



On the complexity of matching cut for graphs of bounded radius and H -free graphs



Felicia Lucke^a, Daniël Paulusma^{b,*}, Bernard Ries^a

^a University of Fribourg, Department of Informatics, Fribourg, Switzerland

^b Durham University, Durham, UK

ARTICLE INFO

Article history:

Received 14 April 2022

Accepted 15 September 2022

Available online 21 September 2022

Communicated by D.-Z. Du

Keywords:

Matching cut

Radius

Complexity dichotomy

H -free

ABSTRACT

For a connected graph $G = (V, E)$, a matching $M \subseteq E$ is a matching cut of G if $G - M$ is disconnected. It is known that for an integer d , the corresponding decision problem MATCHING CUT is polynomial-time solvable for graphs of diameter at most d if $d \leq 2$ and NP-complete if $d \geq 3$. We prove the same dichotomy for graphs of bounded radius. For a graph H , a graph is H -free if it does not contain H as an induced subgraph. As a consequence of our result, we can solve MATCHING CUT in polynomial time for P_6 -free graphs, extending a recent result of Feghali for P_5 -free graphs. We then extend our result to hold even for $(sP_3 + P_6)$ -free graphs for every $s \geq 0$ and initiate a complexity classification of MATCHING CUT for H -free graphs.

© 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Let $G = (V, E)$ be an (undirected) connected graph. A subset of edges $M \subseteq E$ is a *matching* if no two edges in M have a common end-vertex, whilst M is an *edge cut* if V can be partitioned into sets B and R , such that M consists of all the edges with one end-vertex in B and the other one in R . We say that M is a *matching cut* if M is a matching that is also an edge cut; see Fig. 1 for an example. Matching cuts have applications in number theory [12], graph drawing [20], graph homomorphisms [11], edge labellings [1] and ILFI networks [9]. The corresponding decision problem is defined as follows:

MATCHING CUT

Input: a connected graph G .

Question: does G have a matching cut?

Chvátal [8] proved that MATCHING CUT is NP-complete. This led to an extensive study on the computational complexity of the problem restricted to special graph classes [4,5,7,10,14–16,19]. We discuss some relevant results in this section and in Section 7; see [7] for a more detailed overview of known algorithmic results, including exact and parameterized algorithms [2,3,13,14].

Let G be a connected graph. The *distance* between two vertices u and v in G is the *length* (number of edges) of a shortest path between u and v in G . The *eccentricity* of a vertex u is the maximum distance between u and any other vertex of G . The *diameter* of G is the maximum eccentricity over all vertices of G . Borowiecki and Jesse-Józefczyk [5] proved that

* Corresponding author.

E-mail addresses: felicia.lucke@unifr.ch (F. Lucke), daniel.paulusma@durham.ac.uk (D. Paulusma), bernard.ries@unifr.ch (B. Ries).

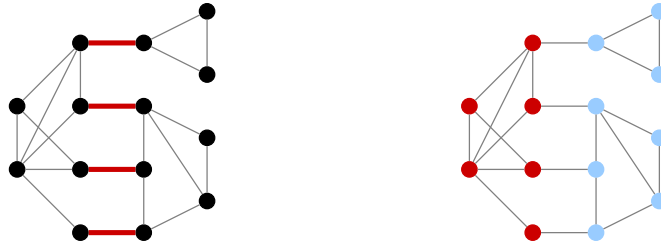


Fig. 1. Left: an example of a matching cut, as indicated by the thick red edges. Right: the corresponding valid red-blue colouring, an equivalent definition given in Section 2. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

MATCHING CUT is polynomial-time solvable for graphs of diameter 2. Le and Le [15] gave a faster polynomial-time algorithm for graphs of diameter 2 and proved the following dichotomy.

Theorem 1 ([15]). *For an integer d , MATCHING CUT for graphs of diameter at most d is polynomial-time solvable if $d \leq 2$ and NP-complete if $d \geq 3$.*

Le and Le [15] also proved that MATCHING CUT for bipartite graphs of diameter at most d is polynomial-time solvable if $d \leq 3$ and NP-complete for $d \geq 4$. Another recent dichotomy is due to Chen et al. [7], who extended results of Le and Randerath [16] and proved that MATCHING CUT for graphs of minimum degree δ is polynomial-time solvable if $\delta = 1$ and NP-complete if $\delta \geq 2$ (note that the problem is trivial if $\delta = 1$).

The radius of a connected graph G is closely related to the diameter; it is defined as the minimum eccentricity over all vertices of G . It is readily seen that for every connected graph G ,

$$\text{radius}(G) \leq \text{diameter}(G) \leq 2 \cdot \text{radius}(G).$$

Complexity dichotomies for graphs of bounded radius have been studied in the literature; for example, Mertzios and Spirakis [18] showed that 3-COLOURING is NP-complete for graphs of diameter 3 and radius 2, whilst 3-COLOURING is trivial for graphs of radius 1.

1.1. Our results

We will prove the following dichotomy for general graphs of bounded radius, which strengthens the polynomial part of Theorem 1 (in order to see this, consider for example an arbitrary star and subdivide each of its edges once; the new graph has radius 2 but its diameter is 4).

Theorem 2. *For an integer r , MATCHING CUT for graphs of radius at most r is polynomial-time solvable if $r \leq 2$ and NP-complete if $r \geq 3$.*

We prove Theorem 2 in Section 4 after giving some more terminology in Section 2. In Section 3 we present some known results that we need as lemmas for proving Theorem 2. In particular, we will use the reduction rules of Le and Le [15], which they used in their polynomial-time algorithms for graphs of diameter 2 and bipartite graphs of diameter 3.

A graph H is an induced subgraph of G if H can be obtained from G after removing all vertices of $V(G) \setminus V(H)$. A graph G is H -free if G does not contain an induced subgraph isomorphic to H . We let P_r denote the path on r vertices. Feighali [10] recently proved that MATCHING CUT is polynomial-time solvable for P_5 -free graphs and that there exists an integer r , such that MATCHING CUT is NP-complete for P_r -free graphs. In a recent paper [17], we showed that the constant r in [10] is equal to 27. In the same paper [17] we proved that MATCHING CUT is NP-complete even for $(4P_5, P_{19})$ -free graphs (by a slight modification of the construction from [10]).

As a consequence of the polynomial part of Theorem 2 we can show the following result.

Corollary 1. *MATCHING CUT is polynomial-time solvable for P_6 -free graphs.*

We prove Corollary 1 in Section 4 as well. In Section 5 we prove that if MATCHING CUT is polynomial-time solvable on a class of H -free graphs, then it is so on the class of $(P_3 + H)$ -free graphs (here, the graph $P_3 + H$ denotes the disjoint union of the graphs P_3 and H). This means in particular that MATCHING CUT is polynomial-time solvable even for $(sP_3 + P_6)$ -free graphs for every $s \geq 0$.

In Section 6 we show some new hardness results of MATCHING CUT for H -free graphs. In the same section, we also combine all our new results with known results to give a state-of-the-art summary of MATCHING CUT for H -free graphs. We finish our paper with a number of open problems in Section 7.

2. Preliminaries

Let $G = (V, E)$ be a graph. For a vertex u , we let $N(u) = \{v \mid uv \in E\}$ denote the *neighbourhood* of u in G . Let $S \subseteq V$. The *neighbourhood* of S in G is the set $N(S) = \bigcup_{u \in S} N(u) \setminus S$. We let $G[S]$ denote the subgraph of G induced by S , that is, $G[S]$ can be obtained from G after deleting the vertices of S . Moreover, S is a *dominating set* of G if every vertex of $V \setminus S$ has at least one neighbour in S . In that case we also say that $G[S]$ *dominates* G . The *domination number* of a graph G is the size of a smallest dominating set of G .

Recall that we denote the path on r vertices by P_r . We let C_s denote the cycle on s vertices. A bipartite graph with non-empty partition classes V_1 and V_2 is *complete* if there exists an edge between every vertex of V_1 and every vertex of V_2 . We let K_{n_1, n_2} denote the complete bipartite graph with partition classes of size n_1 and n_2 , respectively. The graph K_{1, n_2} denotes the *star* on $n_2 + 1$ vertices.

We denote the *disjoint union* of two graphs G_1 and G_2 by $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We denote the disjoint union of s copies of a graph G by sG .

When constructing matching cuts we will often use an equivalent definition in terms of vertex colourings (see also [7,8,10]). Let $G = (V, E)$ be a connected graph. A *red-blue colouring* of G colours every vertex of G either red or blue. A red-blue colouring of G is *valid* if every blue vertex has at most one red neighbour; every red vertex has at most one blue neighbour; and both colours red and blue are used at least once. We refer to Fig. 1 for an example.

For a red-blue colouring, we let R and B denote the sets that consist of all vertices coloured red or blue, respectively (so $V = R \cup B$). We call R the *red set* and B the *blue set* of the red-blue colouring. We let R' consist of all vertices in R that have a (unique) blue neighbour, and similarly, we let B' consist of all vertices of B that have a (unique) red neighbour. We call R' the *red interface* and B' the *blue interface* of the red-blue colouring.

From a matching cut M of a connected graph G we can construct a valid red-blue colouring by colouring the vertices of one connected component of $G - M$ red and all other vertices of G blue. Similarly, from a valid red-blue colouring we can construct a matching cut by taking all edges with one end-vertex in the red interface and one end-vertex in the blue interface. Hence, we can make the following observation, which as mentioned is well known.

Observation 1. *A connected graph G has a matching cut if and only if G admits a valid red-blue colouring.*

3. Auxiliary results

In the remainder of our paper, we use Observation 1 and mainly search for valid red-blue colourings. We need the following lemma, which has been used (implicitly) to prove other results for MATCHING CUT as well, such as the result for P_5 -free graphs [10]. We include a short proof for completeness.

Lemma 1. *For every integer g , it is possible to find in $O(2^g n^{g+2})$ time a valid red-blue colouring (if it exists) of an n -vertex graph with domination number g .*

Proof. Let $g \geq 1$ be an integer, and let G be a graph with domination number at most g . Hence, G has a dominating set D of size at most g . We consider all options of colouring the vertices of D red or blue; note that this number is $2^{|D|} \leq 2^g$. For every red vertex of D with no blue neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours blue (and thus all of its other neighbours will be coloured red). Similarly, for every blue vertex of D with no red neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours red (and thus all of its other neighbours will be coloured blue). Finally, for every red vertex in D with already one blue neighbour in D , we colour all its yet uncoloured neighbours red. Similarly, for every blue vertex in D with already one red neighbour in D , we colour all its yet uncoloured neighbours blue.

As D is a dominating set, the above means that we guessed a red-blue colouring of the whole graph G . We can check in $O(n^2)$ time if a red-blue colouring is valid. Moreover, the total number of red-blue colourings that we must consider is $O(2^g n^g)$. \square

Consider a red-blue colouring of a graph $G = (V, E)$. A subset $S \subseteq V$ is *monochromatic* if every vertex of S has the same colour. We need the following known lemma (see e.g. [10] which uses it implicitly); again we added a short proof for completeness.

Lemma 2. *Let D be a dominating set of a connected graph G . It is possible to check in polynomial time if G has a valid red-blue colouring in which D is monochromatic.*

Proof. Consider a valid red-blue colouring of G , in which D is monochromatic. Assume without loss of generality that every vertex of D is coloured red. Let C be a connected component of $G - D$. If C is not monochromatic, then C has an edge uv where u is red and v is blue. However, now v has at least two red neighbours, namely u and a neighbour in D (such

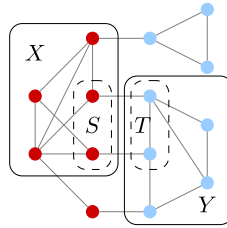


Fig. 2. An example of a graph G with a valid red-blue (S, T, X, Y) -colouring.

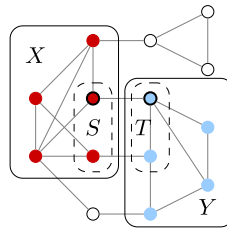


Fig. 3. An example of an application of rules R1–R3 that results in a 4-tuple (S', T', X, Y) . The set S' consists of the thick red vertex and the set T' consists of the thick blue vertex. Note that $S' \subseteq S \subseteq X$ and $T' \subseteq T \subseteq Y$, and that every valid red-blue (S', T', S', T') -colouring is a valid red-blue (S, T, X, Y) -colouring.

a neighbour exists, as D is a dominating set of G). This is a contradiction. Hence, the vertex set of every connected component is monochromatic. Moreover, we may assume without loss of generality that the vertices of exactly one connected component of $G - D$ are coloured blue (else we can safely recolour the vertices of some connected component from blue to red). Hence, we can check all $O(n)$ options of choosing this unique blue connected component. For each option we check in polynomial time if the obtained red-blue colouring is valid. \square

Let $G = (V, E)$ be a connected graph and $S, T, X, Y \subseteq V$ be four non-empty sets with $S \subseteq X$, $T \subseteq Y$ and $X \cap Y = \emptyset$. A red-blue (S, T, X, Y) -colouring of G is a red-blue colouring with a red set containing X ; a blue set containing Y ; a red interface containing S and a blue interface containing T . Note that $V \setminus (X \cup Y)$ might be non-empty. Moreover, the red and blue interfaces may also contain vertices not in S and T , respectively. We refer to Fig. 2 for an example.

Now, let S' and T' be two non-empty subsets of V with $S' \cap T' = \emptyset$ such that every vertex of S' is adjacent to exactly one vertex of T' , and vice versa. We call (S', T') a starting pair of G .

For a starting pair, Le and Le [15] define five propagation rules that we give below – in our terminology – as rules R1–R3. The goal of these rules is to extend S' and T' by finding as many vertices as possible whose colour must be either red or blue in any valid red-blue colouring with a red interface containing S' and a blue interface containing T' . We will place any newly found red vertices in a set X that already contains S' and any newly found blue vertices in a set Y that already contains T' . We let S be the subset of X consisting of red vertices with a blue neighbour in Y , and we let T be the subset of Y consisting of blue vertices with a red neighbour in X . So it holds that $S' \subseteq S \subseteq X$ and $T' \subseteq T \subseteq Y$. The vertices of S will belong to the red interface and the vertices of T will belong to the blue interface of the valid red-blue colouring that we are trying to construct. Note that the vertices of $X \setminus S$ and $Y \setminus T$ may or may not belong to the red or blue interface (as this depends on the colour of the uncoloured vertices in $V \setminus (X \cup Y)$, which we still have to determine).

We now state the propagation rules R1–R3. Initially we set $X := S := S'$ and $Y := T := T'$. As mentioned, R1–R3 try to put vertices of $Z = V \setminus (X \cup Y)$ into one of the sets S , $X \setminus S$, T or $Y \setminus T$. They are defined as follows (see Fig. 3 for an example):

- R1.** Return no (that is, G has no valid red-blue (S', T', S', T') -colouring) if a vertex $v \in Z$ is
 - adjacent to a vertex in S and to a vertex in T , or
 - adjacent to a vertex in S and to two vertices in $Y \setminus T$, or
 - adjacent to a vertex in T and to two vertices in $X \setminus S$, or
 - adjacent to two vertices in $X \setminus S$ and to two vertices in $Y \setminus T$.
- R2.** Assume $v \in Z$ and R1 does not apply. If v is adjacent to a vertex in S or to two vertices of $X \setminus S$, then move v from Z to X . If moreover v is adjacent to a (unique) vertex w in Y , then also add v to S and w to T .
- R3.** Assume $v \in Z$ and R1 does not apply. If v is adjacent to a vertex in T or to two vertices of $Y \setminus T$, then move v from Z to Y . If moreover v is adjacent to a (unique) vertex w in X , then also add v to T and w to S .

Le and Le [15] proved the following two lemmas. The first lemma shows that rules R1–R3 are safe and is not difficult to verify, whereas the second lemma is proven by a reduction to 2-SATISFIABILITY. We slightly changed the formulation of their

lemma so that it can be applied to the case where the graph G may also have a valid red-blue colouring in which not every connected component of $G - (X \cup Y)$ is monochromatic.

Lemma 3 ([15]). *Let G be a graph with a starting pair (S', T') . Assume that exhaustively applying rules R1–R3 did not lead to a no-answer but to a 4-tuple (S, T, X, Y) . The following holds:*

- (i) $S' \subseteq S \subseteq X$ and $T' \subseteq T \subseteq Y$ and $X \cap Y = \emptyset$,
- (ii) G has a valid red-blue (S', T', S', T') -colouring if and only if G has a valid red-blue (S, T, X, Y) -colouring (note that the backward implication holds by definition), and
- (iii) every vertex in $V \setminus (X \cup Y)$ has no neighbour in $S \cup T$; at most one neighbour in $X \setminus S$ and at most one neighbour in $Y \setminus T$.

Moreover, the 4-tuple (S, T, X, Y) can be obtained in polynomial time.

Lemma 4 ([15]). *Let G be a graph with a starting pair (S', T') . Assume that exhaustively applying rules R1–R3 did not lead to a no-answer but to a 4-tuple (S, T, X, Y) . It can be decided in $O(mn)$ time if G has a valid red-blue (S, T, X, Y) -colouring (or equivalently, a valid red-blue (S', T', S', T') -colouring) in which every connected component of $G - (X \cup Y)$ is monochromatic.*

Finally, we need one more result from the literature (which has been strengthened in [6]).

Theorem 3 ([21]). *A graph $G = (V, E)$ on n vertices is P_6 -free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph of G in $O(n^3)$ time.*

4. The proofs of Theorem 2 and Corollary 1

We first prove Theorem 2, which we restate below.

Theorem 2 (restated). *For an integer r , MATCHING CUT is polynomial-time solvable for graphs of radius at most r if $r \leq 2$ and NP-complete for graphs of radius at most r if $r \geq 3$.*

Proof. The case where $r \geq 3$ follows from Theorem 1 after observing that the class of graphs of diameter at most 3 is contained in the class of graphs of radius at most 3. So assume now that $r \leq 2$. Let G be a graph of radius at most r . If $r = 1$, then G has a vertex that is adjacent to all other vertices. In this case G has a matching cut if and only if G has a vertex of degree 1; we can check the latter condition in polynomial time. From now on, assume that $r = 2$. Then G has a dominating star H , say H has centre u and leaves v_1, \dots, v_s for some $s \geq 1$. By Observation 1 it suffices to check if G has a valid red-blue colouring.

We first check if G has a valid red-blue colouring in which $V(H)$ is monochromatic. By Lemma 2 this can be done in polynomial time. Suppose we find no such red-blue colouring. Then we may assume without loss of generality that a valid red-blue colouring of G (if it exists) colours u red and exactly one of v_1, \dots, v_s blue. That is, G has a valid red-blue colouring if and only if G has a valid red-blue $(\{u\}, \{v_i\}, \{u\}, \{v_i\})$ -colouring for some $i \in \{1, \dots, s\}$. We consider all $O(n)$ options of choosing which v_i is coloured blue.

For each option we do as follows. Let v_i be the vertex of v_1, \dots, v_s that we coloured blue. We define the starting pair (S', T') with $S' = \{u\}$ and $T' = \{v_i\}$ and apply rules R1–R3 exhaustively. The latter takes polynomial time by Lemma 3. If this exhaustive application leads to a no-answer, then by Lemma 3 we may discard the option. Suppose we obtain a 4-tuple (S, T, X, Y) . By again applying Lemma 3, we find that G has a valid red-blue $(\{u\}, \{v_i\}, \{u\}, \{v_i\})$ -colouring if and only if G has a valid red-blue (S, T, X, Y) -colouring. By R2 and the fact that $u \in S' \subseteq S$ we find that $\{v_1, \dots, v_s\} \setminus \{v_i\}$ belongs to X .

Suppose that G has a valid red-blue (S, T, X, Y) -colouring c such that $G - (X \cup Y)$ has a connected component D that is not monochromatic. Then D must contain an edge uv , where u is coloured red and v is coloured blue. Note that v cannot be adjacent to v_i , as otherwise v would have been in Y by R3 (since $v_i \in T' \subseteq T$). As H is dominating, this means that v must be adjacent to a vertex $w \in V(H) \setminus \{v_i\} = \{u, v_1, \dots, v_s\} \setminus \{v_i\}$. As $u \in S' \subseteq S \subseteq X$ and $\{v_1, \dots, v_s\} \setminus \{v_i\} \subseteq X$, we find that $w \in X$ by R2 and thus will be coloured red. However, now v being coloured blue is adjacent to two red vertices (namely u and w), contradicting the validity of c .

From the above we conclude that for any valid (S, T, X, Y) -colouring (if it exists), every connected component $G - (X, Y)$ is monochromatic. Hence, we can apply Lemma 4 to find in polynomial time whether or not G has a valid red-blue (S, T, X, Y) -colouring, or equivalently, if G has a valid red-blue $(\{u\}, \{v_i\}, \{u\}, \{v_i\})$ -colouring.

The correctness of our algorithm follows from the above arguments. As we branch $O(n)$ times and each branch takes polynomial time to process, the total running time of our algorithm is polynomial. \square

We now prove Corollary 1, which we restate below.

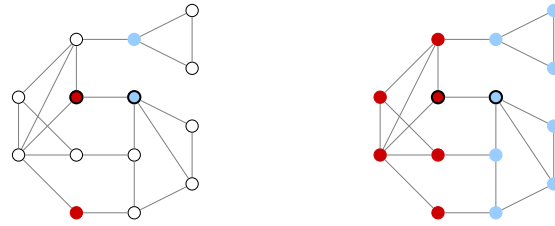


Fig. 4. Left: an example of a generalized starting pair (S', T') with core (S'', T'') , where S' consists of the two red vertices, S'' consists of the thick red vertex, T' consists of the two blue vertices and T'' consists of the thick blue vertex. Right: the application of rules R1-R3 on (S', T') . Note that the resulting four tuple (S, T, X, Y) immediately results in a valid red-blue colouring. Hence, having some vertices in $S' \setminus S''$ and $T' \setminus T''$, which are adjacent to a vertex with an opposite colour, can help significantly.

Corollary 1 (restated). MATCHING CUT is polynomial-time solvable for P_6 -free graphs.

Proof. Let G be a connected P_6 -free graph. By Theorem 3, we find that G has a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph $K_{r,s}$. By Observation 1 it suffices to check if G has a valid red-blue colouring.

If G has a dominating induced C_6 , then G has domination number at most 6. In that case we apply Lemma 1 to find in polynomial time if G has a valid red-blue colouring. Suppose that G has a dominating complete bipartite graph H with partition classes $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_s\}$. We may assume without loss of generality that $r \leq s$.

If $r \geq 2$ and $s \geq 3$, then it is readily seen that applying rules R1-R3 on any starting pair $(\{u_i\}, \{v_j\})$ yields a no-answer. Hence, $V(H)$ is monochromatic for any valid red-blue colouring of G . This means that we can check in polynomial time by Lemma 2 if G has a valid red-blue colouring.

Now assume that $r = 1$ or $s \leq 2$. In the first case, G has a (not necessarily induced) dominating star and thus G has radius 2, and we apply Theorem 2. In the second case, $r \leq s \leq 2$, and thus G has domination number at most 4, and we apply Lemma 1. Hence, in both cases, we find in polynomial time whether or not G has a valid red-blue colouring. \square

5. Extending Corollary 1

In this section we slightly generalize the framework of Le and Le [15] in order to obtain new algorithms for MATCHING CUT on H -free graphs.

First we slightly generalize the definition of a starting pair (S', T') of a graph G . We still let S' and T' be two non-empty subsets of V with $S' \cap T' = \emptyset$. However, now we only require that every vertex of S' is adjacent to at most one vertex of T' , and vice versa, whilst at least one vertex of S' must be adjacent to a vertex of T' . We let $S'' \subseteq S'$ consists of those vertices of S' that have exactly one neighbour in T' . Similarly, we let T'' consists of those vertices of T' that have exactly one neighbour in S' . We call (S', T') a *generalized starting pair* of G with core (S'', T'') . Note that by definition, $|S''| = |T''| \geq 1$. See Fig. 4 for an example.

When we apply rules R1-R3, we first initiate by setting $S := S''$; $X := S'$; $T := T''$; and $Y := T'$; see also Fig. 4. The following two lemmas can be readily checked by mimicking the proofs of Lemmas 3 and 4 given in [15].

Lemma 5. Let G be a graph with a generalized starting pair (S', T') with core (S'', T'') . Assume that exhaustively applying rules R1-R3 did not lead to a no-answer but to a 4-tuple (S, T, X, Y) . The following holds:

- (i) $S'' \subseteq S \subseteq X$; $S' \subseteq X$; $T'' \subseteq T \subseteq Y$; $T' \subseteq Y$; and $X \cap Y = \emptyset$,
- (ii) G has a valid red-blue (S'', T'', S', T') -colouring if and only if G has a valid red-blue (S, T, X, Y) -colouring, and
- (iii) every vertex in $V \setminus (X \cup Y)$ has no neighbour in $S \cup T$; at most one neighbour in $X \setminus S$ and at most one neighbour in $Y \setminus T$.

Moreover, the 4-tuple (S, T, X, Y) can be obtained in polynomial time.

Lemma 6. Let G be a graph with a generalized starting pair (S', T') with core (S'', T'') . Assume that exhaustively applying rules R1-R3 did not lead to a no-answer but to a 4-tuple (S, T, X, Y) . It can be decided in $O(mn)$ time if G has a valid red-blue (S, T, X, Y) -colouring (or equivalently, a valid red-blue (S'', T'', S', T') -colouring) in which every connected component of $G - (X \cup Y)$ is monochromatic.

We are now ready to prove the main result of this section.

Theorem 4. Let H be a graph. If MATCHING CUT is polynomial-time solvable for H -free graphs, then it is so for $(H + P_3)$ -free graphs.

Proof. Assume that MATCHING CUT is polynomial-time solvable for H -free graphs. Let G be a connected $(H + P_3)$ -free graph with n vertices and m edges. We first check if G has a matching cut of size at most 2. We can do this in polynomial time

by considering all $O(m^2)$ options of choosing two edges. From now on we assume that G has no matching cut of size at most 2; in particular this implies that G has no vertex of degree 1.

We may also assume that G has an induced subgraph G' that is isomorphic to H ; else we are done by our assumption. Let G^* be the graph obtained from G after removing every vertex of $V(G') \cup N(V(G'))$. As G' is isomorphic to H and G is $(H + P_3)$ -free, G^* is P_3 -free.

We continue as follows. We first consider all $O(m)$ options of choosing an edge from $E(G)$, one of whose end-vertices we colour red and the other one blue. Afterwards, for each (uncoloured) vertex in G' we consider all options of colouring it either red or blue. As G' is isomorphic to H , there are $2^{|V(H)|}$ options of doing this. As H is a fixed graph, this is a constant number. Now, for every red vertex u of G' with no blue neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours blue (and thus all other not yet coloured neighbours of u will be coloured red). Similarly, for every blue vertex v of G' with no red neighbour, we consider all $O(n)$ options of colouring at most one of its neighbours red (and thus all other neighbours of v will be coloured blue). Note that afterwards each vertex of $V(G') \cup N(V(G'))$ is either coloured red or blue.

There are $O(m2^{|V(H)|}n^{|V(H)|})$ options in total of colouring the end-vertices of an edge in G and the vertices of G' . In each option, we have at least one red vertex and at least one blue vertex. We now consider the options one by one.

Consider an option as described above. In particular, let $e = uv$ be the chosen edge whose end-vertices we coloured differently, say we coloured u red and v blue. We first check in polynomial time if every red vertex in this option has at most one blue neighbour and if every blue vertex has at most one red neighbour. If one of these two conditions does not hold, we discard the option. Now let S' consist of u and all red vertices of $V(G') \cup N(V(G'))$, and let T' consists of v and all blue vertices of $V(G') \cup N(V(G'))$. We let $S'' \subseteq S'$ consist of all red vertices that have (exactly) one blue neighbour, and we let $T'' \subseteq T'$ consist of all blue vertices that have (exactly) one red neighbour. By construction (recall that we started with picking an edge whose end-vertices we coloured differently), $|S''| = |T''| \geq 1$. Hence, we can consider (S', T') as a generalized starting pair with core (S'', T'') .

Our algorithm will now check if G has a valid (S'', T'', S', T') red-blue colouring by applying rules R1–R3 exhaustively. If we find a no-answer, then we can discard the option by Lemma 5. Otherwise, we found in polynomial time, again by Lemma 5, a 4-tuple (S, T, X, Y) , for which the following holds:

- (i) $S'' \subseteq S \subseteq X$; $S' \subseteq X$; $T'' \subseteq T \subseteq Y$; $T' \subseteq Y$; and $X \cap Y = \emptyset$,
- (ii) G has a valid red-blue (S'', T'', S', T') -colouring if and only if G has valid red-blue (S, T, X, Y) -colouring, and
- (iii) every vertex in $V \setminus (X \cup Y)$ has no neighbour in $S \cup T$; at most one neighbour in $X \setminus S$ and at most one neighbour in $Y \setminus T$.

We now prove the following claim.

Claim. *All connected components of $G - (X \cup Y)$ are monochromatic in every valid red-blue (S, T, X, Y) -colouring of G .*

We prove the claim as follows. For a contradiction, assume G has a valid red-blue (S, T, X, Y) -colouring, for which at least one connected component F of $G - (X \cup Y)$ is not monochromatic. As $V(G') \cup N(V(G')) \subseteq S' \cup T'$ and $S' \subseteq X$ and $T' \subseteq Y$, we find that $V(F)$ belongs to G^* (recall that G^* is the P_3 -free graph obtained from G after deleting the vertices of $V(G') \cup N(V(G'))$). Hence, F is P_3 -free and thus as F is connected, F must be a complete graph. If F consists of one vertex or at least three vertices, then F must be monochromatic. Hence, F consists of exactly two vertices x and y .

By statement (iii), both x and y have no neighbour in $S \cup T$; at most one neighbour in $X \setminus S$ and at most one neighbour in $Y \setminus T$. If x and y have a common neighbour, then F must be monochromatic. If one of them, say x , has a neighbour in both $X \setminus S$ and a neighbour in $Y \setminus T$, then y must be coloured with the same colour as x and thus again F must be monochromatic. As G has minimum degree 2, this means that x and y each have exactly one neighbour in $X \cup Y$, and these two vertices in $X \cup Y$ must be different.

Let x' be the unique neighbour of x in $X \cup Y$, and let y' be the unique neighbour of y in $X \cup Y$, so $x' \neq y'$. However, now we find that G has a matching cut of size 2, namely the set $\{xx', yy'\}$, a contradiction. This completes the proof of the claim. \diamond

Due to the above claim, we can now check in polynomial time, by using Lemma 6, whether G has a valid red-blue (S, T, X, Y) -colouring in which every connected component in $G - (X \cup Y)$ is monochromatic. If so, then we are done, and else we discard the option.

The correctness of our algorithm follows from its description. As the total number of branches is $O(m2^{|V(H)|}n^{|V(H)|})$ and we can process each branch in polynomial time, the total running time of our algorithm is polynomial. Hence, we have proven the theorem. \square

Bonsma [4] proved that MATCHING CUT is polynomial-time solvable for the class of claw-free graphs, that is, $K_{1,3}$ -free graphs. By combining, respectively, Corollary 1 and Bonsma's result with s applications of Theorem 4 we obtain the following result.

Theorem 5. For every integer $s \geq 0$, MATCHING CUT is polynomial-time solvable for $(sP_3 + P_6)$ -free graphs and for $(sP_3 + K_{1,3})$ -free graphs.

6. A partial complexity classification for H-free graphs

The girth of a graph that is not a tree is the length of a shortest cycle in it. Bonsma [4] proved that MATCHING CUT is NP-complete for planar graphs of girth 5, and thus for C_r -free graphs with $r \in \{3, 4\}$. Hence, MATCHING CUT is NP-complete for H -free graphs whenever H contains a C_3 or C_4 . Le and Randerath [16] proved that MATCHING CUT is NP-complete for bipartite graphs of minimum degree 3 and maximum degree 4. Consequently, MATCHING CUT is NP-complete for H -free graphs whenever H contains an odd cycle. We use a result of Moshi [19] to prove the same for the case where H has a not necessarily odd cycle.

Let uv be an edge in a graph G . We replace the edge by two new vertices w_1 and w_2 and edges uw_1, uw_2, vw_1 and vw_2 . We call this operation a $K_{2,2}$ -replacement and denote the resulting graph by G_{uv} ; see also Fig. 5. We can now state the following lemma.

Lemma 7 ([19]). For any edge uv of a graph G , the graph G has a matching cut if and only if G_{uv} has a matching cut.

We can show the following result.

Lemma 8. For every graph H that is not a forest, MATCHING CUT is NP-complete for H -free graphs.

Proof. Let H be a graph that is not a forest. Then H has a cycle C_ℓ . As C_ℓ -free graphs form a subclass of H -free graphs, we only need to prove the lemma for C_ℓ -free graphs. As MATCHING CUT is NP-complete for bipartite graphs (of diameter at most 4) due to the aforementioned result of Le and Le [15], we may assume that ℓ is even, say $\ell = 2r$ for some $r \geq 2$.

We prove by induction that MATCHING CUT is NP-complete for C_{2r} -free graphs for every $r \geq 2$. If $r = 2$, this follows immediately from the aforementioned NP-completeness result of Bonsma [4] for planar graphs of girth 5, which are C_4 -free. Assume that $r \geq 3$. We reduce from MATCHING CUT for C_r -free graphs, for which the problem is NP-complete: if r is odd this follows from the above; and if r is even this follows from the induction hypothesis. Let G be a C_r -free graph. Apply a $K_{2,2}$ -replacement on every edge of G . The resulting graph is C_{2r} -free and by Lemma 7, it has a matching cut if and only if G has a matching cut. \square

For two graphs G_1 and G_2 we write $G_1 \subseteq_i G_2$ if G_1 is an induced subgraph of G_2 . Let H^* be the “H”-graph, which is the graph with vertices $a_1, b_1, c_1, a_2, b_2, c_2$ and edges $a_i b_i, b_i c_i$ ($i = 1, 2$) and $b_1 b_2$. We can now show the following summary theorem; see also Fig. 6.

Theorem 6. For a graph H , MATCHING CUT on H -free graphs is

- polynomial-time solvable if $H \subseteq_i sP_3 + K_{1,3}$ or $sP_3 + P_6$ for some $s \geq 0$, and
- NP-complete if $H \supseteq_i C_r$ for some $r \geq 3, K_{1,4}, P_{19}, 4P_5$ or H^* .

Proof. The polynomial-time solvable cases follow from Theorem 5. If H has a cycle, then we apply Lemma 8. Now suppose that H has no cycle so H is a forest. First suppose that H is a forest that contains a vertex of degree at least 4. Then H contains an induced $K_{1,4}$, and thus the class of H -free graphs contains the class of $K_{1,4}$ -free graphs. The result then follows from the aforementioned result of Chvátal [8], which in fact shows that MATCHING CUT is NP-complete even for $K_{1,4}$ -free graphs, as observed by Bonsma [4] and Kratch and Le [14]. The remaining three results are proven in [17]. \square

Every forest that is P_r -free for some positive constant r and that has maximum degree at most 4 has a constant-bounded diameter, so has constant size. Hence, Theorem 6 has the following consequence.

Corollary 2. There only exists a finite number of connected graphs H for which the computational complexity of MATCHING CUT is open when restricted on H -free graphs.

7. Conclusions

We gave a complexity dichotomy for MATCHING CUT for graphs of bounded radius and proved a number of new results on the complexity of MATCHING CUT for H -free graphs. We summarized all the known results for H -free graphs in Theorem 6 and showed that although there still exists an infinite number of unresolved cases, the number of open cases where H is a connected graph is finite.

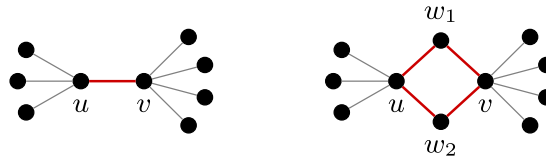


Fig. 5. The $K_{2,2}$ -replacement applied on edge uv .

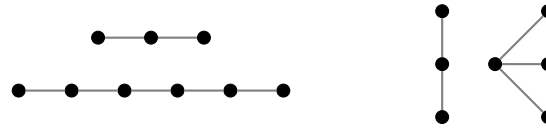


Fig. 6. The graphs $P_3 + P_6$ (left) and $P_3 + K_{1,3}$ (right).

We finish our paper with a number of open problems. Recall that Le and Le [15] showed that MATCHING CUT for bipartite graphs of diameter at most d is polynomial-time solvable if $d \leq 3$ and NP-complete for $d \geq 4$. Their hardness construction has radius 4, and we therefore pose the following open problem:

Open Problem 1. Determine the complexity of MATCHING CUT for bipartite graphs of radius 3.

A standard ingredient of determining the complexity of a problem for H -free graphs is to first consider classes of large girth. If a problem is NP-complete for graphs of girth at least g , for every fixed integer $g \geq 3$, then it is NP-complete for H -free graphs whenever H has a cycle; just take g to be larger than the length of a largest cycle in H . However, to prove Lemma 8 we could only use the result of Bonsma [4] for (planar) graphs of girth 5 and had to rely on the construction of Moshi [19] to show hardness if H has a cycle of length at least 5. Hence, we believe the following open problem of Le and Le [15] is interesting.

Open Problem 2 ([15]). Determine for every $g \geq 6$, the complexity of MATCHING CUT for graphs of girth g .

For $K_{1,t}$ -free graphs, the complexity of MATCHING CUT is fully determined (see Theorem 6) with the problem becoming NP-complete for $t \geq 4$. On the positive side, Kratsch and Le [14] proved that MATCHING CUT can be solved in polynomial time for $K_{1,4}$ -free graphs that in addition are also $(K_{1,4} + e)$ -free, where $K_{1,4} + e$ is the graph obtained from $K_{1,4}$ by adding an edge between two of its leaves. We finish our paper with the following open problem, for which we identified some borderline cases; the *chair* is the graph obtained from the claw $K_{1,3}$ after subdividing one of its edges exactly once.

Open Problem 3. Complete the classification of MATCHING CUT for H -free graphs; in particular what is the complexity of MATCHING CUT for *chair*-free graphs, $2P_4$ -free graphs and P_7 -free graphs?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

- [1] J. Araújo, N. Cohen, F. Giroire, F. Havet, Good edge-labelling of graphs, *Discrete Appl. Math.* 160 (2012) 2502–2513.
- [2] N.R. Aravind, S. Kalyanasundaram, A.S. Kare, On structural parameterizations of the matching cut problem, in: *Proc. COCOA 2017*, in: LNCS, vol. 10628, 2017, pp. 475–482.
- [3] N.R. Aravind, R. Saxena, An FPT algorithm for matching cut and d -cut, in: *Proc. IWOCOA 2021*, in: LNCS, vol. 12757, 2021, pp. 531–543.
- [4] P.S. Bonsma, The complexity of the matching-cut problem for planar graphs and other graph classes, *J. Graph Theory* 62 (2009) 109–126.
- [5] M. Borowiecki, K. Jesse-Józefczyk, Matching cutsets in graphs of diameter 2, *Theor. Comput. Sci.* 407 (2008) 574–582.
- [6] E. Camby, O. Schaudt, A new characterization of P_t -free graphs, *Algorithmica* 75 (2016) 205–217.
- [7] C. Chen, S. Hsieh, H. Le, V.B. Le, S. Peng, Matching cut in graphs with large minimum degree, *Algorithmica* 83 (2021) 1238–1255.
- [8] V. Chvátal, Recognizing decomposable graphs, *J. Graph Theory* 8 (1984) 51–53.
- [9] A.M. Farley, A. Proskurowski, Networks immune to isolated line failures, *Networks* 12 (1982) 393–403.
- [10] C. Feghali, A note on matching-cut in P_t -free graphs, *Inf. Process. Lett.* 179 (2023) 106294.
- [11] P.A. Golovach, D. Paulusma, J. Song, Computing vertex-surjective homomorphisms to partially reflexive trees, *Theor. Comput. Sci.* 457 (2012) 86–100.
- [12] R.L. Graham, On primitive graphs and optimal vertex assignments, *Ann. N.Y. Acad. Sci.* 175 (1970) 170–186.
- [13] C. Komusiewicz, D. Kratsch, V.B. Le, Matching cut: kernelization, single-exponential time fpt, and exact exponential algorithms, *Discrete Appl. Math.* 283 (2020) 44–58.
- [14] D. Kratsch, V.B. Le, Algorithms solving the matching cut problem, *Theor. Comput. Sci.* 609 (2016) 328–335.

- [15] H. Le, V.B. Le, A complexity dichotomy for matching cut in (bipartite) graphs of fixed diameter, *Theor. Comput. Sci.* 770 (2019) 69–78.
- [16] V.B. Le, B. Randerath, On stable cutsets in line graphs, *Theor. Comput. Sci.* 301 (2003) 463–475.
- [17] F. Lucke, D. Paulusma, B. Ries, Finding matching cuts in H -free graphs, in: *Proc. ISAAC 2022, LIPIcs*, in press.
- [18] G.B. Mertzios, P.G. Spirakis, Algorithms and almost tight results for 3-colorability of small diameter graphs, *Algorithmica* 74 (2016) 385–414.
- [19] A.M. Moshi, Matching cutsets in graphs, *J. Graph Theory* 13 (1989) 527–536.
- [20] M. Patrignani, M. Pizzonia, The complexity of the matching-cut problem, in: *Proc. WG 2001*, in: *LNCS*, vol. 2204, 2001, pp. 284–295.
- [21] P. van 't Hof, D. Paulusma, A new characterization of P_6 -free graphs, *Discrete Appl. Math.* 158 (2010) 731–740.