-1})e_1 - \sqrt{2}\langle\partial_1 L^j_{k-1}, e_1\rangle(e_1 + e_j)$ for $j = 2\dots d$,

$P^j_k = \mathrm{sym}\nabla L^j_{k-1} - \mathrm{sym}(L^j_k \otimes \eta_{1j})$.

Multiplying the above by $(-1)/\lambda^k$ and replacing with the first term in the right hand side of (2.8), provides the identity (2.5). The recursive formulas for L^j_k, P^j_k imply the propagation of the properties (i), (ii). The proof is done. □

Finally, we recall a version of *Källén's decomposition*, which originates from an iterative argument in [22]. In comparison with Lemma 2.2, the decomposition below additionally absorbs the non-oscillatory component $\frac{1}{\lambda^2}\nabla a \otimes \nabla a$ of the last defect term on the right hand side of (2.4). The statement below is taken from [27, Lemma 6.1] and we point out that a similar result appeared in [19, Proposition 3.1] with only one extra term of the type $\frac{1}{\lambda^2}\nabla a \otimes \nabla a$, while [7, Lemma 2.2] featured more absorbed terms, as in (2.10). The matrix field H in (2.10) should be thought of as the scaled defect $\mathcal{D}$ in the proof of Theorem 1.3.

Lemma 2.5. *Fix two integers $M, N \geq 1$ and let $H \in \mathcal{C}^{M+N}(\mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfy:*

$$\|H - H_0\|_0 \leq \frac{r_0}{2} \quad \text{and} \quad \|\nabla^{(m)} H\|_0 \leq \bar{C}\mu^m \quad \text{for all } m = 1\dots M+N, \tag{2.9}$$

for some given $\mu, \bar{C} > 1$ and with r_0 as in Lemma 2.2. Then, there exists $\sigma_0 \geq 1$ depending only on d, M, N, such that the following holds. Let $\sigma \geq \sigma_0$ and define:

$$\lambda_{1j} = \sigma^j\mu \quad \text{for } j = 1\dots d.$$

There exist $\{a_{ij} \in \mathcal{C}^{M+1}(\mathbb{R}^d, \mathbb{R})\}_{i,j:1\dots d,\, i\leq j}$ such that, writing:

$$H = \sum_{i\leq j} a_{ij}^2 \eta_{ij}\otimes\eta_{ij} + \sum_{j=1}^d \frac{1}{\lambda_{1j}^2}\nabla a_{1j}\otimes\nabla a_{1j} + \mathcal{F}_0, \tag{2.10}$$

with $\{\eta_{ij} \in \mathbb{R}^d\}_{i,j:1\dots d,\, i\leq j}$ as in Lemma 2.2, there hold the estimates:

$$\begin{aligned}
&\frac{1}{2} \leq a_{ij}^2 \leq \frac{3}{2} \quad \text{and} \quad \frac{1}{2} \leq a_{ij} \leq \frac{3}{2} \quad \text{for all } i \leq j,\\
&\|\nabla^{(m)} a_{ij}^2\|_0 \leq C\mu^m \quad \text{and} \quad \|\nabla^{(m)} a_{ij}\|_0 \leq C\mu^m \quad \text{for } m = 1\dots M+1,\ i \leq j,\\
&\|\nabla^{(m)}\mathcal{F}_0\|_0 \leq C\frac{\mu^m}{\sigma^{2N}} \quad \text{for } m = 0\dots M.
\end{aligned} \tag{2.11}$$

The constant C above depends only on $d, M, N, \bar{C}$.

We note that if $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, then $\{a_{ij}\}_{i\leq j}$ and $\mathcal{F}$ are smooth as well, and their $(M+1)$-th and M-th order derivatives, respectively, depend on the at most $M+N$ derivatives of H.

3. The stage construction and a proof of Theorem 1.3

We will now complete the *stage* in the present formulation of the convex integration algorithm. Recall that, given any $v \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $w \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^d)$, $A \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, we denote:

$$\mathcal{D}(A, v, w) \doteq A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big).$$

As explained in the introduction section 1, the proof below inductively defines K intermediate subsolutions $\{(v_k, w_k)\}_{k=1}^K$, over 13 steps. The initial pair (v_0, w_0) is constructed in step 1, where we also modify A to A_0. The final pair, obtained in step 13, satisfies $(\tilde{v}, \tilde{w}) = (v_K, w_K)$.
The construction of (v_{k+1}, w_{k+1}) from (v_k, w_k) is carried out in steps 3-13. We successively add corrugations according to Lemma 2.3, with the aim of canceling the d_* rank-one primitive defects $a^2\eta\otimes\eta$, see (2.4). These primitive defects arise from the decomposition of $\mathcal{D}(A_0, v_k, w_k)$ in Lemma 2.2, performed in step 3. In steps 4-8, we first use codimension $n = 1$ to cancel the primitive defects corresponding to all η_{1j}, through a higher tier of intermediate subsolution pairs $(V_{1j}, W_{1j})_{j=1}^d$. Here, Lemma 2.4 is applied N times to reduce the order of the resulting error terms. A prior application of Lemma 2.5 allows one to additionally absorb into $\mathcal{D}$ those portions of the error whose oscillation profiles Γ have nonzero mean and would otherwise be of too large an order under the integration (2.6).
In steps 9-12, the codimensions $n = 2 \dots \bar{d}$ are used to cancel the remaining primitive defects, namely those corresponding to η_{ij} with $2 \le i \le j \le d$ (see the relation of n and (i, j) in (3.27)), thereby constructing the next intermediate subsolution pairs $(V_n, W_n)_{n=2}^{\bar{d}}$. The main observation is that the first error term on the right-hand side of (2.4), which involves $\nabla^{(2)} v_k^n$, can be written as a sum of terms of the same structure, now involving only $\nabla^{(2)} v_0$, together with an expansion of oscillatory contributions remaining from previous induction steps. All these terms oscillate in a fixed direction and can therefore be further decomposed and reduced by an argument similar to (2.5), but without the P_i^j components. This observation enables the bounds established in step 12, which constitute the main new ingredient of our analysis. We then conclude the induction step by setting $(v_{k+1}, w_{k+1}) = (V_{\bar{d}}, W_{\bar{d}})$.

Proof of Theorem 1.3. 1. (Mollification and the initial quantities) Let N, K, v, w, A be as in the statement of the theorem, together with $\mathcal{M}$, σ satisfying (1.8), where the relative frequency threshold σ_0 will be determined in the course of the proof. Recall the definition of $\mathcal{D}$ in (1.7) and denote:

$$\delta_0 \doteq \|\mathcal{D}\|_0, \qquad \mu_0 \doteq \frac{\mathcal{M}}{\delta_0^{1/2}}.$$

We now define the mollified fields $v_0 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $w_0 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$, $A_0 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, relative to the kernel ϕ_l as in Lemma 2.1, with $l = 1/\mu_0$, namely:

$$v_0 \doteq v * \phi_{1/\mu_0}, \quad w_0 \doteq w * \phi_{1/\mu_0}, \quad A_0 \doteq A * \phi_{1/\mu_0}.$$

Denote the initial defect in the stage construction by:

$$\mathcal{D}_0 \doteq \mathcal{D}(A_0, v_0, w_0) = A_0 - \big(\frac{1}{2}(\nabla v_0)^T \nabla v_0 + \mathrm{sym}\nabla w_0\big).$$

We collect the following initial bounds:

$$\|v_0 - v\|_1 + \|w_0 - w\|_1 \leq C\frac{\mathcal{M}}{\mu_0} = C\delta_0^{1/2}, \tag{3.1}_1$$

$$\|A_0 - A\|_0 \leq C_A \frac{\|A\|_{r,\beta}}{\mu_0^{r+\beta}} = C_A \frac{\|A\|_{r,\beta}}{\mathcal{M}^{r+\beta}} \delta_0^{(r+\beta)/2}, \tag{3.1}_2$$

$$\begin{aligned}\|\nabla^{(m+2)} v_0\|_0 + \|\nabla^{(m+2)} w_0\|_0 \\ \leq C\mathcal{M}\mu_0^m = C\delta_0^{1/2}\mu_0^{m+1} \quad \text{for } m = 0 \dots K(2N+4)+2,\end{aligned} \tag{3.1}_3$$

$$\|\nabla^{(m)}\mathcal{D}_0\|_0 \leq C_0\delta_0\mu_0^m \qquad \text{for } m = 0 \dots K(2N+4), \tag{3.1}_4$$

where the constants $C, C_0 \geq 1$ depend only on ω, N, K and C_A on ω, r, β, as above. Indeed, $(3.1)_1$, $(3.1)_2$ result from $(2.1)_2$ in Lemma 2.1, while $(3.1)_3$ follows by applying $(2.1)_1$ to $\nabla^{(2)} v$ and $\nabla^{(2)} w$ with the differentiability exponent $m-1$. Since:

$$\mathcal{D}_0 = \mathcal{D} * \phi_{1/\mu_0} - \frac{1}{2}\big((\nabla v_0)^T \nabla v_0 - ((\nabla v)^T \nabla v) * \phi_{1/\mu_0}\big),$$

we get $(3.1)_4$ by applying $(2.1)_1$ to $\mathcal{D}$, and $(2.1)_3$ to ∇v.

2. (Setting up the induction) In steps 3-12, we will construct the intermediate fields $\{v_k \in \mathcal{C}^{(K-k)(2N+4)+2}(\bar{\omega}, \mathbb{R}^{\bar{d}})\}_{k=1}^K$ and $\{w_k \in \mathcal{C}^{(K-k)(2N+4)+2}(\bar{\omega}, \mathbb{R}^d)\}_{k=1}^K$, with their defects:

$$\mathcal{D}_k \doteq \mathcal{D}(A_0, v_k, w_k) = A_0 - \big(\frac{1}{2}(\nabla v_k)^T \nabla v_k + \mathrm{sym}\nabla w_k\big),$$

with respect to the progression of frequencies and the defect magnitudes (see Fig. 1) in:

$$\begin{aligned}&\lambda_k \doteq \mu_{k-1}\sigma^d, \qquad \mu_k \doteq \mu_0 \sigma^{d+N+(d+\frac{N}{2})(k-1)}, \\ &\delta_k \doteq \frac{\delta_0}{\sigma^{Nk}} \quad \text{for all } k = 1 \dots K.\end{aligned} \tag{3.2}$$

We will prove the following estimates, valid for all $k = 1 \dots K$:

$$\|v_k - v\|_1 \leq C\delta_0^{1/2}, \qquad \|w_k - w\|_1 \leq C\delta_0^{1/2}\big(1 + \|\nabla v\|_0\big), \tag{3.3}_1$$

$$\|\nabla^{(m+2)} v_k^1\|_0 \leq C\delta_{k-1}^{1/2}\lambda_k^{m+1} \qquad \text{for all } m = 0 \dots (K-k)(2N+4), \tag{3.3}_2$$

$$\|\nabla^{(m+2)} v_k^n\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^{m+1} \qquad \text{for all } n = 2 \dots \bar{d},\ m = 0 \dots (K-k)(2N+4), \tag{3.3}_3$$

$$\|\nabla^{(m+2)} w_k\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^{m+1}\big(1 + \|\nabla v\|_0\big) \qquad \text{for all } m = 0 \dots (K-k)(2N+4), \tag{3.3}_4$$

$$\|\nabla^{(m)}\mathcal{D}_k\|_0 \leq C_k\delta_k\mu_k^m \qquad \text{for all } m = 0 \dots (K-k)(2N+4). \tag{3.3}_5$$

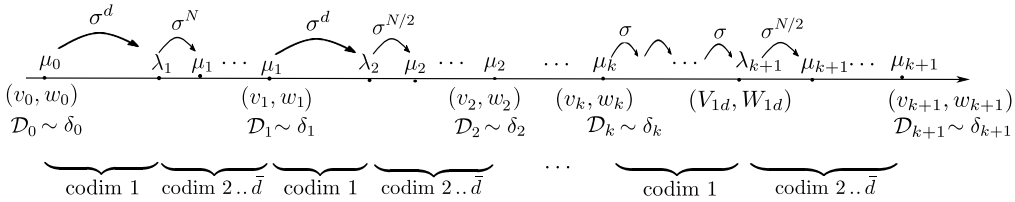


Fig. 1. Progression of frequencies, intermediate fields, defects and codimensions used in the proof of Theorem 1.3.

The fields $\{(v_k, w_k)\}_{k=1}^K$ will be defined by applying Lemma 2.3 with frequencies as in (3.2), and with the corresponding amplitudes $\{a_{ij}^k \in \mathcal{C}^{(K-k)(2N+4)+N+5}(\bar{\omega}, \mathbb{R})\}_{i,j:1\dots d,\, i\le j}$, $\{b_{ij}^k \in \mathcal{C}^{(K-k)(2N+4)+3}(\bar{\omega}, \mathbb{R})\}_{i,j:2\dots d,\, i\le j}$ that obey the bounds below, valid for all $k = 1 \dots K$:

$$\|\nabla^{(m)} a_{ij}^k\|_0 \le C\delta_{k-1}^{1/2}\mu_{k-1}^m \qquad \text{for } m = 0\dots(K-k)(2N+4)+N+5 \text{ and } i \le j, \tag{3.4}_1$$

$$\|\nabla^{(m)} b_{ij}^k\|_0 \le C\delta_{k-1}^{1/2}\lambda_k^m \qquad \text{for } m = 0\dots(K-k)(2N+4)+3 \text{ and } 2 \le i \le j. \tag{3.4}_2$$

The constants $C, C_k \ge 1$ in $(3.3)_1$-$(3.3)_5$ and $(3.4)_1$-$(3.4)_2$ will depend only on ω, N, K. From now on, we fix $k = 0\dots K-1$. If $k \ge 1$, we assume that $(3.3)_1$-$(3.3)_5$ and $(3.4)_1$, $(3.4)_2$ hold for the counter values up to and including k, in addition to the bounds $(3.1)_1$-$(3.1)_4$ at $k = 0$. In the steps 3-12 below, we will validate $(3.3)_1$-$(3.3)_5$ and $(3.4)_1$, $(3.4)_2$ at $k+1$. In the final step 13, we will deduce the statement of the theorem.

3. (The defect decomposition) Define the intermediate frequencies at the present k-counter value of the induction, noting that λ_{1d} equals λ_{k+1} in (3.2):

$$\lambda_{1j} \doteq \mu_k \sigma^j \qquad \text{for all } j = 1\dots d.$$

We apply Lemma 2.5 to the relative frequency σ and the following quantities:

$$H = H_0 + \frac{r_0}{2C_k\delta_k}\mathcal{D}_k, \quad \mu = \mu_k, \quad \sigma,$$

involving H_0 given in (2.2). Consequently, we obtain the positive coefficients $a_{ij} = \bar{a}_{ij}^{1/2}$ in the decomposition of H above, together with the error field $\mathcal{F}_0$. Multiplying these by $(2C_k\delta_k/r_0)^{1/2}$ and $2C_k\delta_k/r_0$, respectively, we obtain the scaled fields:

$$\{a_{ij}^{k+1} \in \mathcal{C}^{(K-k)(2N+4)-N+1}(\bar{\omega}, \mathbb{R})\}_{i,j=1\dots d,\, i\le j}, \quad \mathcal{F} \in \mathcal{C}^{(K-k)(2N+4)-N}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}}),$$

in the following decomposition of $\mathcal{D}_k$:

$$\mathcal{D}_k = \sum_{i\le j}(a_{ij}^{k+1})^2\eta_{ij}\otimes\eta_{ij} + \sum_{j=1}^d \frac{1}{\lambda_{1j}^2}\nabla a_{1j}^{k+1}\otimes\nabla a_{1j}^{k+1} + \mathcal{F} - \frac{2C_k\delta_k}{r_0}H_0, \tag{3.5}$$

that satisfy the scaled version of (2.11):

$$
\begin{aligned}
&\frac{C_k\delta_k}{r_0} \le (a_{ij}^{k+1})^2 \le \frac{3C_k\delta_k}{r_0} \quad \text{and} \quad \frac{1}{2}\Big(\frac{2C_k\delta_k}{r_0}\Big)^{1/2} \le a_{ij}^{k+1} \le \frac{3}{2}\Big(\frac{2C_k\delta_k}{r_0}\Big)^{1/2} \quad \text{for all } i \le j,\\
&\|\nabla^{(m)}(a_{ij}^{k+1})^2\|_0 \le C\delta_k\mu_k^m \quad \text{and} \quad \|\nabla^{(m)}a_{ij}^{k+1}\|_0 \le C\delta_k^{1/2}\mu_k^m\\
&\qquad\qquad \text{for all } m = 1\dots(K-k)(2N+4)-N+1,\ i \le j,\\
&\|\nabla^{(m)}\mathcal{F}\|_0 \le C\delta_k\frac{\mu_k^m}{\sigma^{2N}} \qquad \text{for all } m = 0\dots(K-k)(2N+4)-N.
\end{aligned}
\tag{3.6}
$$

The constants C, C_k depend only on ω, K, N. In particular, there follows $(3.4)_1$ at $k+1$.

4. (Adding corrugations in the first codimension) We recursively define the fields: $\{V_{1j} \in \mathcal{C}^{(K-k)(2N+4)-N+1}(\bar\omega, \mathbb{R}^{\bar d})\}_{j=1}^d$, $\{W_{1j} \in \mathcal{C}^{(K-k)(2N+4)-2N-1}(\bar\omega, \mathbb{R}^d)\}_{j=1}^d$, by setting:

$$
(V_{1,0}, W_{1,0}) = (v_k, w_k)
$$

and further, utilizing construction (2.3) in Lemma 2.3 for each $j = 1\dots d$, with $a = a_{1j}$ (we suppress the superscript $k+1$ to ease the notation), $\lambda = \lambda_{1j}$, $\eta = \eta_{1j}$ and $n = 1$:

$$
\begin{aligned}
V_{1j} &\doteq V_{1,j-1} + \frac{a_{1j}}{\lambda_{1j}}\Gamma(\lambda_{1j}t_{\eta_{1j}})e_1,\\
W_{1j} &\doteq W_{1,j-1} - \frac{a_{1j}}{\lambda_{1j}}\Gamma(\lambda_{1j}t_{\eta_{1j}})\nabla V_{1,j-1}^1 + \frac{(a_{1j})^2}{\lambda_{1j}}\bar\Gamma(\lambda_{1j}t_{\eta_{1j}})\eta_{1j}\\
&\quad + \frac{a_{1j}}{(\lambda_{1j})^2}\bar{\bar\Gamma}(\lambda_{1j}t_{\eta_{1j}})\nabla a_{1j} - \bar\Theta_{1j}.
\end{aligned}
\tag{3.7}
$$

Note the additional perturbation term in W_{1j}, that is given by:

$$
\begin{aligned}
\bar\Theta_{11} &\doteq \Theta_{11} + \frac{2C_k\delta_k}{r_0}H_0x, \qquad \bar\Theta_{1j} \doteq \Theta_{1j} \quad \text{for } j = 2\dots d,\\
\Theta_{1j} &= -\sum_{i=0}^N(-1)^i\frac{\Gamma_{i+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+2}}L_i^j(a_{1j}\nabla^{(2)}V_{1,j-1}^1)\\
&\quad + \sum_{i=0}^N(-1)^i\frac{\bar{\bar\Gamma}_{i+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+3}}L_i^j(a_{1j}\nabla^{(2)}a_{1j})\\
&\quad + \sum_{i=0}^N(-1)^i\frac{(\bar{\bar{\bar\Gamma}}-1)_{i+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+3}}L_i^j(\nabla a_{1j}\otimes\nabla a_{1j}),
\end{aligned}
\tag{3.8}
$$

where $\{L_i^j\}_{i=0}^N$ are the operators defined in Lemma 2.4. By (2.5) and (2.4), we get:

$$
\begin{aligned}
&\mathcal{D}(A_0, V_{1j}, W_{1j}) - \mathcal{D}(A_0, V_{1,j-1}, W_{1,j-1})\\
&= -\Big(\big(\frac{1}{2}(\nabla V_{1j})^T\nabla V_{1j} + \mathrm{sym}\nabla W_{1j}\big) - \big(\frac{1}{2}(\nabla V_{1,j-1})^T\nabla V_{1,j-1} + \mathrm{sym}\nabla W_{1,j-1}\big)\Big)
\end{aligned}
$$

$$= -(a_{1j})^2\eta_{1j}\otimes\eta_{1j} + \frac{\Gamma(\lambda_{1j}t_{\eta_{1j}})}{\lambda_{1j}}a_{1j}\nabla^{(2)}V^1_{1,j-1} - \frac{\bar{\bar{\Gamma}}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^2}a_{1j}\nabla^{(2)}a_{1j}$$
$$- \frac{(\bar{\bar{\Gamma}}-1)(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^2}\nabla a_{1j}\otimes\nabla a_{1j} - \frac{1}{(\lambda_{1j})^2}\nabla a_{1j}\otimes\nabla a_{1j} + \mathrm{sym}\nabla\bar{\Theta}_{1j}.$$

The above formula can be rewritten as:

$$\mathcal{D}(A_0, V_{1j}, W_{1j}) - \mathcal{D}(A_0, V_{1,j-1}, W_{1,j-1})$$
$$= \Big(-(a_{1j})^2\eta_{1j}\otimes\eta_{1j} - \frac{1}{(\lambda_{1j})^2}\nabla a_{1j}\otimes\nabla a_{1j}\Big) - \mathcal{F}_{1j} - G_{1j} + \begin{cases} \frac{2C_k\delta_k}{r_0}H_0 & \text{if } j=1 \\ 0 & \text{if } j\geq 2, \end{cases} \tag{3.9}$$

where the error fields $\{\mathcal{F}_{1j}\}_{j=1}^d$ collect the following terms:

$$\begin{aligned}\mathcal{F}_{1j} \doteq\ & (-1)^{N+1}\frac{\Gamma_{N+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{N+2}}\mathrm{sym}\nabla L^j_N(-a_{1j}\nabla^{(2)}V^1_{1,j-1}) \\ &+ (-1)^{N+1}\frac{\bar{\bar{\Gamma}}_{N+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{N+3}}\mathrm{sym}\nabla L^j_N(a_{1j}\nabla^{(2)}a_{1j}) \\ &+ (-1)^{N+1}\frac{(\bar{\bar{\Gamma}}-1)_{N+1}(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{N+3}}\mathrm{sym}\nabla L^j_N(\nabla a_{1j}\otimes\nabla a_{1j}),\end{aligned} \tag{3.10}$$

while $\{G_{1j}\}_{j=1}^d$ are given by means of the operators $\{P_i^j\}_{i=0}^N$ in Lemma 2.4:

$$\begin{aligned}G_{1j} =\ & \sum_{i=0}^N(-1)^i\frac{\Gamma_i(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+1}}P_i^j(-a_{1j}\nabla^{(2)}V^1_{1,j-1}) \\ &+ \sum_{i=0}^N(-1)^i\frac{\bar{\bar{\Gamma}}_i(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+2}}P_i^j(a_{1j}\nabla^{(2)}a_{1j}) \\ &+ \sum_{i=0}^N(-1)^i\frac{(\bar{\bar{\Gamma}}-1)_i(\lambda_{1j}t_{\eta_{1j}})}{(\lambda_{1j})^{i+2}}P_i^j(\nabla a_{1j}\otimes\nabla a_{1j}).\end{aligned} \tag{3.11}$$

Recall that there holds:

$$G_{1j}(x)\in\mathrm{span}\{\eta_{st}\otimes\eta_{st};\, 2\leq s\leq t\} \quad \text{for all } j=1\dots d, \quad x\in\bar{\omega}. \tag{3.12}$$

Inserting (3.9) into (3.5), we observe that:

$$\mathcal{D}(A_0, V_{1d}, W_{1d}) = \mathcal{D}_k + \sum_{j=1}^d\Big(\mathcal{D}(A_0, V_{1j}, W_{1j}) - \mathcal{D}(A_0, V_{1,j-1}, W_{1,j-1})\Big)$$

$$= \sum_{2\le i\le j} (a_{ij}^{k+1})^2 \eta_{ij}\otimes\eta_{ij} + \Big(\mathcal{F} - \sum_{j=1}^{d}\mathcal{F}_{1j}\Big) - \sum_{j=1}^{d} G_{1j}. \tag{3.13}$$

5. (Bounds on V_{1j}) Recalling the definition (3.7), the newly constructed fields V_{1j} satisfy:

$$V_{1j}^1 = v_k + \sum_{i=1}^{j}(V_{1i}^1 - V_{1,i-1}^1).$$

Further, by (3.6) there holds, for all $m = 0\dots(K-k)(2N+4) - N + 1$:

$$\begin{aligned}\|\nabla^{(m)}(V_{1j}^1 - V_{1,j-1}^1)\|_0 &\le C\sum_{p+q=m}(\lambda_{1j})^{p-1}\|\nabla^{(q)}a_{1j}\|_0\\ &\le C\sum_{p+q=m}(\lambda_{1j})^{p-1}\delta_k^{1/2}\mu_k^q \le C\delta_k^{1/2}(\lambda_{1j})^{m-1}.\end{aligned}$$

Applying $(3.1)_1$ when $k=0$ and $(3.3)_1$ if $k\ge 1$, it consequently transpires that:

$$\|V_{1j}^1 - v^1\|_1 \le \|v_k - v\|_1 + \sum_{i=1}^{j}\|V_{1i}^1 - V_{1,i-1}^1\|_1 \le C\delta_0^{1/2} + C\delta_k^{1/2} \le C\delta_0^{1/2}, \tag{3.14}$$

while for all $m = 0\dots(K-k)(2N+4) - N + 1$ we get:

$$\begin{aligned}\|\nabla^{(m+2)}V_{1j}^1\|_0 &\le \|\nabla^{(m+2)}v_k^1\|_0 + \sum_{i=1}^{j}\|\nabla^{(m+2)}(V_{1i}^1 - V_{1,i-1}^1)\|_0\\ &\le \|\nabla^{(m+2)}v_k^1\|_0 + C\delta_k^{1/2}(\lambda_{1j})^{m+1} \le C\delta_k^{1/2}(\lambda_{1j})^{m+1}.\end{aligned} \tag{3.15}$$

The final inequality above follows by noting that $\{\delta_i^{1/2}\mu_i\}_{i=0}^K$ is an increasing sequence:

$$\frac{\delta_{i+1}^{1/2}\mu_{i+1}}{\delta_i^{1/2}\mu_i} = \frac{1}{\sigma^{N/2}}\frac{\mu_{i+1}}{\mu_i} \ge \frac{\sigma^{d+\frac{N}{2}}}{\sigma^{N/2}} \ge 1 \quad \text{for } i = 0\dots K-1. \tag{3.16}$$

In conclusion, we get (3.15) at $k=0$ by $(3.1)_3$, because:

$$\|\nabla^{(m+2)}v_0\|_0 \le C\delta_0^{1/2}\mu_0^{m+1} \le C(\delta_0^{1/2}\mu_0)(\lambda_{1j})^m \le C(\delta_k^{1/2}\mu_k)(\lambda_{1j})^m \le C\delta_0^{1/2}(\lambda_{1j})^{m+1},$$

while at $k\ge 1$, we use the inductive assumption $(3.3)_2$ in:

$$\begin{aligned}\|\nabla^{(m+2)}v_k^1\|_0 &\le C\delta_{k-1}^{1/2}(\lambda_k)^{m+1} \le C(\delta_{k-1}^{1/2}\lambda_k)(\lambda_{1j})^m = C(\delta_k^{1/2}\sigma^{N/2}\lambda_k)(\lambda_{1j})^m\\ &\le C(\delta_k^{1/2}\mu_k)(\lambda_{1j})^m \le C\delta_0^{1/2}(\lambda_{1j})^{m+1}.\end{aligned}$$

In particular, since v_{k+1} will be defined to differ from V_{1d} only on its $n = 2\dots\bar{d}$ coordinates, the bounds (3.14), (3.15) readily imply a partial bound in $(3.3)_1$ and $(3.3)_2$ at $k+1$, since:

$$\begin{aligned}&\|v^1_{k+1} - v^1\|_1 \le C\delta_0^{1/2},\\ &\|\nabla^{(m+2)} v^1_{k+1}\|_0 \le C\delta_k^{1/2}\lambda_{k+1}^{m+1} \quad \text{for all } m = 0\dots(K-k)(2N+4) - N + 1.\end{aligned} \tag{3.17}$$

6. (Bounds on W_{1j}) Similarly as for V_{1j}, we write:

$$W_{1j} = w_k + \sum_{i=1}^{j}(W_{1i} - W_{1,i-1}),$$

and, recalling the definition (3.7), prepare the estimate:

$$\begin{aligned}\|\nabla^{(m)}(W_{1j} - W_{1,j-1})\|_0 \le\ & C \sum_{p+q+r=m} (\lambda_{1j})^{p-1}\|\nabla^{(q)} a_{1j}\|_0 \|\nabla^{(r+1)} V^1_{1,j-1}\|_0\\ &+ C\sum_{p+q=m}(\lambda_{1j})^{p-1}\|\nabla^{(q)}(a_{1j})^2\|_0\\ &+ C\sum_{p+q+r=m}(\lambda_{1j})^{p-2}\|\nabla^{(q)} a_{1j}\|_0\|\nabla^{(r+1)} a_{1j}\|_0 + \|\nabla^{(m)}\bar{\Theta}_{1j}\|_0\end{aligned} \tag{3.18}$$

To carry out estimating the four terms in the right hand side above, we first use (3.15), $(3.3)_2$, $(3.1)_3$ to check that for $m = 0\dots(K-k)(2N+4) - N - 1$:

$$\|\nabla^{(m+2)} V^1_{1,j-1}\|_0 \le C\left\{\begin{array}{ll}\delta_k^{1/2}(\lambda_{1,j-1})^{m+1} & \text{if } j \ge 2,\ k \ge 0\\ \delta_{k-1}^{1/2}(\lambda_k)^{m+1} & \text{if } j = 1,\ k \ge 1\\ \delta_0^{1/2}\mu_0^{m+1} & \text{if } j \ge 1,\ k = 0\end{array}\right\} \le C\delta_k^{1/2}(\lambda_{1,j-1})^{m+1}, \tag{3.19}$$

where the final inequality above is due, as before, to (3.16), and with the convention that $\lambda_{1,0} \doteq \mu_k$. Consequently, in virtue of (3.6) and (3.14), we get that for all $m = 0\dots(K-k)(2N+4) - N$:

$$\begin{aligned}&\sum_{p+q+r=m}(\lambda_{1j})^{p-1}\|\nabla^{(q)} a_{1j}\|_0\|\nabla^{(r+1)} V^1_{1,j-1}\|_0\\ &\quad\le C\sum_{p+q=m}(\lambda_{1j})^{p-1}\delta_k^{1/2}\mu_k^q\big(\delta_0^{1/2} + \|\nabla v\|_0\big) + C\sum_{p+q+r=m,\ r\ge 1}(\lambda_{1j})^{p-1}\delta_k^{1/2}\mu_k^q\delta_k(\lambda_{1,j-1})^r\\ &\quad\le C\delta_k^{1/2}(\lambda_{1j})^{m-1}\big(1 + \|\nabla v\|_0\big),\\ &\sum_{p+q=m}(\lambda_{1j})^{p-1}\|\nabla^{(q)}(a_{1j})^2\|_0 + \sum_{p+q+r=m}(\lambda_{1j})^{p-2}\|\nabla^{(q)} a_{1j}\|_0\|\nabla^{(r+1)} a_{1j}\|_0\\ &\quad\le C\sum_{p+q=m}(\lambda_{1j})^{p-1}\delta_k\mu_k^q + C\sum_{p+q+r=m}(\lambda_{1j})^{p-2}\delta_k\mu_k^{q+r+1} \le C\delta_k(\lambda_{1j})^{m-1}.\end{aligned}$$

Finally, recalling the definition (3.8) and Lemma 2.4, we note that:

$$\|\nabla^{(m)}\Theta_{1j}\|_0 \le C\sum_{i=0}^{N}\Big(\sum_{p+q=m}(\lambda_{1j})^{p-(i+2)}\|\nabla^{(q+i)}(a_{1j}\nabla^{(2)}V^1_{1,j-1})\|_0 + \sum_{p+q=m}(\lambda_{1j})^{p-(i+3)}\|\nabla^{(q+i)}(a_{1j}\nabla^{(2)}a_{1,j})\|_0 + \sum_{p+q=m}(\lambda_{1j})^{p-(i+3)}\|\nabla^{(q+i)}(\nabla a_{1j}\otimes\nabla a_{1j})\|_0\Big),$$

which yields for all $m = 0\dots(K-k)(2N+4)-2N-1$:

$$\begin{aligned}\|\nabla^{(m)}\Theta_{1j}\|_0 &\le C\sum_{i=0}^{N}\sum_{p+q=m}\Big(\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}V^1_{1,j-1}\|_0 + \sum_{r+t=q+i}(\lambda_{1j})^{p-(i+3)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}a_{1j}\|_0 + \sum_{r+t=q+i}(\lambda_{1j})^{p-(i+3)}\|\nabla^{(r+1)}a_{1j}\|_0\|\nabla^{(t+1)}a_{1j}\|_0\Big)\\ &\le C\sum_{i=0}^{N}\sum_{p+q=m}\Big(\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\delta_k^{1/2}\mu_k^r\delta_k^{1/2}(\lambda_{1,j-1})^{t+1} + \sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\delta_k\mu_k^{r+t+2}\Big)\\ &\le C\sum_{i=0}^{N}\sum_{p+q=m}\Big((\lambda_{1j})^{p-(i+2)}\delta_k(\lambda_{1,j-1})^{q+i+1} + (\lambda_{1j})^{p-(i+2)}\delta_k\mu_k^{q+i+2}\Big)\\ &\le C\delta_k(\lambda_{1j})^{m-1}\sum_{i=0}^{N}\Big(\big(\frac{\lambda_{1,j-1}}{\lambda_{1j}}\big)^{i+1} + \big(\frac{\mu_k}{\lambda_{1j}}\big)^{i+2}\Big) \le C\delta_k(\lambda_{1j})^{m-1}.\end{aligned}$$

In conclusion, the bound (3.18) becomes:

$$\|\nabla^{(m)}(W_{1j}-W_{1,j-1})\|_0 \le C\delta_k^{1/2}(\lambda_{1j})^{m-1}\big(1+\|\nabla v\|_0\big) + C\delta_k\|\nabla^{(m)}(H_0x)\|_0 \quad \text{for all } m = 0\dots(K-k)(2N+4)-2N-1.$$

Therefore, by means of $(3.1)_1$, $(3.1)_3$, $(3.3)_1$, $(3.3)_4$:

$$\|W_{1d}-w\|_1 \le \|w_k-w\|_1 + \sum_{j=1}^{d}\|W_{1j}-W_{1,j-1}\|_1 \le C\delta_0^{1/2}\big(1+\|\nabla v\|_0\big),$$

$$\begin{aligned}\|\nabla^{(m+2)}W_{1d}\|_0 &\le C\|\nabla^{(m+2)}w_k\|_0 + \sum_{j=1}^{d}\|\nabla^{(m+2)}(W_{1j}-W_{1,j-1})\|_0\\ &\le C\left\{\begin{array}{ll}\delta_{k-1}^{1/2}\mu_k^{m+1}\big(1+\|\nabla v\|_0\big) & \text{if } k\ge 1\\ \delta_0^{1/2}\mu_0^{m+1} & \text{if } k=0\end{array}\right\} + C\delta_k^{1/2}\lambda_k^{m+1}\big(1+\|\nabla v\|_0\big)\end{aligned}$$

$$\leq C\delta_k^{1/2}\mu_{k+1}^{m+1}\big(1+\|\nabla v\|_0\big) \quad \text{for all } m=0\dots(K-k)(2N+4)-2N-3, \tag{3.20}$$

where, similarly to (3.16), we have noted that the sequence $\{\delta_i^{1/2}\mu_{i+1}\}_{i=0}^{K-1}$ is increasing:

$$\frac{\delta_{i+1}^{1/2}\mu_{i+2}}{\delta_i^{1/2}\mu_{i+1}} = \frac{1}{\sigma^{N/2}}\frac{\mu_{i+2}}{\mu_{i+1}} = \frac{\sigma^{d+\frac{N}{2}}}{\sigma^{N/2}} \geq 1 \quad \text{for } i=0\dots K-2. \tag{3.21}$$

7. (Bounds on $\mathcal{F}_{1j}$) Recalling the definition (3.10) and Lemma 2.4(i), we proceed to bound:

$$\begin{aligned}\|\nabla^{(m)}\mathcal{F}_{1j}\|_0 \leq C\sum_{p+q=m}\Big(&\sum_{r+t=q+N+1}(\lambda_{1j})^{p-(N+2)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}V^1_{1,j-1}\|_0\\ &+\sum_{r+t=q+N+1}(\lambda_{1j})^{p-(N+3)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}a_{1j}\|_0\\ &+\sum_{r+t=q+N+1}(\lambda_{1j})^{p-(N+3)}\|\nabla^{(r+1)}a_{1j}\|_0\|\nabla^{(t+1)}a_{1j}\|_0\Big).\end{aligned}$$

From (3.6) and (3.19), it hence follows that for all $m=0\dots(K-k)(2N+4)-2N-2$:

$$\begin{aligned}\|\nabla^{(m)}\mathcal{F}_{1j}\|_0 &\leq C\sum_{p+q=m}\Big(\sum_{r+t=q+N+1}(\lambda_{1j})^{p-(N+2)}\delta_k^{1/2}\mu_k^r\delta_k^{1/2}(\lambda_{1,j-1})^{t+1}\\ &\qquad+\sum_{r+t=q+N+1}(\lambda_{1j})^{p-(N+3)}\delta_k\mu_k^{r+t+2}\Big)\\ &\leq C\delta_k\sum_{p+q=m}\Big((\lambda_{1j})^{p-(N+2)}(\lambda_{1,j-1})^{q+N+2}+(\lambda_{1j})^{p-(N+3)}(\lambda_{1,j-1})^{q+N+3}\Big)\\ &= C\delta_k(\lambda_{1j})^m\Big(\big(\frac{\lambda_{1,j-1}}{\lambda_{1j}}\big)^{N+2}+\big(\frac{\lambda_{1,j-1}}{\lambda_{1j}}\big)^{N+3}\Big)\leq C\frac{\delta_k}{\sigma^{N+2}}\lambda_{k+1}^m.\end{aligned} \tag{3.22}$$

8. (Bounds on G_{1j} and the updated decomposition coefficients b_{ij}) As in the previous step, we use Lemma 2.4(i) to bound the terms in (3.11):

$$\begin{aligned}\|\nabla^{(m)}G_{1j}\|_0 \leq C\sum_{i=0}^{N}\sum_{p+q=m}\Big(&\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+1)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}V^1_{1,j-1}\|_0\\ &+\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\|\nabla^{(r)}a_{1j}\|_0\|\nabla^{(t+2)}a_{1j}\|_0\\ &+\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\|\nabla^{(r+1)}a_{1j}\|_0\|\nabla^{(t+1)}a_{1j}\|_0\Big).\end{aligned}$$

From (3.6) and (3.19) we get that, for all $m=0\dots(K-k)(2N+4)-2N-1$:

$$\begin{aligned}\|\nabla^{(m)} G_{1j}\|_0 &\le C\sum_{i=0}^{N}\sum_{p+q=m}\Big(\sum_{r+t=q+i}(\lambda_{1j})^{p-(i+1)}\delta_k^{1/2}\mu_k^r\delta_k^{1/2}(\lambda_{1,j-1})^{t+1} \\ &\qquad + \sum_{r+t=q+i}(\lambda_{1j})^{p-(i+2)}\delta_k\mu_k^{r+t+2}\Big) \\ &\le C\delta_k\sum_{i=0}^{N}\sum_{p+q=m}\Big((\lambda_{1j})^{p-(i+1)}(\lambda_{1,j-1})^{q+i+1} + (\lambda_{1j})^{p-(i+2)}(\lambda_{1,j-1})^{q+i+2}\Big) \\ &= C\delta_k(\lambda_{1j})^m\sum_{i=0}^{N}\Big(\big(\frac{\lambda_{1,j-1}}{\lambda_{1j}}\big)^{i+1} + \big(\frac{\lambda_{1,j-1}}{\lambda_{1j}}\big)^{i+2}\Big) \le C\frac{\delta_k}{\sigma}\lambda_{k+1}^m.\end{aligned}$$

Recall the property (3.12). Denoting $c_{ij} = \bar{a}_{ij}(\sum_{j=1}^{d} G_{1j})$, Lemma 2.2 implies that:

$$\sum_{j=1}^{d} G_{1j} = \sum_{i,j=2\ldots d,\, i\le j} c_{ij}\eta_{ij}\otimes\eta_{ij}, \tag{3.23}$$

where the coefficients $\{c_{ij}\}_{2\le i\le j}$ obey the same bounds as those derived for $\sum_{j=1}^{d} G_{1j}$, namely:

$$\|\nabla^{(m)} c_{ij}\|_0 \le C\frac{\delta_k}{\sigma}\lambda_{k+1}^m \quad \text{for all } m = 0\ldots(K-k)(2N+4)-2N-1. \tag{3.24}$$

In view of (3.6), we note that if $\sigma \ge \sigma_0$ has been taken large enough, then:

$$\frac{C_k\delta_k}{2r_0} \le (a_{ij})^2 - c_{ij} \le \frac{4C_k\delta_k}{r_0},$$

and therefore the following fields are well defined:

$$\{b_{ij}^{k+1} \in C^{(K-k)(2N+4)-2N-1}(\bar{\omega},\mathbb{R})\}_{i,j=2\ldots d,\, i\le j}, \qquad b_{ij}^{k+1} \doteq \big((a_{ij}^{k+1})^2 - c_{ij}\big)^{1/2}.$$

By (3.24) and (3.6), we note that:

$$\begin{gathered}\|\nabla^{(m)}(b_{ij}^{k+1})^2\|_0 \le C\delta_k\mu_k^m + C\frac{\delta_k}{\sigma}\lambda_{k+1}^m \le C\delta_k\lambda_{k+1}^m \\ \text{for all } m = 0\ldots(K-k)(2N+4)-2N-1.\end{gathered}$$

Applying Faádi Bruno's formula, we also arrive at, for $m \neq 0$:

$$\begin{aligned}\|\nabla^{(m)} b_{ij}^{k+1}\|_0 &\le C\Big\|\sum_{p_1+2p_2+\ldots mp_m=m}(b_{ij}^{k+1})^{2(1/2-p_1-\ldots-p_m)}\prod_{t=1}^{m}\big|\nabla^{(t)}(b_{ij}^{k+1})^2\big|^{p_t}\Big\|_0 \\ &\le C\|b_{ij}^{k+1}\|_0\sum_{p_1+2p_2+\ldots mp_m=m}\prod_{t=1}^{m}\Big(\frac{\delta_k\lambda_{k+1}^t}{C_k\delta_k/(2r_0)}\Big)^{p_t} \le C\delta_k^{1/2}\lambda_{k+1}^m.\end{aligned}$$

This already yields the bound $(3.4)_2$ at $k+1$:

$$\|\nabla^{(m)} b_{ij}^{k+1}\|_0 \leq C\delta_k^{1/2}\lambda_{k+1}^m \quad \text{for all } m = 0\dots(K-k)(2N+4)-2N-1. \tag{3.25}$$

Finally, inserting (3.23) in (3.13), we take note of the current defect decomposition:

$$\begin{aligned}\mathcal{D}(A_0, V_{1d}, W_{1d}) &= \sum_{2\leq i\leq j}(a_{ij}^{k+1})^2\eta_{ij}\otimes\eta_{ij} - \sum_{j=1}^{d} G_{1j} + \Big(\mathcal{F} - \sum_{j=1}^{d}\mathcal{F}_{1j}\Big)\\ &= \sum_{2\leq i\leq j}(b_{ij}^{k+1})^2\eta_{ij}\otimes\eta_{ij} + \Big(\mathcal{F} - \sum_{j=1}^{d}\mathcal{F}_{1j}\Big).\end{aligned} \tag{3.26}$$

9. (Adding corrugations in the remaining codimensions $n = 2\dots\bar{d}$) We now recursively define the remaining vector fields $\{V_n \in \mathcal{C}^{(K-k)(2N+4)-2N-2}(\bar{\omega}, \mathbb{R}^{\bar{d}})\}_{n=2}^{\bar{d}}$, $\{W_n \in \mathcal{C}^{(K-k)(2N+4)-2N-2}(\bar{\omega}, \mathbb{R}^{d})\}_{n=2}^{\bar{d}}$, by first setting:

$$(V_1, W_1) \doteq (V_{1d}, W_{1d})$$

and then using (2.3) in Lemma 2.3 for each $n = 2\dots\bar{d}$, with $a = b_{ij}$ (we suppress the superscript $k+1$), $\lambda = \mu_{k+1}$, and $\eta = \eta_{ij}$. We also define the one-to-one correspondence between the pairs of indices ij and the single index n as above, through the formula:

$$n = \frac{(i-2)(2d-1-i)}{2} + j, \qquad n = 2\dots\bar{d}, \quad 2\leq i\leq j, \tag{3.27}$$

so that when $n = 2$ then $ij = (2,2)$, and when $n = \bar{d}$ then $ij = (d,d)$. We declare:

$$\begin{aligned} V_n &\doteq V_{n-1} + \frac{b_{ij}}{\mu_{k+1}}\Gamma(\mu_{k+1}t_{\eta_{ij}})e_n,\\ W_n &\doteq W_{n-1} - \frac{b_{ij}}{\mu_{k+1}}\Gamma(\mu_{k+1}t_{\eta_{ij}})\nabla V_{n-1}^n + \frac{(b_{ij})^2}{\mu_{k+1}}\bar{\Gamma}(\mu_{k+1}t_{\eta_{ij}})\eta_{ij}\\ &\quad + \frac{b_{ij}}{\mu_{k+1}^2}\bar{\bar{\Gamma}}(\mu_{k+1}t_{\eta_{ij}})\nabla b_{ij} + \Theta_n,\end{aligned} \tag{3.28}$$

where the additional perturbation term Θ_n in W_n is:

$$\begin{aligned}\Theta_n \doteq\ & 2\frac{\Gamma_1(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^2}b_{ij}\sum_{s=1}^{k}\Gamma'(\mu_s t_{\eta_{ij}})\nabla b_{ij}^s + \frac{\Gamma_1(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^2}b_{ij}\sum_{s=1}^{k}\mu_s\Gamma''(\mu_s t_{\eta_{ij}})b_{ij}^s\eta_{ij}\\ &- \frac{\Gamma_2(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^3}\nabla\Big(b_{ij}\sum_{s=1}^{k}\mu_s\Gamma''(\mu_s t_{\eta_{ij}})b_{ij}^s\Big).\end{aligned} \tag{3.29}$$

We set the principal fields at the consecutive counter $k+1$ as

$$v_{k+1} \doteq V_{\bar{d}}, \quad w_{k+1} \doteq W_{\bar{d}}, \tag{3.30}$$

and use (2.4) in Lemma 2.3 to get:

$$\begin{aligned}
&\mathcal{D}(A_0, v_{k+1}, w_{k+1}) - \mathcal{D}(A_0, V_{1,d}, W_{1,d}) \\
&= -\sum_{n=2}^{\bar{d}} \Big(\big(\frac{1}{2}(\nabla V_n)^T \nabla V_n + \mathrm{sym}\nabla W_n\big) - \big(\frac{1}{2}(\nabla V_{n-1})^T \nabla V_{n-1} + \mathrm{sym}\nabla W_{n-1}\big)\Big) \\
&= -\sum_{n=2}^{\bar{d}} \Big((b_{ij})^2 \eta_{ij} \otimes \eta_{ij} - \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \nabla^{(2)} V_{n-1}^n + \frac{\bar{\bar{\Gamma}}(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}^2} b_{ij} \nabla^{(2)} b_{ij} \\
&\qquad + \frac{\bar{\bar{\bar{\Gamma}}}(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}^2} \nabla b_{ij} \otimes \nabla b_{ij} + \mathrm{sym}\nabla \Theta_n \Big).
\end{aligned}$$

Inserting the above in (3.26), we arrive at the final defect decomposition bound:

$$\mathcal{D}(A_0, v_{k+1}, w_{k+1}) = \Big(\mathcal{F} - \sum_{j=1}^{d} \mathcal{F}_{1j} \Big) + \sum_{n=2}^{\bar{d}} \mathcal{H}_n - \mathcal{G}, \tag{3.31}$$

where the error fields $\{\mathcal{H}_n\}_{n=2}^{\bar{d}}$ and $\mathcal{G}$ are given in:

$$\begin{aligned}
\mathcal{H}_n &\doteq \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \nabla^{(2)} V_{n-1}^n - \mathrm{sym}\nabla \Theta_n, \\
\mathcal{G} &\doteq \sum_{n=2}^{\bar{d}} \Big(\frac{\bar{\bar{\Gamma}}(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}^2} b_{ij} \nabla^{(2)} b_{ij} + \frac{\bar{\bar{\bar{\Gamma}}}(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}^2} \nabla b_{ij} \otimes \nabla b_{ij} \Big).
\end{aligned} \tag{3.32}$$

10. (Bounds on V_n) Note first that by (3.30) there holds:

$$v_{k+1} = V_{1d} + \sum_{n=2}^{\bar{d}} (V_n - V_{n-1}), \qquad v_{k+1}^n = V_n^n = v_k^n + \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \quad \text{for } n = 2 \dots \bar{d}.$$

Now, in virtue of (3.6), we estimate for all $m = 0 \dots (K-k)(2N+4) - 2N - 2$:

$$\|\nabla^{(m)}(V_n - V_{n-1})\|_0 \leq C \sum_{p+q=m} \mu_{k+1}^{p-1} \|\nabla^{(q)} b_{ij}\|_0 \leq C \sum_{p+q=m} \mu_{k+1}^{p-1} \delta_k^{1/2} \lambda_{k+1}^q \leq C \delta_k^{1/2} \mu_{k+1}^{m-1}.$$

Applying (3.14), (3.15), and further $(3.1)_1$, $(3.3)_1$, followed by $(3.1)_3$, $(3.3)_3$, we observe:

$$\begin{aligned}
&\|v_{k+1}^n - v^n\|_1 \leq \|v_k - v\|_1 + C\delta_k^{1/2} \leq C\delta_0^{1/2}, \\
&\|\nabla^{(m+2)} v_{k+1}^n\|_0 \leq \|\nabla^{(m+2)} v_k^n\|_0 + C\delta_k^{1/2} \mu_{k+1}^{m+1}
\end{aligned}$$

$$\leq C \left\{ \begin{matrix} \delta_{k-1}^{1/2} \mu_k^{m+1} & \text{if } k \geq 1 \\ \delta_0^{1/2} \mu_0^{m+1} & \text{if } k = 0 \end{matrix} \right\} + C \delta_k^{1/2} \mu_{k+1}^{m+1} \leq C \delta_k^{1/2} \mu_{k+1}^{m+1}$$

$$\text{for all } m = 0 \dots (K-k)(2N+4) - 2N - 4, \quad n = 2 \dots \bar{d}.$$

In the last inequality above we argued as in the proof of (3.20), utilizing (3.21). By (3.17) we hence conclude the two inductive bounds $(3.3)_1$, $(3.3)_3$ at the counter $k+1$.

11. (Bounds on W_n) Similarly, as for V_n, we write by (3.30):

$$w_{k+1} = W_{1d} + \sum_{n=2}^{\bar{d}} (W_n - W_{n-1}).$$

Recalling the definition (3.28) and noting that $V_{n-1}^n = v_k^n$, we write:

$$\begin{aligned} &\|\nabla^{(m)}(W_n - W_{n-1})\|_0 \leq C \sum_{p+q+r=m} \mu_{k+1}^{p-1} \|\nabla^{(q)} b_{ij}\|_0 \|\nabla^{(r+1)} v_k^n\|_0 \\ &+ C \sum_{p+q=m} \mu_{k+1}^{p-1} \|\nabla^{(q)} (b_{ij})^2\|_0 + C \sum_{p+q+r=m} \mu_{k+1}^{p-2} \|\nabla^{(q)} b_{ij}\|_0 \|\nabla^{(r+1)} b_{ij}\|_0 + \|\nabla^{(m)} \Theta_n\|_0. \end{aligned} \tag{3.33}$$

To estimate the first three terms in the right hand side, we use (3.25), $(3.1)_1$, $(3.1)_3$, together with the induction assumptions $(3.3)_1$, $(3.3)_3$. Namely, for all $m = 0 \dots (K-k)(2N+4) - 2N - 2$:

$$\begin{aligned} &\sum_{p+q+r=m} \mu_{k+1}^{p-1} \|\nabla^{(q)} b_{ij}\|_0 \|\nabla^{(r+1)} v_k^n\|_0 \\ &\leq C \sum_{p+q=m} \mu_{k+1}^{p-1} \delta_k^{1/2} \lambda_{k+1}^q \big(\delta_0^{1/2} + \|\nabla v\|_0\big) + C \sum_{p+q+r=m,\, r \geq 1} \mu_{k+1}^{p-1} \delta_k^{1/2} \lambda_{k+1}^q \delta_{k-1}^{1/2} \mu_k^r \\ &\leq C \delta_k^{1/2} \mu_{k+1}^{m-1} \big(\delta_0^{1/2} + \|\nabla v\|_0\big) + C \delta_k \mu_{k+1}^{m-1} \\ &\leq C \delta_k^{1/2} \mu_{k+1}^{m-1} \big(1 + \|\nabla v\|_0\big), \end{aligned}$$

where we also applied the fact that $\|\nabla^{(m+2)} v_k^n\|_0 \leq C \delta_{k-1}^{1/2} \mu_k^{m+1} \leq C \delta_k^{1/2} \mu_{k+1}^{m+1}$ for all $k \geq 1$ by (3.21), whereas at $k = 0$ the same bound holds trivially in view of $(3.1)_3$. Similarly:

$$\begin{aligned} &\sum_{p+q=m} \mu_{k+1}^{p-1} \|\nabla^{(q)} (b_{ij})^2\|_0 + \sum_{p+q+r=m} \mu_{k+1}^{p-2} \|\nabla^{(q)} b_{ij}\|_0 \|\nabla^{(r+1)} b_{ij}\|_0 \\ &\leq C \sum_{p+q=m} \mu_{k+1}^{p-1} \delta_k \lambda_{k+1}^q + C \sum_{p+q+r=m} \mu_{k+1}^{p-2} \delta_k \lambda_{k+1}^{q+r+1} \leq C \delta_k \mu_{k+1}^{m-1}. \end{aligned}$$

Also, by definition (3.29) and the inductive assumption $(3.4)_2$:

$$\|\nabla^{(m)}\Theta_n\|_0 \le C\sum_{s=1}^{k}\Big(\sum_{p+q+r+t=m}\mu_{k+1}^{p-2}\|\nabla^{(q)}b_{ij}\|_0\mu_s^r\|\nabla^{(t+1)}b_{ij}^s\|_0 + \sum_{p+q+r+t=m}\mu_{k+1}^{p-2}\|\nabla^{(q)}b_{ij}\|_0\mu_s^{r+1}\|\nabla^{(t)}b_{ij}^s\|_0 + \sum_{p+q=m}\sum_{r+t+u=q+1}\mu_{k+1}^{p-3}\|\nabla^{(r)}b_{ij}\|_0\mu_s^{t+1}\|\nabla^{(u)}b_{ij}^s\|_0\Big),$$

whereas, for all $m = 0\dots(K-k)(2N+4)-2N-2$:

$$\begin{aligned}
&\|\nabla^{(m)}\Theta_n\|_0\\
&\le C\sum_{s=1}^{k}\Big(\sum_{p+q+r+t=m}\mu_{k+1}^{p-2}\delta_k^{1/2}\lambda_{k+1}^q\mu_s^r\delta_{s-1}^{1/2}\lambda_s^{t+1} + \sum_{p+q+r+t=m}\mu_{k+1}^{p-2}\delta_k^{1/2}\lambda_{k+1}^q\mu_s^{r+1}\delta_{s-1}^{1/2}\lambda_s^t\\
&\qquad + \sum_{p+q=m}\sum_{r+t+u=q+1}\mu_{k+1}^{p-3}\delta_k^{1/2}\lambda_{k+1}^r\mu_s^{t+1}\delta_{s-1}^{1/2}\lambda_s^u\Big)\\
&\le C\mu_{k+1}^{m-1}\sum_{s=1}^{k}\delta_k^{1/2}\delta_{s-1}^{1/2}\Big(\frac{\lambda_s}{\mu_{k+1}} + \frac{\mu_s}{\mu_{k+1}} + \frac{\lambda_{k+1}\mu_s}{\mu_{k+1}^2}\Big)\\
&\le C\delta_k\mu_{k+1}^{m-1}\sum_{s=1}^{k}\sigma^{(k-s+1)N/2}\frac{\mu_s}{\mu_{k+1}} \le C\delta_k\mu_{k+1}^{m-1},
\end{aligned}$$

as $s = 1\dots k$, there holds $\mu_{k+1}/\mu_s \ge \sigma^{(k+1-s)N/2}$. In summary, (3.33) becomes:

$$\|\nabla^{(m)}(W_n - W_{n-1})\|_0 \le C\delta_k^{1/2}\mu_{k+1}^{m-1}\big(1+\|\nabla v\|_0\big) \qquad \text{for all } m = 0\dots(K-k)(2N+4)-2N-2.$$

It now follows, through (3.20), the inductive bounds $(3.3)_1$, $(3.3)_4$ at the counter value $k+1$:

$$\begin{aligned}
&\|w_{k+1} - w\|_1 \le \|w_k - w\|_1 + \sum_{n=2}^{\bar d}\|W_n - W_{n-1}\|_1 \le C\delta_0^{1/2}\big(1+\|\nabla v\|_0\big),\\
&\|\nabla^{(m+2)}w_{k+1}\|_0 \le \|\nabla^{(m+2)}W_{1d}\|_0 + \sum_{n=2}^{\bar d}\|\nabla^{(m+2)}(W_n - W_{n-1})\|_0\\
&\qquad \le C\delta_k^{1/2}\mu_{k+1}^{m+1}\big(1+\|\nabla v\|_0\big) \quad \text{for all } m = 0\dots(K-k)(2N+4)-2N-4.
\end{aligned}$$

12. (Bounds on $\mathcal{G}$ and $\mathcal{H}_n$) Recall the definition (3.32). From (3.25) we directly see that:

$$\|\nabla^{(m)}\mathcal{G}\|_0 \le C\sum_{p+q=m}\mu_{k+1}^{p-2}\delta_k\lambda_{k+1}^{q+2} \le C\delta_k\mu_{k+1}^m\Big(\frac{\lambda_{k+1}}{\mu_{k+1}}\Big)^2 \le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m \qquad \text{for all } m = 0\dots(K-k)(2N+4)-2N-3. \tag{3.34}$$

Since $V^n_{n-1} = v^n_k$, and since, by our construction:

$$v^n_k = v^n_0 + \sum_{s=1}^{k} \frac{\Gamma(\mu_s t_{\eta_{ij}})}{\mu_s} b^s_{ij},$$

we can rewrite the formula for $\mathcal{H}_n$ as:

$$\begin{aligned}
\mathcal{H}_n &= \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \nabla^{(2)} \Big(v^n_0 + \sum_{s=1}^{k} \frac{\Gamma(\mu_s t_{\eta_{ij}})}{\mu_s} b^s_{ij} \Big) - \mathrm{sym}\nabla\Theta_n \\
&= \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \nabla^{(2)} v^n_0 + \sum_{s=1}^{k} \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \frac{\Gamma(\mu_s t_{\eta_{ij}})}{\mu_s} \nabla^{(2)} b^s_{ij} \\
&\quad + 2 \sum_{s=1}^{k} \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \Gamma'(\mu_s t_{\eta_{ij}}) \, \mathrm{sym}\big(\nabla b^s_{ij} \otimes \eta_{ij} \big) \\
&\quad + \sum_{s=1}^{k} \frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \eta_{ij} \otimes \eta_{ij} - \mathrm{sym}\nabla\Theta_n.
\end{aligned} \tag{3.35}$$

The components of the third term in the right hand side above can be decomposed as:

$$\begin{aligned}
&\frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \Gamma'(\mu_s t_{\eta_{ij}}) \, \mathrm{sym}\big(\nabla b^s_{ij} \otimes \eta_{ij} \big) \\
&= \mathrm{sym}\nabla\Big(\frac{\Gamma_1(\mu_{k+1} t_{\eta_{ij}})}{\mu^2_{k+1}} b_{ij} \Gamma'(\mu_s t_{\eta_{ij}}) \nabla b^s_{ij} \Big) - \frac{\Gamma_1(\mu_{k+1} t_{\eta_{ij}})}{\mu^2_{k+1}} \mathrm{sym}\nabla\Big(b_{ij} \Gamma'(\mu_s t_{\eta_{ij}}) (\nabla b^s_{ij}) \Big),
\end{aligned}$$

and, similarly, we express components of the fourth term as:

$$\begin{aligned}
&\frac{\Gamma(\mu_{k+1} t_{\eta_{ij}})}{\mu_{k+1}} b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \eta_{ij} \otimes \eta_{ij} \\
&= \mathrm{sym}\nabla\Big(\frac{\Gamma_1(\mu_{k+1} t_{\eta_{ij}})}{\mu^2_{k+1}} b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \eta_{ij} \Big) \\
&\quad - \frac{\Gamma_1(\mu_{k+1} t_{\eta_{ij}})}{\mu^2_{k+1}} \mathrm{sym}\Big(\nabla\big(b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \big) \otimes \eta_{ij} \Big) \\
&= \mathrm{sym}\nabla\Big(\frac{\Gamma_1(\mu_{k+1} t_{\eta_{ij}})}{\mu^2_{k+1}} b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \eta_{ij} \Big) \\
&\quad - \mathrm{sym}\nabla\Big(\frac{\Gamma_2(\mu_{k+1} t_{\eta_{ij}})}{\mu^3_{k+1}} \nabla\big(b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \big) \Big) \\
&\quad + \frac{\Gamma_2(\mu_{k+1} t_{\eta_{ij}})}{\mu^3_{k+1}} \nabla^{(2)} \Big(b_{ij} \mu_s \Gamma''(\mu_s t_{\eta_{ij}}) b^s_{ij} \Big).
\end{aligned}$$

Recalling the formula for Θ_n in (3.29), the equation (3.35) hence becomes:

$$\begin{aligned}\mathcal{H}_n = &\frac{\Gamma(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}} b_{ij}\nabla^{(2)}v_0^n + \sum_{s=1}^{k}\frac{\Gamma(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}} b_{ij}\frac{\Gamma(\mu_s t_{\eta_{ij}})}{\mu_s}\nabla^{(2)}b_{ij}^s \\ &- 2\sum_{s=1}^{k}\frac{\Gamma_1(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^2}\mathrm{sym}\nabla\Big(b_{ij}\Gamma'(\mu_s t_{\eta_{ij}})\nabla b_{ij}^s\Big) \\ &+ \sum_{s=1}^{k}\frac{\Gamma_2(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^3}\nabla^{(2)}\Big(b_{ij}\mu_s\Gamma''(\mu_s t_{\eta_{ij}})b_{ij}^s\Big).\end{aligned} \tag{3.36}$$

We finally estimate the four terms above, together with their derivatives of orders $m = 0\dots(K-k)(2N+4)-2N-3$. For the first term, we use (3.25) and $(3.1)_3$ to get:

$$\begin{aligned}&\Big\|\nabla^{(m)}\Big(\frac{\Gamma(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}}b_{ij}\nabla^{(2)}v_0^n\Big)\Big\|_0 \le C\sum_{p+q+r=m}\mu_{k+1}^{p-1}\|\nabla^{(q)}b_{ij}\|_0\|\nabla^{(r+2)}v_0\|_0 \\ &\le C\sum_{p+q+r=m}\mu_{k+1}^{p-1}\delta_k^{1/2}\lambda_{k+1}^q\delta_0^{1/2}\mu_0^{r+1} \le C\delta_k\sigma^{Nk/2}\mu_{k+1}^m\frac{\mu_0}{\mu_{k+1}} \le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m,\end{aligned}$$

in view of $\mu_{k+1}/\mu_0 \ge \sigma^{(k+2)N/2}$. This is the only estimate necessitating the larger increase from λ_1 to μ_1 than from all the subsequent λ_{k+1} to μ_{k+1}. For the second, third and fourth terms in (3.36), we fix $s = 1\dots k$ and use (3.25) together with the inductive assumption $(3.4)_2$, to bound the derivatives of b_{ij}^s. Thus, the components of the second term obey:

$$\begin{aligned}&\Big\|\nabla^{(m)}\Big(\frac{\Gamma(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}}b_{ij}\frac{\Gamma(\mu_s t_{\eta_{ij}})}{\mu_s}\nabla^{(2)}b_{ij}^s\Big)\Big\|_0 \le C\sum_{p+q+r+t=m}\mu_{k+1}^{p-1}\|\nabla^{(q)}b_{ij}\|_0\mu_s^r\|\nabla^{(t+2)}b_{ij}^s\|_0 \\ &\le C\sum_{p+q+r+t=m}\mu_{k+1}^{p-1}\delta_k^{1/2}\lambda_{k+1}^q\mu_s^{r-1}\delta_{s-1}^{1/2}\lambda_s^{t+2} \le C\delta_k\sigma^{(k-s+1)N/2}\mu_{k+1}^m\frac{\lambda_s^2}{\mu_{k+1}\mu_s} \le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m,\end{aligned}$$

as $\mu_{k+1}\mu_s/\lambda_s^2 \ge \sigma^{N/2}\sigma^{((k+1)-(s-1))N/2} = \sigma^{(k-s+3)N/2}$. For the third term in (3.36), we get:

$$\begin{aligned}&\Big\|\nabla^{(m)}\Big(\frac{\Gamma_1(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^2}\mathrm{sym}\nabla\big(b_{ij}\Gamma'(\mu_s t_{\eta_{ij}})\nabla b_{ij}^s\big)\Big)\Big\|_0 \\ &\le C\sum_{p+q=m}\sum_{r+t+u=q+1}\mu_{k+1}^{p-2}\|\nabla^{(r)}b_{ij}\|_0\mu_s^t\|\nabla^{(u+1)}b_{ij}^s\|_0 \\ &\le C\sum_{p+q=m}\sum_{r+t+u=q+1}\mu_{k+1}^{p-2}\delta_k^{1/2}\lambda_{k+1}^r\mu_s^t\delta_{s-1}^{1/2}\lambda_s^{u+1} \le C\delta_k^{1/2}\delta_{s-1}^{1/2}\sum_{p+q=m}\mu_{k+1}^{p-2}\lambda_{k+1}^{q+1}\lambda_s \\ &\le C\delta_k\sigma^{(k-s+1)N/2}\mu_{k+1}^m\frac{\lambda_{k+1}\lambda_s}{\mu_{k+1}^2} \le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m,\end{aligned}$$

as $\mu_{k+1}^2/(\lambda_{k+1}\lambda_s) \ge \sigma^{N/2}\sigma^{((k+1)-(s-1))N/2} = \sigma^{(k-s+3)N/2}$. For the last term in (3.36), we get:

$$
\begin{aligned}
&\Big\|\nabla^{(m)}\Big(\frac{\Gamma_2(\mu_{k+1}t_{\eta_{ij}})}{\mu_{k+1}^3}\nabla^{(2)}\big(b_{ij}\mu_s\Gamma''(\mu_s t_{\eta_{ij}})b_{ij}^s\big)\Big)\Big\|_0\\
&\le C\sum_{p+q=m}\ \sum_{r+t+u=q+2}\mu_{k+1}^{p-3}\|\nabla^{(r)}b_{ij}\|_0\mu_s^{t+1}\|\nabla^u b_{ij}^s\|_0\\
&\le C\sum_{p+q=m}\ \sum_{r+t+u=q+2}\mu_{k+1}^{p-3}\delta_k^{1/2}\lambda_{k+1}^r\mu_s^{t+1}\delta_{s-1}^{1/2}\lambda_s^u \le C\delta_k^{1/2}\delta_{s-1}^{1/2}\sum_{p+q=m}\mu_{k+1}^{p-3}\lambda_{k+1}^{q+2}\mu_s\\
&\le C\delta_k\sigma^{(k-s+1)N/2}\mu_{k+1}^m\frac{\lambda_{k+1}^2\mu_s}{\mu_{k+1}^3}\le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m,
\end{aligned}
$$

since $\mu_{k+1}^3/(\lambda_{k+1}^2\mu_s)\ge\sigma^N\sigma^{((k+1)-s))N/2}=\sigma^{(k-s+3)N/2}$. This finally implies:

$$
\|\nabla^{(m)}\mathcal{H}_n\|_0\le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m\qquad\text{for all } m=0\dots(K-k)(2N+4)-2N-3. \tag{3.37}
$$

In view of the decomposition (3.31) and the bounds on the derivatives of the fields $\mathcal{F}$, $\{\mathcal{F}_{1j}\}_{j=1}^d$, $\mathcal{G}$, $\{\mathcal{H}_n\}_{n=2}^{\bar d}$, given in (3.6), (3.22), (3.34), (3.37), we arrive at:

$$
\begin{aligned}
\|\nabla^{(m)}\mathcal{D}_{k+1}\|_0\le C\Big(\frac{\delta_k}{\sigma^{2N}}\mu_k^m+\frac{\delta_k}{\sigma^{N+2}}\lambda_{k+1}^m+\frac{\delta_k}{\sigma^N}\mu_{k+1}^m\Big)\le C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m\\
\text{for all } m=0\dots(K-k)(2N+4)-2N-3,
\end{aligned}
$$

proving the final inductive estimate $(3.3)_5$ at the counter $k+1$.

13. (End of proof) After the total of K steps, we declare:

$$
\tilde v\doteq v_K,\qquad \tilde w\doteq w_K,\qquad \tilde{\mathcal{D}}=(A-A_0)+\mathcal{D}_K=A-\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v+\mathrm{sym}\nabla\tilde w\big).
$$

Recalling $(3.3)_1$-$(3.3)_4$ and (3.2), we conclude that:

$$
\begin{aligned}
&\|\tilde v-v\|_1\le C\delta_0^{1/2}=C\|\mathcal{D}\|_0^{1/2},\\
&\|\tilde w-w\|_1\le C\delta_0^{1/2}\big(1+\|\nabla v\|_0\big)=C\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big),\\
&\|\nabla^{(2)}\tilde v\|_0\le C\delta_{K-1}^{1/2}\mu_K=C\frac{\delta_0^{1/2}}{\sigma^{(K-1)N/2}}\mu_0\sigma^{d+N+(d+\frac{N}{2})(K-1)}\\
&\qquad\qquad=C\delta_0^{1/2}\mu_0\sigma^{d+N+d(K-1)}=C\mathcal{M}\sigma^{dK+N},\\
&\|\nabla^{(2)}\tilde w\|_0\le C\delta_{K-1}^{1/2}\mu_K\big(1+\|\nabla v\|_0\big)=C\mathcal{M}\sigma^{dK+N}\big(1+\|\nabla v\|_0\big),
\end{aligned}
$$

while $(3.1)_2$ and $(3.3)_5$ yield:

$$
\begin{aligned}
\|\tilde{\mathcal{D}}\|_0\le\|A-A_0\|_0+\|\mathcal{D}_K\|_0\le C_A\frac{\|A\|_{r,\beta}}{\mathcal{M}^{r+\beta}}\delta_0^{(r+\beta)/2}+C\delta_K\\
=(C+C_A)\Big(\frac{\|A\|_{r,\beta}}{\mathcal{M}^{r+\beta}}\|\mathcal{D}\|_0^{(r+\beta)/2}+\frac{\|\mathcal{D}\|_0}{\sigma^{NK}}\Big).
\end{aligned}
$$

These are precisely the claimed bounds $(1.9)_1$-$(1.9)_3$. The proof is done. □

4. The Nash-Kuiper scheme and a proof of Theorem 1.1

The proof of Theorem 1.1 relies on iterating Theorem 1.3 via "induction on stages" which is the essence of the so-called *Nash-Kuiper scheme*. We now quote the result in [24, Theorem 5.1], which is similar to the argument in [12, section 6]. Recall our notation:

$$\mathcal{D}(A, v, w) \doteq A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big).$$

Theorem 4.1. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, d-dimensional set, and let $A \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $r \in \{0, 1\}$, $\beta \in (0, 1]$, $r + \beta < 2$. Assume that $k, J, S \geq 1$ are such that the statement of Theorem 1.3 holds true with k replacing the specific codimension $\bar{d}$, J replacing the exponent $dK + N$ in (1.9)$_2$, and S replacing the exponent KN in (1.9)$_3$, namely:*

There exists $\sigma_0 > 1$ depending only on ω, k, J, S such that, given any $v \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$ and any $\mathcal{M}$, σ with:

$$\mathcal{D} \doteq \mathcal{D}(A, v, w) \quad \text{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$
$$\mathcal{M} \geq \max\big\{\|v\|_2, \|w\|_2, 1\big\}, \qquad \sigma \geq \sigma_0,$$

there exists $\tilde{v} \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^k)$, $\tilde{w} \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$, obeying the bounds:

$$\|\tilde{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \qquad \|\tilde{w} - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}\big(1 + \|\nabla v\|_0\big),$$
$$\|\nabla^2 \tilde{v}\|_0 \leq C\mathcal{M}\sigma^J, \qquad \|\nabla^2 \tilde{w}\|_0 \leq C\mathcal{M}\sigma^J\big(1 + \|\nabla v\|_0\big),$$
$$\|\mathcal{D}(A, \tilde{v}, \tilde{w})\|_0 \leq C\Big(\frac{\|A\|_{r,\beta}}{\mathcal{M}^{r+\beta}}\|\mathcal{D}\|_0^{(r+\beta)/2} + \frac{\|\mathcal{D}\|_0}{\sigma^S}\Big),$$

with constants C depending only on $\omega, k, r, \beta, J, S$.

Then, for every $v \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$ such that:

$$\mathcal{D} \doteq \mathcal{D}(A, v, w) \quad \text{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

and for every α in the range:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, \frac{1}{1 + 2J/S}\Big\},$$

there exist $\tilde{v} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^k)$ and $\tilde{w} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^d)$ with the following properties:

$$\|\tilde{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde{w} - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0),$$
$$A - \big(\frac{1}{2}(\nabla \tilde{v})^T \nabla \tilde{v} + \mathrm{sym}\nabla \tilde{w}\big) = 0 \quad \text{in } \bar{\omega},$$

where the constants C depend only on $\omega, k, r, \beta, J, S$.

We remark that [24, Theorem 5.1] is stated as above, but with $r = 0$. The proof, allowing now for $r = 1$, follows by a straightforward inspection of the proof in [24].
Clearly, Theorem 4.1 and Theorem 1.3 yield together the next result below, because:

$$\frac{1}{1+2J/S} = \frac{S}{S+2J} = \frac{KN}{KN+(dK+N)} \to 1 \quad \text{as } K, N \to \infty.$$

Corollary 4.2. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded set, and let $A \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\text{sym}})$ with $r \in \{0,1\}$, $\beta \in (0,1]$, $r+\beta < 2$. Fix α as in (1.1). Then, given $v \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $w \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$ such that:*

$$\mathcal{D} \doteq \mathcal{D}(A, v, w) \quad \text{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

there exist $\tilde{v} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{\bar{d}})$, $\tilde{w} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^d)$ with the following properties:

$$\|\tilde{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde{w} - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0),$$
$$A - \big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \text{sym}\nabla\tilde{w}\big) = 0 \quad \text{in } \bar{\omega}.$$

The constants C depend only on ω, r, β and α.

The proof of Theorem 1.1 proceeds in the same manner as [24, Theorem 1.1]: we first use the basic stage-like construction to reduce the defect magnitude $\|\mathcal{D}\|_0$ below 1, after which Corollary 4.2 yields the final result. □

5. Extension to the isometric immersion system

Using the same ideas underlying the proof of Theorem 1.1, one can show the following full flexibility result for the isometric immersion system (1.5):

Theorem 5.1. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded set, diffeomorphic to the unit ball $B_1 \subset \mathbb{R}^d$, let $g \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\text{sym},>})$ be a Riemannian metric on $\bar{\omega}$, and k be a codimension satisfying:*

$$k \geq \bar{d}.$$

Then, for every immersion $u \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{d+k})$ such that:

$$(\nabla u)^T\nabla u < g \quad \text{in } \bar{\omega},$$

for every $\epsilon > 0$, and for every regularity exponent α in:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, 1\Big\},$$

there exists $\tilde{u} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{d+k})$ such that:

$$\|\tilde{u} - u\|_0 \leq \epsilon \quad \text{and} \quad (\nabla\tilde{u})^T\nabla\tilde{u} = g \quad \text{in } \bar{\omega}.$$

We now outline the differences between the Nash-Kuiper scheme for the Monge-Ampère system (1.2) and the isometric immersion system (1.5), arising in the proof of Theorem 5.1:

1. A version of Theorem 4.1 remains valid; see [27, Theorem 1.3] in case $d = k = 2$. Thus, it suffices to show a conclusion similar to Theorem 1.3. The defect is now defined as:

$$\mathcal{D}(g, u) = g - (\nabla u)^T \nabla u.$$

Roughly speaking, assuming that $\|\mathcal{D}\|_0$ is sufficiently small, and given a sufficiently large σ, we construct an updated immersion $\tilde{u}$ with the properties:

$$\|\tilde{u} - u\|_1 \leq C\|\mathcal{D}(g, u)\|_0^{1/2}, \qquad \|\nabla^{(2)}\tilde{u}\|_0 \leq C\sigma^{dK+N},$$
$$\|\mathcal{D}(g, \tilde{u})\|_0 \leq C\Big(\frac{\|\mathcal{D}(g, u)\|_0}{\sigma^{KN}} + \|g\|_{r,\beta}\Big),$$

where constants C depend on g, N, K and also on $\|u\|_2$ and the immersion constant of u. In practice, since the system (1.5) is fully nonlinear and one cannot take advantage of the linear component $\mathrm{sym}\nabla w$ in (1.2), the defect needs to be calculated and decreased with respect to a scaled version of H_0, whereas the last estimate above takes the following form with the following initial assumption (compare [27, Theorem 1.2]):

$$\big\|\mathcal{D}\big(g - \frac{\delta}{\sigma^{NK}} H_0, \tilde{u}\big)\big\|_0 \leq \frac{r_0}{5}\frac{\delta}{\sigma^{NK}} + C\|g\|_{r,\beta} \quad \text{with} \quad \|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \frac{r_0}{4}\delta.$$

2. A version of Kuiper's corrugation in Lemma 2.3 is as follows (see [7,27]):

$$\tilde{u} = u + \frac{\Gamma(\lambda t_\eta)}{\lambda} a(x) E_u(x) + T_u(x)\Big(\frac{\bar{\Gamma}(\lambda t_\eta)}{\lambda} a(x)^2 \eta + \frac{\bar{\bar{\Gamma}}(\lambda t_\eta)}{\lambda^2} a(x)\nabla u(x) + \Theta\Big),$$
$$\text{where } T_u = (\nabla u)\big((\nabla u)^T \nabla u\big)^{-1},$$

with $\Gamma(t) = \sqrt{2}\sin t$, $\bar{\Gamma}(t) = -\frac{1}{4}\sin(2t)$ and $\bar{\bar{\Gamma}}(t) = \frac{1}{4}\cos(2t)$. Note the vector E_u which is unit normal to the surface $u(\omega)$, and replaces the previously constant codimension direction e_n; and the w-corrugation in (2.3), appearing as the tangential component of $\tilde{u} - u$, carried by the tangent frame T_u. The additional perturbation Θ arises from the anticipated application of Lemma 2.4. The formula (2.4) consequently becomes:

$$\begin{aligned}&(\nabla\tilde{u})^T\nabla\tilde{u} - (\nabla u)^T\nabla u - a(x)^2\eta\otimes\eta \\ &= -2\frac{\Gamma(\lambda t_\eta)}{\lambda} a[\langle \partial_s\partial_t u, E_u\rangle]_{s,t=1\ldots d} + 2\frac{\bar{\bar{\Gamma}}(\lambda t_\eta)}{\lambda^2} a\nabla^2 a + \frac{\bar{\bar{\bar{\Gamma}}}(\lambda t_\eta)}{\lambda^2}\nabla a\otimes\nabla a + \mathcal{R},\end{aligned} \tag{5.1}$$

where $\bar{\bar{\bar{\Gamma}}}(t) = 1 - \frac{1}{2}\cos(2t)$. The additional error term $\mathcal{R}$ involves derivatives of a, E_u, T_u, and Θ; see [27, Lemma 2.5] for a precise version of the above calculation without the $a\nabla u$ component in $\tilde{u}$. In practice, $\mathcal{R}$ is of higher order, appearing in the proofs as powers of the defect measure δ, greater than 1, see [7, Lemma 3.1]. Since δ can be assumed to be

sufficiently small relative to a fixed but possibly large power of $\|\nabla^{(2)}u\|_0$, the term $\mathcal{R}$ never competes with the three leading order terms in (5.1).

3. While translating steps 1-12 in the proof of Theorem 1.3, the intermediate immersions $\{u_k\}_{k=0}^K$ are constructed with the uniform immersion constants. In this construction, the codimension vectors $\{e_n\}_{n=1\dots\bar{d}}$ are replaced by a frame of normal vectors $\{E_u^n\}_{n=1\dots\bar{d}}$. The initial frame, normal to the surface $u_0(\omega)$, can be chosen independently as in [8]. However, along the iteration in $k = 1 \dots K$, the subsequent frames must be updated relative to the previous ones so that the differences $(E^n_{u_{k+1}} - E^n_{u_k})$, together with their derivatives, satisfy bounds consistent with those on $(\nabla u_{k+1} - \nabla u_k)$. This general update procedure has been carried out in [27, Section 3] and applies here with minor modifications, due to $d, \bar{d} \geq 2$ rather than the case $d = k = 2$. See also the discussion in [17] in the context of codimension $k = 1$ isometric immersions.

Data availability

No data was used for the research described in the article.

References

[1] Y. Borisov, The parallel translation on a smooth surface. I-IV, Vestn. Leningr. Univ. 13 (4) (1958) 160–171, and 13 (19) (1958) 45–54, and 14 (1) (1959) 34–50, and 14 (13) (1959) 20–26.

[2] Y. Borisov, $C^{1,\alpha}$-isometric immersions of Riemannian spaces, Dokl. Akad. Nauk SSSR 163 (1965) 869–871.

[3] Y. Borisov, Irregular $C^{1,\beta}$-surfaces with analytic metric, Sib. Mat. Zh. 45 (1) (2004) 25–61.

[4] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr, V. Vicol, Onsager's conjecture for admissible weak solutions, Commun. Pure Appl. Math. 72 (2) (2019) 229–274.

[5] T. Buckmaster, V. Vicol, Convex integration constructions in hydrodynamics, Bull. Am. Math. Soc. (N.S.) 58 (1) (2021) 1–44.

[6] W. Cao, J. Hirsch, D. Inauen, $C^{1,1/3-}$ very weak solutions to the two dimensional Monge-Ampère equation, Calc. Var. 64 (2025) 160.

[7] W. Cao, J. Hirsch, D. Inauen, A Nash-Kuiper theorem for isometric immersions beyond Borisov's exponent, arXiv: 2503.13867.

[8] W. Cao, D. Inauen, Rigidity and flexibility of isometric extension, Comment. Math. Helv. 99 (1) (2024) 39–80.

[9] W. Cao, L. Székelyhidi Jr., Very weak solutions to the two-dimensional Monge-Ampère equation, Sci. China, Math. 62 (6) (2019) 1041–1056.

[10] W. Cao, L. Székelyhidi Jr., On the isometric version of Whitney's strong embedding theorem, Adv. Math. 460 (2025) 110040.

[11] W. Cao, L. Székelyhidi Jr., Global Nash-Kuiper theorem for compact manifolds, J. Differ. Geom. 122 (1) (2022) 35–68.

[12] S. Conti, C. De Lellis, L. Székelyhidi Jr., h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings, in: Proceedings of the Abel Symposium, in: Nonlinear Partial Differential Equations, 2010, pp. 83–116.

[13] C. De Lellis, D. Inauen, L. Székelyhidi Jr., A Nash-Kuiper theorem for $C^{1,\frac{1}{5}-\delta}$ immersions of surfaces in 3 dimensions, Rev. Mat. Iberoam. 34 (3) (2018) 1119–1152.

[14] C. De Lellis, D. Inauen, $C^{1,\alpha}$ isometric embeddings of polar caps, Adv. Math. 363 (2020) 106996.

[15] M. Gromov, Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 9, Springer-Verlag, Berlin, 1986.

[16] M. Günther, Isometric embeddings of Riemannian manifolds, in: Proceedings of the International Congress of Mathematicians, vol. I, II, Kyoto, 1990, Math. Soc. Japan, Tokyo, 1991, pp. 1137–1143.

[17] D. Inauen, Flexibility of codimension one $C^{1,\theta}$ isometric immersions, arXiv:2603.08382.

[18] D. Inauen, M. Lewicka, The Monge-Ampère system in dimension two and codimension three, arXiv:2501.12474, to appear in Rev. Mat. Iberoam.

[19] D. Inauen, M. Lewicka, The Monge-Ampére system in dimension two is fully flexible in codimension two, arXiv: 2504.03582.

[20] P. Isett, A proof of Onsager's conjecture, Ann. Math. (2) 188 (3) (2018) 871–963.
[21] H. Jacobowitz, Implicit function theorems and isometric embeddings, Ann. Math. 95 (2) (1972) 191–225.
[22] A. Källén, Isometric embedding of a smooth compact manifold with a metric of low regularity, Ark. Mat. 16 (1) (1978) 29–50.
[23] N. Kuiper, On C^1-isometric imbeddings. I, II, Indag. Math. 17 (1955) 545–556 and 683–689, Nederl. Akad. Wetensch. Proc. Ser. A 58.
[24] M. Lewicka, The Monge-Ampère system: convex integration in arbitrary dimension and codimension, SIAM J. Math. Anal. 57 (1) (2025) 601–636.
[25] M. Lewicka, The Monge-Ampère system: convex integration with improved regularity in dimension two and arbitrary codimension, Differ. Integral Equ. 38 (3/4) (2025) 141–174.
[26] M. Lewicka, The Monge-Ampère system in dimension two: a regularity improvement, J. Funct. Anal. 289 (8) (2025) 111064.
[27] M. Lewicka, Full flexibility of isometric immersions of metrics with low Hölder regularity in Poznyak theorem's dimension, arXiv:2511.16305.
[28] M. Lewicka, M. Pakzad, Convex integration for the Monge-Ampère equation in two dimensions, Anal. PDE 10 (3) (2017) 695–727.
[29] J. Nash, The imbedding problem for Riemannian manifolds, Ann. Math. 63 (1956) 20–63.
[30] J. Nash, C^1 isometric imbeddings, Ann. Math. 60 (1954) 383–396.
[31] È.G̃. Poznjak, Isometric imbedding of two-dimensional Riemannian metrics in Euclidean spaces, Usp. Mat. Nauk 28 (4) (1973) 47–76.