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On blockers and transversals of maximum independent sets in co-comparability graphs

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ABSTRACT

In this paper, we consider the following two problems: (i) Deletion Blocker(α) where we are given an undirected graph $G = (V, E)$ and two integers $k, d \geq 1$ and ask whether there exists a subset of vertices $S \subseteq V$ with $|S| \leq k$ such that $\alpha(G-S) \leq \alpha(G)-d$, that is the independence number of G decreases by at least d after having removed the vertices from S; (ii) Transversal(α) where we are given an undirected graph $G = (V, E)$ and two integers $k, d \geq 1$ and ask whether there exists a subset of vertices $S \subseteq V$ with $|S| \leq k$ such that for every maximum independent set I we have $|I \cap S| \geq d$. We show that both problems are polynomial-time solvable in the class of co-comparability graphs by reducing them to the well-known Vertex Cut problem. Our results generalise a result of Chang et al. (2001) and a recent result of Hoang et al. (2023).

1. Introduction

Graph parameters like for instance the independence number, the clique number, the chromatic number, and the domination number have been intensively studied in the literature. While one is usually interested in maximising or minimising such parameters, another interesting question one may ask is by how much one can decrease the value of a graph parameter by using a limited number of some predefined graph operations (like for instance vertex deletions or edge contractions). This leads to so-called blocker problems which are defined as follows. For a fixed set S of graph operations, a given graph $G = (V, E)$, two integers k and d, and some graph parameter π, we want to know if we can transform G into a graph G' by using at most k operations from the set S and such that $\pi(G') \leq \pi(G) - d$. The integer d is called the *threshold*.

Over the last years, blocker problems have been investigated intensively in the literature (see for instance [2–6,8,9,11–15,17–19,21–23,25]) and have relations to many other well known problems (like for instance Hadwiger Number, Bipartite Contraction, and Maximum Induced Bipartite Subgraph; see [9] for more examples). The graph parameters that have been considered were for instance the matching number (see [25]), the chromatic number (see [21,22]), the (total or semitotal) domination number (see [12]), the length of a longest path (see [18]), the clique number (see [21]), the weight of a minimum dominating set (see [20]), the vertex cover number (see [3]), and the independence number (see [8,9]). Regarding the graph operations, the set S always consisted in a single operation: vertex deletion, edge deletion, edge contraction or edge addition.

A related problem to the blocker problem is the so-called transversal problem. Given a graph $G = (V, E)$, a property $\mathcal{P}$, and two integers d and k, we want to know if there exists a set $V' \subseteq V$ (resp. a set $E' \subseteq E$) of size at most k that

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Fig. 1. A 2-transversal of the graph P_5 (left) and a 2-deletion blocker of P_5 (right), highlighted with light blue vertices. Both are of minimum size. Note that the 2-deletion blocker is a 2-transversal but the 2-transversal is not a 2-deletion blocker. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

intersects each set of vertices (resp. set of edges) satisfying property $\mathcal{P}$ on at least d vertices (resp. d edges). For example, if $\mathcal{P}$ corresponds to "being a maximum independent set", the transversal problem consists in asking whether one can find a set of at most k vertices which intersects every maximum independent set on at least d vertices. Another example is the well-known Feedback Vertex Set problem. Here, we are interested in finding a subset of vertices of size at most k which intersects every cycle on at least one vertex. Thus, it is a transversal problem with $\mathcal{P}$ corresponding to "being a cycle" and $d = 1$.

Transversal problems have also received much attention in the literature over the last years. Properties considered were for instance "being a maximum independent set" (see [4,5]), "being a maximum matching" (see [23,25]), "being a maximal clique" (see [16]), and "being a cycle" (see [1]).

In this paper, we will focus on the blocker problem with π being the independence number α and S consisting in a single operation, namely vertex deletion, as well as on the transversal problem with $\mathcal{P}$ corresponding to "being a maximum independent set". We formally define our problems as follows.

Deletion Blocker(α)

Instance: A graph $G = (V, E)$ and two integers $d, k \geq 1$.

Question: Is there a set $S \subseteq V$ of cardinality $|S| \leq k$ such that $\alpha(G - S) \leq \alpha(G) - d$?

We denote by d-Deletion Blocker(α) the problem Deletion Blocker(α), when d is fixed. A *d-deletion blocker* of $G = (V, E)$ of size k is a set $S \subseteq V$ with $|S| \leq k$ such that $\alpha(G - S) \leq \alpha(G) - d$.

Transversal(α)

Instance: A graph $G = (V, E)$ and two integers $d, k \geq 1$.

Question: Is there a set $S \subseteq V$ of cardinality $|S| \leq k$ such that for every maximum independent set I we have $|I \cap S| \geq d$?

We denote by d-Transversal(α) the problem Transversal(α), when d is fixed. A *d-transversal* of $G = (V, E)$ of size k is a set $S \subseteq V$ with $|S| \leq k$ such that for every maximum independent set I we have $|I \cap S| \geq d$.

Notice that for $d = 1$, the problems 1-Transversal(α) and 1-Deletion Blocker(α) are equivalent. This does not hold for $d > 1$, as can be seen in Fig. 1. If we consider the clique number ω, respectively the property "being a maximum clique", we can define in an analogous way the problems Deletion Blocker(ω) and Transversal(ω). Since an independent set in a graph G corresponds to a clique in the complement graph of G, it follows immediately that from any computational complexity result for Deletion Blocker(α) (resp. Transversal(α)) in some graph class $\mathcal{G}$, we can deduce a corresponding computational complexity result for Deletion Blocker(ω) (resp. Transversal(ω)) in the complement graph class of $\mathcal{G}$ and vice versa.

Since both problems are easily seen to be difficult in general graphs, much effort has been put on special graph classes. For Deletion Blocker(α), it has been shown that it is NP-complete in split graphs (see [8]), and thus in chordal and perfect graphs. A recent improvement of this result states that it is NP-complete in chordal, and thus in perfect graphs, even if $d = 1$ (see [17]), i.e. that 1-Deletion Blocker(α) is NP-complete in chordal graphs. On the positive side, Deletion Blocker(α) is polynomial-time solvable in trees (see [3,8]), bipartite graphs (see [3,8]), cobipartite graphs (see [9]), and cographs (see [3]). For split graphs, it has been shown that d-Deletion Blocker(α) is polynomial-time solvable (see [8]). In a recent paper, Hoang et al. show that it can be solved in polynomial time for interval graphs (see [15]).

Regarding Transversal(α), it was shown by Bentz et al. that this problem is solvable in polynomial time on trees (see [4]) which was then improved to bipartite graphs in [5]. Notice that the authors considered here even weighted maximum independent sets. From the fact that 1-Transversal(α) and 1-Deletion Blocker(α) are equivalent, it follows from [17] that 1-Transversal(α) is NP-complete for chordal graphs, and thus for perfect graphs. Also, it follows from [7] that 1-Transversal(α) is polynomial-time solvable in co-comparability graphs.

In this paper, we will consider the class of co-comparability graphs, which is a subclass of the class of perfect graphs, and show that both problems Transversal(α) and Deletion Blocker(α) are polynomial-time solvable in this graph class.

This generalises the results of [7,15]. Notice that, as explained above, our results directly imply that Transversal(ω) and Deletion Blocker(ω) can be solved in polynomial time in comparability graphs. In order to show our results, we will reduce both problems to the Vertex Cut problem which can be solved in polynomial time (see [10]).

Our paper is structured as follows. Section 2 contains notations and terminology. In Section 3, we present some important properties of maximum independent sets in co-comparability graphs. Section 4 contains one of our main results stating that Transversal(α) is solvable in polynomial time in co-comparability graphs. Sections 5 and 6 deal with the Deletion Blocker(α) problem. After introducing some more properties of independent sets in co-comparability graphs (Section 5), we show our second main result, namely that Deletion Blocker(α) is polynomial-time solvable in co-comparability graphs in Section 6. We finish with a conclusion and further research directions in Section 7.

2. Preliminaries

In this paper, we only consider finite graphs without self-loops and multiple edges. Unless specified otherwise, all graphs will be undirected and not necessarily connected. Let $G = (V, E)$ be a graph. For $U \subseteq V$, we denote by $G[U]$ the subgraph of G induced by U, i.e. the graph with vertex set U and edge set $\{uv \in E | u, v \in U\}$. We write $G - U = G[V \setminus U]$. We denote by $\overline{G}$ the *complement* of G, that is the graph with vertex set V and edge set $\overline{E} = \{uv | u, v \in V, uv \notin E\}$. We define $N_G(v)$ as the *neighbourhood* of a vertex $v \in V$ in G, i.e. the set of vertices $w \in V$ such that $vw \in E$.

A vertex $v \in V$ is said to be *complete* to some vertex set $U \subseteq V$, if $U \subseteq N_G(v)$. A *clique* is a set of pairwise adjacent vertices and the *clique number* ω denotes the size of a maximum clique of G. We call a vertex $v \in V$ *independent* to some vertex set $U \subseteq V$, if $U \cap N_G(v) = \varnothing$. An *independent set* in G is a set of pairwise non-adjacent vertices of G. The *independence number* α is the size of a maximum independent set of G. For an independent set S, another independent set I is called an *extension* of S if $S \subseteq I$. The set S is then called extendable. If I is a maximum independent set, we say it is a *maximum extension* of S and S is *max-extendable*.

We denote by $[\![d]\!]$, with $d \in \mathbb{N}^*$, the set $\{1, 2, \ldots, d\}$.

A *transitive ordering* $\prec$ of V is an ordering of the vertices such that if $u \prec v \prec w$ and $uv, vw \in E$, then $uw \in E$. A *comparability* graph is a graph admitting a transitive ordering of its vertices. The complement of a comparability graph is called a *co-comparability* graph. A transitive ordering of a comparability graph gives an ordering on its complement with the following property.

Property 1. *Let $G = (V, E)$ be a co-comparability graph and let $\prec$ be a transitive ordering of V in the comparability graph $\overline{G}$. Then, $\prec$ is an ordering of V in G such that if $u \prec v \prec w$ and $uv, vw \notin E$, then $uw \notin E$.*

In the rest of the paper, whenever we consider an ordering $\prec$ of the vertices of a co-comparability graph G, we mean an ordering satisfying Property 1, that is, it is a transitive ordering of its complementary comparability graph. From Property 1, we can directly deduce the following property.

Property 2. *Let $G = (V, E)$ be a co-comparability graph and let $\prec$ be a transitive ordering of V in the comparability graph $\overline{G}$. Let $u, v, w \in V$ with $v \prec w$, $uv, vw \notin E$ and $uw \in E$. Then $v \prec u$.*

Consider a graph $G = (V, E)$ and an ordering $\prec$ of its vertex set V. Let $U \subseteq V$ and $v \in V \setminus U$. We say that $U \prec v$, if $u \prec v$ for every $u \in U$.

Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$ and let I be an independent set in G. We denote by $\text{pos}_I(v)$, the *position of vertex v in I*, that is $\text{pos}_I(v) = |\{u \in I | u \prec v\}| + 1$. A *left extension* (resp. *right extension*) of $v \in V$ is an independent set $I \subseteq V$, containing only vertices $u \in V$ with $u \prec v$ (resp. $v \prec u$) and such that v is independent to I. We make the following easy observations.

Observation 3. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$ and $u, v \in V$ with $u \prec v$ and $uv \notin E$.*

- *Let I_ℓ^u be a left extension of u. Then, I_ℓ^u is a left extension of v.*
- *Let I_r^v be a right extension of v. Then, I_r^v is a right extension of u.*

Let $G = (V, A)$ be a directed graph and let $s, t \in V$. We call a directed path from s to t an *s-t-path*. An *s-t-cut* in G is a vertex set $C \subseteq V \setminus \{s, t\}$, such that there is no s-t-path in $G - C$. We will use the following problem.

Vertex Cut

Instance: A directed graph $G = (V, A)$ with two specified vertices s, t and an integer $k \geq 0$.

Question: Is there an s-t-cut $C \subseteq V \setminus \{s, t\}$ with $|C| \leq k$?

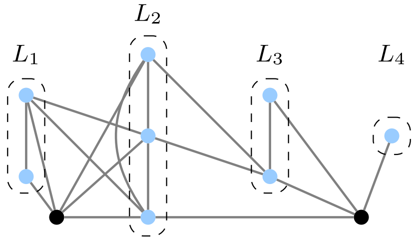

Fig. 2. A co-comparability graph and the sets $L_1, \ldots, L_4$. The black vertices are not contained in any maximum independent set.

3. Some properties of independent sets in co-comparability graphs

In this section, we will present some structural properties of independent sets in co-comparability graphs. These properties are crucial for our proofs in Section 4, where we prove that TRANSVERSAL(α) is polynomial-time solvable in this graph class.

Lemma 4. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let I_1, I_2 be two maximum independent sets in G. Let $\alpha = \alpha(G)$ and let $I_1 = \{u_1, \ldots, u_\alpha\}$ and $I_2 = \{v_1, \ldots, v_\alpha\}$, where $u_1 \prec \cdots \prec u_\alpha$ and $v_1 \prec \cdots \prec v_\alpha$. Assume $\exists\, i, j \in [\![\alpha]\!]$, such that $u_i = v_j$. Then $i = j$.*

Proof. Assume for a contradiction that there are $i, j \in [\![\alpha]\!]$, $i < j$, such that $u_i = v_j$. From Property 1, it follows that, since there is no edge vu_i with $v \in \left\{v_1, \ldots, v_{j-1}\right\}$ and no edge u_iu with $u \in \{u_{i+1}, \ldots, u_\alpha\}$, there is no edge vu with $v \in \left\{v_1, \ldots, v_{j-1}\right\}$ and $u \in \{u_{i+1}, \ldots, u_\alpha\}$. Hence, $\left\{v_1, \ldots, v_{j-1}, u_i, \ldots, u_\alpha\right\}$ is an independent set of size at least $\alpha + 1$, a contradiction. ◂

So Lemma 4 tells us that in a co-comparability graph $G = (V, E)$, the position of a vertex v, which belongs to some maximum independent set, is the same in every maximum independent set containing v. We denote by $\mathcal{I} \subseteq V$ the set of vertices which are contained in some maximum independent set. For any $v \in \mathcal{I}$, we then denote by pos(v) the position of v in every maximum independent set it belongs to. Thus, for a vertex $v \in \mathcal{I}$, we have $\text{pos}(v) = \text{pos}_I(v)$, for any maximum independent set I containing v.

Lemma 4 allows us to partition the vertices in $\mathcal{I} \subseteq V$ into sets $L_1, \ldots, L_\alpha \subseteq \mathcal{I}$, where $\alpha = \alpha(G)$, such that for any $v \in \mathcal{I}$ we have $v \in L_p$, $p \in [\![\alpha]\!]$ if and only if there exists an independent set $\left\{u_1, \ldots, u_{p-1}, u_p = v, u_{p+1}, \ldots, u_\alpha\right\}$ with $u_1 \prec \cdots \prec u_\alpha$. In other words, $L_p = \{v \in \mathcal{I} \mid \text{pos}(v) = p\}$, for $p \in [\![\alpha]\!]$ (see also Fig. 2). These sets L_p satisfy the following property.

Property 5. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let L_p, for $p \in [\![\alpha]\!]$, be as defined above. Then, L_p is a clique.*

Proof. Assume for a contradiction that there exist $u, v \in L_p$ with $u \prec v$ such that $uv \notin E$. Let I_r be a right extension of v of maximum size. Let I_ℓ be a left extension of u of maximum size. Then $v \prec I_r$ and $I_\ell \prec u$. From Property 1, we get that $I_\ell \cup \{u, v\} \cup I_r$ is an independent set which has size $(p - 1) + 2 + (\alpha - p) = \alpha + 1$, a contradiction. ◂

Property 6. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let $u, v \in \mathcal{I}$ with $uv \notin E$. Then $\text{pos}(u) < \text{pos}(v)$ if and only if $u \prec v$.*

Proof. Suppose for a contradiction that there are $u, v \in V$ with $uv \notin E$, $\text{pos}(u) < \text{pos}(v)$ and $v \prec u$. Let I_ℓ^v be a maximum left extension of v and I_ℓ^u be a maximum left extension of u. From Observation 3 we know that I_ℓ^v is also a left extension of u. Since $\text{pos}(u) < \text{pos}(v) = |I_\ell^v| + 1 \leq |I_\ell^u| + 1 = \text{pos}(u)$ we get a contradiction. ◂

Our algorithm that we will present in Section 4 relies on the sets L_p defined above. Thus, it is important to be able to determine those sets in a co-comparability graph. The following result shows that this can be done in polynomial time.

Lemma 7. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Then, the sets $L_1, \ldots, L_\alpha$, where $\alpha = \alpha(G)$, can be found in polynomial time.*

Proof. First notice that it follows from Property 1 that if I_ℓ^u is a left extension of some vertex $u \in V$ and I_r^u is a right extension of this same vertex $u \in V$, then $I_\ell^u \cup \{u\} \cup I_r^u$ is an independent set in G containing u. Furthermore, any independent set containing u consists of a left extension of u, a right extension of u, and u itself. Thus, the size of a maximum independent set in G containing some vertex u can be obtained by determining the maximum left respectively right extension of u. Let us define the two functions leftext($\cdot$) and rightext($\cdot$) as follows: leftext(v), for $v \in V$, is the size of a maximum left extension of v; and rightext(v), for $v \in V$, is the size of a maximum right extension of v. From the above, it follows that $\text{leftext}(v) + \text{rightext}(v) + 1$ corresponds to the size of a maximum independent set of G which contains v. Furthermore, the value $\text{leftext}(v) + 1$ gives us the position of v in a maximum independent set I containing v, i.e. $\text{pos}_I(v) = \text{leftext}(v) + 1$. We can also determine α, since $\alpha = \max_{v\in V}(\text{leftext}(v) + \text{rightext}(v) + 1)$.

From the above, it follows that once we determined leftext(v) and rightext(v) for every vertex $v \in V$, we obtain the sets $L_1, \ldots, L_\alpha$, since

$$L_i = \{v \in V |\ \text{leftext}(v) + \text{rightext}(v) + 1 = \alpha \textit{ and } \text{leftext}(v) + 1 = i\}.$$

Thus, it remains to show how we can compute the functions leftext($\cdot$) and rightext($\cdot$). Herefore, we use the ordering $\prec$ of the vertices and we get that

$$\text{leftext}(v) = \max_{u\in V, u\prec v, uv\notin E}(\text{leftext}(u) + 1)$$
$$\text{rightext}(v) = \max_{u\in V, v\prec u, uv\notin E}(\text{rightext}(u) + 1)$$

Suppose $V = \{v_1, \ldots, v_n\}$ and $v_1 \prec v_2 \prec \cdots \prec v_n$. Clearly, $\text{leftext}(v_1) = \text{rightext}(v_n) = 0$. By iterating through the vertices in increasing order with respect to $\prec$, we can calculate the values of leftext(v), and similarly by iterating in decreasing order we get rightext(v), for all vertices $v \in V$. Both functions can therefore clearly be computed for all vertices in $\mathcal{O}(|V|^2)$ time. Hence we can find the partition in polynomial time. ◀

4. Transversals in co-comparability graphs

In this section, we present a polynomial-time algorithm to solve Transversal(α) in co-comparability graphs. Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$, independence number $\alpha = \alpha(G)$ and let $d > 0$ be an integer. We will construct a directed graph $G' = (V', A')$ such that (G', k) is a Yes-instance of Vertex Cut if and only if (G, d, k) is a Yes-instance of Transversal(α) for some integer $k > 0$. This equivalence will be shown in Theorem 16. Since G' can be constructed in polynomial time (see Theorem 16) and since we know that Vertex Cut can be solved in polynomial time (see [10]), we obtain our result.

We first make an important observation.

Observation 8. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let $S \subseteq V$. Then S is a d-transversal of G if and only if it contains at least one vertex from every max-extendable independent set of size $\alpha - d + 1$ of G.*

Observation 8 tells us that, in order to obtain a d-transversal in G, we must intersect all max-extendable independent sets of size $\alpha - d + 1$ in at least one vertex. Therefore, in the following, we will construct a directed graph G' with a source s and a sink t, such that every s-t-path in G' corresponds to a max-extendable independent set of size $\alpha - d + 1$ in G. This one-to-one correspondence will be proven in Lemma 13. An s-t-cut in G' will then correspond to a d-transversal in G (see Theorem 16).

Let us now describe the construction of $G' = (V', A')$. Let $\mathcal{I} = \bigcup_{p\in[\![\alpha]\!]} L_p$ be the set of vertices contained in a maximum independent set. The vertex set V' of G' consists of d copies $U_1, \ldots, U_d$ of $\mathcal{I}$ and two additional vertices s, t, that is, $V' = \bigcup_{\ell\in[\![d]\!]} U_\ell \cup \{s, t\}$. We denote by $L_{p,\ell}$ the set of vertices in U_ℓ that correspond to vertices of L_p in G, for $p \in [\![\alpha]\!]$ and $\ell \in [\![d]\!]$, and we say that ℓ is the *level* of the vertices in U_ℓ, denoted by $\text{level}(x) = \ell$ for $x \in U_\ell$. Recall that pos(v), for $v \in \mathcal{I}$, is the position of v in every maximum independent set it belongs to in G. For simplicity, we will adopt this same notion for all vertices in $V' \setminus \{s, t\}$, i.e. for every vertex $x \in V' \setminus \{s, t\}$ that corresponds to some vertex $v \in L_p$, for $p \in [\![\alpha]\!]$, we will also use pos(x) and call it the position of x in order to actually refer to the position of v, the vertex that x corresponds to in G. Furthermore, we set $\text{pos}(s) = 0$ and $\text{level}(s) = 1$ as well as $\text{pos}(t) = \alpha + 1$ and $\text{level}(t) = d$, in order to simplify the readability of our proofs.

Let $x, y \in V' \setminus \{s, t\}$, where $x \in L_{p,\ell}$ and $y \in L_{p',\ell'}$, with $p, p' \in [\![\alpha]\!]$ and $\ell, \ell' \in [\![d]\!]$. Let $u, v \in \mathcal{I}$, where x corresponds to u and y corresponds to v. We add an arc (x, y), if $\{u, v\}$ is max-extendable in G and $p' = p + g + 1$, $\ell' = \ell + g$ for some integer $g \geq 0$. Finally, for any vertex $x \in L_{p,\ell}$, with $p \in [\![\alpha]\!]$ and $\ell \in [\![d]\!]$, we add an arc (s, x) if $p = \text{pos}(s) + g + 1 = g + 1$, $\ell = \text{level}(s) + g = g + 1$, for some integer $g \geq 0$, and we add an arc (x, t) if $\text{pos}(t) = \alpha + 1 = p + g + 1$, $\text{level}(t) = d = \ell + g$ for some integer $g \geq 0$.

We make the following observations which immediately follow from the definition of $G' = (V', A')$.

Observation 9. *Let $G' = (V', A')$ be the directed graph constructed from a co-comparability graph G as described above. For any arc $(x, y) \in A'$, we have*

(a) $\text{pos}(y) > \text{pos}(x)$;
(b) $\text{level}(y) \geq \text{level}(x)$;
(c) $\text{pos}(y) - \text{pos}(x) - 1 = \text{level}(y) - \text{level}(x)$.

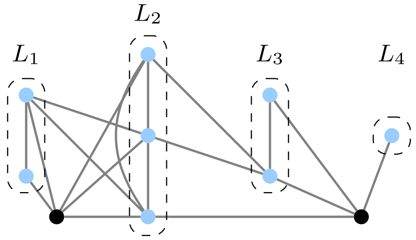

Fig. 3. The graph G' constructed from the graph G from Fig. 2 for $d = 1$. The level of all vertices is 1.

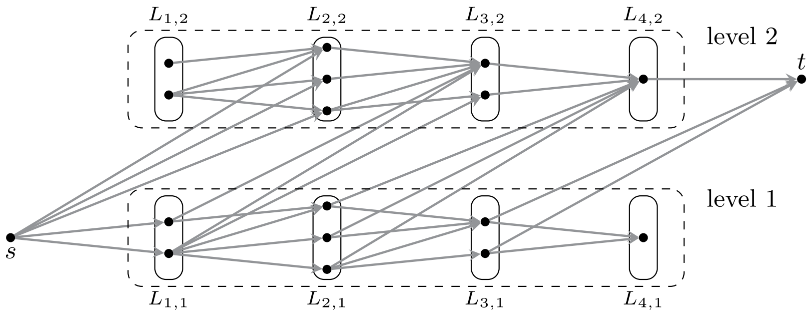

Fig. 4. The graph G' constructed from the graph G from Fig. 2 for $d = 2$.

Figs. 3 and 4 show the graph G' constructed from the graph G in Fig. 2 for $d = 1$ (Fig. 3) and $d = 2$ (Fig. 4).

Before we show the one-to-one correspondence between max-extendable independent sets of size $\alpha - d + 1$ in G and s-t-paths in G', we present some useful properties.

Property 10. *Let P be an s-t-path in G' with vertices $s, x_1, \ldots, x_h, t$ in that order. There exist exactly $d - 1$ distinct integers in $[\![\alpha]\!]$, say $g_1, \ldots, g_{d-1}$, such that $\mathrm{pos}(x_i) \notin \{g_1, \ldots, g_{d-1}\}$ for all $i \in [\![h]\!]$. For any other integer $g \in [\![\alpha]\!] \setminus \{g_1, \ldots, g_{d-1}\}$, there exists exactly one vertex $x \in V(P) \setminus \{s, t\}$ such that $\mathrm{pos}(x) = g$.*

Proof. Let P be an s-t-path in G' with vertices $s, x_1, \ldots, x_h, t$ in that order. Consider an arc (x_i, x_{i+1}) in P. Assume that $x_i \in L_{p,\ell}$ and $x_{i+1} \in L_{p',\ell'}$, with $p, p' \in [\![\alpha]\!]$ and $\ell, \ell' \in [\![d]\!]$. From Observation 9(c), it follows that $p' = p + 1 + \ell' - \ell$ and by Observation 9(b), we know that $\ell' - \ell \geq 0$. Hence, we skip $\ell' - \ell$ positions between x_i and x_{i+1}, which will not be used by any vertex in P, that is, there are $\ell' - \ell$ integers between p and p' which do not correspond to any position of some vertex in P. Since we start at level 1 (recall that $\mathrm{level}(s) = 1$) and we end at level d (recall that $\mathrm{level}(t) = d$), we get that there are exactly $d-1$ distinct integers in $[\![\alpha]\!]$, say $g_1, \ldots, g_{d-1}$, such that $\mathrm{pos}(x_j) \notin \{g_1, \ldots, g_{d-1}\}$ for all $j \in [\![h]\!]$. Furthermore, since $\mathrm{pos}(x_{j'}) > \mathrm{pos}(x_j)$, for any $j' > j$, with $j, j' \in [\![h]\!]$ (see Observation 9(a)), it is obvious to see that for any other integer $g \in [\![\alpha]\!] \setminus \{g_1, \ldots, g_{d-1}\}$, there exists exactly one vertex $x \in V(P) \setminus \{s, t\}$ such that $\mathrm{pos}(x) = g$. ◀

The next property gives the exact number of vertices in any s-t-path in G'.

Property 11. *Every s-t-path P in G' contains $\alpha - d + 3$ vertices.*

Proof. From Property 10, we know that there are $d-1$ positions that do not correspond to any position of some vertex in P and that all other positions do correspond each to a different position of some vertex in $V(P) \setminus \{s, t\}$. Since there are in total α possible positions of which none corresponds to the positions of s and t (recall that $\mathrm{pos}(s) = 0$ and $\mathrm{pos}(t) = \alpha + 1$), there are $\alpha - d + 1$ vertices in $V(P) \setminus \{s, t\}$, and hence, P contains exactly $\alpha - d + 3$ vertices. ◀

The following property immediately follows from the definition of the position of a vertex in $\mathcal{I}$ and Observation 9(a).

Property 12. *Let $I = \{v_1, \ldots, v_{\alpha-d+1}\}$ be a max-extendable independent set in G. There exist exactly $d - 1$ distinct integers in $[\![\alpha]\!]$, say $g_1, \ldots, g_{d-1}$ such that $\mathrm{pos}(v_i) \notin \{g_1, \ldots, g_{d-1}\}$ for all $i \in [\![\alpha - d + 1]\!]$.*

We are now ready to show the one-to-one correspondence between s-t-paths in G' and max-extendable independents sets of size $\alpha - d + 1$ in G.

Lemma 13. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let $\alpha = \alpha(G)$, $d > 0$ be an integer and consider $G' = (V', A')$, constructed as described above. Then, every max-extendable independent set of size $\alpha - d + 1$ in G corresponds to an s-t-path in G' and vice versa.*

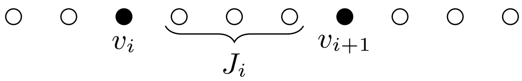

Fig. 5. An extension of v_i and v_{i+1} to a maximum independent set. The empty circles represent all vertices of the maximum extension J_i', while the set $J_i \subseteq J_i'$ contains exactly the vertices whose corresponding vertices have positions between those of v_i and v_{i+1}. We assume the vertices to be ordered from left to right according to the ordering $\prec$.

Proof. Let us first consider an s-t-path $P = s, x_1, x_2, \ldots, x_h, t$ in G'. We know from Property 11 that this path consists of exactly $\alpha - d + 3$ vertices, hence $h = \alpha - d + 1$. Let $v_i \in V$ be the vertex in G corresponding to $x_i \in V'$, for $i \in [\![\alpha - d + 1]\!]$. Notice that by Observation 9(a), we have $\mathrm{pos}(v_1) < \mathrm{pos}(v_2) < \cdots < \mathrm{pos}(v_{\alpha-d+1})$.

Claim 13.1. *$\{v_1, \ldots, v_{\alpha-d+1}\}$ is an independent set in G.*

Proof. By construction, if there is an arc (x_i, x_{i+1}) in G', for $i \in [\![\alpha - d]\!]$, then v_i and v_{i+1} are necessarily non-adjacent. Since $\mathrm{pos}(v_i) < \mathrm{pos}(v_{i+1})$, we have by Property 6 that $v_i \prec v_{i+1}$. Thus, we obtain that $v_i \prec v_{i+1} \prec \cdots \prec v_{i+j}$. Now using Property 1, we conclude that v_i, v_{i+j} are non-adjacent, for $i, i+j \in [\![\alpha - d + 1]\!]$, $j > 1$, such that $x_i, x_{i+j} \in V(P)$. Thus, $\{v_1, \ldots, v_{\alpha-d+1}\}$ is an independent set in G. $\triangleleft$

Claim 13.2. *The independent set $\{v_1, \ldots, v_{\alpha-d+1}\}$ is max-extendable in G.*

Proof. As mentioned before, by construction, we know that if there is an arc (x_i, x_{i+1}) in G', then $\{v_i, v_{i+1}\}$ is max-extendable in G. Let now $J_i' \subseteq V$, for $i \in [\![\alpha - d]\!]$, be a maximum extension of $\{v_i, v_{i+1}\}$. Let further $J_i = \left\{w \in J_i' \mid \mathrm{pos}(v_i) < \mathrm{pos}(w) < \mathrm{pos}(v_{i+1})\right\}$ (see Fig. 5). We also adapt this definition to the first vertex v_1, and denote by J_0' a maximum extension of $\{v_1\}$ (recall that $v_1 \in \mathcal{I}$). We choose $J_0 = \left\{w \in J_0' \mid \mathrm{pos}(w) < \mathrm{pos}(v_1)\right\}$. Similarly, we denote by $J_{\alpha-d+1}'$ a maximum extension of $\{v_{\alpha-d+1}\}$ and $J_{\alpha-d+1} = \left\{w \in J_{\alpha-d+1}' \mid \mathrm{pos}(v_{\alpha-d+1}) < \mathrm{pos}(w)\right\}$ (recall that $v_{\alpha-d+1} \in \mathcal{I}$). The set $\bigcup_{h=0}^{\alpha-d+1} J_h \cup \{v_1, \ldots, v_{\alpha-d+1}\}$ is an independent set, since $\{v_i, v_{i+1}\} \cup J_i$ is an independent set, for $i \in [\![\alpha - d + 1]\!]$, and we can combine them using Property 1. From the fact that a maximum extension of $\{v_i, v_{i+1}\}$, with $i \in \{1, \ldots, \alpha - d\}$, is a maximum independent set I, it follows that for any $p \in [\![\alpha]\!]$, there is a vertex in $u \in I$ with $\mathrm{pos}(u) = p$. Thus, we get that $|J_i| = \mathrm{pos}(v_{i+1}) - \mathrm{pos}(v_i) - 1$, for $i \in [\![\alpha - d]\!]$. Further, we have $|J_0| = \mathrm{pos}(v_1) - 1$ and $|J_{\alpha-d+1}| = \alpha - \mathrm{pos}(v_{\alpha-d+1})$. Thus,

$$\left|\{v_1, \ldots, v_{\alpha-d+1}\} \cup \bigcup_{h=0}^{\alpha-d+1} J_h\right|$$
$$= (\alpha - d + 1) + (\mathrm{pos}(v_1) - 1) + \sum_{h=1}^{\alpha-d}\big(\mathrm{pos}(v_{h+1}) - \mathrm{pos}(v_h) - 1\big) + (\alpha - \mathrm{pos}(v_{\alpha-d+1}))$$
$$= (\alpha - d + 1) + \alpha - (\alpha - d) - 1 = \alpha.$$

Hence, it follows that $\{v_1, \ldots, v_{\alpha-d+1}\} \cup \bigcup_{h=0}^{\alpha-d+1} J_h$ is a maximum independent set in G, which shows that $\{v_1, \ldots, v_{\alpha-d+1}\}$ is max-extendable. $\triangleleft$

Let us now prove the converse, i.e. that a max-extendable independent set of size $\alpha - d + 1$ in G corresponds to an s-t-path in G'. Consider a max-extendable independent set $I = \{v_1, \ldots, v_{\alpha-d+1}\}$ of size $\alpha - d + 1$ in G. We may assume that $\mathrm{pos}(v_1) < \mathrm{pos}(v_2) < \cdots < \mathrm{pos}(v_{\alpha-d+1})$.

Let $g_1, \ldots, g_{d-1}$ be as in Property 12. For $i \in [\![\alpha - d + 1]\!]$, let $x_i \in V'$ be the vertex corresponding to v_i, with $x_i \in L_{\mathrm{pos}(v_i),\ell}$, where $\ell = |\{g_k \mid g_k < \mathrm{pos}(v_i), k \in [\![d-1]\!]\}| + 1$. By Property 12, we know that $1 \le \ell \le d$, and since $1 \le \mathrm{pos}(v_i) \le \alpha$, we get that x_i exists. To show the existence of a path P with vertices $s, x_1, \ldots, x_{\alpha-d+1}, t$ in that order, it remains to show that the arcs (s, x_1), (x_i, x_{i+1}), for $i \in [\![\alpha - d]\!]$, and $(x_{\alpha-d+1}, t)$ exist.

Claim 13.3. *The arcs (s, x_1), (x_i, x_{i+1}), for $i \in [\![\alpha - d]\!]$, and $(x_{\alpha-d+1}, t)$ exist in G'.*

Proof. For the arc (s, x_1), notice that $|\{g_k \mid g_k < \mathrm{pos}(v_1), k \in [\![d-1]\!]\}| + 1 = \mathrm{pos}(v_1)$, and hence the arc exists by definition. Let $i \in [\![\alpha - d]\!]$. Let $\ell_i = |\{g_k \mid g_k < \mathrm{pos}(v_i), k \in [\![d-1]\!]\}| + 1$ be the level of x_i and let $\ell_{i+1} = |\{g_k \mid g_k < \mathrm{pos}(v_{i+1}), k \in [\![d-1]\!]\}| + 1$ be the level of x_{i+1}, as defined above. Recall that by definition the arc (x_i, x_{i+1}) exists, if $\mathrm{pos}(x_{i+1}) = \mathrm{pos}(x_i) + g + 1$ and $\ell_{i+1} = \ell_i + g$, for some $g \ge 0$, and $\{v_i, v_{i+1}\}$ is max-extendable. It follows from the above that $\ell_{i+1} - \ell_i = |\{g_k \mid \mathrm{pos}(v_i) \le g_k < \mathrm{pos}(v_{i+1}), k \in [\![d-1]\!]\}|$, that is, the number of positions between $\mathrm{pos}(v_i)$ and $\mathrm{pos}(v_{i+1})$ that are not used by any vertex in I, and hence we get $\mathrm{pos}(x_{i+1}) = \mathrm{pos}(x_i) + \ell_{i+1} - \ell_i + 1$. Furthermore, $\{v_i, v_{i+1}\}$ is clearly max-extendable since both belong to I. We conclude that the arc (x_i, x_{i+1}) necessarily exists. If we consider $x_{\alpha-d+1}$, we get that $\mathrm{level}(x_{\alpha-d+1}) = |\{g_k \mid g_k < \mathrm{pos}(v_{\alpha-d+1}), k \in [d-1]\}| + 1 = d - 1 - (\alpha - \mathrm{pos}(v_{\alpha-d+1})) + 1 = \mathrm{pos}(v_{\alpha-d+1}) + d - \alpha$. Hence, the arc $(x_{\alpha-d+1}, t)$ exists by definition. $\triangleleft$

It follows that $P = s, x_1, \ldots, x_{\alpha-d+1}, t$ is a path in G'. This concludes the proof of our lemma. ◂

Let us show two more properties that we will need in our main theorem of the section.

Property 14. *Let G be a co-comparability graph with vertex ordering $\prec$ and let G' be the corresponding directed graph, constructed as described above. Let I be a max-extendable independent set of G and let P be the corresponding s-t-path in G'. Consider $v \in I$ and its corresponding vertex $x \in V(P)$. Then, $\text{pos}_I(v) = \text{pos}(v) - \text{level}(x) + 1$.*

Proof. Let $g_1, \ldots, g_{d-1}$ be as in Property 12. Recall from the proof of Lemma 13 that $\text{level}(x) = |\{g_k \mid g_k < \text{pos}(v_i), k \in [\![d-1]\!]\}| + 1$. Hence, the result follows. ◂

Property 15. *Let G be a co-comparability graph with vertex ordering $\prec$ and let G' be the corresponding directed graph, constructed as described above. Let I_1, I_2 be two max-extendable independent sets of size $\alpha - d + 1$ in G, and let P_1, P_2 be their corresponding paths in G'. Let $v \in I_1 \cap I_2$ such that the corresponding vertices $x_1 \in P_1, x_2 \in P_2$ are different. Assume without loss of generality that $\text{level}(x_1) < \text{level}(x_2)$. Then, $\text{pos}_{I_1}(v) > \text{pos}_{I_2}(v)$.*

Proof. Since, by Property 14, $\text{pos}_{I_1}(v) = \text{pos}(v) - \text{level}(x_1) + 1$ and $\text{pos}_{I_2}(v) = \text{pos}(v) - \text{level}(x_2) + 1$, and since we assume that $\text{level}(x_1) < \text{level}(x_2)$, it follows that $\text{pos}_{I_1}(v) > \text{pos}_{I_2}(v)$. ◂

Theorem 16. Transversal(α) *is polynomial-time solvable for co-comparability graphs.*

Proof. Let $G = (V, E)$ be a co-comparability graph and let (G, d, k) be an instance of Transversal(α). We construct the graph $G' = (V', A')$ as described above. We will show that (G, d, k) is a Yes-instance of Transversal(α) if and only if (G', k) is a Yes-instance of Vertex Cut. Let (G', k) be a Yes-instance of Vertex Cut and let C be an s-t-cut of G' of size at most k. We want to prove that (G, d, k) is a Yes-instance of Transversal(α).

For every vertex in the cut $C \subseteq V'$, we add the corresponding vertex in G to a set S. We assume for a contradiction that there is an independent set I in $G - S$ of size $\alpha - d + 1$ which is max-extendable in G. By Lemma 13, we know that there is a path P from s to t in G' representing I. Since $I \subseteq V \setminus S$, we get that $P \cap C = \varnothing$ and hence, we can find an s-t-path in $G' - C$, a contradiction. Thus, such an independent set I does not exist, and so by Observation 8, we deduce that (G, d, k) is a Yes-instance of Transversal(α).

Let now (G, d, k) be a Yes-instance of Transversal(α). We want to show that (G', k) is a Yes-instance of Vertex Cut. Let $S \subseteq V$, with $|S| \leq k$, be a d-transversal of G. We may assume that S is minimal.

We iteratively construct a set C using Algorithm 1, and we will prove that C is an s-t-cut in G' with $|S| = |C|$. For each vertex in S, the algorithm chooses the corresponding vertex in G' belonging to the lowest level such that there is an s-t-path in $G' - C$ containing this vertex, and then adds it to C. We will find such a vertex for every vertex in S, since otherwise S would not be minimal. Hence, it is clear that $|S| = |C|$.

Algorithm 1

Input: The graph G' constructed from a co-comparability graph G,
  a minimal d-transversal S in G.
Output: An s-t-cut $C \subseteq V'$ with $|S| = |C|$.
  Let $S = \{u_1, \ldots, u_{|S|}\}$, $u_i \prec u_j$ for $i < j$, $i, j \in [\![|S|]\!]$.
  Let $C = \varnothing$.
  for i from $1 \to |S|$ **do**
    Let $u = u_i$.
    Let $y_1, \ldots, y_d \in V'$ be the vertices corresponding to u, sorted by increasing level.
    for j from $1 \to d$ **do**
      if $\exists$ s-t-path in $G' - C$ containing y_j **then**
        $C = C \cup y_j$
        break
      else
        continue
      end if
    end for
  end for

To prove that C, which is constructed by applying Algorithm 1, is indeed an s-t-cut in G', we assume for a contradiction that there exists an s-t-path P_1 in $G' - C$. Let $I_1 \subseteq V$ be the max-extendable independent set in G of size $\alpha - d + 1$ corresponding to P_1 (I_1 exists by Lemma 13). Let $I_1^e \subseteq V$ be a maximum extension of I_1 in G. Since S is a d-transversal of G, we know that $S \cap I_1 \neq \varnothing$. Let $v \in S \cap I_1$, and if $|S \cap I_1| > 1$, we choose the rightmost vertex according to $\prec$ in $S \cap I_1$ for

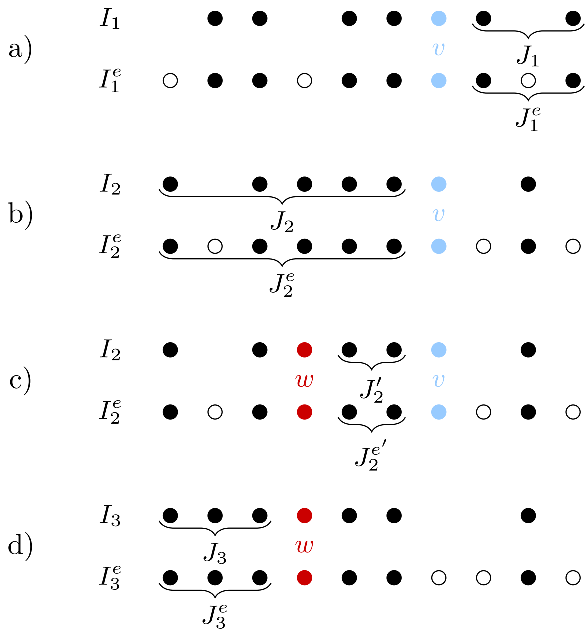

Fig. 6. The different independent sets, considered in the proof. Note that the empty vertices are only part of the corresponding extended version of the independent set (e.g. in I_1^e but not in I_1), while the other vertices are contained in both sets.

v. Let $y_j \in V'$, for $j \in [\![d]\!]$, be the copy of v in P_1. Since P_1 is a path in $G' - C$, we know that $y_j \notin C$. Hence, there is some other copy of v in G', say y_i, $i \in [\![d]\!]$, such that Algorithm 1 added y_i to C. Due to the procedure we use in Algorithm 1 to choose the vertices in C, we have $i < j$.

Since y_i was added to C, there exists a path P_2 in G' containing y_i. Let I_2 be the max-extendable independent set of size $\alpha - d + 1$ in G corresponding to P_2 (I_2 exists by Lemma 13), and let I_2^e be a maximum extension of I_2 in G. Let $J_1 \subseteq I_1$ be the set of the $\alpha - d + 1 - \text{pos}_{I_1}(v)$ rightmost vertices in I_1, i.e. those vertices u in I_1 such that $\text{pos}(v) < \text{pos}(u)$ (see Fig. 6(a)). We know that $J_1 \cap S = \varnothing$ by the choice of v. Let $J_1^e \supseteq J_1$ be the set of the $\alpha - \text{pos}(v)$ rightmost vertices in I_1^e, i.e. those vertices u in I_1^e such that $v \prec u$ (see Fig. 6(a)). Similarly, let $J_2 \subseteq I_2$ be the set of the first $\text{pos}_{I_2}(v) - 1$ vertices in I_2, i.e. those vertices u in I_2 such that $\text{pos}(u) < \text{pos}(v)$ (see Fig. 6(b)), and let $J_2^e \supseteq J_2$ be the set of the first $\text{pos}(v) - 1$ vertices in I_2^e, i.e. those vertices u in I_2^e such that $\text{pos}(u) < \text{pos}(v)$ (see Fig. 6(b)). It then follows from Property 1 that $J_2^e \cup \{v\} \cup J_1^e$ is an independent set in G of size $(\alpha - \text{pos}(v)) + 1 + (\text{pos}(v) - 1) = \alpha$, i.e. a maximum independent set in G. Thus, $J_2 \cup J_1$ is a max-extendable independent set in G. It follows from Property 15, that $\text{pos}_{I_1}(v) < \text{pos}_{I_2}(v)$. We conclude that $|J_2 \cup J_1| = (\text{pos}_{I_2}(v) - 1) + (\alpha - d + 1 - \text{pos}_{I_1}(v)) > \alpha - d$. Hence, $J_2 \cup J_1$ either is a max-extendable independent set in $G - S$ of size at least $\alpha - d + 1$, or $J_2 \cap S \neq \varnothing$. In the first case, we directly get a contradiction to our assumption that S is a d-transversal in G. So, we may assume that $J_2 \cap S \neq \varnothing$.

Let $w \in J_2 \cap S$, and if $|J_2 \cap S| > 1$, we take the rightmost vertex w in $J_2 \cap S$ with respect to $\prec$ such that $\text{pos}(w) < \text{pos}(v)$. Let y_h, $h \in [\![d]\!]$, be the vertex in P_2 corresponding to w. Then, $y_h \notin C$, since otherwise Algorithm 1 would not have added y_i to C. Thus, as before, there exists some vertex y_g, $g \in [\![d]\!]$, with $g < h$, such that y_g corresponds to w and $y_g \in C$. Therefore, there exists a path P_3 in G' containing y_g. Let I_3 be the max-extendable independent set of size $\alpha - d + 1$ in G corresponding to P_3, and let I_3^e be its extension (see Fig. 6(d)). We define $J_3 = \left\{u \in I_3 |\, \text{pos}_{I_3}(u) < \text{pos}_{I_3}(w)\right\}$, as well as $J_3^e = \{u \in I_3 |\, \text{pos}(u) < \text{pos}(w)\} \supseteq J_3$. Furthermore, we consider the set $J_2' = \left\{u \in I_2 |\, \text{pos}_{I_2}(w) < \text{pos}_{I_2}(u) < \text{pos}_{I_2}(v)\right\}$ as well as $J_2^{e'} = \{u \in I_2 |\, \text{pos}(w) < \text{pos}(u) < \text{pos}(v)\} \supseteq J_2'$ (see Fig. 6(c)). Thus, $J_3 \cup J_2' \cup J_1$ is a subset of $J_3^e \cup \{w\} \cup J_2^{e'} \cup \{v\} \cup J_1^e$, which by Property 1 is an independent set in G of size $(\text{pos}(w) - 1) + 1 + (\text{pos}(v) - \text{pos}(w) - 1) + 1 + (\alpha - \text{pos}(v)) = \alpha$, i.e. a maximum independent set. Thus, $J_3 \cup J_2' \cup J_1$ is a max-extendable independent set in G. It follows from Property 15, that $\text{pos}_{I_2}(w) < \text{pos}_{I_3}(w)$. Since in addition $\text{pos}_{I_1}(v) < \text{pos}_{I_2}(v)$ (see above), we conclude that $|J_3 \cup J_2' \cup J_1| = (\text{pos}_{I_3}(w) - 1) + (\text{pos}_{I_2}(v) - \text{pos}_{I_2}(w) - 1) + (\alpha - d + 1 - \text{pos}_{I_1}(v)) > \alpha - d$. Hence, we obtain again that either $J_3 \cup J_2' \cup J_1$ is a max-extendable independent set in $G - S$ of size at least $\alpha - d + 1$, or that $J_3 \cap S \neq \varnothing$. As before, the first case gives us a contradiction to our assumption that S is a d-transversal in G. So, we may assume that $J_3 \cap S \neq \varnothing$.

By repeatedly using these arguments, we can always find a new vertex in S. But since S is finite, this case cannot always occur. Hence, we will necessarily get a contradiction and thus, there is no s-t-path P in $G' - C$. So we conclude that C is an s-t-cut in G'.

Let us now consider the complexity of our algorithm. From [24], we know that for a graph with n vertices, we can solve Vertex Cut in $\mathcal{O}(n^3)$. Since the graph G' has $\mathcal{O}(d|V|)$ vertices, computing a Vertex Cut in G' can be done in time

$\mathcal{O}(d^3|V|^3)$. We still need to consider the time we need to construct G'. Using Lemma 7, we know that we can find the partition of $\mathcal{I}$ into sets L_p, for $p \in [\![d]\!]$, in time $\mathcal{O}(|V|^2)$. For every pair of vertices in G', we can check in $\mathcal{O}(|V|)$ time if we introduce an arc between them in G'. Hence, G' can be constructed in $\mathcal{O}(d^2|V|^3)$. We conclude that Transversal(α) can be solved in time $\mathcal{O}(d^3|V|^3)$. ◀

5. More properties of independent sets in co-comparability graphs

To solve Deletion Blocker(α), we will use an approach similar to the one in Section 4. This requires an extension of the structural results from Section 3 for independent sets that are not necessarily maximum. We sometimes omit proofs since they are very similar to the ones we presented in the previous sections.

Recall that Lemma 4 allowed us to partition the vertices in a maximum independent set by using their position. The same property holds for all vertices with respect to the largest independent set in which they are contained.

Lemma 17. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let $v \in V$ and let I_1, I_2 be two independent sets of G such that both contain v and they are maximum among the independent sets of G containing v. Let $\beta = |I_1| = |I_2|$. Let $I_1 = \{u_1, \dots, u_{i-1}, v = u_i, u_{i+1}, \dots, u_\beta\}$, with $u_1 \prec \cdots \prec u_\beta$, and $I_2 = \{v_1, \dots, v_{j-1}, v = v_j, v_{j+1}, \dots, v_\beta\}$, with $v_1 \prec \cdots \prec v_\beta$. Then $i = j$.*

Proof. Suppose that $i < j$, then similar to the proof of Lemma 4, we can find an independent set $\{v_1, \dots, v_{j-1}, v, u_{i+1}, \dots, u_\beta\}$ which has size $\beta + 1$ and contains v, a contradiction. ◀

Definition 18. Let $G = (V, E)$ be a co-comparability graph with $\alpha = \alpha(G)$. Let $L_1, \dots, L_\alpha$ be as in Lemma 7. Let $\mathcal{I}_\beta$ be the set of vertices that occur in an independent set of size β, but not of size $\beta + 1$, $\beta \in [\![\alpha]\!]$. We define $L_{p,\beta}$, $p \in [\![\beta]\!]$, as the set of vertices $v \in \mathcal{I}_\beta$ such that for an independent set I of size β containing $v \in V$ we have that $\mathrm{pos}_I(v) = p$. From now on, we refer to the sets L_p from Lemma 7 as $L_{p,\alpha}$. We say that $\mathrm{pos}(v) = p$.

Note that we defined the position of a vertex in two different ways. We will consider in the following both the relative position pos_I of a vertex, which depends on the independent set I and the absolute position pos of a vertex, which is independent of any specific independent set. Fig. 7 gives an example of a co-comparability graph. We can see the assignment of the vertices in $\mathcal{I}_\alpha$ and $\mathcal{I}_{\alpha-1}$ to the sets $L_{p,\beta}$. The set $L_{3,\alpha-1}$ is empty.

Observation 19. *Let $G = (V, E)$ be a co-comparability graph and let $v \in L_{p,\beta}$, $\beta \in [\![\alpha]\!]$, $p \in [\![\beta]\!]$. Consider a maximum left extension I_ℓ^v of v and a maximum right extension I_r^v of v. Then we have that $|I_\ell^v| = p - 1$ and $|I_r^v| = \beta - |I_\ell^v| - 1 = \beta - (p - 1) - 1 = \beta - p$.*

From Property 5 we know that each of the sets $L_{p,\alpha}$ for $p \in [\![\alpha]\!]$ is a clique. We will generalise this to the sets $L_{p,\beta}$, for $\beta \in [\![\alpha]\!]$.

Property 20. *Let $G = (V, E)$ be a co-comparability graph with $\alpha = \alpha(G)$. Let $L_{p,\beta}$, with $\beta \in [\![\alpha]\!]$, $p \in [\![\beta]\!]$ be as in Definition 18. Let $v \in L_{p,\beta}$. Then, v is adjacent to all vertices in*

$$\bigcup_{j \in \{\beta,\dots,\alpha\}, i \in \{0,\dots,j-\beta\}} L_{p+i,j}.$$

Proof. Let $v \in L_{p,\beta}$. Assume there exists $u \in L_{q,\gamma}$, for $\gamma \in \{\beta, \dots, \alpha\}$, $q \in \{p, \dots, p + \gamma - \beta\}$ such that $uv \notin E$. We assume now that $v \prec u$. The case $u \prec v$ can be handled in a similar way. Let I_ℓ^v be a maximum left extension of v and let I_r^u be a maximum right extension of u. From Observation 3 we obtain that I_ℓ^v is a left extension of u and I_r^u is a right extension of v. This gives us $I_\ell^v \prec v \prec u \prec I_r^u$ and thus, by Property 1, $I = I_\ell^v \cup \{u, v\} \cup I_r^u$ is an independent set. Since $u \in L_{q,\gamma}$ we have, by Observation 19, that $|I_r^u| = \gamma - q \geq \gamma - (p + \gamma - \beta) = \beta - p$. Furthermore, Observation 19 tells us that $|I_\ell^v| = p - 1$, since $v \in L_{p,\beta}$. Thus, $|I| \geq p - 1 + 2 + \beta - p = \beta + 1$, a contradiction to $v \in I_\beta$. ◀

We can see in Fig. 7 that vertex $v \in L_{1,\alpha-1}$ is complete to $L_{1,\alpha}$ and $L_{2,\alpha}$.

Lemma 21. *Let $G = (V, E)$ be a co-comparability graph with vertex ordering $\prec$ and $\alpha = \alpha(G)$. Let $u, v \in V$, $uv \notin E$ and $\mathrm{pos}(u) < \mathrm{pos}(v)$. Then $u \prec v$.*

Proof. Let I_ℓ^v be a maximum left extension of v and I_r^u be a maximum right extension of u. Suppose for a contradiction that $v \prec u$. By definition of a left extension (resp. right extension) we get that $I_\ell^v \prec v$ (resp. $u \prec I_r^u$). Since $v \prec u$ it follows from Property 1 that $I_\ell^v \prec v \prec u \prec I_r^u$. Further, since $I_\ell^v \cup v$ and $u \cup I_r^u$ are both independent sets and $uv \notin E$ the set $I = I_\ell^v \cup v \cup u \cup I_r^u$ is an independent set. Let β_u be the size of a maximum independent set in G containing u. From Observation 19 we get that

$$|I| = |I_\ell^v| + 2 + |I_r^u| = \mathrm{pos}(v) - 1 + 2 + \beta_u - \mathrm{pos}(u) = \mathrm{pos}(v) - \mathrm{pos}(u) + 1 + \beta_u \geq \beta_u + 1$$

a contradiction to β_u being the size of a maximum independent set containing u. Hence, $u \prec v$. ◀

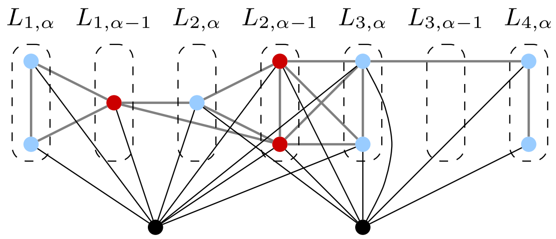

Fig. 7. The figure shows a co-comparability graph. The blue vertices are those in $\mathcal{I}_\alpha$, the red ones those in $\mathcal{I}_{\alpha-1}$. The black vertices are contained in neither $\mathcal{I}_\alpha$ nor in $\mathcal{I}_{\alpha-1}$. Note that in this example $\alpha = \alpha(G) = 4$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Lemma 22. *Let $G = (V, E)$ be a co-comparability graph with vertex ordering $\prec$ and $\alpha = \alpha(G)$. Let $u, v \in V$, $uv \notin E$ and $u \prec v$. Then $\text{pos}(u) < \text{pos}(v)$.*

Proof. Suppose for a contradiction that $\text{pos}(u) \geq \text{pos}(v)$. Consider first the case where $\text{pos}(u) = \text{pos}(v) = p$, for some $p \in [\![\alpha]\!]$. There are $\beta, \beta' \in \{p, \dots, \alpha\}$ such that $u \in L_{p,\beta}$ and $v \in L_{p,\beta'}$. Without loss of generality we may assume that $\beta < \beta'$. Then from Property 20 it follows that $uv \in E$, a contradiction. Thus, $\text{pos}(u) > \text{pos}(v)$. Further, Lemma 21 tells us that $v \prec u$, a contradiction. ◂

Lemma 23. *The partition of V into sets $L_{p,\beta}$ can be found in $\mathcal{O}(|V|^2)$.*

Proof. To see that we can find this partition in polynomial time, recall the proof of Lemma 7. We defined $\text{leftext}(v)$ respectively $\text{rightext}(v)$ as the size of a left respectively right extension of v of maximum size for any vertex $v \in V$. We showed that we can calculate both functions in $\mathcal{O}(|V|^2)$. From Property 1, we know that for a vertex $v \in V$, we can combine every left extension I_ℓ of v and every right extension I_r of v together with v to an independent set $I_\ell \cup \{v\} \cup I_r$. Consider a maximum independent set containing v. This consists of v together with a maximum left extension and a maximum right extension of v. Thus, for $v \in V$ we can conclude that it is contained in $L_{p,\beta}$, where $p = \text{leftext}(v) + 1$ and $\beta = \text{leftext}(v) + 1 + \text{rightext}(v)$. Hence, after precomputing $\text{leftext}(v)$ and $\text{rightext}(v)$ for all $v \in V$ as described above, we can find the partition in $\mathcal{O}(|V|)$ time and we get a total running time of $\mathcal{O}(|V|^2)$. ◂

6. Deletion blocker in co-comparability graphs

In this section, we will show that Deletion Blocker(α) can be solved in polynomial time in co-comparability graphs. Let $G = (V, E)$ be a co-comparability graph with vertex ordering $\prec$, $\alpha = \alpha(G)$, and $d > 0$ be an integer. We will construct a directed graph $G' = (V', A')$ such that (G', k) is a Yes-instance of Vertex Cut if and only if (G, d, k) is a Yes-instance of Deletion Blocker(α), for some integer $k > 0$. The construction is similar to the one in Section 4. We will show the equivalence between the two instances in Theorem 31. We can make an observation similar to Observation 8.

Observation 24. *Let $G = (V, E)$ be a co-comparability graph with a vertex ordering $\prec$. Let $S \subseteq V$. S is a d-deletion blocker of G if and only if it contains at least one vertex from every independent set of size $\alpha - d + 1$ of G.*

By Observation 24 it suffices to consider independent sets of size $\alpha - d + 1$. Thus, in the following, we construct a directed graph G' with source s and sink t, such that every s-t-path in G' corresponds to an independent set of size $\alpha - d + 1$ in G. The one-to-one correspondence will be proven in Lemma 28. An s-t-cut will then correspond to a d-deletion blocker in G (see Theorem 31).

Let us now describe the construction of $G' = (V', A')$. Recall that $\mathcal{I}_\beta = \bigcup_{p\in[\![\beta]\!]} L_{p,\beta}$ is the set of vertices which are contained in an independent set of size β but not of size $\beta + 1$, for $\beta \in [\![\alpha]\!]$. The vertex set V' of G' consists of d copies $U_{1,\beta}, \dots, U_{d,\beta}$ of $\mathcal{I}_\beta$, for all $\beta \in \{\alpha - d + 1, \dots, \alpha\}$, and two additional vertices s, t. That is

$$V' = \bigcup_{\beta\in\{\alpha-d+1,\dots,\alpha\},\ell\in[\![d]\!]} U_{\ell,\beta} \cup \{s, t\}\,.$$

We denote by $L_{p,\ell,\beta}$ the set $L_{p,\beta}$ in $U_{\ell,\beta}$, for $p \in [\![\alpha]\!]$, $\ell \in [\![d]\!]$, and $\beta \in \{\alpha - d + 1, \dots, \alpha\}$. We say that ℓ is the *level* of the vertices in $U_{\ell,\beta}$, denoted by $\text{level}(x) = \ell$ for $x \in U_{\ell,\beta}$. For simplicity, we adopt the notion of positions for all vertices in $V' \setminus \{s, t\}$, that is, for every vertex $x \in V' \setminus \{s, t\}$ that corresponds to some vertex $v \in L_p$, for $p \in [\![\alpha]\!]$, we will also use $\text{pos}(x)$ in order to actually refer to the position of v, the vertex that x corresponds to in G. Furthermore, we set $\text{pos}(s) = 0$ and $\text{level}(s) = 1$ and for t we have $\text{pos}(t) = \alpha + 1$ and $\text{level}(t) = d$.

Let $x, y \in V' \setminus \{s, t\}$, where $x \in L_{p,\ell,\beta}$ and $y \in L_{p',\ell',\beta'}$, with $p, p' \in [\![\alpha]\!]$, $\ell, \ell' \in [\![d]\!]$, and $\beta, \beta' \in \{\alpha - d + 1, \dots, \alpha\}$. Let $u, v \in V$, where x corresponds to u and y corresponds to v. We add an arc (x, y) if $uv \notin E$ and $p' = p + g + 1$, $\ell' = \ell + g$

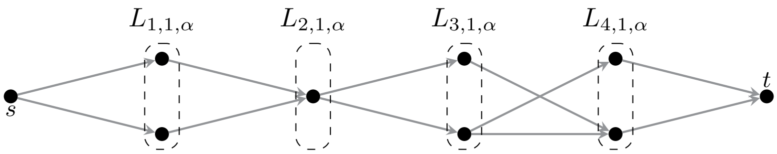

Fig. 8. The graph G' corresponding to the graph G from Fig. 7 for $d = 1$.

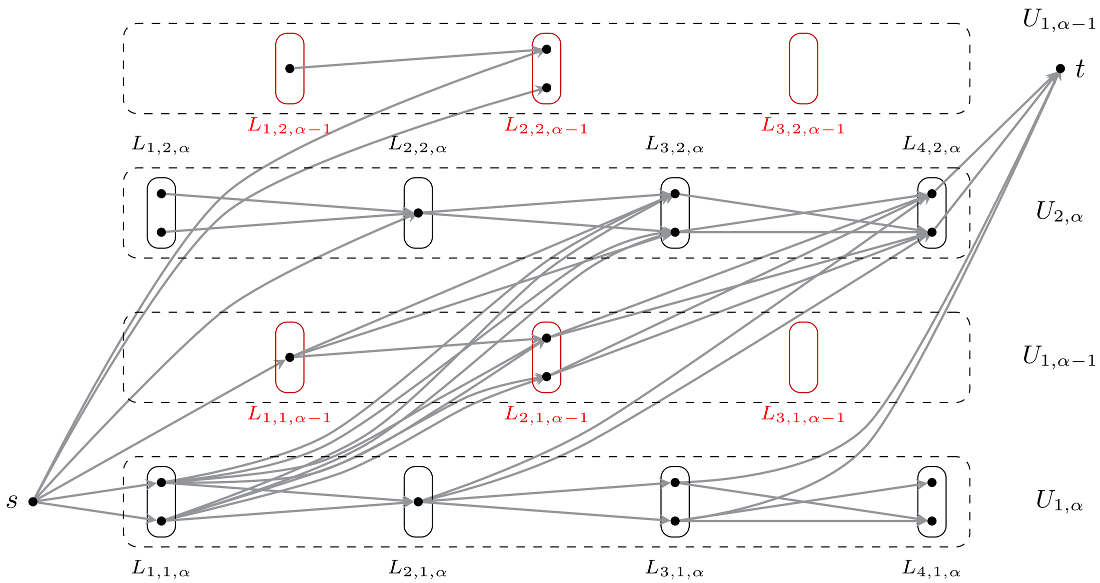

Fig. 9. The graph G' corresponding to the graph G from Fig. 7 for $d = 2$.

for some $g \geq 0$. Similar to the construction in Section 4, we add an arc (s, y) if $p' = g + 1$, $\ell' = g + 1$, for some $g \geq 0$, and we add an arc (x, t) if $\text{pos}(t) = p + g + 1$, $\text{level}(t) = \ell + g$, for some $g \geq 0$. We can see that Observation 9 holds for this construction as well.

Figs. 8 and 9 give examples for the graph G' constructed from the graph in Fig. 7 for $d = 1$ and $d = 2$. Note that some sets $L_{p,\ell,\beta}$ in Fig. 9 are not contained in any s-t-path and thus could be removed from the graph. More such sets appear when $d > 2$. We kept them to make the definitions and proofs easier.

We give three properties similar to Properties 10–12 which will help us to prove the one-to-one correspondence between paths in G' and independent sets in G, which we show in Lemma 28. We omit the proofs due to their similarity to the proofs of Properties 10–12. Note that Property 25 follows from the proof of Property 10 since β has no influence on whether an arc exists or not.

Property 25. *Let $P = s, x_1, \ldots, x_h, t$ be an s-t-path in G'. There exist exactly $d - 1$ distinct integers in $[\![\alpha]\!]$, say $g_1, \ldots, g_{d-1}$, such that $\text{pos}(x_i) \notin \{g_1, \ldots, g_{d-1}\}$ for all $i \in [\![h]\!]$. For any other integer $g \in [\![\alpha]\!] \setminus \{g_1, \ldots, g_{d-1}\}$, there exists exactly one vertex $x \in P - \{s, t\}$ such that $\text{pos}(x) = g$.*

Property 26 follows from the proof of Property 11 by applying Property 25 instead of Property 10.

Property 26. *Every s-t-path P in G' has $\alpha - d + 3$ vertices.*

Property 27. *Let $I = \{v_1, \ldots, v_{\alpha-d+1}\}$ be an independent set in G. There exist exactly $d - 1$ distinct integers in $[\![\alpha]\!]$, say $g_1, \ldots, g_{d-1}$ such that $\text{pos}(v_i) \notin \{g_1, \ldots, g_{d-1}\}$ for all $i \in [\![\alpha - d + 1]\!]$.*

Using Properties 25–27 we can now prove the one-to-one correspondence between s-t-paths in G' and independent sets of size $\alpha - d + 1$ in G.

Lemma 28. *Let $G = (V, E)$ be a co-comparability graph, $\alpha = \alpha(G)$, $d > 0$ an integer, and G' constructed as above. Every independent set of size $\alpha - d + 1$ in G corresponds to an s-t-path in G' and vice versa.*

Proof. We first consider an s-t-path $P = s, x_1, x_2, \ldots, x_h, t$ in G'. We know from Property 26 that this path consists of exactly $\alpha - d + 3$ vertices, hence $h = \alpha - d + 1$. Let $v_1, \ldots, v_{\alpha-d+1}$ be the vertices in G corresponding to $x_1, x_2, \ldots, x_{\alpha-d+1}$.

Notice that for any two sets $L_{p,\ell,\beta}$ and $L_{p',\ell',\beta'}$, there exists an arc from $x_i \in L_{p,\ell,\beta}$ to $x_j \in L_{p',\ell',\beta'}$ in G' only if $p < p'$, $\ell \le \ell'$. Thus, $\text{pos}(v_1) = \text{pos}(x_1) < \cdots < \text{pos}(x_{\alpha-d+1}) = \text{pos}(v_{\alpha-d+1})$.

Claim 28.1. $\{v_1, \ldots, v_{\alpha-d+1}\}$ *is an independent set in G.*

Proof. The proof to this claim can be obtained by replacing Property 6 by Lemma 21 in the proof of Claim 13.1. ◁

We will now prove the converse, that is, we show that an independent set of size $\alpha - d + 1$ in G corresponds to an s-t-path in G'. Let $I = \{v_1, \ldots, v_{\alpha-d+1}\}$ be an independent set of size $\alpha - d + 1$ in G. We assume that we have $v_1 \prec v_2 \prec \cdots \prec v_{\alpha-d+1}$. From Lemma 22, it follows that $\text{pos}(v_1) < \text{pos}(v_2) < \cdots < \text{pos}(v_{\alpha-d+1})$.

Let $g_1, \ldots, g_{d-1}$ be as in Property 27. For $i \in [\![\alpha - d + 1]\!]$, let $x_i \in V'$ be the copy of v_i in $L_{\text{pos}(v_i),\ell,\beta}$, where $\ell = |\{g_k \mid g_k < \text{pos}(v_i), k \in [\![d-1]\!]\}| + 1$ and $\beta \in \{\alpha - d + 1, \ldots, \alpha\}$ such that $v_i \in I_\beta$. We see that x_i exists, since $\ell \le d$.

To get the existence of a path $P = s, x_1, \ldots, x_{\alpha-d+1}, t$, it remains to show that the arcs (s, x_1), (x_i, x_{i+1}), for $i \in [\![\alpha - d]\!]$, and $(x_{\alpha-d+1}, t)$ exist.

Claim 28.2. *The arcs* (s, x_1), (x_i, x_{i+1}), *for* $i \in [\![\alpha - d]\!]$, *and* $(x_{\alpha-d+1}, t)$ *exist.*

Proof. This proof is the same as the proof of Claim 13.3. ◁

It follows that $P = s, x_1, \ldots, x_{\alpha-d+1}, t$ is a path in G'. This concludes the proof of the lemma. ◀

Recall that $\text{pos}(v)$ denotes the position of v in a maximum independent set containing v, while $\text{pos}_I(v)$ denotes the position of v in a specific, not necessarily maximum independent set I. To prove our second main result we need two more properties.

Property 29. *Let G be a co-comparability graph with vertex ordering ≺ and let G′ be the corresponding directed graph, constructed as described above. Let I be an independent set of size* $\alpha - d + 1$ *of G and let P be the corresponding s-t-path in G′. Consider* $v \in I$ *and its corresponding vertex* $x \in V(P)$. *Then,* $\text{pos}_I(v) = \text{pos}(v) - \text{level}(x) + 1$.

Proof. Let $g_1, \ldots, g_{d-1}$ be as in Property 27. Recall from the proof of Lemma 28 that $\text{level}(x) = |\{g_k \mid g_k < \text{pos}(x), k \in [\![d-1]\!]\}| + 1$. Hence, the result follows. ◀

Property 30. *Let G be a co-comparability graph with vertex ordering ≺, G′ as constructed above. Let* I_1, I_2 *be independent sets of size* $\alpha - d + 1$ *in G and let* P_1, P_2 *be their corresponding paths in G′. Suppose there is* $v \in I_1 \cap I_2$ *such that the corresponding vertices* $x_1 \in P_1, x_2 \in P_2$ *are different. Assume without loss of generality that* $\text{level}(x_1) < \text{level}(x_2)$. *Then,* $\text{pos}_{I_1}(v) > \text{pos}_{I_2}(v)$.

Proof. Since, by Property 29, $\text{pos}_{I_1}(v) = \text{pos}(v) - \text{level}(x_1) + 1$ and $\text{pos}_{I_2}(v) = \text{pos}(v) - \text{level}(x_2) + 1$, and since we assume that $\text{level}(x_1) < \text{level}(x_2)$, it follows that $\text{pos}_{I_1}(v) > \text{pos}_{I_2}(v)$. ◀

Theorem 31. Deletion Blocker(α) *is polynomial-time solvable for co-comparability graphs.*

Proof. Let $G = (V, E)$ be a co-comparability graph and let (G, d, k) be an instance of Deletion Blocker(α). We construct the graph $G' = (V', A')$ as described above. We will show that (G, d, k) is a Yes-instance of Deletion Blocker(α) if and only if (G', k) is a Yes-instance of Vertex Cut.

Let (G', k) be a Yes-instance of Vertex Cut and let C be an s-t-cut of G' of size at most k. We want to prove that (G, d, k) is a Yes-instance of Deletion Blocker(α).

For every vertex in the cut $C \subseteq V'$, we add the corresponding vertex in G to a set S. We assume for a contradiction that there is an independent set I in $G - S$ of size $\alpha - d + 1$. By Lemma 28, we know that there is an s-t-path P in G' representing I. Since $I \subseteq V \setminus S$, we get that $P \cap C = \varnothing$ and hence, we can find an s-t-path in $G' - C$, a contradiction. Thus, such an independent set I does not exist, and so by Observation 24, we deduce that (G, d, k) is a Yes-instance of Deletion Blocker(α).

Let now (G, d, k) be a Yes-instance of Deletion Blocker(α). We want to show that (G', k) is a Yes-instance of Vertex Cut. Let $S \subseteq V$ with $|S| \le k$ be such a d-deletion blocker of G. We may assume that S is minimal.

We iteratively construct a set C using again Algorithm 1, with the difference that we get as input the graph G' and d-blocker S in G instead of a d-transversal. We will prove that C is an s-t-cut in G' with $|S| = |C|$. For each vertex in S, the algorithm chooses the corresponding vertex in G' belonging to the lowest level such that there is an s-t-path in $G' - C$ containing this vertex and then adds it to C. We will find such a vertex for every vertex in S, since otherwise S would not be minimal. Hence, it is clear that $|S| = |C|$.

To prove that C, which is constructed by application of Algorithm 1, is indeed an s-t-cut in G', we assume for a contradiction that there exists an s-t-path P_1 in $G' - C$. Let $I_1 \subseteq V$ be the independent set in G corresponding to P_1, which exists by Lemma 28. Since S is a d-deletion blocker of G, we know that $S \cap I_1 \ne \varnothing$. Let $v \in S \cap I_1$ and if $|S \cap I_1| > 1$,

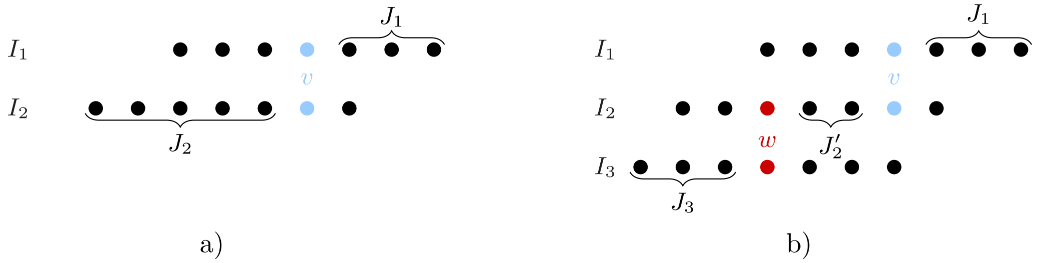

Fig. 10. The different independent sets considered in the proof.

we choose the rightmost vertex according to $\prec$ in $S \cap I_1$ for v. Let $y_j \in V'$, $j \in [\![d]\!]$, be the copy of v in P_1. Since P_1 is a path in $G' - C$, we know that $y_j \notin C$. Hence, there is some other copy of v, say y_i, $i \in [\![d]\!]$, such that Algorithm 1 added y_i to C. Due to the procedure we used in Algorithm 1 to choose the vertices in C, we have $i < j$.

Since y_i was added to C, there exists a path P_2 in G' containing y_i. Let I_2 be the independent set in G corresponding to P_2, which exists by Lemma 28. From Property 30 we know that $\text{pos}_{I_1}(v) < \text{pos}_{I_2}(v)$. Let $J_1 \subseteq I_1$ be the set of the $\alpha - d + 1 - \text{pos}_{I_1}(v)$ rightmost vertices in I_1, i.e. those vertices u in I_1 such that $v \prec u$ (see Fig. 10(a)). We know that $J_1 \cap S = \varnothing$ by the choice of v. Let J_2 be the set of the $\text{pos}_{I_2}(v) - 1$ leftmost vertices from I_2 (see Fig. 10(a)), i.e. those vertices u in I_2 with $u \prec v$. Since $J_2 \prec v \prec J_1$, it follows from Property 1 that $J_2 \cup J_1$ is an independent set. Since $\text{pos}_{I_1}(v) < \text{pos}_{I_2}(v)$, we get that $|J_2 \cup J_1| \geq \alpha - d + 1$. Hence, $J_2 \cup J_1$ either is an independent set of size at least $\alpha - d + 1$ in $G - S$ or $J_2 \cap S \neq \varnothing$. In the first case, we directly get a contradiction to our assumption that S is a d-deletion blocker in G. So, we may assume that $J_2 \cap S \neq \varnothing$.

Let $w \in J_2 \cap S$, and if $|J_2 \cap S| > 1$, we take the rightmost vertex w in $J_2 \cap S$ with respect to $\prec$ such that $w \prec v$. Let y_h, $h \in [\![d]\!]$ be the vertex in P_2 corresponding to w. Then $y_h \notin C$, since otherwise Algorithm 1 would not have added y_i to C. Thus, as before, there exists some vertex y_g, $g \in [\![d]\!]$, with $g < h$, such that y_g corresponds to w and $y_g \in C$. Therefore, there exists an s-t-path P_3 in G' containing y_g with corresponding independent set I_3. I_3 contains more vertices to the left of w than I_2, since by Property 30 we have that $\text{pos}_{I_2}(w) < \text{pos}_{I_3}(w)$ (see Fig. 10(b)). Remember that $J_2 \cup J_1$ is an independent set. We consider the set $J_2' = \left\{u \in I_2 |\, \text{pos}_{I_2}(w) < \text{pos}_{I_2}(u) < \text{pos}_{I_2}(v)\right\} = \{u \in I_2 | w \prec u \prec v\}$ which is a subset of the set J_2. Thus, $J_2' \cup J_1$ is still an independent set. We define a new set $J_3 = \left\{u \in I_3 |\, \text{pos}_{I_3}(u) < \text{pos}_{I_3}(w)\right\} \subseteq I_3$, which is an independent set, since I_3 is an independent set. From the fact that J_3, respectively $J_2' \cup J_1$, contains only vertices on the left, respectively right, of w and that both are independent to w we get that $J_3 \cup J_2' \cup J_1$ is an independent set. Since $\text{pos}_{I_1}(v) < \text{pos}_{I_2}(v)$ and $\text{pos}_{I_2}(w) < \text{pos}_{I_3}(w)$, we get that $|J_3 \cup J_2' \cup J_1| \geq \alpha - d + 1$. Hence, in $G - S$ we get again either an independent set $J_3 \cup J_2' \cup J_1$ of size at least $\alpha - d + 1$ or $J_3 \cap S \neq \varnothing$.

By repeatedly using these arguments, we can always find a new vertex in S. But since S is finite, this case cannot always occur. Hence, we will necessarily get a contradiction and thus, there is no s-t-path P in $G' - C$. So we conclude that C is an s-t-cut in G'.

Let us now consider the complexity of our algorithm. From [24] we know that for a graph with n vertices we can solve Vertex Cut in $\mathcal{O}(n^3)$. Since the graph G' has $\mathcal{O}(d|V|)$ vertices, computing a Vertex Cut in G' can be done in time $\mathcal{O}(d^3|V|^3)$. We still need to consider the time to construct G'. Using Lemma 23, we know that we can find the partition of $\mathcal{I}$ into sets $L_{p,\beta}$ in $\mathcal{O}(|V|^2)$. For every pair of vertices in G', we can check in $\mathcal{O}(|V|)$ time if we introduce an arc between them in G'. Hence, G' can be constructed in $\mathcal{O}(d^2|V|^3)$. We conclude that Deletion Blocker(α) can be solved in time $\mathcal{O}(d^3|V|^3)$. ◂

7. Conclusion

In this paper, we showed that Deletion Blocker(α) and Transversal(α) are polynomial-time solvable in the class of co-comparability graphs by reducing them to the well-known Vertex Cut problem. This generalises results of [7,15]. We believe that our approach (reduction to the Vertex Cut problem) can also be used to solve the weighted version of our problems in polynomial time, i.e., where we consider maximum weighted independent sets. The same holds for Deletion Blocker(γ) and Transversal(γ), i.e., the blocker and the transversal problems with respect to minimum dominating sets. We leave this as open questions.

Data availability

No data was used for the research described in the article.

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