Dichotomies for Maximum Matching Cut: H-freeness, bounded diameter, bounded radius [☆,☆☆]

Felicia Lucke [a], Daniël Paulusma [b], Bernard Ries [a,*]

[a] *Department of Informatics, University of Fribourg, Fribourg, Switzerland*
[b] *Department of Computer Science, Durham University, Durham, UK*

A B S T R A C T

The (PERFECT) MATCHING CUT problem is to decide if a graph G has a (perfect) matching cut, i.e., a (perfect) matching that is also an edge cut of G. Both MATCHING CUT and PERFECT MATCHING CUT are known to be NP-complete. A perfect matching cut is also a matching cut with maximum number of edges. To increase our understanding of the relationship between the two problems, we perform a complexity study for the MAXIMUM MATCHING CUT problem, which is to determine a largest matching cut in a graph. Our results yield full dichotomies of MAXIMUM MATCHING CUT for graphs of bounded diameter, bounded radius and H-free graphs. A disconnected perfect matching of a graph G is a perfect matching that contains a matching cut of G. We also show how our new techniques can be used for finding a disconnected perfect matching with a largest matching cut for special graph classes. In this way we can prove that the decision problem DISCONNECTED PERFECT MATCHING is polynomial-time solvable for $(P_6 + sP_2)$-free graphs for every $s \geq 0$, extending a known result for P_5-free graphs (Bouquet and Picouleau, 2020).

1. Introduction

A *matching M* (i.e., a set of pairwise disjoint edges) of a connected graph $G = (V, E)$ is a *matching cut* if V can be partitioned into a set of blue vertices B and a set of red vertices R, such that M consists of all the edges with one end-vertex in B and the other one in R. Graphs with matching cuts were introduced in 1970 by Graham [24] (as *decomposable* graphs) to solve a problem on cube numbering. Other relevant applications include ILFI networks [15], WDM networks [1], graph drawing [39] and surjective graph homomorphisms [21].

The decision problem is called MATCHING CUT: does a given connected graph have a matching cut? In 1984, Chvátal [11] proved that it is NP-complete even for graphs of maximum degree at most 4. Afterwards, parameterized and exact algorithms were given [2,8,20,23,29,30]. A variant called DISCONNECTED PERFECT MATCHING "does a connected graph have a perfect matching that contains a matching cut?" has also been studied [7,17,37] (see Section 1.2 for more on this problem). Moreover, MATCHING CUT was generalized, for every $d \geq 1$, to d-CUT "does a connected graph have an edge cut where each vertex has at most d neighbours across the cut?" [3,23]. In particular, many results have appeared where the input for MATCHING CUT was restricted to some special graph
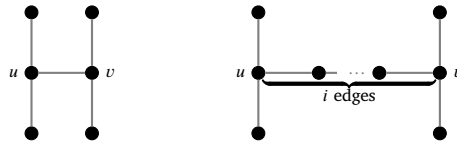

Fig. 1. The graphs H_1^* (left) and H_i^* (right).

class, and this is what we do in our paper as well. We first discuss related work, restricting ourselves mainly to those classes relevant to our paper (see, for example, [8] for a more comprehensive overview):

- graphs of bounded diameter;
- graphs of bounded radius;
- hereditary graph classes; in particular H-free graphs.

The *distance* between two vertices u and v in a connected graph G is the *length* (number of edges) of a shortest path between u and v in G. The *eccentricity* of a vertex u is the maximum distance between u and any other vertex of G. The *diameter*, denoted by diameter(G), and *radius*, denoted by radius(G), are the maximum and minimum eccentricity, respectively, over all vertices of G; note that radius$(G) \leq$ diameter$(G) \leq 2 \cdot$ radius(G) for every graph G.

The MATCHING CUT problem is polynomial-time solvable for graphs of diameter at most 2 [6,31]. This result was extended to graphs of radius at most 2 [35]. In contrast, the problem is NP-complete for graphs of diameter at most 3 [31], yielding two dichotomies:

Theorem 1 *([31,35]). For an integer $d \geq 1$,* MATCHING CUT *for graphs of diameter d and for graphs of radius d is polynomial-time solvable if $d \leq 2$ and NP-complete if $d \geq 3$.*

A class of graphs is *hereditary* if it is closed under vertex deletion. Hereditary graph classes include many well-known classes, such as those that are H-free for some graph H. A graph G is H-*free* if G does not contain H as an *induced* subgraph, that is, G cannot be modified into H by a sequence of vertex deletions. For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. If $\mathcal{H} = \{H_1, \ldots, H_p\}$ for some $p \geq 1$, we also say that G is $(H_1, \ldots, H_p)$-*free*. Note that a class of graphs $\mathcal{G}$ is hereditary if and only if there is a set of graphs $\mathcal{H}$, such that every graph in $\mathcal{G}$ is $\mathcal{H}$-free. Hence, for a *systematic* complexity study, it is natural to first focus on the case where $\mathcal{H}$ has size 1; see, e.g., [9,10,12,19,27,41].

For an integer $r \geq 1$, let P_r denote the path on r vertices, $K_{1,r}$ the star on $r+1$ vertices, and $K_{1,r} + e$ the graph obtained from $K_{1,r}$ by adding one edge (between two leaves). The graph $K_{1,3}$ is also known as the *claw*. For $s \geq 3$, let C_s denote the cycle on s vertices. Let H_1^* be the graph that looks like the letter "H", and for $i \geq 2$, let H_i^* be the graph obtained from H_1^* by subdividing the middle edge of H_1^* exactly $i - 1$ times; see also Fig. 1. We denote the *disjoint union* of two graphs G_1 and G_2 by $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We denote by sG the disjoint union of s copies of G, for $s \geq 1$.

Polynomial-time algorithms for MATCHING CUT exist for *subcubic* graphs (graphs of maximum degree at most 3) [11], $K_{1,3}$-free graphs [5], P_6-free graphs [35], $(K_{1,4}, K_{1,4} + e)$-free graphs [30] and *quadrangulated* graphs, i.e., $(C_5, C_6, \ldots)$-free graphs [38]; the latter class contains the class of *chordal* graphs, i.e., $(C_4, C_5, C_6, \ldots)$-free graphs. Moreover, if MATCHING CUT is polynomial-time solvable for H-free graphs, then it is so for $(H + P_3)$-free graphs [35]. The problem is NP-complete even for graphs of maximum degree at most 4 [11]; $K_{1,4}$-free graphs [11] (see [5,30]); planar graphs of girth 5 [5]; $K_{1,5}$-free bipartite graphs [33]; graphs of girth at least g, for every $g \geq 3$ [17]; $(3P_5, P_{15})$-free graphs [37] (improving a result of [16]); bipartite graphs where the vertices in one bipartition class all have degree exactly 2 [38] and thus for H_i^*-free graphs for every odd $i \geq 1$; and for H_i^*-free graphs for every even $i \geq 2$ [17]. Recently, Le and Le [32] proved that MATCHING CUT is NP-complete even for $(3P_6, 2P_7, P_{14})$-free graphs. In fact, their hardness gadget also works for PERFECT MATCHING CUT (defined below) and DISCONNECTED PERFECT MATCHING and can be readily checked to have diameter 4 and radius 3.

The above results imply the following partial complexity classification, which leaves open only a number of cases where H is a forest, each connected component of which is either a path or a *subdivided claw* (tree with one vertex of degree 3 and all other vertices of degree at most 2). For two graphs H and H', we write $H \subseteq_i H'$ if H is an induced subgraph of H'.

Theorem 2 *([5,11,17,32,37,35,38]). For a graph H,* MATCHING CUT *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_3 + K_{1,3}$ or $sP_3 + P_6$ for some $s \geq 0$, and*
- *NP-complete if $H \supseteq_i K_{1,4}$, P_{14}, $2P_7$, $3P_5$, C_r for some $r \geq 3$ or H_j^* for some $j \geq 1$.*

1.1. Our focus

We already mentioned the known generalization of MATCHING CUT (i.e. 1-CUT) to d-CUT. In our paper, we consider a different kind of generalization, namely MAXIMUM MATCHING CUT, which is to determine a *largest* matching cut of a connected graph (if a matching cut exists). So far, it is known that MATCHING CUT is fixed-parameter tractable when parameterized by the size of the

Fig. 2. The graph P_6 with a matching cut of size 2 that is not contained in a (disconnected) perfect matching (left), a disconnected perfect matching with a matching cut of size 2 (middle) and a perfect matching cut (of size 3) (right). In each figure, thick edges denote matching cut edges.

cut; this even holds for d-CUT for every $d \geq 1$ [3,23]. However, for special graph classes, MAXIMUM MATCHING CUT has only been studied for the extreme case, where the task is to decide if a connected graph has a *perfect* matching cut which is a matching cut that is even a *perfect* matching, i.e., that saturates every vertex; see also Fig. 2. This variant was introduced as PERFECT MATCHING CUT by Heggernes and Telle [26], who proved that it is NP-complete. We briefly discuss some very recent results for PERFECT MATCHING CUT on special graph classes below.

It is readily seen that the gadget in the NP-hardness reduction of Heggernes and Telle [26] has diameter 6 and radius 3. We recall that the NP-hardness gadget of Le and Le [32] for $(3P_6, 2P_7, P_{14})$-free graphs even has diameter 4 (and radius 3). It is also known that PERFECT MATCHING CUT is polynomial-time solvable for graphs of radius (and thus also diameter) at most 2 [37]. Hence, we only obtain a dichotomy for graphs of bounded radius but in this case, only a partial complexity classification for graphs of bounded diameter.

Theorem 3 ([26,32,37]). *For integers d and r,* PERFECT MATCHING CUT *for graphs of diameter d and for graphs of radius r is polynomial-time solvable if $d \leq 2$ or $r \leq 2$, respectively, and* NP-*complete if $d \geq 4$ or $r \geq 3$, respectively.*

For $1 \leq h \leq i \leq j$, the graph $S_{h,i,j}$ is the tree of maximum degree 3 with exactly one vertex u of degree 3, whose leaves are at distance h, i and j, respectively, from u; note $S_{1,1,1} = K_{1,3}$.

It is known that PERFECT MATCHING CUT is polynomial-time solvable for $S_{1,2,2}$-free graphs (and thus for $K_{1,3}$-free graphs) [34]; P_6-free graphs [37]; and for pseudo-chordal graphs [34] (and thus for chordal graphs, i.e., $(C_4, C_5, \ldots)$-free graphs). Moreover, PERFECT MATCHING CUT is polynomial-time solvable for $(H + P_4)$-free graphs if it is polynomial-time solvable for H-free graphs [37]. It is also known that PERFECT MATCHING CUT is NP-complete even for 3-connected cubic planar bipartite graphs [4], $(3P_6, 2P_7, P_{14})$-free graphs [32], $K_{1,4}$-free bipartite graphs of girth g for every $g \geq 3$ [34] and for H_i^*-free graphs for every $i \geq 1$ [17]. This gives us a partial complexity classification:

Theorem 4 ([17,32,34,37]). *For a graph H,* PERFECT MATCHING CUT *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_4 + S_{1,2,2}$ or $sP_4 + P_6$ for some $s \geq 0$, and*
- NP-*complete if $H \supseteq_i K_{1,4}$, P_{14}, $2P_7$, $3P_6$, C_r for some $r \geq 3$ or H_j^* for some $j \geq 1$.*

From Theorem 4 it can be seen that again only cases where H is a forest, each connected component of which is either a path or a subdivided claw, remain open. However, the number of open cases is smaller than for MATCHING CUT, as can be seen from Theorem 2. So far, all known complexities for MATCHING CUT and PERFECT MATCHING CUT on special graph classes coincide except for (sub)cubic graphs.

We note that whenever MAXIMUM MATCHING CUT is polynomial-time solvable for some graph class, then so are MATCHING CUT and PERFECT MATCHING CUT. Similarly, if one of the latter two problems is NP-complete, then MAXIMUM MATCHING CUT is NP-hard. For instance, this immediately yields a complexity dichotomy for graphs of maximum degree at most Δ. Namely, as MAXIMUM MATCHING CUT is trivial if $\Delta = 2$ and PERFECT MATCHING CUT is NP-complete if $\Delta = 3$, we have a complexity jump from $\Delta = 2$ to $\Delta = 3$, just like PERFECT MATCHING CUT; recall that for MATCHING CUT this jump appears from $\Delta = 3$ to $\Delta = 4$. We consider the following research question:

For which graph classes is MAXIMUM MATCHING CUT *harder than* MATCHING CUT *and* PERFECT MATCHING CUT *and for which graph classes do the complexities coincide?*

1.2. Our results for Maximum Matching Cut

In Section 4 we show that MAXIMUM MATCHING CUT is NP-hard for $2P_3$-free quadrangulated graphs of diameter 3 and radius 2. We note that the restrictions to radius 2 and diameter 3 are not redundant: consider, for example, the P_6, which is $2P_3$-free but which has radius 3 and diameter 5. In the same section, we also show NP-hardness for subcubic line graphs of triangle-free graphs, or equivalently, subcubic $(K_{1,3},$ diamond)-free graphs (the diamond is obtained from the K_4 after removing an edge). These NP-hardness results are in stark contrast to the situation for MATCHING CUT and PERFECT MATCHING CUT, as evidenced by Theorems 1–4 and to the aforementioned result of Moshi [38] that MATCHING CUT is polynomial-time solvable for quadrangulated graphs.

Before proving these results, we first show in Section 3 that MAXIMUM MATCHING CUT is polynomial-time solvable for graphs of diameter 2, generalizing the known polynomial-time algorithms for MATCHING CUT and PERFECT MATCHING CUT for graphs of diameter at most 2. Hence, all three problems have the same dichotomies for graphs of bounded diameter.

We also prove in Section 3 that MAXIMUM MATCHING CUT is polynomial-time solvable for P_6-free graphs, generalizing the previous polynomial-time results for MATCHING CUT and PERFECT MATCHING CUT for P_6-free graphs. Due to the hardness result for $2P_3$-free graphs, we cannot show polynomial-time solvability for "$+P_4$" (as for PERFECT MATCHING CUT) or "$+P_3$" (as for MATCHING CUT). However, we can prove that if MAXIMUM MATCHING CUT is polynomial-time solvable for H-free graphs, then it is so for $(H + P_2)$-free graphs; again, see Section 3. The common proof technique for our polynomial-time results is as follows:

1. Translate the problem into a colouring problem. We pre-colour some vertices either red or blue, and try to extend the pre-colouring to a red-blue colouring of the whole graph via reduction rules. This technique has been used for Matching Cut and Perfect Matching Cut, but our analysis is different. In particular, the algorithms for Matching Cut and Perfect Matching Cut on P_6-free graphs use an algorithm for graphs of radius at most 2 as a subroutine. We cannot do this for Maximum Matching Cut, as we will show NP-hardness for radius 2.

2. Reduce the set of uncoloured vertices, via a number of branching steps, to an independent set, and then translate the problem into a matching problem. This is a new proof ingredient. The matching problem is to find a largest matching that saturates every vertex of the independent set of uncoloured vertices. Plesník [40] gave a polynomial time algorithm for this,[1] which we will use as subroutine.

The above polynomial-time and NP-hardness results yield the following three dichotomies for Maximum Matching Cut shown in Section 5; in particular we have obtained a complete complexity classification of Maximum Matching Cut for H-free graphs (whereas such a classification is only partial for the other two problems, as shown in Theorems 2 and 4).

Theorem 5. *For an integer d,* Maximum Matching Cut *on graphs of diameter d is*

- *polynomial-time solvable if $d \leq 2$, and*
- NP-*hard if $d \geq 3$.*

Theorem 6. *For an integer r,* Maximum Matching Cut *on graphs of radius r is*

- *polynomial-time solvable if $r \leq 1$, and*
- NP-*hard if $r \geq 2$.*

Theorem 7. *For a graph H,* Maximum Matching Cut *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_2 + P_6$ for some $s \geq 0$, and*
- NP-*hard if $H \supseteq_i K_{1,3}$, $2P_3$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

1.3. A second application of our proof techniques

In Section 6 we apply our techniques on the optimization variant of the problem Disconnected Perfect Matching. A *disconnected perfect matching* is a perfect matching that contains a matching cut. Disconnected perfect matchings were initially studied for cubic graphs from a graph-structural point of view [13,18]. The Disconnected Perfect Matching problem, which asks whether a given graph has a disconnected perfect matching, was introduced more recently, by Bouquet and Picouleau [7] (under a different name[2]). The problem is closely related to Perfect Matching Cut. Namely, every perfect matching cut is a disconnected perfect matching. However, the reverse might not be true, as illustrated by the C_6, which has a disconnected perfect matching but no perfect matching cut.

Bouquet and Picouleau [7] proved that Disconnected Perfect Matching can be solved in polynomial time for graphs of diameter 2 and is NP-complete for graphs of diameter 3 (and thus for graphs of radius 3). As the problem is trivial for graphs of radius 1, this leads to the following classification (in which the case where the radius is 2 remains open).

Theorem 8 ([7]). *For integers d and r,* Disconnected Perfect Matching *for graphs of diameter d and for graphs of radius r is polynomial-time solvable if $d \leq 2$ or $r \leq 1$, respectively, and* NP-*complete if $d \geq 3$ or $r \geq 3$, respectively.*

Bouquet and Picouleau [7] also proved that Disconnected Perfect Matching is polynomial-time solvable for bipartite graphs of diameter 3, $K_{1,3}$-free graphs and P_5-free graphs, and that it is NP-complete for bipartite graphs of diameter 4, $K_{1,4}$-free planar graphs, planar graphs of maximum degree 4, planar graphs of girth 5, and bipartite 5-regular graphs. As one of their open problems, they asked about the complexity for P_r-free graphs for $r \geq 6$. In [37] we showed that the problem is NP-complete for $(3P_7, P_{19})$-free graphs. This result was latter improved by Le and Le [32] to $(3P_6, 2P_7, P_{14})$-free graphs (we recall that they used the same gadget to prove the complexity of three problems simultaneously). Finally, NP-completeness has recently been shown for graphs of girth at least g for all fixed $g \geq 3$ [17], and thus for C_s-free graphs for all $s \geq 3$.

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[1] The polynomial-time algorithm of Plesník [40] solves a more general problem. It takes as input a graph G with an edge weighting w, a vertex subset S and two integers a and b. It then finds a maximum weight matching over all matchings that saturate S and whose cardinality is between a and b.

[2] Bouquet and Picouleau [7] use the name Perfect Matching-Cut instead of Disconnected Perfect Matching. To avoid confusion with Perfect Matching Cut we follow the terminology of Le and Telle [34] and use the name Disconnected Perfect Matching instead of Perfect Matching-Cut. We also note that in the literature the slightly similar name Disconnected Matching appears [22,25]. However, this name stands for the problem of determining the size of a largest matching whose vertex set induces a disconnected graph, so it is used for a different problem that does not involve edge cuts.

We now introduce the MAXIMUM DISCONNECTED PERFECT MATCHING problem. This problem is to determine a disconnected perfect matching of a connected graph G with a largest matching cut over all disconnected perfect matchings of G. This problem might seem artificial at first sight, but turns out to be highly useful for obtaining results for DISCONNECTED PERFECT MATCHING; note that polynomial-time results from the optimization version immediately carry over to the original variant.

By making minor modifications to our proofs, we can show exactly the same results for MAXIMUM DISCONNECTED PERFECT MATCHING as for MAXIMUM MATCHING CUT. So, in particular we prove that MAXIMUM DISCONNECTED PERFECT MATCHING is polynomial-time solvable for P_6-free graphs and for $(H + P_2)$-free graphs, if it is so for H-free graphs. Hence, combining these two results with the aforementioned result of [7] for $K_{1,3}$-free graphs, we immediately find that DISCONNECTED PERFECT MATCHING is polynomial-time solvable for $(K_{1,3} + sP_2)$-free graphs and $(P_6 + sP_2)$-free graphs. This means that we made further progress on the aforementioned open problem of [7]. By combining our new results with the above results from [7,17,32], we can now update the state-of-art summary from [17]:

Theorem 9. *For a graph H,* DISCONNECTED PERFECT MATCHING *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_2 + K_{1,3}$ or $sP_2 + P_6$ for some $s \geq 0$, and*
- NP-*complete if $H \supseteq_i K_{1,4}$, P_{14}, $3P_6$, $2P_7$, C_r for some $r \geq 3$ or H_j^* for some $j \geq 1$.*

Our new results for MAXIMUM DISCONNECTED PERFECT MATCHING, proven in Section 6, also lead to the following three dichotomies, as we will show in Section 6 as well.

Theorem 10. *For an integer d,* MAXIMUM DISCONNECTED PERFECT MATCHING *on graphs of diameter d is*

- *polynomial-time solvable if $d \leq 2$, and*
- NP-*hard if $d \geq 3$.*

Theorem 11. *For an integer r,* MAXIMUM DISCONNECTED PERFECT MATCHING *on graphs of radius r is*

- *polynomial-time solvable if $r \leq 1$, and*
- NP-*hard if $r \geq 2$.*

Theorem 12. *For a graph H,* MAXIMUM DISCONNECTED PERFECT MATCHING *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_2 + P_6$ for some $s \geq 0$, and*
- NP-*hard if $H \supseteq_i K_{1,3}$, $2P_3$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

In Section 7 we conclude our paper by stating a number of open problems.

2. Preliminaries

We consider finite, undirected graphs without multiple edges and self-loops. Let $G = (V, E)$ be a connected graph. For $u \in V$, the set $N(u) = \{v \in V \mid uv \in E\}$ is the *neighbourhood* of u in G, where $|N(u)|$ is the *degree* of u. For $S \subseteq V$, the *neighbourhood* of S is the set $N(S) = \bigcup_{u \in S} N(u) \setminus S$. The graph $G[S]$ is the subgraph of G *induced* by $S \subseteq V$, that is, $G[S]$ is the graph obtained from G after deleting the vertices not in S. We say that S is a *dominating* set of G, and that $G[S]$ *dominates* G if every vertex of $V \setminus S$ has at least one neighbour in S. The *domination number* of G is the size of a smallest dominating set of G. The set S is an *independent set* if no two vertices in S are adjacent and S is a *clique* if every two vertices in S are adjacent. A matching M is S-*saturating* if every vertex in S is an end-vertex of an edge in M. An S-saturating matching is *maximum* if there is no S-saturating matching of G with more edges. We will use the following result.

Theorem 13 ([40]). *For a graph G and set $S \subseteq V(G)$, it is possible in polynomial time to find a maximum S-saturating matching or conclude that G has no S-saturating matching.*

The *line graph* of a graph G is the graph $L(G)$ whose vertices are the edges of G, such that for every two vertices e and f, there exists an edge between e and f in $L(G)$ if and only if e and f share an end-vertex in G. A *linear forest* is a forest, each connected component of which is a path. A bipartite graph with non-empty partition classes V_1 and V_2 is *complete* if there is an edge between every vertex of V_1 and every vertex of V_2. If $|V_1| = k$ and $|V_2| = \ell$, then we denote the complete bipartite graph by $K_{k,\ell}$. We will need the following theorem.

Theorem 14 ([42]). *A graph G on n vertices is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. We can find such a dominating subgraph of G in $O(n^3)$ time.*

A *red-blue colouring* of a connected graph G colours every vertex of G either red or blue. If every vertex of a set $S \subseteq V$ has the same colour (red or blue), then S, and also $G[S]$, are called *monochromatic*. An edge with a blue and a red end-vertex is called *bichromatic*. A red-blue colouring is *valid* if every blue vertex has at most one red neighbour; every red vertex has at most one blue neighbour; and both colours red and blue are used at least once. A valid red-blue colouring is *perfect-extendable* if there is a perfect matching in G containing all bichromatic edges. For a valid red-blue colouring of G, we let R be the *red* set consisting of all vertices coloured red and B be the *blue* set consisting of all vertices coloured blue (so $V(G) = R \cup B$). Moreover, the *red interface* is the set $R' \subseteq R$ consisting of all vertices in R with a (unique) blue neighbour, and the *blue interface* is the set $B' \subseteq B$ consisting of all vertices in B with a (unique) red neighbour in R. The *value* of a valid red-blue colouring is its number of bichromatic edges, or equivalently, the size of its red (or blue) interface. A valid red-blue colouring is *maximum* if there is no valid red-blue colouring of the graph with a larger value. Similarly, a perfect-extendable red-blue colouring is *maximum* if there is no perfect-extendable red-blue colouring of the graph with a larger value.

We can now make the following observations, which can be easily verified (the notion of red-blue colourings has been used before; see, for example, [16,35]).

Observation 15. For every connected graph G and integer k, it holds that

- G has a matching cut with at least k edges if and only if G has a valid red-blue colouring of value at least k.
- G has a disconnected perfect matching with at least k edges belonging to a matching cut if and only if G has a perfect-extendable red-blue colouring of value at least k.

Observation 16. Every complete graph K_r with $r \geq 3$ and every complete bipartite graph $K_{r,s}$ with $\min\{r,s\} \geq 2$ and $\max\{r,s\} \geq 3$ is monochromatic.

3. Polynomial-time results for Maximum Matching Cut

In this section we prove three polynomial-time results that we need for obtaining the three dichotomies for MAXIMUM MATCHING CUT, as shown in Theorems 5–7. We first explain our general approach and some helpful lemmas.

The proof of our first lemma for MAXIMUM MATCHING CUT is very similar to the proofs of corresponding lemmas for MATCHING CUT [16] and PERFECT MATCHING CUT [37]. We include this proof for completeness. On an aside, the lemma implies that MAXIMUM MATCHING CUT is in XP when parameterized by the domination number of a graph.

Lemma 17. *For a connected n-vertex graph G with domination number g, it is possible to find a maximum red-blue colouring (if a red-blue colouring exists) in $O(2^g n^{g+2})$ time.*

Proof. Let D be a dominating set of G with $|D| = g$. We consider all $2^{|D|} = 2^g$ options of colouring the vertices of D red or blue. For every red vertex of D with no blue neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours blue (and thus all of its other neighbours will be coloured red). Similarly, for every blue vertex of D with no red neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours red (and thus all of its other neighbours will be coloured blue). Finally, for every red vertex in D with already one blue neighbour in D, we colour all its yet uncoloured neighbours red. Similarly, for every blue vertex in D with already one red neighbour in D, we colour all its yet uncoloured neighbours blue.

As D is a dominating set, the above means that we guessed a red-blue colouring of the whole graph G. We can check in $O(n^2)$ time if a red-blue colouring is valid and count its number of bichromatic edges. We take the valid red-blue colouring with largest value. The total number of red-blue colourings that we must consider is $O(2^g n^g)$. $\square$

Our general approach is to guess some "partial" red-blue colouring that we then try to extend to a maximum valid red-blue colouring of a graph. To explain this approach we first modify some terminology from [37] for matching cuts to work for maximum matching cuts as well.

Let $G = (V, E)$ be a connected graph and $S, T, X, Y \subseteq V$ be four non-empty sets with $S \subseteq X$, $T \subseteq Y$ and $X \cap Y = \emptyset$. A *red-blue (S,T,X,Y)-colouring* of G is a red-blue colouring of G, with a red set containing X; a blue set containing Y; a red interface containing S and a blue interface containing T. To obtain a red-blue (S,T,X,Y)-colouring, we start with two disjoint subsets S'' and T'' of V, called a *starting pair*, such that

(i) every vertex of S'' is adjacent to at most one vertex of T'', and vice versa, and
(ii) at least one vertex in S'' is adjacent to a vertex in T''.

Let S^* consist of all vertices of S'' with a (unique) neighbour in T'', and let T^* consist of all vertices of T'' with a (unique) neighbour in S''; so, every vertex in S^* has a unique neighbour in T^*, and vice versa. We call (S^*, T^*) the *core* of (S'', T''). Note that $|S^*| = |T^*| \geq 1$.

We now colour every vertex in S'' red and every vertex in T'' blue. Propagation rules will try to extend S'' to a set X, and T'' to a set Y, by finding new vertices whose colour must always be either red or blue. That is, we place new red vertices in the set X,
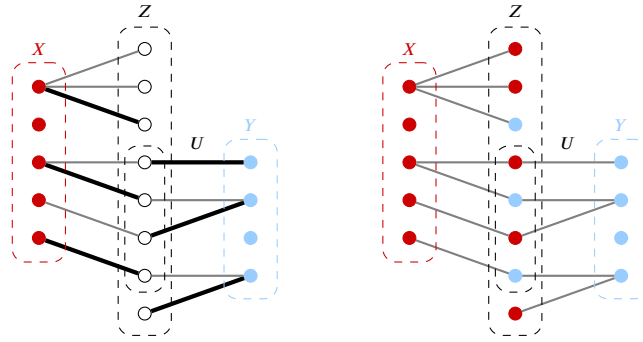
Fig. 3. An example (from [37]) of a red-blue (S', T', X', Y')-colouring of a graph with an intermediate 4-tuple (S', T', X', Y'). (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

which already contains S'', and new blue vertices in the set Y, which already contains T''. If a red and blue vertex are adjacent, then we add the red one to a set $S \subseteq X$ and the blue one to a set $T \subseteq Y$. So initially, $S := S^*$, $T := T^*$, $X := S''$ and $Y := T''$. We let $Z := V \setminus (X \cup Y)$.

Our task is to try to extend the partial red-blue colouring on $X \cup Y$ to a *maximum* valid red-blue (S, T, X, Y)-colouring of G, that is, a valid red-blue (S, T, X, Y)-colouring that has largest value over all valid red-blue (S, T, X, Y)-colourings of G. In order to do this, we present three propagation rules, which indicate necessary implications of previous choices.

We start with rules R1 and R2, which together correspond to the five rules from [31]. Rule R1 detects cases where we cannot extend the partial red-blue colouring defined on $X \cup Y$. Rule R2 tries to extend the sets S, T, X, Y as much as possible. While the sets S, T, X, Y grow, Rule R2 ensures that we keep constructing a (maximum) valid red-blue (S, T, X, Y)-colouring (assuming G has a valid red-blue (S, T, X, Y)-colouring).

R1. Return no (i.e., G has no red-blue (S, T, X, Y)-colouring) if a vertex $v \in Z$ is
   (i) adjacent to a vertex in S and to a vertex in T, or
   (ii) adjacent to a vertex in S and to two vertices in $Y \setminus T$, or
   (iii) adjacent to a vertex in T and to two vertices in $X \setminus S$, or
   (iv) adjacent to two vertices in $X \setminus S$ and to two vertices in $Y \setminus T$.

R2. Let $v \in Z$.
   (i) If v is adjacent to a vertex in S or to two vertices of $X \setminus S$, then move v from Z to X. If v is also adjacent to a vertex w in Y, then add v to S and w to T.
   (ii) If v is adjacent to a vertex in T or to two vertices of $Y \setminus T$, then move v from Z to Y. If v is also adjacent to a vertex w in X, then add v to T and w to S.

Suppose that exhaustively applying rules R1 and R2 on a starting pair (S'', T'') does not lead to a no-answer but to a tuple (S', T', X', Y'). Then, we call (S', T', X', Y') an *intermediate tuple*; see also Fig. 3. A propagation rule is *safe* if for every integer v the following holds: G has a valid red-blue (S, T, X, Y)-colouring of value v before the application of the rule if and only if G has a valid red-blue (S, T, X, Y)-colouring of value v after the application of the rule. Le and Le [31] proved the following lemma, which shows that R1 and R2 can be used safely and which is not difficult to verify. The fact that the value v is preserved in Lemma 18 (ii) below is implicit in their proof.

Lemma 18 ([31]). *Let G be a connected graph with a starting pair (S'', T'') with core (S^*, T^*), and with an intermediate tuple (S', T', X', Y'). The following three statements hold:*

   (i) $S^* \subseteq S'$, $S'' \subseteq X'$ and $T^* \subseteq T'$, $T'' \subseteq Y'$ and $X' \cap Y' = \emptyset$,
   (ii) *for every integer v, G has a valid red-blue (S^*, T^*, S'', T'')-colouring of value v if and only if G has a valid red-blue (S', T', X', Y')-colouring of value v (note that the backward implication holds by definition), and*
   (iii) *every vertex in S' has exactly one neighbour in Y', which belongs to T'; every vertex in T' has exactly one neighbour in X', which belongs to S'; every vertex in $X' \setminus S'$ has no neighbour in Y'; every vertex in $Y' \setminus T'$ has no neighbour in X'; and every vertex of $V \setminus (X' \cup Y')$ has no neighbour in $S' \cup T'$, at most one neighbour in $X' \setminus S'$, and at most one neighbour in $Y' \setminus T'$.*

Moreover, (S', T', X', Y') is obtained in polynomial time.

Let (S', T', X', Y') be an intermediate tuple of a graph G. Let $Z = V \setminus (X' \cup Y')$. A red-blue (S', T', X', Y')-colouring of G is called *monochromatic* if all connected components of $G[Z]$ are monochromatic. We say that an intermediate tuple (S', T', X', Y') is *monochromatic* if every connected component of $G[V \setminus (X' \cup Y')]$ is monochromatic in every valid red-blue (S', T', X', Y')-colouring of G. A propagation rule is *mono-safe* if for every integer v the following holds: G has a valid monochromatic red-blue (S, T, X, Y)-colouring of value v before the application of the rule if and only if G has a valid monochromatic red-blue (S, T, X, Y)-colouring of value v after the application of the rule.

We now present Rule R3 (which is used implicitly in [31]) and prove that R3 is mono-safe.

Fig. 4. A U-saturating matching (left) and the corresponding valid red-blue colouring (right). Note that not every vertex in $X \cup Y$ belongs to W.

R3. If there are two distinct vertices u and v in a connected component D of $G[Z]$ with a common neighbour $w \in X \cup Y$, then colour every vertex of D with the colour of w.

Lemma 19. *Rule R3 is mono-safe.*

Proof. Say $w \in X \cup Y$ is in X, so w is red. Then, at least one of u and v must be coloured red. Hence, as D must be monochromatic, every vertex of D must be coloured red. Note that the value of a maximum monochromatic red-blue (S, T, X, Y)-colouring (if it exists) is not affected. □

Suppose that exhaustively applying rules R1–R3 on an intermediate tuple (S', T', X', Y') does not lead to a no-answer but to a tuple (S, T, X, Y). We call (S, T, X, Y) the *final* tuple. The following lemma can be proved by a straightforward combination of the arguments of the proof of Lemma 18 with Lemma 19 and the observation that an application of R3 takes polynomial time, just as a check to see if R3 can be applied.

Lemma 20. *Let G be a connected graph with a monochromatic intermediate tuple (S', T', X', Y') and a resulting final tuple (S, T, X, Y). The following three statements hold:*

(i) $S' \subseteq S$, $X' \subseteq X$, $T' \subseteq T$, $Y' \subseteq Y$, *and* $X \cap Y = \emptyset$,
(ii) *For every integer v, G has a valid (monochromatic) red-blue (S', T', X', Y')-colouring of value v if and only if G has a valid monochromatic red-blue (S, T, X, Y)-colouring of value v (note that the backward implication holds by definition), and*
(iii) *every vertex in S has exactly one neighbour in Y, which belongs to T; every vertex in T has exactly one neighbour in X, which belongs to S; every vertex in $X \setminus S$ has no neighbour in Y and no two neighbours in the same connected component of $G[V \setminus (X \cup Y)]$; every vertex in $Y \setminus T$ has no neighbour in X and no two neighbours in the same connected component of $G[V \setminus (X \cup Y)]$; and every vertex of $V \setminus (X \cup Y)$ has no neighbour in $S \cup T$, at most one neighbour in $X \setminus S$, and at most one neighbour in $Y \setminus T$.*

Moreover, (S, T, X, Y) is obtained in polynomial time.

The following lemma will be the final step in each of our polynomial-time results. It is an application of Theorem 13.

Lemma 21. *Let $G = (V, E)$ be a connected graph with a monochromatic intermediate tuple (S', T', X', Y') and a final tuple (S, T, X, Y). If $V \setminus (X \cup Y)$ is an independent set, then it is possible to find in polynomial time either a maximum valid red-blue (S, T, X, Y)-colouring of G, or conclude that G has no valid red-blue (S, T, X, Y)-colouring.*

Proof. Let $Z = V \setminus (X \cup Y)$. Let $W = N(Z)$. Recall that Z is independent. Hence, by Lemma 20-(iii), every vertex of W belongs to $(X \setminus S) \cup (Y \setminus T)$. Let $U \subseteq Z$ consist of all vertices of Z that have a neighbour in both $X \setminus S$ and $Y \setminus T$. We claim that the set of bichromatic edges of every valid red-blue (S, T, X, Y)-colouring is the union of a U-saturating matching in $G[W \cup Z]$ (if it exists) and the set of edges with one end-vertex in S and the other one in T.

First suppose that $G[W \cup Z]$ has a U-saturating matching M. We colour every vertex in X red and every vertex in Y blue. Let $z \in Z$. First assume that z is incident to an edge $zw \in M$. If $w \in X \setminus S$, then colour z blue. If $w \in Y \setminus T$, then colour z red. Now suppose z is not incident to an edge in M. Then $z \notin U$, as M is U-saturating. Hence, either every neighbour of z belongs to $X \setminus S$ and is coloured red, in which case we colour z red, or every neighbour of z belongs to $Y \setminus T$ and is coloured blue, in which case we colour z blue. This gives us a valid red-blue (S, T, X, Y)-colouring of G. See also Fig. 4.

Now suppose that G has a valid red-blue (S, T, X, Y)-colouring. By definition, every vertex of X is coloured red, and every vertex of Y is coloured blue. By Lemma 20-(iii), every edge with an end-vertex in S and the other one in T is bichromatic, and there are no other bichromatic edges in $G[X \cup Y]$. Let M be the set of other bichromatic edges. Then, every vertex of M has one vertex in Z

and the other one in W. Moreover, if $z \in U$, then z has a red neighbour (its neighbour in $X \setminus S$) and a blue neighbour (its neighbour in $Y \setminus T$). Hence, no matter what colour z has itself, z is incident to a bichromatic edge of M. We conclude that M is U-saturating, and the claim is proven.

From the above claim, it follows that all we have to do is to find a maximum U-saturating matching in $G[W \cup Z]$. By Theorem 13, this takes polynomial time. □

We are now ready to prove our first polynomial-time result.

Theorem 22. MAXIMUM MATCHING CUT *is solvable in polynomial time for* P_6*-free graphs.*

Proof. Let $G = (V, E)$ be a connected P_6-free graph. By Observation 15 it suffices to find a maximum valid red-blue colouring of G. By Theorem 14, we find in polynomial time either a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph $K_{r,s}$ in G.

If G has a dominating induced C_6, then G has domination number at most 6, and we apply Lemma 17. Suppose that G has a dominating complete bipartite graph F with partition classes $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_s\}$. We may assume without loss of generality that $r \leq s$. If $s \leq 2$, then G has domination number at most 4, and we apply Lemma 17 again. So we assume that $s \geq 3$.

If $r \geq 2$, then $V(F)$ must be monochromatic in any valid red-blue colouring of G by Observation 16. In this case we colour every vertex of $V(F)$ blue. If $r = 1$, then we may assume without loss of generality that $N(u_1) = \{v_1, \ldots, v_s\}$. In this case we colour u_1 blue, and we branch over all $O(n)$ options of colouring at most one vertex of $N(u_1)$ red.

So, now we consider a red-blue colouring of F. It might be that F is monochromatic (in particular, this will be the case if $r \geq 2$). If F is monochromatic, then every vertex of F is blue. In order to get a starting pair with a non-empty core, we branch over all $O(n^2)$ options of choosing a bichromatic edge (one end-vertex of which may belong to F). Let D be the set of all coloured vertices, that is, D contains $V(F)$ and possibly one or two other vertices. By construction, exactly one vertex of D is coloured red, and all other vertices of D are blue.

Let $S^* = S''$ be the set containing the red vertex of D. Let T^* be the singleton set containing the blue neighbour of the vertex in S^*. Let T'' be the set of blue vertices, so $T^* \subseteq T''$. We exhaustively apply rules R1 and R2 on the starting pair (S'', T''). By Lemma 18 we either find in polynomial time that G has no valid red-blue (S^*, T^*, S'', T'')-colouring, and we discard the branch, or we obtain an intermediate tuple (S', T', X', Y') of G. Suppose the latter case holds. We prove the following two claims for the set $Z' = V \setminus (X' \cup Y')$ of uncoloured vertices.

Claim 22.1. *Every vertex* $z \in Z'$ *has a neighbour in* $Y' \setminus T'$ *that belongs to* F.

Proof. As F is dominating, z has a neighbour in F. Since $D \supseteq V(F)$ contains exactly one red vertex x, which has a blue neighbour in D, all neighbours of x in $G - D$ are coloured red, that is, belong to X. As $z \in G - D$ belongs to Z', this means that x and z are non-adjacent. So, the neighbour of z in F must belong to $Y' \setminus T'$ (as else we could have applied R2). □

Claim 22.2. *The intermediate tuple* (S', T', X', Y') *is monochromatic.*

Proof. Suppose for a contradiction that there is an edge $uv \in E(G[Z'])$ such that u is blue and v is red. Then v has two blue neighbours by Claim 22.1, a contradiction. □

Since Claim 22.2 holds, we may now exhaustively apply R1–R3 to the intermediate tuple (S', T', X', Y'). By Lemma 20 we either find in polynomial time that G has no valid red-blue (S', T', X', Y')-colouring, and thus no valid red-blue (S^*, T^*, S', T')-colouring, and we discard the branch, or we obtain a final tuple (S, T, X, Y) of G. Again, we let $Z = V \setminus (X \cup Y)$. By the same lemma and Claim 22.1, the following holds for every (uncoloured) vertex $z \in Z$:

- z has at most one neighbour in $X \setminus S$,
- z has exactly one neighbour in $Y \setminus T$, which belongs to F, and
- if z' is in the same connected component of $G[Z]$ as z, then z and z' do not share a neighbour in $G - Z$.

Claim 22.3. *In any valid red-blue* (S, T, X, Y)*-colouring at most one red component of* $G[Z]$ *may have more than one vertex.*

Proof. Let c be a valid red-blue (S, T, X, Y)-colouring of G. For a contradiction, assume that Z_1 and Z_2 are connected components of size at least 2 that are both coloured red. For $i = 1, 2$, let z_i and z_i' be two adjacent vertices in Z_i, and let w_i be the blue neighbour of z_i in F (which exists due to Claim 22.1). As c is valid, no blue vertex of G has two red neighbours. Hence, we find that w_1 and w_2 are distinct vertices, and also that w_1 is not adjacent to any vertex of $\{z_1', z_2, z_2'\}$, and w_2 is not adjacent to any vertex of $\{z_1, z_1', z_2'\}$. Hence, if w_1 and w_2 are adjacent, then $z_1' z_1 w_1 w_2 z_2 z_2'$ is an induced P_6; see also Fig. 5 (left side). As G is P_6-free, this is not possible. Hence, w_1 and w_2 are not adjacent.

We now use the fact that w_1 and w_2 both belong to F and that F is a complete bipartite graph. As $w_1 w_2 \notin E$, the latter means that there exists a vertex $w_3 \in V(F)$ that is adjacent to both w_1 and w_2, so w_3 is blue as well. As z_1' and z_2' are both coloured red, at
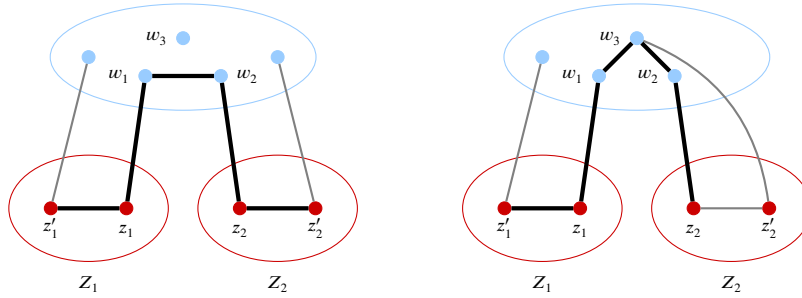
Fig. 5. The situation in Claim 22.3 where two connected components Z_1, Z_2 of $G[Z]$, each with at least two vertices, are both coloured red. This will always yield an induced path on at least six vertices, even if w_1 and w_2 are not adjacent, as at most one of z_1', z_2' is adjacent to w_3.

most one of z_1', z_2' can be adjacent to w_3. Hence, we may assume without loss of generality that w_3 is not adjacent to z_1'. As z_1 and z_2 have w_1 and w_2, respectively, as their matching partner, w_3 is adjacent neither to z_1 nor to z_2. Now, $z_1' z_1 w_1 w_3 w_2 z_2$ is an induced P_6, a contradiction. See also Fig. 5 (right side). □

Due to Claim 22.3, we can now branch over all $O(n)$ options to colour at most one connected component of $G[Z]$ of size at least 2 red, and all other components of size at least 2 blue. We then exhaustively apply rules R1-R3 again. This takes polynomial time. In essence, we merely pre-coloured some more vertices red. So, in the end we either find a new tuple of G with the same properties as those listed in Lemma 20, or we find that G has no such tuple, in which case we discard the branch. Suppose we have not discarded the branch. Now the set of uncoloured vertices form an independent set. Hence, we can apply Lemma 21 to find in polynomial time a red-blue colouring of G that is a maximum red-blue (S^*, T^*, S'', T'')-colouring due to Lemmas 18-(ii) and 20-(ii).

If somewhere in the above process we discarded a branch, that is, if G has no valid red-blue (S^*, T^*, S'', T'')-colouring, we consider the next one. If we did not discard the branch, then we remember the value of the maximum red-blue (S^*, T^*, S'', T'')-colouring that we found. Afterwards, we pick one with the largest value to obtain a maximum valid red-blue colouring of G.

The correctness of our branching algorithm follows from its description. The running time is polynomial: each branch takes polynomial time to process, and the number of branches is $O(n^3)$. This completes our proof. □

The proof of our second polynomial-time result combines Lemma 21 with arguments used in the proof that MATCHING CUT is polynomial-time solvable for $(H + P_3)$-free graphs if it is so for H-free graphs [35].

Theorem 23. *Let H be a graph. If* MAXIMUM MATCHING CUT *is polynomial-time solvable for H-free graphs, then it is so for $(H + P_2)$-free graphs.*

Proof. Assume that MAXIMUM MATCHING CUT is polynomial-time solvable for H-free graphs. Let $G = (V, E)$ be a connected $(H + P_2)$-free graph on n vertices. If G is H-free, we are done by assumption. Suppose G has an induced subgraph G' isomorphic to H. Let G^* be the graph obtained from G after removing the vertices of $V(G') \cup N(V(G'))$. Since G' is isomorphic to H and G is $(H + P_2)$-free, G^* is P_2-free. Hence, $V(G^*)$ is an independent set. By Observation 15 it suffices to find a maximum valid red-blue colouring of G. Below we explain how to do this.

We first branch over all options of colouring every $u \in V(G')$ red or blue, and colouring at most one neighbour of every $u \in V(G')$ with a different colour than u. If in a branch we only used one colour, we branch over all $O(n^2)$ options of choosing a bichromatic edge. In this way we obtain, for each branch, a starting pair with a non-empty core.

Consider a branch with a starting pair (S'', T'') and core (S^*, T^*). We apply rules R1 and R2 exhaustively. If we obtain a no-answer, we may discard the branch due to Lemma 18. Else, we obtain an intermediate tuple (S, T, X, Y). Note that every vertex in $Z = V \setminus (X \cup Y)$ belongs to G^*. Hence, Z is an independent set, and thus (S, T, X, Y) is a final tuple. This means that we may apply Lemma 21. Then, in polynomial time, we either find that G has no valid red-blue (S, T, X, Y)-colouring, in which case we may discard the branch due to Lemma 18, or we find a maximum valid red-blue (S, T, X, Y)-colouring. The latter is also a maximum valid red-blue (S^*, T^*, S'', T'')-colouring, again due to Lemma 18. We remember its value. In the end, after the last branch, we output a colouring with largest value as a maximum valid red-blue colouring of G.

The correctness of our branching algorithm follows from its description. The running time is polynomial: each branch takes polynomial time to process, and the number of branches is $O(2^{|V(H)|} n^{|V(H)|}) + O(n^2)$. This completes our proof. □

We now show our third polynomial-time result. Again, the idea is to branch over a polynomial number of options, each of which reduces to the setting where we can apply Lemma 21.

Theorem 24. MAXIMUM MATCHING CUT *is solvable in polynomial time for graphs with diameter at most* 2.

Proof. Let $G = (V, E)$ be a graph of diameter at most 2. If G has diameter 1, then the problem is trivial to solve. Assume that G has diameter 2. By Observation 15 it suffices to find a maximum valid red-blue colouring of G. By definition, such a colouring has at least one bichromatic edge (has value at least 1). We branch over all $O(n^2)$ options of choosing the bichromatic edge.

Consider a branch, where $e = uv$ is the bichromatic edge, say u is blue and v is red. Now all other neighbours of u must be coloured blue. We let $D = \{u\} \cup N(u)$ and note that D dominates G, as G has diameter 2.

We set $S^* = \{v\}$, $T^* = \{u\}$, $S'' = \{v\}$, and $T'' = D \setminus \{v\}$. This gives us a starting pair (S'', T'') with core (S^*, T^*). We exhaustively apply rules R1 and R2 on (S'', T''). By Lemma 18 we either find in polynomial time that G has no valid red-blue (S^*, T^*, S'', T'')-colouring, and we discard the branch, or we obtain an intermediate tuple (S', T', X', Y') of G. Suppose the latter case holds. We prove the following two claims for the set $Z' = V \setminus (X' \cup Y')$ of uncoloured vertices.

Claim 24.1. *Every vertex $z \in Z'$ has a neighbour in $Y' \setminus T'$ that belongs to D.*

Proof. As $z \in Z'$, we have that $z \notin D$. As D is dominating, z has a neighbour b in D. As every neighbour of u belongs to D and z is not in D, we find that $b \neq u$. Since D contains exactly one red vertex v, which has a blue neighbour in D (namely u), all neighbours of v in $G - D$ are coloured red, that is, belong to X. As z belongs to $G - D$ and z is not coloured red, this means that v and z are non-adjacent, and thus $b \neq v$. So, b must belong to $T'' \setminus \{u\}$, and thus to $Y' \setminus T'$, as else we could have applied R2. $\square$

Claim 24.2. *The intermediate tuple (S', T', X', Y') is monochromatic.*

Proof. Suppose for a contradiction that there is an edge $pq \in E(G[Z'])$ such that p is blue and q is red. Then q has two blue neighbours by Claim 24.1, a contradiction. $\square$

Since Claim 24.2 holds, we may exhaustively apply R1–R3 to the intermediate tuple (S', T', X', Y'). By Lemma 20 we either find in polynomial time that G has no valid red-blue (S', T', X', Y')-colouring, and thus no valid red-blue (S^*, T^*, S', T')-colouring, and we discard the branch, or we obtain a final tuple (S, T, X, Y) of G. Again, we let $Z = V \setminus (X \cup Y)$. By the same lemma and Claim 24.1, the following holds for every (uncoloured) vertex $w \in Z$:

  (i) w has at most one neighbour in $X \setminus S$,
 (ii) w has exactly one neighbour in $Y \setminus T$, which belongs to D, and
(iii) if w' is in the same connected component of $G[Z]$ as w, then w and w' do not share a neighbour in $G - Z$.

We strengthen (i) by proving the following claim.

Claim 24.3. *Every vertex $w \in Z$ has exactly one neighbour in $X \setminus S$.*

Proof. By (i), we find that w has at most one neighbour in $X \setminus S$. For a contradiction, suppose that w has no neighbours in $X \setminus S$. We also know that w has no neighbours in S, as else we could have applied R1 or R2. Recall that v was the only red vertex of D. As v has a blue neighbour, namely u, all the other neighbours of v are coloured red due to R2. Hence, w is adjacent neither to v nor to any vertex in $N(v) \setminus \{u\}$. As all neighbours of u that are not equal to v are coloured blue and $w \in Z$ is uncoloured, we find that w is not adjacent to u either. Hence, the distance between v and w is at least 3, contradicting our assumption that G has diameter 2. $\square$

We continue by proving the following claim.

Claim 24.4. *If $G[Z]$ contains two connected components F_1 and F_2 of size at least 2, then $G[Z] = F_1 + F_2$.*

Proof. Let F_1 contain u_1 and u_2. Let F_2 contain v_1 and v_2. By combining Claim 24.3 with (ii) and (iii), we find that the vertices u_1 and u_2 have each a different red (respectively, blue) neighbour and the same holds for v_1 and v_2. However, as G has diameter 2, it holds that u_1 and u_2 each have a common neighbour with both v_1 and v_2. Thus, without loss of generality, u_1 and v_1 have a red common neighbour, u_1 and v_2 a blue one, while u_2 and v_1 have a blue common neighbour, u_2 and v_2 a red one. See also Fig. 6 (a).

For a contradiction, assume that $G[Z]$ contains a third connected component F_3. Let w be a vertex in F_3. Then w has a common neighbour with each of u_1, u_2, v_1 and v_2. Furthermore, w has exactly one red and one blue neighbour. As can be seen in Figs. 6 (b) and (c), there do not exist vertices $x \in X$ and $y \in Y$ such that $\{u_1, u_2, v_1, v_2\} \subseteq N_G(\{x, y\})$. Hence, w has no common neighbour with some vertex of $\{u_1, u_2, v_1, v_2\}$, contradicting our assumption that G has diameter 2. $\square$

From Claim 24.4, it follows that $G[Z]$ has at most two components with more than one vertex, which are both monochromatic in every valid red-blue (S, T, X, Y)-colouring of G (if such a colouring exists) due to Claim 24.2. Hence, we can branch over all possible colourings of these connected components (there are at most four branches).

For each branch, we propagate the obtained partial red-blue colouring by exhaustively applying rules R1–R3. This takes polynomial time. In essence, we merely pre-coloured some more vertices red or blue. So, in the end we either find a new tuple of G with the same
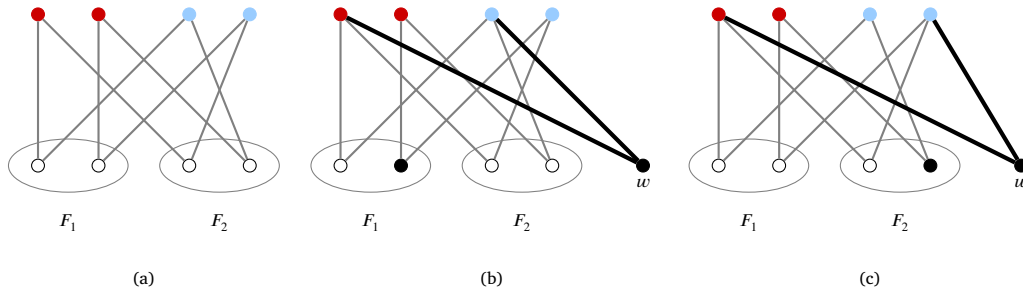
Fig. 6. The unique way (up to symmetry) to connect two components F_1 and F_2 of $G[Z]$ of size 2 (a) and the two options to connect a vertex w in a third component F_3 to the coloured part of the graph (b) and (c). We can see that there is always an uncoloured vertex without a common neighbour with w.
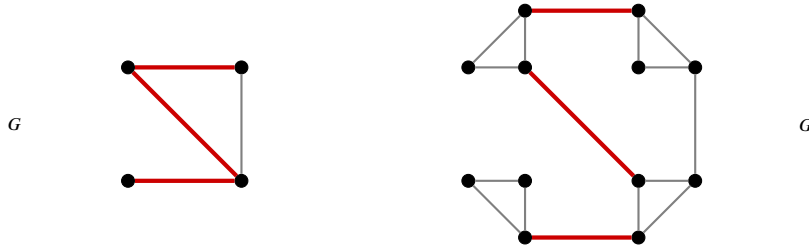

Fig. 7. A graph G (left) where the tick red edges form a maximum edge cut, and the graph G' (right) from the proof of Theorem 25, where the thick red edges form a maximum matching cut.

properties as those listed in Lemma 20, or we find that G has no such tuple, in which case we discard the branch. Suppose we have not discarded the branch. Now the set of uncoloured vertices form an independent set. Hence, we can apply Lemma 21 to find in polynomial time a red-blue colouring of G that is a maximum red-blue (S^*, T^*, S'', T'')-colouring due to Lemmas 18-(ii) and 20-(ii).

If somewhere in the above process we discarded a branch, that is, if G has no valid red-blue (S^*, T^*, S'', T'')-colouring, we consider the next one. If we did not discard the branch, then we remember the value of the maximum red-blue (S^*, T^*, S'', T'')-colouring that we found. Afterwards, we pick one with the largest value to obtain a maximum valid red-blue colouring of G.

The correctness of our branching algorithm follows from its description. The running time is polynomial: each branch takes polynomial time to process, and the number of branches is $O(n^2)$. This completes our proof. □

4. Hardness results for Maximum Matching Cut

In the following we will prove that MAXIMUM MATCHING CUT is NP-hard for subcubic line graphs and $2P_3$-free quadrangulated graphs of diameter 3 and radius 2. To prove the first hardness result, we reduce from MAXIMUM CUT. The problem takes as input a graph G and an integer k. The question is whether G has an edge cut of size at least k. This problem is well known to be NP-complete even for subcubic graphs, as shown by Yannakakis [43].

Theorem 25. MAXIMUM MATCHING CUT *is NP-hard for subcubic line graphs of triangle-free graphs.*

Proof. Let (G, k) be an instance of MAXIMUM CUT, where G is a subcubic graph. From G, we construct a graph G' as follows. First replace every vertex $v \in V(G)$ by a triangle C_v. Next, for every edge $uv \in E(G)$, add an edge between a vertex in C_v and a vertex in C_u, such that every vertex in C_v has at most one neighbour outside of C_v. This is possible since G is subcubic. See Fig. 7 for an example. The graph G' is subcubic, as every vertex in G' has two neighbours inside a triangle and at most one neighbour outside. Moreover, G is $(K_{1,3}, \text{diamond})$-free, or equivalently, the line graph of a triangle-free graph.

We claim that G has an edge cut of size at least k if and only if G' has a matching cut of size at least k.

First suppose that G has an edge cut M of size at least k. So, $V(G)$ can be partitioned into sets R and B, such that for every $e \in E(G)$, it holds that $e \in C$ if and only if e has one end-vertex in R and the other one in B. We define the edge set

$$M' = \{u'v' \in E(G') \mid u' \in C_u, v' \in C_v, uv \in M\}.$$

Note that $|M'| = |M| \geq k$. Moreover, M' contains no edge from any triangle C_u, so M' is a matching. For every $v \in V(G)$, we put all vertices of C_v in a set B' if $v \in B$, and else we put all vertices of C_v in a set R'. We now find that for every edge $e \in E(G')$, it holds that e belongs to M' if and only if e has one end-vertex in R' and the other one in B'. Hence, M' is an edge cut, and thus a matching cut, of G' with $|M'| \geq k$.

Now suppose that G' has a matching cut M' of size at least k. Let R' and B' be the corresponding sets of red and blue vertices, respectively. We define the edge set
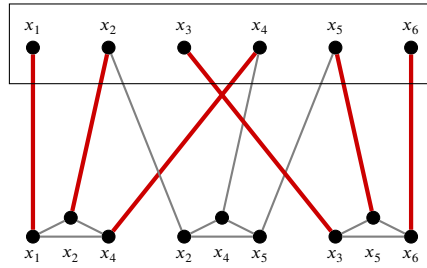
Fig. 8. The graph G for $X = \{x_1, \ldots, x_6\}$ and $S = \{\{x_1, x_2, x_4\}, \{x_2, x_4, x_5\}, \{x_3, x_5, x_6\}\}$. The vertices in the rectangle form a clique, whose edges we did not draw for readability. For the same reason, we also omitted the superscripts of the vertices in the three triangles. The set $S' = \{\{x_1, x_2, x_4\}, \{x_3, x_5, x_6\}\}$ is an exact 3-cover of X. The thick red edges in the graph show the corresponding matching cut of size $3q = 6$.

$$M = \{uv \in E(G) \mid u'v' \in M', u' \in C_u, v' \in C_v\}.$$

Note that $|M| = |M'| \geq k$. Due to Lemma 16, every triangle C_u is monochromatic. For every $u \in V(G)$, we put u in a set R if C_u is coloured red, else we put u in a set B. We now find for every edge $e \in E(G)$ that e belongs to M if and only if one end-vertex of e belongs to R and the other one to B. Hence, M is an edge cut in G of size at least k. ☐

For our next NP-hardness result, we reduce from the following problem. An *exact 3-cover* of a set X is a collection C of 3-element subsets of X, such that every $x \in X$ is in exactly one 3-element subset of C. The EXACT 3-COVER problem has as input a set X with $3q$ elements and a collection S of 3-element subsets of X. The question is if S contains an *exact 3-cover* of X (which will be of size q). This problem is well known to be NP-complete (see [28]).

Theorem 26. MAXIMUM MATCHING CUT *is NP-hard for* $2P_3$*-free quadrangulated graphs of radius at most* 2 *and diameter at most* 3.

Proof. Let (X, S) be an instance of EXACT 3-COVER where $X = \{x_1, \ldots, x_{3q}\}$ and $S = \{S_1, \ldots, S_k\}$, such that each S_i contains exactly three elements of X. From (X, S) we construct a graph G. We first define a clique $K_X = \{x_1, \ldots, x_{3q}\}$. For each $S \in S$, we do as follows. Let $S = \{x_h, x_i, x_j\}$. We add a triangle K_S on vertices x_h^S, x_i^S and x_j^S. We add an edge between a vertex $x_i \in K_X$ and a vertex $u \notin K_X$ if and only if $u = x_i^S$ for some $S \in S$. This completes the construction of G. See Fig. 8 for an example.

As every induced P_3 must contain at least one vertex from the clique K_X, we find that G is $2P_3$-free. As G is not only $2P_3$-free, but also (C_5, C_6)-free, G is quadrangulated. Consider some $x_i \in K_X$. Then every other vertex is of distance at most 2 from x_i. Consider some $x_i^S \in K_S$ for some $S \in S$. Then every other vertex is of distance at most 3 from x_i^S. Hence, the radius of G is at most 2 and the diameter of G is at most 3.

We claim that S contains an exact 3-cover of X if and only if G has a matching cut of size $3q$. First suppose that S contains an exact 3-cover C of X. We colour every vertex of K_X red. We colour a triangle K_S blue if $S \in S$ and otherwise we colour it red. This yields a valid red-blue colouring of value $3q$, and thus a matching cut of size $3q$.

Now suppose that G has a matching cut M of size $3q$. As K_X is a clique of size $3q \geq 3$, the corresponding valid red-blue colouring assigns every vertex of K_X the same colour, say red. As every triangle K_S is monochromatic, this means that exactly q triangles must be coloured blue. Moreover, no two blue triangles have a common red neighbour in K_X. Hence, the blue triangles correspond to an exact 3-cover of X. See again Fig. 8. ☐

5. Dichotomies for Maximum Matching Cut

In this section we prove our three dichotomy results, which we restate below.

Theorem 5 (restated). *For an integer* d, MAXIMUM MATCHING CUT *on graphs of diameter* d *is*

- *polynomial-time solvable if* $d \leq 2$, *and*
- NP-*hard if* $d \geq 3$.

Proof. The two results follow from Theorems 24 and 26, respectively. ☐

Theorem 6 (restated). *For an integer* r, MAXIMUM MATCHING CUT *on graphs of radius* r *is*

- *polynomial-time solvable if* $r \leq 1$, *and*
- NP-*hard if* $r \geq 2$.

Proof. A graph of radius 1 has a dominating vertex, and thus it has a matching cut if and only if it has a vertex of degree 1. This can be checked in polynomial time and thus proves the first result. The second result follows from Theorem 26. ☐

Theorem 7 (restated). *For a graph H,* MAXIMUM MATCHING CUT *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i sP_2 + P_6$ for some $s \geq 0$, and*
- *NP-hard if $H \supseteq_i K_{1,3}, 2P_3$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

Proof. Let H be a graph. If H contains a cycle, then MATCHING CUT, and thus MAXIMUM MATCHING CUT, is NP-hard due to Theorem 2. Now suppose that H has no cycle, so H is a forest. If H contains a vertex of degree at least 3, then the class of H-free graphs contains the class of $K_{1,3}$-free graphs. The latter class contains the class of line graphs, and thus we can apply Theorem 25.

Now suppose that H is a forest of maximum degree at most 2, that is, H is a linear forest. If $H \subseteq_i sP_2 + P_6$ for some $s \geq 0$, then we apply Theorem 22 in combination with s applications of Theorem 23. Else H contains an induced $2P_3$ and we apply Theorem 26. This completes the proof. □

6. Dichotomies for Maximum Disconnected Perfect Matching

In this section we prove the results of Section 1.3. We first need a similar lemma as Lemma 17, which is proven by copying the arguments of the proof of Lemma 17 and using the fact that we can check if a graph has a perfect matching in polynomial time by using, for instance, Edmonds' Blossom algorithm [14].

Lemma 27. *For a connected n-vertex graph G with domination number g, it is possible to find a maximum perfect-extendable red-blue colouring (if a red-blue colouring exists) in $O(2^g n^{g+2})$ time.*

Note that Lemmas 18 and 20 can be adapted to perfect-extendable red-blue colourings in a straightforward way, since the propagation rules R1–R3 hold for valid red-blue colourings and every perfect-extendable red-blue colouring is valid (that is, the propagation of any partial red-blue colouring does not influence if the resulting partial red-blue colouring is perfect-extendable or not). In other words, we immediately obtain the following two lemmas.

Lemma 28. *Let G be a connected graph with a starting pair (S'', T'') with core (S^*, T^*), and with an intermediate tuple (S', T', X', Y'). The following three statements hold:*

(i) *$S^* \subseteq S'$, $S'' \subseteq X'$ and $T^* \subseteq T'$, $T'' \subseteq Y'$ and $X' \cap Y' = \emptyset$,*

(ii) *for every integer v, G has a perfect-extendable red-blue (S^*, T^*, S'', T'')-colouring of value v if and only if G has a perfect-extendable red-blue (S', T', X', Y')-colouring of value v (note that the backward implication holds by definition), and*

(iii) *every vertex in S' has exactly one neighbour in Y', which belongs to T'; every vertex in T' has exactly one neighbour in X', which belongs to S'; every vertex in $X' \setminus S'$ has no neighbour in Y'; every vertex in $Y' \setminus T'$ has no neighbour in X'; and every vertex of $V \setminus (X' \cup Y')$ has no neighbour in $S' \cup T'$, at most one neighbour in $X' \setminus S'$, and at most one neighbour in $Y' \setminus T'$.*

Moreover, (S', T', X', Y') is obtained in polynomial time.

Lemma 29. *Let G be a connected graph with a monochromatic intermediate tuple (S', T', X', Y') and a resulting final tuple (S, T, X, Y). The following three statements hold:*

(i) *$S' \subseteq S$, $X' \subseteq X$, $T' \subseteq T$, $Y' \subseteq Y$, and $X \cap Y = \emptyset$,*

(ii) *For every integer v, G has a perfect-extendable (monochromatic) red-blue (S', T', X', Y')-colouring of value v if and only if G has a perfect-extendable monochromatic red-blue (S, T, X, Y)-colouring of value v (note that the backward implication holds by definition), and*

(iii) *every vertex in S has exactly one neighbour in Y, which belongs to T; every vertex in T has exactly one neighbour in X, which belongs to S; every vertex in $X \setminus S$ has no neighbour in Y and no two neighbours in the same connected component of $G[V \setminus (X \cup Y)]$; every vertex in $Y \setminus T$ has no neighbour in X and no two neighbours in the same connected component of $G[V \setminus (X \cup Y)]$; and every vertex of $V \setminus (X \cup Y)$ has no neighbour in $S \cup T$, at most one neighbour in $X \setminus S$, and at most one neighbour in $Y \setminus T$.*

Moreover, (S, T, X, Y) is obtained in polynomial time.

We prove our next lemma by similar but simpler arguments as in the proof of Lemma 21.

Lemma 30. *Let $G = (V, E)$ be a connected graph with a final tuple (S, T, X, Y). If $V \setminus (X \cup Y)$ is an independent set, then it is possible to find in polynomial time either a maximum perfect-extendable red-blue (S, T, X, Y)-colouring of G, or conclude that G has no perfect-extendable red-blue (S, T, X, Y)-colouring.*

Proof. Let $Z = V \setminus (X \cup Y)$. Let $W = N(Z)$. Recall that Z is independent. Hence, by Lemma 29-(iii), every vertex of W belongs to $(X \setminus S) \cup (Y \setminus T)$. By combining this observation with Lemma 29-(iii), we find that every bichromatic edge of any perfect-extendable red-blue (S, T, X, Y)-colouring (if it exists) is

(i) either an edge with one end-vertex in S and the other one in T, or

(ii) an edge with one end-vertex in Z and the other one in either $X \setminus S$ or $Y \setminus T$.

By Lemma 29-(iii), the subgraph of G induced by $S \cup T$ has a perfect matching that consist of every edge with one end-vertex in S and the other one in T. Hence, a maximum perfect-extendable red-blue (S, T, X, Y)-colouring (if it exists) is the union of

(i) the set of edges with one end-vertex in S and the other one in T; and

(ii) a perfect matching in $G \setminus (S \cup T)$ that contains as many edges with one end-vertex in Z (and the other one in either $X \setminus S$ or $Y \setminus T$) as possible.

We first check in polynomial time if $G \setminus (S \cup T)$ has a perfect matching (for example, by using Edmonds' Blossom algorithm [14]). If not, then G has no perfect-extendable red-blue (S, T, X, Y)-colouring, and we stop. Otherwise, we found a perfect matching M of $G \setminus (S \cup T)$, and we continue as follows.

We colour every vertex in X red and every vertex in Y blue. By Lemma 29-(iii), every vertex in Z has at most one neighbour in X, which belongs to $X \setminus S$, and at most one neighbour in Y, which belongs to $Y \setminus T$. As Z is independent, we can do as follows for every $u \in Z$. If u has degree 1 in G and a neighbour $x \in X$, then ux must belong to M, and we colour u blue. If u has degree 1 in G and a neighbour $y \in Y$, then uy must belong to M, and we colour u red. If u has degree 2 in G, and a neighbour $x \in X$ and a neighbour $y \in Y$, then we colour u blue if $ux \in M$ and red if $uy \in M$. This takes polynomial time. As every edge of M with one end-vertex in Z and the other one in either $X \setminus S$ or $Y \setminus T$ is monochromatic, we found, in polynomial time, a maximum perfect-extendable red-blue (S, T, X, Y)-colouring of G. $\square$

The following result is proven in exactly the same way as the proof of Theorem 22 after replacing Lemma 17 by Lemma 27; Lemma 18 by Lemma 28; Lemma 20 by Lemma 29; and Lemma 21 by Lemma 30.

Theorem 31. MAXIMUM DISCONNECTED PERFECT MATCHING *is solvable in polynomial time for P_6-free graphs.*

Our following result can be proven in the same way as Theorem 23 after replacing Lemma 18 by Lemma 28; and Lemma 21 by Lemma 30.

Theorem 32. *Let H be a graph. If* (MAXIMUM) DISCONNECTED PERFECT MATCHING *is polynomial-time solvable for H-free graphs, then it is so for $(H + P_2)$-free graphs.*

The following result is proven in exactly the same way as the proof of Theorem 24 after replacing Lemma 18 by Lemma 28; Lemma 20 by Lemma 29; and Lemma 21 by Lemma 30.

Theorem 33. MAXIMUM DISCONNECTED PERFECT MATCHING *is solvable in polynomial time for graphs with diameter at most* 2.

We now show the following result by modifying the proof of Theorem 25; note that the maximum degree bound is no longer 3 but 6.

Theorem 34. MAXIMUM DISCONNECTED PERFECT MATCHING *is* NP-*hard for graphs of maximum degree* 6 *that are line graphs of triangle-free graphs.*

Proof. We make the following changes in the hardness construction of Theorem 25. First, we replace every vertex u of the input graph G by a clique C_u of size 6 instead of a triangle. Then, for each edge $uv \in E(G)$, we add two edges between C_u and C_v, such that (again) every vertex in C_v has at most one neighbour outside C_v. The resulting graph G' is still $(K_{1,3}, \text{diamond})$-free but has maximum degree is 6. We can now show that G has an edge cut of size at least k if and only if G' has a matching cut of size at least k, using the same arguments as in the proof of Theorem 25. The proof follows from the observation that we can extend a matching cut M of G' to a perfect matching by the fact that the number of vertices in a clique C_u that are not incident with any edge of M' is even. $\square$

We now show the following result by modifying the proof of Theorem 26.

Theorem 35. MAXIMUM DISCONNECTED PERFECT MATCHING *is* NP-*hard for $2P_3$-free quadrangulated graphs of radius at most* 2 *and diameter at most* 3.

Proof. We reduce from EXACT 4-COVER instead of EXACT 3-COVER. This allows us to modify the construction of the proof of Theorem 26, such that the triangles K_S become cliques of size 4. This does not change the size of the matching cut. Moreover, all vertices in the clique K_X still need to be matched to the cliques K_S to obtain a matching cut of maximum size. However, all previously unmatched vertices, which exist only inside the cliques K_S, may now be matched to a vertex inside K_S. $\square$

We are now ready to prove Theorems 10–12, which we restate below.

Theorem 10 (restated). *For an integer d,* MAXIMUM DISCONNECTED PERFECT MATCHING *on graphs of diameter d is*

- *polynomial-time solvable if $d \leq 2$, and*
- NP-*hard if $d \geq 3$.*

Proof. The two results follow from Theorems 33 and 35, respectively. $\square$

Theorem 11 (restated). *For an integer r,* MAXIMUM DISCONNECTED PERFECT MATCHING *on graphs of radius r is*

- *polynomial-time solvable if $r \leq 1$, and*
- NP-*hard if $r \geq 2$.*

Proof. Recall that a graph G of radius 1 has a dominating vertex u, and hence the only matching cuts are of the form $\{uv\}$, where v is a vertex of degree 1 in G. Notice that such an edge uv must belong to any perfect matching of G. Hence, we just need to check if G has a perfect matching and if G contains a vertex of degree 1. Both can be checked in polynomial time and thus proves the first result. The second result follows from Theorem 35. $\square$

Theorem 12 (restated). *For a graph H,* MAXIMUM DISCONNECTED PERFECT MATCHING *on H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i s P_2 + P_6$ for some $s \geq 0$, and*
- NP-*hard if $H \supseteq_i K_{1,3}$, $2P_3$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

Proof. Let H be a graph. First suppose that H has a cycle. Recall that DISCONNECTED PERFECT MATCHING is NP-complete for C_s-free graphs [17] and thus for H-free graphs. Hence, the same holds for MAXIMUM DISCONNECTED PERFECT MATCHING. If H is not a cycle, then H is a forest. If H contains a vertex of degree at least 3, then the class of H-free graphs contains the class of $K_{1,3}$-free graphs, which contains the class of line graphs, so we apply Theorem 34. Otherwise H is a linear forest. If $H \subseteq_i s P_2 + P_6$ for some $s \geq 0$, then we apply Theorem 31 in combination with s applications of Theorem 32. Else H contains an induced $2P_3$ and we apply Theorem 35. $\square$

7. Conclusions

We considered the optimization version MAXIMUM MATCHING CUT of the classical MATCHING CUT problem after first observing that the PERFECT MATCHING CUT problem is a special case of the former problem. We generalized known algorithms for graphs of diameter at most 2 and P_6-free graphs from MATCHING CUT and PERFECT MATCHING CUT to MAXIMUM MATCHING CUT. We also showed that the latter problem is computationally harder (assuming $P \neq NP$) than MATCHING CUT and PERFECT MATCHING CUT for various graph classes. Our results led to three new dichotomy results, namely a computational complexity classification of MAXIMUM MATCHING CUT for H-free graphs, and complexity classifications for graphs of bounded diameter and bounded radius. Classification for H-free graphs is still unsettled for MATCHING CUT and PERFECT MATCHING CUT, as can be observed from Theorems 2 and 4. We also pose the following open problem, which is the missing case from Theorem 3.

Open Problem 1. Determine the complexity of PERFECT MATCHING CUT for graphs of diameter 3.

To prove the dichotomies for MAXIMUM MATCHING CUT, we showed that MAXIMUM MATCHING CUT is NP-hard for $2P_3$-free quadrangulated graphs of diameter 3 and radius 2, whereas MATCHING CUT is known to be polynomial-time solvable for quadrangulated graphs [38]. We recall an open problem of Le and Telle [34] who asked, after proving polynomial-time solvability for chordal graphs, the following question (a graph is k-*chordal* for some $k \geq 3$ if it is $(C_{k+1}, C_{k+2}, \ldots)$-free, so 3-chordal graphs are the chordal graphs).

Open Problem 2 *([34])*. Determine the complexity of PERFECT MATCHING CUT for quadrangulated graphs, or more general, k-chordal graphs for $k \geq 4$.

We also showed how our proofs could be adapted to hold for MAXIMUM DISCONNECTED PERFECT MATCHING, the optimization version of DISCONNECTED PERFECT MATCHING. This led to exactly the same dichotomies for the former problem as for MAXIMUM MATCHING CUT for bounded diameter, bounded radius and H-free graphs. Moreover, it implied new results for DISCONNECTED PERFECT MATCHING as well, including a polynomial-time algorithm for P_6-free graphs. The complexity classification of DISCONNECTED PERFECT MATCHING for H-free graphs is still not complete (see Theorem 9). We also pose the following open problem, which is the missing case from Theorem 8.

Open Problem 3. Determine the complexity of DISCONNECTED PERFECT MATCHING for graphs of radius 2.

Our final open problem is related to H-free graphs.

Open Problem 4. For every graph H, is DISCONNECTED PERFECT MATCHING polynomial-time solvable for $(H + P_3)$-free graphs if it is polynomial-time solvable for H-free graphs?

We now know from Theorem 9 that the above result holds for P_2, while for MATCHING CUT we have this result for P_3 [35] and for PERFECT MATCHING CUT even for P_4 [37].

CRediT authorship contribution statement

Felicia Lucke: Writing – review & editing, Writing – original draft. **Daniël Paulusma:** Writing – review & editing, Writing – original draft. **Bernard Ries:** Writing – review & editing, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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