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Research paper

Examples of tangent cones of non-collapsed Ricci limit spaces

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ABSTRACT

We give new examples of manifolds that appear as cross sections of tangent cones of non-collapsed Ricci limit spaces. It was shown by Colding–Naber that the homeomorphism types of the tangent cones of a fixed point of such a space do not need to be unique. In fact, they constructed an example in dimension 5 where two different homeomorphism types appear at the same point. In this note, we extend this result and construct limit spaces in all dimensions at least 5 where any finite collection of manifolds that admit *core metrics*, a type of metric introduced by Perelman and Burdick to study Riemannian metrics of positive Ricci curvature on connected sums, can appear as cross sections of tangent cones of the same point.

1. Introduction and main results

In this note, we consider pointed Gromov–Hausdorff limits (Y, d_Y, y) of sequences of pointed n-dimensional Riemannian manifolds (M_i, g_i, x_i) with a lower Ricci curvature bound, i.e.

$$\mathrm{Ric}(g_i) \geq -(n-1)$$

for all i. Additionally, we require that the limit is non-collapsed, i.e. there exists $v > 0$ such that

$$\mathrm{vol}(B_1(p_i)) \geq v$$

for all i. We call such a space (Y, d_Y, p) a *non-collapsed Ricci limit space*. The structure of non-collapsed Ricci limit spaces, or Ricci limit spaces in general, has been studied extensively, see e.g. [1–10], and the references therein.

When studying the structure of the limit space Y, a central role is played by its *tangent cones*. A tangent cone at $x \in Y$ is the pointed Gromov–Hausdorff limit of a sequence $(Y, R_i^{-1} d_Y, x)$, where $R_i \to 0$ as $i \to \infty$. By Gromov's precompactness theorem, any such sequence has a converging subsequence. Moreover, if Y is non-collapsed, it was shown by Cheeger–Colding [3] that every tangent cone of Y is a metric cone, and it follows from work of Ketterer [11] that the cross section of the cone satisfies the curvature dimension condition $\mathrm{CD}(n-2, n-1)$.

However, even in the non-collapsed case, the tangent cones at a point obtained from different sequences R_i do not need to coincide. This was first demonstrated by Perelman [12] and Cheeger–Colding [3], where they constructed a family of metrics on S^3 whose cones all appear as tangent cones of the same point of a non-collapsed Ricci limit space. Subsequently, Colding–Naber [10] gave further examples, including cones over S^{n-1} that isometrically split off precisely $\mathbb{R}^k$ for all $0 \leq k \leq n-2$, and cones whose cross sections are not even homeomorphic. The latter was realised by a 5-dimensional limit space that contains a point with two tangent cones whose cross sections are given by S^4 and $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$, respectively. We note that this is in strong contrast to the situation

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of non-collapsed limits of manifolds with a lower sectional curvature bound, where all tangent cones are unique (see [13,14]) and all spaces of directions are homeomorphic to spheres (see [15]).

The goal of this note is to provide further examples of non-collapsed Ricci limit spaces with non-homeomorphic tangent cones. To formulate our main result, we first need to recall the notion of *core metrics*.

Definition 1.1. A Riemannian metric g of positive Ricci curvature on an n-dimensional smooth manifold M is a *core metric* if there exists an embedded disc $D^n \subseteq M$ such that the induced metric on the boundary of $M \setminus D^{n\circ}$ is the round metric on S^{n-1} and its second fundamental form is strictly positive.

Based on work by Perelman [16], core metrics were introduced by Burdick [17] in the context of preserving positive Ricci curvature along connected sums. The known examples of manifolds with core metrics are given as follows:

(1) The sphere S^n and the compact rank one symmetric spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$ (see [16,17]),
(2) Linear sphere bundles and projective bundles with fibre $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{O}P^2$ over manifolds with core metrics (see [18–20]),
(3) Products of manifolds with core metrics (see [20]),
(4) Connected sums of manifolds with core metrics (see [21]),
(5) Manifolds obtained as boundaries of certain plumbings (see [19,22]),
(6) Certain manifolds that decompose as the union of two disc bundles, such as the Wu manifold W^5 (see [20]).

Note also that a closed manifold that admits a core metric is simply-connected by [23].

Our main result is the following.

Theorem A. *Let $M_1^n, \dots, M_\ell^n$ be closed, smooth, n-dimensional manifolds that admit core metrics. Then there exist a non-collapsed Ricci limit space (Y^{n+1}, d_Y, y) and (non-smooth) metrics d_i on each M_i such that the cones $C(M_i, d_i)$ all are tangent cones of Y at y.*

Each of the metrics d_i is in fact a Riemannian metric with $(2\ell - 1)$ singular points which isometrically contains the core metric on M_i with the embedded disc removed. The limit space Y is homeomorphic to the cone over the connected sum

$$M = M_1 \# \dots \# M_\ell \# (-M_1) \# \dots \# (-M_\ell)$$

equipped with a Riemannian metric with a singularity at the origin. The construction of the metric is based on a technique developed by Colding–Naber [10] to construct metrics with non-unique tangent cones on a (topological) cone. One of the main requirements of this technique is the construction of a *Ricci closable* metric (see Definition 2.1 below) on the cross section. To prove Theorem A, we will give a general criterion for the existence of a Ricci closable metric on a manifold with an isometric $\mathbb{Z}/2$-action and apply it to the $\mathbb{Z}/2$-action on M that interchanges each M_i with $(-M_i)$.

Theorem A shows in particular that for any closed manifold M that admits a core metric there exists a (non-smooth) metric d such that the cone $C(M, d)$ is a non-collapsed Ricci limit space, since every tangent cone of the limit space is the limit of an appropriate rescaling of the original sequence. In general, it is open on which closed manifold M there exists a metric d such that the cone $C(M, d)$ is a non-collapsed Ricci limit space. For example, it was shown in [24] that this is the case for all 3-dimensional spherical space forms. Moreover, we give further examples in Section 4 below (which, in contrast to the manifolds in Theorem A, may also be non-simply-connected). On the other hand, by [25] or [1] there is no such metric on $\mathbb{R}P^2$. Further obstructions are given in [26] if one additionally assumes that the converging sequence is orientable. This is in contrast to the collapsed case, where it follows from the work of Sha–Yang [27] that for every closed Riemannian n-manifold (M, g) of Ricci curvature at least $(n-1)$ the cone $C(M, g)$ is a collapsed Ricci limit space, see Theorem A.1 below. By using Theorem A, we obtain a similar statement in the non-collapsed case when restricting to simply-connected 4-manifolds.

Corollary B. *Let M^4 be a closed, smooth, simply-connected 4-manifold that admits a Riemannian metric of positive scalar curvature. Then there exists a (non-smooth) metric d on M such that the cone $C(M, d)$ is a non-collapsed Ricci limit space. Moreover, if M is the boundary of a compact, oriented, smooth 5-manifold, then, after possibly changing the smooth structure on M, we can assume that d is a smooth Riemannian metric.*

This article is organised as follows. In Section 2 we recall the construction of [10] to construct non-collapsed Ricci limit spaces with prescribed tangent cones, as well as basic results on Ricci curvature. Further, in Section 3 we establish a criterion for a given Riemannian metric to be Ricci closable and apply this to prove Theorem A. In Section 4 we give further examples of Ricci closable metrics that are based on known construction methods for Riemannian metrics of positive Ricci curvature. Finally, in the Appendix we consider the collapsed case and recall the construction of [27].

2. Preliminaries

Following [10], for a non-collapsed Ricci limit space (Y, d_Y, y) we define $\overline{\Omega}_{Y,y}$ as the family of metric spaces $\{(X_s, d_s)\}$ such that $C(X_s, d_s)$ is a tangent cone of Y at y.

Definition 2.1 (*[10]*)**.** A Riemannian manifold (M^n, g) is *Ricci closable* if for every $\varepsilon > 0$ there exists a pointed open Riemannian manifold $(N_\varepsilon^{n+1}, h_\varepsilon, y_\varepsilon)$ of non-negative Ricci curvature such that the annulus $A_{1,\infty}(y_\varepsilon) \subseteq N_\varepsilon$ is isometric to the annulus $A_{1,\infty}(o) \subseteq C(M, (1-\varepsilon)g)$.

Note that a Ricci closable Riemannian manifold in particular has positive Ricci curvature.

The main result of [10] is now given as follows.

Theorem 2.2 (*[10, Theorem 1.1]*). *Let Ω be a connected manifold and X^{n-1} a closed manifold with $n \geq 3$. Let $\{g_s\}_{s\in\Omega}$ be a smooth family of Riemannian metrics on X such that*

(1) $\mathrm{Vol}(X, g_s) = V \leq \mathrm{Vol}(S^{n-1}, ds^2_{n-1})$,
(2) $\mathrm{Ric}_{g_s} \geq n-2$,
(3) There exists $s_0 \in \Omega$ such that g_{s_0} is Ricci closable.

Then there exists a non-collapsing sequence of pointed complete Riemannian manifolds $(M^n_\alpha, g_\alpha, p_\alpha)$ of $\mathrm{Ric} \geq -(n-1)$ that Gromov–Hausdorff converges to a pointed metric space (Y, d_Y, y) with $\overline{\Omega}_{Y,y} = \overline{\{(X, g_s)\}}$.

We will also need the following gluing result of Perelman [16].

Theorem 2.3 (*[16, Section 4], see also [28, Section 2]*). *Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds of $\mathrm{Ric} > 0$ with compact boundaries such that there exists an isometry $\phi : \partial M_1 \to \partial M_2$. Assume that the sum of second fundamental forms $\mathrm{I\!I}_{\partial M_1} + \phi^*\mathrm{I\!I}_{\partial M_2}$ is non-negative. Then the C^0-metric $g_1 \cup_\phi g_2$ on $M_1 \cup_\phi M_2$ can be smoothed in an arbitrarily small neighbourhood of the gluing area into a smooth metric of $\mathrm{Ric} > 0$.*

Here we use the convention that the second fundamental form of the boundary of a Riemannian manifold (M, g) is given by

$$\mathrm{I\!I}(u, v) = g(\nabla_u \nu, v),$$

where $u, v \in T\partial M$ and ν is the outward unit normal field of ∂M. We say the boundary is *convex* if $\mathrm{I\!I}$ is positive definite.

Finally, we recall the following formulae for the Ricci curvatures of a metric on a cylinder, see e.g. [20, Lemma 2.1]

Lemma 2.4. *Let M be a manifold, I an interval and let $g = dt^2 + h_t$ be a Riemannian metric on $I \times M$, where h_t is a smooth family of metrics on M. Let h'_t and h''_t denote the first and second derivative of h_t in t-direction, respectively. Then the second fundamental form of a slice $\{t\} \times M$ with respect to the normal vector ∂_t is given by*

$$\mathrm{I\!I} = \frac{1}{2} h'_t,$$

and the Ricci curvature of the metric g are given as follows:

$$\begin{aligned}
\mathrm{Ric}(\partial_t, \partial_t) &= -\frac{1}{2}\mathrm{tr}_{h_t} h''_t + \frac{1}{4}\|h'_t\|^2_{h_t},\\
\mathrm{Ric}(v, \partial_t) &= -\frac{1}{2} v(\mathrm{tr}_{h_t} h'_t) + \frac{1}{2}\sum_i (\nabla^{h_t}_{e_i} h'_t)(v, e_i),\\
\mathrm{Ric}(u, v) &= \mathrm{Ric}^{h_t}(u, v) - \frac{1}{2} h''_t(u, v) + \frac{1}{2}\sum_i h'_t(u, e_i) h'_t(v, e_i) - \frac{1}{4} h'_t(u, v)\mathrm{tr}_{h_t} h'_t.
\end{aligned}$$

Here $u, v \in T_x M$ and $\{e_i\}$ is an orthonormal basis of $T_x M$ with respect to h_t.

In the special case of a doubly warped product metric, we obtain the following.

Lemma 2.5. *Let $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ be Riemannian manifolds and let $f_1, f_2 : [0, t_0] \to (0, \infty)$ be smooth functions for some $t_0 > 0$. Then the Ricci curvatures of the metric*

$$dt^2 + f_1(t)^2 g_1 + f_2(t)^2 g_2$$

on $[0, t_0] \times M_1 \times M_2$ are given by

$$\begin{aligned}
\mathrm{Ric}(\partial_t, \partial_t) &= -n_1 \frac{f''_1}{f_1} - n_2 \frac{f''_2}{f_2},\\
\mathrm{Ric}(\tfrac{v_1}{f_1}, \tfrac{v_1}{f_1}) &= -\frac{f''_1}{f_1} + \frac{\mathrm{Ric}^{g_1}(v_1, v_1) - (n_1 - 1) f'^2_1}{f_1^2} - n_2 \frac{f'_1 f'_2}{f_1 f_2},\\
\mathrm{Ric}(\tfrac{v_2}{f_2}, \tfrac{v_2}{f_2}) &= -\frac{f''_2}{f_2} + \frac{\mathrm{Ric}^{g_2}(v_2, v_2) - (n_2 - 1) f'^2_2}{f_2^2} - n_1 \frac{f'_1 f'_2}{f_1 f_2}\\
\mathrm{Ric}(\partial_t, v_1) &= \mathrm{Ric}(\partial_t, v_2) = \mathrm{Ric}(v_1, v_2) = 0,
\end{aligned}$$

where $v_i \in TM_i$ are unit vectors with respect to g_i. Further, the second fundamental form of a slice $\{t\} \times M_1 \times M_2$ with respect to the unit normal ∂_t is given by

$$\mathrm{I\!I}(\tfrac{v_i}{f_i}, \tfrac{v_j}{f_j}) = \frac{f'_i}{f_i}\delta_{ij}.$$

3. A criterion for Ricci closability

In this section we give a criterion for a Riemannian metric to be Ricci closable (Proposition 3.2) and use this to prove Theorem A. We first slightly reformulate the condition of Ricci closability.

Lemma 3.1. *Let (M^n, g) be a closed Riemannian manifold of positive Ricci curvature and suppose that there exists a compact Riemannian manifold (N, g_N) with convex boundary isometric to (M, g). Then there exists $c > 0$ such that $(M, c^2\, g)$ is Ricci closable.*

Proof. For $c > 0$ we attach a cylinder $([0,\infty)\times M, dt^2 + f(t)^2\, g)$ to N, where $f : [0,\infty) \to (0,\infty)$ is a smooth function satisfying $f'(0) = 2c$, $f''|_{[0,1)} < 0$ and $f'|_{[1,\infty)} \equiv c$. Then, for c sufficiently small, the metric $dt^2 + f(t)^2 g$ has positive Ricci curvature near the boundary $\{0\}\times M$ and the sum of second fundamental forms of g_N and $dt^2 + f(t)^2 g$ is positive on M by Lemma 2.5. Hence, we can apply Theorem 2.3 to smooth this metric in a small neighbourhood of the gluing area while preserving positive Ricci curvature in this neighbourhood. Outside a neighbourhood of the manifold N, this metric is then of the form $dt^2 + (ct + c_0)^2 g$. Thus, by appropriately rescaling this metric we obtain that the metric $c^2 g$ is Ricci closable. □

Our criterion is now given as follows.

Proposition 3.2. *Let (M^n, g) be a closed Riemannian manifold of positive Ricci curvature that admits an isometric action of $\mathbb{Z}/2$ such that the fixed point set $Y \subseteq M$ is a non-empty hypersurface with trivial normal bundle. Then there exists $c > 0$ such that $(M, c^2\, g)$ is Ricci closable.*

Proof. Since Y is the fixed point set of an isometric action, it is totally geodesic. Moreover, since the normal bundle of Y is trivial, cutting M along Y results in a manifold with two totally geodesic boundary components isometric to Y. By [23] these two boundary components cannot lie in the same connected component of $\overline{M\setminus Y}$, so M decomposes into $M_1 \cup_Y M_2$, where $M_1, M_2 \subset M$ are submanifolds with common boundary Y, and the action of $\mathbb{Z}/2$ yields an isometry $\phi : (M_1, g|_{M_1}) \to (M_2, g|_{M_2})$.

We now slightly deform the metric $g_1 = g|_{M_1}$ near $\partial M_1 = Y$ as follows. First, we rescale the metric g so that the Ricci curvatures of the induced metric on Y are bounded from below by $-(n-2)$. A neighbourhood of the boundary ∂M_1 can be identified with $[0,\varepsilon]\times Y$ such that the metric g_1 is of the form $dt^2 + h_t$ on this part, where h_t is a smoothly varying family of metrics on Y. By Lemma 2.4, since Y is totally geodesic, we have $\partial_t h_t = 0$ at $t = 0$.

Now let $f : [0,\varepsilon] \to (0,\infty)$ be a smooth function satisfying

$$-\frac{f''}{f} - (n-2)\frac{1+f'^2}{f^2} > 0, \tag{3.1}$$

and such that $f(0) = 1$ and f is an even function at $t = 0$. Such a function can for example be obtained as the solution of the initial value problem

$$f'' = -f - (n-2)\frac{1+f'^2}{f}, \quad f(0) = 1,\ f'(0) = 0,$$

where we possibly need to choose ε smaller to ensure that a solution exists, and subsequently smoothing the function $t \mapsto f(|t|)$ on $[-\varepsilon,\varepsilon]$ at $t = 0$ to obtain an odd function at $t = 0$. Since the original function is already C^2 at $t = 0$, the smoothing can be done C^2-close, so that (3.1) is still satisfied.

By Lemma 2.5, the metric $dt^2 + f(t)^2 h_0$ on $[0,\varepsilon]\times Y$ has positive Ricci curvature. Hence, since the 1-jets of the metrics g_1 and $dt^2 + f(t)^2 h_0$ coincide on Y, we can apply the deformation result of [29], see also [30], to deform g_1 through metrics g_s, $s \in [0,1]$, of positive Ricci curvature into a metric g_0 that is of the form $dt^2 + f(t)^2 h_0$ on $[0,\varepsilon']\times Y$ for some $\varepsilon' \in (0,\varepsilon]$. Moreover, since near Y the metric g_s defined in [29] is a convex combination of the metrics g_0 and g_1, and since both the metric g_0 and g_1 define a smooth metric on the double $M_1 \cup_Y M_1$, the same holds true for all metrics g_s.

Next, we consider the space $N = I \times M_1$, where $I = [0,\pi]$. Then

$$\partial N \cong M_1 \cup_Y (I\times Y) \cup_Y M_1 \cong M_1 \cup_Y M_1 \cong M.$$

To obtain a smooth metric on N, we choose a smooth function $k : [0,\varepsilon'] \to [0,\infty)$ such that

(1) k is an odd function at $t = 0$ with $k'(0) = 1$ and $k'''(0) < 0$,
(2) $k(\varepsilon') > 0$ and all derivatives of k vanish at $t = \varepsilon'$,
(3) $k'' < 0$ on $(0,\varepsilon')$.

We then define the metric g_N on N by

$$g_N = \begin{cases} k(t)^2 ds^2 + dt^2 + f(t)^2 h_0, & \text{on } I\times[0,\varepsilon']\times Y,\\ k(\varepsilon')^2 ds^2 + g_0, & \text{else.}\end{cases}$$

Here ds^2 denotes the standard metric on I. Since $k(0) = 0$, the metric g_N is in fact a metric on

$$(I\times M_1) \cup_{I\times Y} (D^2_+ \times Y),$$

which is diffeomorphic to the space obtained from N by smoothing the corners. Here $D^2_+ = (\mathbb{R}\times[0,\infty)) \cap D^2$ is a half-disc and we identify the interior face $(\mathbb{R}\times\{0\}) \cap D^2$ of its boundary with I.

By the boundary conditions of f and k at $t = 0$, doubling the metric g_N along its boundary results in a smooth metric on $(S^1 \times M_1) \cup_{S^1 \times Y} (D^2 \times Y)$, see e.g. [31, Proposition 1.4.7]. In particular, the metric g_N itself is smooth and has totally geodesic boundary. Moreover, by Lemma 2.5, the metric g_N has non-negative Ricci curvature, and strictly positive Ricci curvature on $I \times [0, \varepsilon'] \times Y$. Hence, by [20, Proposition 2.15], we can deform the metric g_N into a metric (which we again denote by g_N) of strictly positive Ricci curvature and convex boundary, while leaving the induced metric on the boundary unchanged.

The induced metric on the boundary is given by the double of the metric g_0. By applying the deformation g_s, we obtain a deformation through metrics of positive Ricci curvature to the double of the metric g_1, i.e. to g. Hence, by [18, Theorem C], we obtain a metric of positive Ricci curvature on N with convex boundary isometric to g. In particular, by Lemma 3.1, there exists $c > 0$ so that $(M, c^2 g)$ is Ricci closable. □

We now apply Proposition 3.2 to the following 2ℓ-fold connected sum.

Proposition 3.3. *Let $M_1^n, \ldots, M_\ell^n$, $n \geq 3$, be closed manifolds that admit core metrics. Then there exists a metric g of positive Ricci curvature on the space*

$$M_1 \# \ldots \# M_\ell \# (-M_1) \# \ldots \# (-M_\ell)$$

such that for each summand there exists an isometric embedding of $\pm M_i \setminus D^{n\circ}$ equipped with the corresponding core metric and such that $c^2 g$ is Ricci closable for some $c > 0$.

Proof. We use Perelman's "docking station" [16], which, for any $\ell \in \mathbb{N}$ and $\nu > 0$ sufficiently small is a metric of positive Ricci curvature on $S^n \setminus \sqcup_\ell D^{n\circ}$ with round boundary components on which the second fundamental form is at least $-\nu$. This metric is the combination of the "neck" of [16, Section 2] and the "ambient space" of [16, Section 3], see also [17, Section 4].

More precisely, the "ambient space" is a doubly warped product metric of positive sectional curvature on the sphere S^n given by

$$dt^2 + \cos(t)^2 dx^2 + R(t)^2 ds_{n-2}^2$$

for $t \in [0, \frac{\pi}{2}]$ and dx^2 denotes the standard metric on S^1. The function R is a smooth function which is odd at $t = 0$ with $R'(0) = 1$ and even at $t = \frac{\pi}{2}$, and satisfies $R'' < 0$. Note that, while the metric is defined a priori on $[0, \frac{\pi}{2}] \times S^1 \times S^{n-2}$, the resulting space is the join of S^1 and S^{n-2}, which is indeed the sphere S^n, by the condition $R(0) = 0$ (as R is odd at $t = 0$).

From this metric one now cuts out ℓ small disjoint geodesic balls along the circle $\{t = 0\}$, and for a suitable choice of R one can glue in ℓ copies of the "neck metric" on $[0, 1] \times S^{n-1}$, that transitions to the round metric and a sufficiently small second fundamental form.

By choosing sufficiently small radii and placing all geodesic balls on one side of the circle, we can restrict this metric to the hemisphere S_+^n defined by $[0, \frac{\pi}{2}] \times S_+^1 \times S^{n-2}$, where $S_+^1 \subseteq S^1$ is a half-circle, and we can still attach ℓ disjoint copies of the "neck". By choosing ν sufficiently small, we can now glue $M_1 \setminus D^{n\circ}, \ldots, M_\ell \setminus D^{n\circ}$ to $S_+^n \setminus \sqcup_\ell D^{n\circ}$ using Theorem 2.3, which results in a metric of positive Ricci curvature on $(M_1 \# \ldots \# M_\ell) \setminus D^{n\circ}$. Moreover, by the form of the original metric, doubling results in a smooth metric on

$$M_1 \# \ldots \# M_\ell \# (-M_1) \# \ldots \# (-M_\ell),$$

and interchanging $M_1 \# \ldots \# M_\ell$ and $-(M_1 \# \ldots \# M_\ell)$ defines an isometric $\mathbb{Z}/2$-action with fixed point set the hypersurface along which we have glued. Hence, the claim follows from Proposition 3.2. □

Lemma 3.4. *Let M^n be a closed manifold that admits a core metric g. Then there exists $\nu > 0$ and a smooth family g_s, $s \in (0, 1]$, of Riemannian metrics on $M \setminus D^{n\circ}$ with $g_1 = g$ such that the following holds:*

1. *The volume and the Ricci curvatures of g_s are bounded from below by positive constants that do not depend on s,*
2. *For all $s \in (0, 1]$ the induced metric on the boundary $S^{n-1} = \partial(M \setminus D^{n\circ})$ is the round metric of radius 1 and the principal curvatures are bounded from below by ν,*
3. *As $s \to 0$ the sequence $(M \setminus D^{n\circ}, g_s)$ Gromov–Hausdorff converges to the disc D^n equipped with a rotationally symmetric Riemannian metric with a singularity at the origin.*

Proof. The proof is an adaptation of [10, Section 4.1]. Let g be a metric of positive Ricci curvature on $M \setminus D^{n\circ}$ with round and convex boundary and let $\nu > 0$ such that the principal curvatures at the boundary are all at least 2ν. For $s \in (0, \frac{\pi}{4\nu})$ we define the metric h_s on $[s, \frac{\pi}{4\nu}] \times S^{n-1}$ by

$$h_s = dt^2 + 2\sin^2(\nu t) ds_{n-1}^2.$$

By Lemma 2.5, the Ricci curvatures of the metric h_s are bounded from below by a positive constant independent of s. The boundary component $\{t = \frac{\pi}{4\nu}\}$ is round with principal curvatures given by ν, while the boundary component $\{t = s\}$ is, after rescaling, round with principal curvatures given by $-\sqrt{2}\nu\cos(\nu s) > -2\nu$. Hence, we can glue the Riemannian manifolds

$$(M \setminus D^{n\circ}, 2\sin^2(\nu s) g) \quad \text{and} \quad ([s, \tfrac{\pi}{4\nu}] \times S^{n-1}, h_s)$$

using Theorem 2.3 preserving the lower bounds on the Ricci curvatures. By the explicit form of the smoothing in Theorem 2.3, see [28, Section 2], we can arrange that the resulting family of metrics is smooth in s. This family is the required family g_s. □

Proof of Theorem A. For $n = 2$, the statement is trivial as all M_i are necessarily S^2 (recall that a closed manifold with a core metric is simply-connected). For $n \geq 3$, we start with the metric g on

$$X = M_1 \# \ldots \# M_\ell \# (-M_1) \# \ldots \# (-M_\ell)$$

constructed in Proposition 3.3. We set $M_{\ell+i} = -M_i$ for $i \in \{1, \ldots, \ell\}$. For $\bar{s} = (s_1, \ldots, s_{2\ell}) \in (0,1]^{2\ell}$ we now define the metric $g_{\bar{s}}$ as the metric obtained from g by replacing each core metric on M_i by the metric g_{s_i} constructed in Lemma 3.4 and smoothing the gluing area using Theorem 2.3. Then, after suitable rescaling, the family $\{g_{\bar{s}}\}$ satisfies the requirements of Theorem 2.2, so that we obtain a non-collapsed Ricci limit space (Y, d_Y, p) satisfying $\Omega_{Y,p} = \overline{\{(X, g_{\bar{s}})\}}$

Moreover, for each $i \in \{1, \ldots, \ell\}$, the sequence of metrics $g_{\bar{s}}$ with $\bar{s} = (s, \ldots, s, 1, s, \ldots, s)$, where the entry 1 is at position i, converges to a metric d_i on M_i with $(2\ell - 1)$ singularities as $s \to 0$. In particular, $(M_i, d_i) \in \Omega_{Y,p}$. □

Proof of Corollary B. As explained in [32], if M^4 admits a Riemannian metric of positive scalar curvature, it is homeomorphic to either

$$\#_\ell (S^2 \times S^2) \quad \text{or} \quad \#_k \mathbb{C}P^2 \#_\ell (-\mathbb{C}P^2).$$

For convenience we outline the argument. First note that, since M is simply-connected, its second Stiefel–Whitney class equals its second Wu class and hence its intersection form is even if and only if it is spin. For the results on 4-manifolds and intersection forms we will use we refer to [33, Section 1.2].

Since in dimension 4 the signature is a multiple of the $\hat{A}$-genus, it follows that M has vanishing signature if it is spin. Hence, the intersection form of M is isomorphic to that of $\#_\ell (S^2 \times S^2)$ as they have the same rank, parity and signature. If M is non-spin, its intersection form is odd, and hence its intersection form is equivalent to that of $\#_k \mathbb{C}P^2 \#_\ell (-\mathbb{C}P^2)$ by Donaldson's theorem (see [34], [33, Theorem 1.2.30]). Hence, by Freedman's theorem (see [35], [33, Theorem 1.2.27]), M is homeomorphic to $\#_\ell (S^2 \times S^2)$ when it is spin, and to $\#_k \mathbb{C}P^2 \#_\ell (-\mathbb{C}P^2)$ when it is non-spin. By [17,20,21] all of these spaces admit core metrics, hence the first claim follows from Theorem A.

For the second claim, assume that M bounds a compact, oriented 5-manifold. Then M has vanishing signature. Further, since the signature is an oriented homeomorphism invariant, it follows that M is homeomorphic to one of $\#_\ell (S^2 \times S^2)$ and $\#_\ell \mathbb{C}P^2 \#_\ell (-\mathbb{C}P^2)$. All these manifolds admit Ricci closable metrics by Proposition 3.3 and Proposition 4.3 below. □

4. Further examples of Ricci closable manifolds

In this section, we collect results to construct Ricci closable manifolds. The proofs directly follow from well-known construction methods for positive Ricci curvature. We therefore omit most of the proofs.

Proposition 4.1. *Let $E \xrightarrow{\pi} B$ be a fibre bundle with fibre F and structure group G such that E is closed. Suppose the following:*

(1) B admits a Riemannian metric g_B of positive Ricci curvature,
(2) F admits a G-invariant metric g_F of positive Ricci curvature,
(3) There exists a compact Riemannian manifold $(\overline{F}, \overline{g})$ of positive Ricci curvature with convex boundary isometric to (F, g_F) such that the G-action on F extends to an isometric action on $\overline{F}$.

Then E admits a Ricci closable metric g_E such that $(E, g_E) \xrightarrow{\pi} (B, c^2 g_B)$ is a Riemannian submersion with totally geodesic fibres isometric to ${c'}^2 g_F$ for some $c, c' > 0$.

This follows for example from the methods of [36, Section 9.G], [37, Theorem 2.7.3], [38] to lift metrics of positive Ricci curvature along fibre bundles.

In particular, the assumptions of Proposition 4.1 are satisfied when π is a linear S^p-bundle with $p \geq 2$ whose base admits a Riemannian metric of positive Ricci curvature. Indeed, by setting $g_F = ds_p^2$ and $\overline{F} = D^{p+1}$ equipped with the induced metric of a geodesic ball in the round sphere of some suitable radius condition (3) is satisfied. In the case of a 1-dimensional fibre we have the following result:

Proposition 4.2. *Let $E \xrightarrow{\pi} B$ be a principal S^1-bundle such that B is closed and admits a Riemannian metric g_B of positive Ricci curvature and E has finite fundamental group. Then E admits a Ricci closable metric g_E such that $(E, g_E) \xrightarrow{\pi} (B, c^2 g_B)$ is a Riemannian submersion for some $c > 0$.*

Proof. As in Proposition 4.1, the corresponding linear D^2-bundle $\overline{E}$ has positive Ricci curvature when we equip it with a submersion metric with totally geodesic fibres equipped with the metric of a sufficiently small round hemisphere (see e.g. [36, 9.59 and 9.70]). Here we can freely choose the principal connection. In particular, the boundary E is totally geodesic as well. If we choose this metric to have harmonic curvature form, the induced metric on the boundary has non-negative Ricci curvature by [39] and this metric can be deformed to have positive Ricci curvature by [40]. Hence, by the deformation of [20, Proposition 2.15], followed by a small deformation that makes the Ricci curvatures on E positive, we obtain a metric on $\overline{E}$ with the required properties. □

Proposition 4.2 can for example be applied to connected sums

$$L(m;1,\overset{n+1}{\cdots},1)\#_\ell(S^2\times S^{2n-1})$$

when m is odd or n is odd, and to

$$L(m;1,\overset{n+1}{\cdots},1)\#_\ell(S^2\mathbin{\tilde{\times}} S^{2n-1})$$

when m and n are even, where $S^2\mathbin{\tilde{\times}} S^{2n-1}$ denotes the total space of the unique non-trivial linear S^{2n-1}-bundle over S^2 and $\ell\in\mathbb{N}_0$. Indeed, by [41, Theorems A and B], these manifolds are total spaces of principal S^1-bundles over $\#_{\ell+1}\mathbb{C}P^n$.

When $m=1$ this shows that the manifold $\#_\ell(S^2\times S^{2n-1})$ admits a Ricci closable metric for all $\ell\geq 0$. By Proposition 3.3 the same holds for connected sums of other products of spheres when ℓ is even. We now extend this to odd values of ℓ and arbitrary dimensions of the spheres involved.

Proposition 4.3. *For $p,q\geq 2$, $\ell\geq 0$, there exists a Ricci closable metric on the connected sum $\#_\ell(S^p\times S^q)$.*

Proof. If ℓ is even, this follows from Proposition 3.3 since there is an orientation-preserving diffeomorphism $S^p\times S^q\cong -(S^p\times S^q)$. Hence, it remains to consider the case where ℓ is odd.

For that we use the construction of Sha–Yang [27] (see also [42, Section 5.5] for a discussion of this result), where $\#_\ell(S^p\times S^q)$ is constructed as the manifold obtained from $(\ell+1)$ surgeries on the second factor of $S^{p+1}\times S^{q-1}$, i.e.

$$\#_\ell(S^p\times S^q)\cong\left(\left(S^{p+1}\setminus\bigsqcup_{\ell+1}D^{p+1^\circ}\right)\times S^{q-1}\right)\cup_{\sqcup_{\ell+1}(S^p\times S^{q-1})}\bigsqcup_{\ell+1}(S^p\times D^q).$$

The starting point for the construction is the round metric on S^{p+1}, from which $(\ell+1)$ pairwise disjoint geodesic balls are removed. After taking the product with S^{q-1} equipped with a round metric of sufficiently small radius, $(\ell+1)$ copies of $(S^p\times D^q)$, equipped with suitable metrics that glue smoothly with the metric on $(S^{p+1}\setminus\sqcup_{\ell+1}D^{p+1^\circ})\times S^{q-1}$, are attached.

Since we can freely choose the positions and radii of the geodesic balls in S^{p+1}, we can arrange that, since $(\ell+1)$ is even, each one gets mapped to another one under the isometric $\mathbb{Z}/2$-action on S^{p+1} given by reflection along the equator. In this way we obtain a metric of non-negative Ricci curvature, and strictly positive Ricci curvature if $q>2$, on the glued space that is invariant under a $\mathbb{Z}/2$-action, whose fixed point set is given by $S^p\times S^{q-1}$ equipped with the product of two round metrics. For $q=2$ one additionally applies the deformation results of [43], which, as explained in [43, p. 20], can be arranged to preserve the $\mathbb{Z}/2$-invariance of the metric. Hence, we can apply Proposition 3.2 to obtain a Ricci closable metric on $\#_\ell(S^p\times S^q)$. □

Finally, we consider Ricci closable metrics with large isometry group.

Proposition 4.4. *Let G/H be a homogeneous space with compact Lie groups $H\subseteq G$ such that the fundamental group of G/H is finite. Suppose that there exists a subgroup $H\subseteq K\subseteq G$ such that G/K has finite fundamental group and K/H is diffeomorphic to a sphere of dimension $d\geq 1$. Then G/H admits a homogeneous metric that is Ricci closable.*

This follows from [44], since under these assumptions the manifold G/H is the boundary of the cohomogeneity one manifold $G\times_K D^{d+1}$, where D^{d+1} is identified with the cone over K/H.

For example, Proposition 4.4 can be applied to the quotient of $S^{4n-1}\subseteq\mathbb{H}^n$ by the standard action of the binary dihedral group $Dic_m\subseteq S^3=Sp(1)$. Indeed, this space is the homogeneous space $G/H=Sp(n)/Sp(n-1)Dic_m$, and the group K is given by $Sp(n-1)Pin(2)$.

Proposition 4.5. *Let M be a closed manifold that admits a cohomogeneity one action of a compact Lie group G such that the orbit space M/G is homeomorphic to the interval $[-1,1]$ and such that both M and its principal orbits have finite fundamental group. Suppose that the associated group diagram $H\subseteq K_\pm\subseteq G$ satisfies $K_+=K_-$. Then M admits a G-invariant Riemannian metric that is Ricci closable.*

See e.g. [44, Section 1] for an introduction to cohomogeneity one manifolds. The proof then directly follows from the construction in [44] together with Proposition 3.2 since the $\mathbb{Z}/2$-action that interchanges the two halves $\pi^{-1}([-1,0])$ and $\pi^{-1}([0,1])$, where $\pi: M\to M/G$ denotes the projection, is isometric and has fixed point set G/H.

For example, Proposition 4.5 can be applied to $\mathbb{C}P^n\#(-\mathbb{C}P^n)$, which has group diagram $H\subseteq K_\pm\subseteq G$ with $H=U(n-1)$, $K_+=K_-=U(n-1)U(1)$ and $G=U(n)$. In fact, since the principal orbits have codimension two, it follows from the construction in [45] that we can construct a Ricci closable metric that has non-negative sectional curvature.

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Appendix. Tangent cones of collapsed Ricci limit spaces

In the context of surgery on manifolds of positive Ricci curvature, Sha–Yang [27] constructed a metric on $S^n \times D^m$ of $\mathrm{Ric} \geq 0$ that is close to the product $C(S^n, ds_n^2) \times (S^{m-1}, R^2 ds_{m-1}^2)$ for some $R > 0$ near the boundary. A consequence of this construction is the following theorem. We include its proof for convenience.

Theorem A.1. *Let (M^n, g) be a closed Riemannian manifold with $\mathrm{Ric} \geq (n-1)$. Then, for any $m \geq 2$ there exists a complete Riemannian metric of $\mathrm{Ric} \geq 0$ on $M \times \mathbb{R}^m$ with asymptotic cone given by $C(M, g)$.*

Proof. Let $\alpha = 2\frac{n-1}{m}$ and let $f : [0, \infty) \to \mathbb{R}$ be the unique solution of the initial value problem

$$f'' = \frac{\alpha}{2} f^{-\alpha-1},$$
$$f(0) = 1,$$
$$f'(0) = 0.$$

Further, let $h : [0, \infty) \to \mathbb{R}$ be the function

$$h = \frac{2}{\alpha} f'$$

and define the metric $g_{f,h}$ on $[0, \infty) \times S^{m-1} \times M$ by

$$g_{f,h} = dt^2 + h(t)^2 ds_{m-1}^2 + f(t)^2\ g.$$

By the initial conditions of f we have $h(0) = 0$ and $h'(0) = 1$, and by the defining equation of f we obtain inductively that at $t = 0$ the function h is odd and f is even. Therefore, the metric $g_{f,h}$ defines a smooth metric on $\mathbb{R}^m \times M$.

By Lemma 2.5, the Ricci curvatures of the metric $g_{f,h}$ are given as follows:

$$\mathrm{Ric}(\partial_t, \partial_t) = -(m-1)\frac{h''}{h} - n\frac{f''}{f},$$
$$\mathrm{Ric}(\tfrac{u}{h}, \tfrac{u}{h}) = -\frac{h''}{h} + (m-2)\frac{1-h'^2}{h^2} - n\frac{h'f'}{hf},$$
$$\mathrm{Ric}(\tfrac{v}{f}, \tfrac{v}{f}) = -\frac{f''}{f} + \frac{\mathrm{Ric}^g(v,v) - (n-1)f'^2}{f^2} - (m-1)\frac{h'f'}{hf}$$
$$\geq -\frac{f''}{f} + (n-1)\frac{1-f'^2}{f^2} - (m-1)\frac{h'f'}{hf},$$
$$\mathrm{Ric}(\partial_t, u) = \mathrm{Ric}(\partial_t, v) = \mathrm{Ric}(u, v) = 0,$$

where $u \in TS^{m-1}$ and $v \in TM$ are unit vectors with respect to ds_{m-1}^2 and g, respectively.

Integrating the equation $f''f' = \frac{\alpha}{2} f^{-\alpha-1} f'$ now shows that

$$f'^2 = 1 - f^{-\alpha}.$$

Further, we have

$$\frac{1-h'^2}{h^2} = \frac{\alpha^2}{4}\frac{1-f^{-2\alpha-2}}{1-f^{-\alpha}} \geq \frac{\alpha^2}{4} f^{-\alpha-2}, \quad \frac{h''}{h} = -\frac{\alpha(\alpha+1)}{2} f^{-\alpha-2}, \quad \frac{h'f'}{hf} = \frac{\alpha}{2} f^{-\alpha-2}.$$

Using this a calculation shows that all Ricci curvatures are non-negative.

Finally, since $f'' > 0$, we have $f(t) \to \infty$ as $t \to \infty$, and therefore $f'(t) \to 1$ and $h(t) \to \frac{2}{\alpha}$ as $t \to \infty$. This shows that $R^2 g_{f,h}$ converges to the cone $C(M, g)$ as $R \to 0$. □

Remark A.2. Similar arguments using the generalisations of [27] given in [19,46] show that $M \times \mathbb{R}^m$ in Theorem A.1 can be replaced by $M \times \mathrm{int}(N)$, where N is a compact manifold with boundary that admits a Riemannian metric of positive Ricci curvature such that on the boundary the second fundamental form is non-negative and the induced metric has positive Ricci curvature (e.g. if $N = N' \setminus D^{m\circ}$ and N' admits a core metric).

Data availability

No data was used for the research described in the article.

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