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On the complexity of the selective graph coloring problem in some special classes of graphs

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Abstract

In this paper, we consider the selective graph coloring problem. Given an integer $k \geq 1$ and a graph $G = (V, E)$ with a partition $V_1, \ldots, V_p$ of V, it consists in deciding whether there exists a set V^* in G such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$, and such that the graph induced by V^* is k-colorable. We investigate the complexity status of this problem in various classes of graphs.

Keywords: Computational complexity, Approximation algorithms, PTAS, Scheduling, Bipartite graphs, Split graphs, Complete q-partite graphs Clustering

1. Introduction and related works

Scheduling problems appearing in real-life situations may often be modeled as graph coloring problems (see [2,26,13,16,20,24,27]). For instance, scheduling problems involving only incompatibility constraints correspond to the classical vertex coloring problem in undirected graphs. If in addition precedence constraints occur, the problem may be handled using the vertex coloring problem in mixed graphs (i.e., graphs containing both undirected and directed edges). Thus many types of graph coloring problems are of interest: precoloring extension, list-coloring, multicoloring, mixed graph coloring, T-coloring, edge coloring, etc.

In this paper, we consider the **selective graph coloring problem**. Consider an undirected graph $G = (V, E)$ and a partition $V_1, \ldots, V_p$ of its vertex set V. For some integer $k \geq 1$, the selective graph coloring problem consists in finding a subset $V^* \subseteq V$ such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$ and such that the graph induced by V^* is k-colorable (see Fig. 1 for an example).

Consider the following scheduling problem: we are given a set of p tasks $t_1, \ldots, t_p$ each of which needs to be executed on one of k identical machines $m_1, \ldots, m_k$; each task t_j has a given length ℓ_j, for $j = 1, 2, \ldots, p$; for each task t_j of length l_j, $j \in \{1, \ldots, p\}$, we are given a list of possible time intervals $I_1(j), \ldots, I_{n_j}(j)$, each of length l_j, during which the task may be executed. Suppose that each machine m_i cannot process more than one task simultaneously, for $i = 1, 2, \ldots, k$. Furthermore, the tasks are supposed to be non preemptive, i.e., once a machine started executing a task, the execution cannot be interrupted temporarily but the task must be finished on that machine. The goal is to determine for each task t_j, $j \in \{1, \ldots, p\}$, one feasible time interval among $I_1(j), \ldots, I_{n_j}(j)$ such that all tasks can be executed using at most k machines.

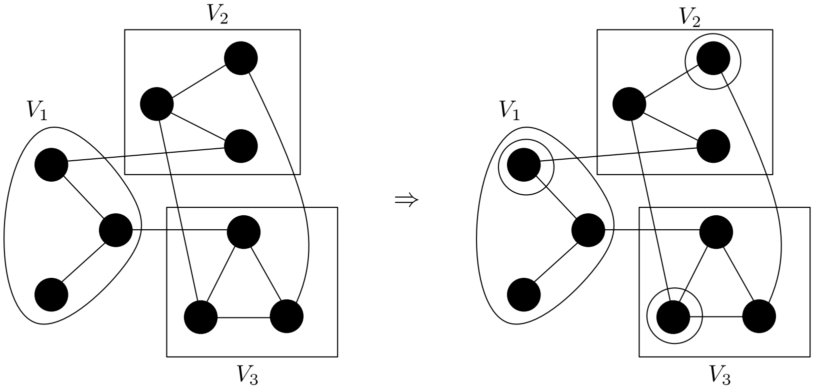

Fig. 1. A graph $G = (V, E)$ with a partition V_1, V_2, V_3 of V and a set V^* (encircled vertices) which induces a 1-colorable graph.

In order to solve this scheduling problem, we may use the selective graph coloring model in an interval graph. Indeed, with each task t_j, $j \in \{1, \dots, p\}$, and each time interval $I_i(j)$, $i = 1, \dots, n_j$, we associate a vertex v_{ij}; then we add an edge between two vertices u, v if the corresponding time intervals have a non empty intersection; thus we obtain an interval graph $G = (V, E)$. Finally we define a partition $V_1, \dots, V_p$ as follows: $V_j = \{v_{1j}, \dots, v_{n_j j}\}$ for $j = 1, \dots, p$. Clearly, there exists a feasible schedule using at most k machines if and only if G admits a selective graph coloring with at most k colors.

Note that the "selective framework" of the graph coloring problem also exists for other combinatorial optimization problems, for instance the Traveling Salesman Problem (TSP). This problem is known as *Generalized* TSP, *Group*-TSP or *One-of-a-set* TSP (see for instance [19,25]) and is defined as follows. A salesman needs to visit n customers $c_1, \dots, c_n$. Each customer c_i, $i \in \{1, \dots, n\}$, specifies some locations $l_1(i), \dots, l_{n_i}(i)$ in which he/she is willing to meet the salesman. The goal is then to find a tour of minimum length such that the salesman visits each customer c_i, $i \in \{1, \dots, n\}$, once and such that the meeting takes place in one of the specified locations $l_1(i), \dots, l_{n_i}(i)$. Thus if each customer specifies exactly one location, we obtain the classical TSP problem. Further combinatorial optimization problems with this selective framework can be found in [12] or [22].

The selective graph coloring problem was introduced in [17] under the name of *partition coloring problem* in the frame of routing and wavelength assignment in networks. For this application, the problem was defined in the edge intersection graph of some preselected paths in the network. Several heuristic methods have been designed in this context (see for instance [23]).

Notice that the selective coloring problem is related to another type of coloring problem, called the *empire coloring problem* (see for instance [8,21]). We are given a planar graph $G = (V, E)$ whose vertex set is partitioned into sets $V_1, \dots, V_p$ such that each such set contains exactly r vertices, for some fixed positive integer r. Then, for some fixed positive integer k, the empire coloring problem (in its decision version) consists in deciding whether there exists a coloring of the vertices of G with at most k colors such that adjacent vertices in different sets get different colors and all the vertices in a same set get a same color, disregarding the adjacencies. Thus this problem can be seen as a kind of generalization of the selective graph coloring problem since instead of coloring exactly one vertex per cluster, we color all the vertices in each cluster.

Another problem which is related to the selective graph coloring problem is the so-called *multicolored clique problem* (Mcc). In this problem, we are given an integer r and a connected graph $G = (V, E)$ as well as a partition of its vertex set $V_1, \dots, V_r$ such that every set V_i induces a stable set. Then the question is whether there exists a clique of size r in G. This problem has been studied for instance in [11] from a parametrized complexity point of view and it was shown to be $W[1]$-hard. Obviously, the Mcc problem in G is equivalent to asking whether the complement of G admits a selective graph coloring using exactly one color.

Finally, note also that the selective graph coloring problem has some natural connections with the *inverse chromatic number problem* (see [7]). For a graph G and an integer k, this problem consists in modifying the graph as little as possible such that the chromatic number of the resulting graph is at most k. For an interval graph, suppose that the possible modifications of the graph correspond to shifting intervals to the left or to the right in the interval representation. Furthermore, we associate a cost with each such modification. Then for each interval, we define a cluster as the set of all possible locations of that interval. Now the problem consists in selecting in each cluster one interval (i.e., one vertex) such that the resulting graph is k-colorable and the total cost of the selected intervals is minimum.

All these close problems or particular cases of the selective graph coloring problem justify considering it in a systematic way from a theoretical point of view. This work is such a first attempt to better understand its complexity status in different classes. Since the classical graph coloring problem is a special case of the selective graph coloring problem when $|V_i| = 1$ for all $i \in \{1, \dots, p\}$, it follows that the selective graph coloring problem is $\mathcal{NP}$-hard in general. To the best of our knowledge, there exists no better hardness result. In this paper, we investigate some classical classes of graphs and determine the complexity status of the selective graph coloring problem in these classes. Furthermore, for some classes of graphs for which the problem is $\mathcal{NP}$-hard, we present polynomial-time approximation algorithms.

Our paper is organized as follows. In Section 2, we give some notations and definitions which will be used throughout the paper. Section 3 deals with split graphs, Section 4 with complete q-partite graphs and Section 5 with bipartite graphs. In Section 6, we consider the case when each set V_i of the partition induces a clique. Finally, in Section 7 we present some further results for some special classes of graphs and a conclusion is given in Section 8, where we also present a table containing all our results of this paper.

2. Preliminaries

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in G that are adjacent to v, i.e., the *neighbors* of v. $N(v)$ is called the *neighborhood* of vertex v.

Let $V' \subseteq V$. We denote by $G[V']$ the *graph induced by* V', i.e., the graph obtained from G by deleting the vertices of $V - V'$ and all edges incident to at least one vertex of $V - V'$. For two graphs H and $G = (V, E)$, G is called H-free if there is no $V' \subseteq V$ such that $G[V']$ is isomorphic to H. G is called $(H_1, \ldots, H_p)$-free if it is H_i-free for any $i = 1, \ldots, p$.

A *stable set* in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of a stable set in a graph G is called the *stability number* of G and is denoted by $\alpha(G)$. A *clique* in a graph $G = (V, E)$ is a set of pairwise adjacent vertices. A *matching* in a graph $G = (V, E)$ is a set of pairwise nonadjacent edges. In a graph $G = (V, E)$, a matching M is said to *saturate* a set $V' \subseteq V$ if for every vertex $v \in V'$ there exists an edge in M incident to v.

We denote by nG the disjoint union of n copies of a graph G. As usual P_n (respectively C_n) denotes the induced path (resp. the induced cycle) on n vertices. A clique on n vertices will be denoted by K_n. Consider two graphs G and H. Then we denote by $G + H$ the disjoint union of G and H.

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all $uv \in E$. The smallest integer k such that G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. Consider now a partition $\mathcal{V} = (V_1, V_2, \ldots, V_p)$ of the vertex set V of G. The sets $V_1, \ldots, V_p$ will be called *clusters*. A *selective k-coloring* of G with respect to $\mathcal{V}$ is a mapping $c : V^* \to \{1, \ldots, k\}$, where $V^* \subseteq V$ with $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$, such that $c(u) \neq c(v)$ for all $uv \in E$. Thus determining a selective k-coloring with respect to $\mathcal{V}$ consists in finding a set $V^* \subseteq V$ such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$ and such that $G[V^*]$ admits a k-coloring. The smallest integer k for which a graph G admits a selective k-coloring with respect to $\mathcal{V}$ is called the *selective chromatic number* of G with respect to $\mathcal{V}$ and is denoted by $\chi_{SEL}(G, \mathcal{V})$. It is obvious to see that $\chi_{SEL}(G, \mathcal{V}) \leq \chi(G)$ for every partition $\mathcal{V}$ of V.

In this paper we will be interested in the following two problems.

Sel-Col
Input: An undirected graph $G = (V, E)$; a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V.
Question: Find a set $V^* \subseteq V$ such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$ and such that $\chi(G[V^*])$ is minimum.

Let $k \geq 1$ be a fixed integer.
k-DSel-Col
Input: An undirected graph $G = (V, E)$; a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V.
Question: Does there exist a set $V^* \subseteq V$ such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$ and such that $G[V^*]$ is k-colorable?

For instance, 1-DSel-Col consists in deciding whether there exists a stable set $V^* \subseteq V$ such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. Clearly, k-DSel-Col and Sel-Col are related problems. Consider a graph class $\mathcal{G}$. If for some fixed k, k-DSel-Col is $\mathcal{NP}$-complete in $\mathcal{G}$, then Sel-Col is $\mathcal{NP}$-hard in $\mathcal{G}$ and if Sel-Col is polynomial-time solvable in $\mathcal{G}$, then k-DSel-Col is polynomial-time solvable in $\mathcal{G}$ for every fixed k. However, these two problems are not equivalent from a complexity point of view since for split graphs and complete q-partite graphs for instance, we will see that Sel-Col is $\mathcal{NP}$-hard whereas k-DSel-Col is polynomial-time solvable for every fixed k.

Consider a minimization (resp. maximization) problem Π and an instance $\mathcal{I}$ of Π. Let S be a solution of $\mathcal{I}$. We denote by $f(\mathcal{I}, S)$ the value of solution S, and by $OPT(\mathcal{I})$ the value of an optimal solution of $\mathcal{I}$. Then an algorithm is said to be a *c-approximation algorithm* for problem Π, where $c \geq 1$ (resp. where $c \leq 1$), if for any instance $\mathcal{I}$ of the problem it gives a solution S such that $f(\mathcal{I}, S) \leq c \cdot OPT(\mathcal{I})$ (resp. $f(\mathcal{I}, S) \geq c \cdot OPT(\mathcal{I})$).

An algorithm A is an *approximation scheme* for a minimization problem Π, if for any instance $\mathcal{I}$ of Π and for any $\varepsilon > 0$, A gives a solution S such that $f(\mathcal{I}, S) \leq (1 + \varepsilon) \cdot OPT(\mathcal{I})$. A is said to be a *polynomial time approximation scheme (PTAS)* if for each fixed $\varepsilon > 0$, its running time is bounded by a polynomial in the size of instance $\mathcal{I}$. If its running time is bounded by a polynomial in the size of $\mathcal{I}$ and $\frac{1}{\varepsilon}$, then A is said to be a *fully polynomial time approximation scheme (FPTAS)*.

Notice that if Sel-Col is $\mathcal{NP}$-hard in a graph class $\mathcal{G}$, then Sel-Col does not admit a FPTAS for $\mathcal{G}$ unless $\mathcal{P} = \mathcal{NP}$. Moreover, if k-DSel-Col is $\mathcal{NP}$-complete in $\mathcal{G}$, then Sel-Col does not admit a $(\frac{k+1}{k} - \varepsilon)$-approximation in $\mathcal{G}$ for every $\varepsilon > 0$ and conversely, if Sel-Col admits a PTAS for $\mathcal{G}$ then k-DSel-Col is polynomial-time solvable in $\mathcal{G}$ for every fixed k. Finally, since Sel-Col contains the usual graph coloring problem (when all clusters have size one), it follows that Sel-Col is $\mathcal{NP}$-hard and 3-DSel-Col is $\mathcal{NP}$-complete in general graphs.

For all graph theoretical terms not defined here the reader is referred to [28] and for all $\mathcal{NP}$-completeness related notions and definitions, the reader is referred to [14].

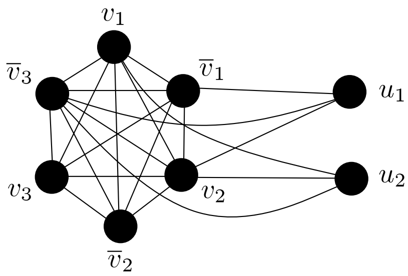

Fig. 2. Example for clauses $u_1 = (x_1 \vee \bar{x}_2 \vee x_3)$, $u_2 = (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$.

3. Split graphs

A *split graph* $G = (V, E)$ is a graph whose vertex set V can be partitioned into two sets: a clique K and a stable set S. Notice that $|K| \leq \chi(G) \leq |K| + 1$. Furthermore, if $\chi(G) = |K|$, then for every vertex $s \in S$, there exists a vertex $u \in K$ which is nonadjacent to s. Since every induced subgraph of G is also a split graph, we conclude that $|V^* \cap K| \leq \chi(G[V^*]) \leq |V^* \cap K| + 1$ for any set $V^* \subseteq V$. Thus, if $V^* \subseteq V$ is a solution of SEL-COL with respect to some partition $\mathcal{V}$ of V, we have $|V^* \cap K| \leq \chi_{SEL}(G, \mathcal{V}) \leq |V^* \cap K| + 1$.

Consider now a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V. Suppose that there exists a cluster V_i, $i \in \{1, \ldots, p\}$, such that $V_i \cap K, V_i \cap S \neq \emptyset$. Consider a solution V^* of Sel-Col such that $V^* \cap V_i \subseteq K$. Let $V^* \cap V_i = \{v\}$ and let $u \in V_i \cap S$. We claim that $V^{*'} = (V^* - \{v\}) \cup \{u\}$ is also a solution of Sel-Col. Indeed, since $N(u) \subseteq N(v)$, we clearly have $\chi(G[V^{*'}]) \leq \chi(G[V^*])$. So it is always a good strategy to choose in each cluster a vertex from S (if possible). Thus we may assume now that for every cluster V_i, $i \in \{1, \ldots, p\}$, of the partition we have either $V_i \subseteq K$ or $V_i \subseteq S$. It follows from the above that if $V_1, \ldots, V_q \subseteq K$ and $V_{q+1}, \ldots, V_p \subseteq S$, then $q \leq \chi_{SEL}(G, \mathcal{V}) \leq q + 1$.

Theorem 3.1. Sel-Col *is $\mathcal{NP}$-hard for split graphs even if the partition $V_1, \ldots, V_p$ satisfies $|V_i| \leq 2$ for all $i \in \{1, \ldots, p\}$.*

Proof. We will use a reduction from 3Sat which is known to be $\mathcal{NP}$-complete (see [14]). This problem is defined as follows: we are given a set U of variables and a collection C of clauses over U such that each clause $c \in C$ satisfies $|c| = 3$; then we ask whether there exists a satisfying truth assignment for C. Consider an instance $\mathcal{I}$ of 3Sat with n variables $x_1, \ldots, x_n$ and m clauses $C_{n+1}, \ldots, C_{n+m}$. We construct a split graph $G = (V, E)$ as follows: with each variable x_i, $i \in \{1, \ldots, n\}$, we associate two vertices v_i and $\bar{v}_i$; with each clause C_j, $j \in \{n+1, \ldots, n+m\}$, we associate a vertex u_j; we add all the edges between the vertices associated with the variables; we add an edge between vertices v_i (resp. $\bar{v}_i$) and u_j if and only if x_i (resp. $\bar{x}_i$) is a literal not appearing in clause C_j. Thus the vertices $v_1, \bar{v}_1, \ldots, v_n, \bar{v}_n$ induce a clique K of size $2n$ and the vertices $u_{n+1}, \ldots, u_{n+m}$ induce a stable set S of size m (see Fig. 2 for an example). Now we define the following partition $\mathcal{V}$ of V: for every vertex v_i, we set $V_i = \{v_i, \bar{v}_i\}$ and for every vertex u_j, we set $V_j = \{u_j\}$, for $i = 1 \ldots, n$ and $j = n+1, \ldots, n+m$. Thus we get an instance $\mathcal{I}'$ of Sel-Col in a split graph G. Notice that it follows from the discussion above that $n \leq \chi_{SEL}(G, \mathcal{V}) \leq n + 1$.

Now suppose that $\mathcal{I}$ is a yes-instance. Then for every clause C_j, consider a literal $x_i \in C_j$ (respectively $\bar{x}_i \in C_j$) which is true and add the vertices u_j and v_i (resp. $\bar{v}_i$) to V_i^*. This clearly gives us a set $V^* = \cup_{i=1,\ldots,n} V_i^*$ such that $|V^* \cap V_\ell| = 1$ for $\ell = 1, \ldots, n+m$. Furthermore $G[V^*]$ is n-colorable since for every $i \in \{1, \ldots, n\}$, V_i^* is a stable set. Thus $\chi_{SEL}(G, \mathcal{V}) = n$.

Conversely, suppose now that $\chi_{SEL}(G, \mathcal{V}) = n$ and let V^* be the corresponding solution. Since we have n clusters contained in the clique K and since each vertex in S represents a cluster, it follows that for every vertex $u_j \in S$ there exists a vertex $v_i \in K$ (respectively $\bar{v}_i \in K$) nonadjacent to u_j and such that $u_j, v_i \in V^*$ (resp. $u_j, \bar{v}_j \in V^*$). Indeed, if this is not the case then $\chi(G[V^*]) \geq n + 1$, a contradiction. Now we recall that a vertex v_i which is nonadjacent to some vertex u_j represents a literal appearing in the clause C_j. Thus, by setting to true every literal x_i (resp. $\bar{x}_i$) such that the corresponding vertex v_i (resp. $\bar{v}_i$) belongs to V^* and to false the remaining literals, we obtain a truth assignment such that every clause C_j contains at least one true literal. Hence $\mathcal{I}$ is a yes-instance. □

Notice that the result given in Theorem 3.1 is the best possible with respect to the maximum size of the clusters. Indeed, if $|V_i| \leq 1$ for all $i \in \{1, \ldots, p\}$, then Sel-Col is equivalent to the usual graph coloring problem which is polynomial-time solvable in split graphs.

Remark 3.1. Notice that if $G = (V, E)$ is a threshold graph, then Sel-Col becomes polynomial-time solvable. Indeed, a threshold graph is a split graph in which the vertices may be ordered $v_1, \ldots, v_n$ with $N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_n)$. Without loss of generality we may assume that $v_1, \ldots, v_\ell \in S$ and $v_{\ell+1}, \ldots, v_n \in K$. Let q be the number of clusters which are contained in the clique K. Recall that $q \leq \chi_{SEL}(G, \mathcal{V}) \leq q + 1$. Thus the answer to Sel-Col is q if and only if there exists a vertex $v \in K$ such that v is nonadjacent to v_ℓ.

Although Sel-Col is $\mathcal{NP}$-hard, we will now show that Sel-Col admits a PTAS if the input graph is a split graph.

Theorem 3.2. *Let $G = (V, E)$ be a split graph. Then* Sel-Col *admits a* PTAS *for G.*

Proof. Consider a split graph $G = (V, E)$ with clique K, stable set S and $|V| = n$. Let $\mathcal{V} = (V_1, \dots, V_p)$ be a partition of V and let $\varepsilon > 0$ be fixed. As mentioned above, we may assume that for every cluster V_i, $i \in \{1, \dots, p\}$, we have either $V_i \subseteq K$ or $V_i \subseteq S$. Without loss of generality, we may assume that $V_1, \dots, V_q \subseteq K$ and $V_{q+1}, \dots, V_p \subseteq S$. Recall that $q \le \chi_{SEL}(G, \mathcal{V}) \le q + 1$. We will distinguish two cases.

(i) If $q < \frac{1}{\varepsilon}$, then we obtain an optimal solution in polynomial time. Indeed, for every possible choice of vertices in $V_1, \dots, V_q$ to be added to V^*, we need to check if every cluster $V_{q+1}, \dots, V_p$ contains at least one vertex which is nonadjacent to some previously chosen vertex in $V_1 \cup \dots \cup V_q$. If this is true, then we get a solution of value q; if not, the optimal solution is $q + 1$, and we obtain such a solution by an arbitrary choice of vertices to be added to V^* and by coloring all vertices of $V^* \cap S$ with a same color. Since $q < \frac{1}{\epsilon}$, the number of clusters contained in K is bounded by a constant and hence the number of choices mentioned above is bounded by a polynomial in n.

(ii) If $q \ge \frac{1}{\varepsilon}$, then let OPT denote the value of an optimal solution of Sel-Col in G. Clearly $\frac{1}{\varepsilon} \le \text{OPT}$ and thus $1 \le \varepsilon \cdot \text{OPT}$. By arbitrarily choosing one vertex in each cluster and adding it to V^*, we obtain (as explained above) a solution for Sel-col of value VAL such that $\text{OPT} \le \text{VAL} \le \text{OPT}+1$. Hence $\text{VAL} \le (1 + \varepsilon)\cdot\text{OPT}$.

Since the algorithm described above is clearly polynomial in n, we conclude that Sel-Col admits a PTAS if G is a split graph. □

The following is an immediate consequence of Theorem 3.2.

Corollary 3.3. *For every $k \ge 1$, k-Dsel-Col is polynomial-time solvable in split graphs.*

4. Complete partite graphs

A graph $G = (V, E)$ is a *complete q-partite graph* if V can be partitioned into q stable sets $L_1, \dots, L_q$ such that there exist all possible edges between any two stable sets L_i, L_j, $i, j \in \{1, \dots, q\}$ with $i \neq j$. These graphs are recognizable in polynomial time because they are exactly the $(K_1 + K_2)$-free graphs.

Consider a partition $\mathcal{V} = (V_1, \dots, V_p)$ of V. Notice that for every $u, v \in L_j$, $j \in \{1, \dots, q\}$, we have $N(u) = N(v)$. Thus we may assume that $|V_i \cap L_j| \le 1$ for every $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Hence $|V_i| \le q$ for every $i \in \{1, \dots, p\}$. Finally, notice that for a complete q-partite graph G we have $1 \le \chi_{SEL}(G, \mathcal{V}) \le q$.

Theorem 4.1. Sel-Col *is polynomial-time solvable for complete q-partite graphs when q is fixed.*

Proof. As mentioned above, for a complete q-partite graph G, we have $1 \le \chi_{SEL}(G, \mathcal{V}) \le q$. In order to determine $\chi_{SEL}(G, \mathcal{V})$, we proceed as follows: for $k = 1, \dots, q$ and for every possible choice of k sets $L_{i_1}, \dots, L_{i_k}$ among $L_1, \dots, L_q$, we color all vertices in L_{i_j} with color j for $j = 1, \dots, k$; if necessary we may uncolor some vertices such that every cluster V_i, $i \in \{1, \dots, p\}$, contains at most one colored vertex; we add all colored vertices to V^* and check if $|V^* \cap V_i| = 1$ for all $i = 1, \dots, p$. Since q is fixed, it follows that the above algorithm determines $\chi_{SEL}(G, \mathcal{V})$ in polynomial time. □

Theorem 4.2. *For every $k \ge 1$, k-Dsel-Col is polynomial-time solvable for complete q-partite graphs.*

Proof. The proof is similar to the one of Theorem 4.1. For every possible choice of k sets $L_{i_1}, \dots, L_{i_k}$ among $L_1, \dots, L_q$, we color all vertices in L_{i_j} with color j for $j = 1, \dots, k$; if necessary we may uncolor some vertices such that every cluster V_i, $i \in \{1, \dots, p\}$, contains at most one colored vertex; we add all colored vertices to V^* and check if $|V^* \cap V_i| = 1$ for all $i = 1, \dots, p$. Since k is fixed, this yields a polynomial-time algorithm. □

While Sel-Col is polynomial-time solvable in complete q-partite graphs when q is fixed, we will show now that it is $\mathcal{NP}$-hard even if the sets L_j and V_i, $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, have all fixed sizes.

Theorem 4.3. Sel-Col *is $\mathcal{NP}$-hard for complete q-partite graphs $G = (L_1, \dots, L_q, E)$ even if $|L_j| = 3$ for all $j \in \{1, \dots, q\}$ and the partition $\mathcal{V} = (V_1, \dots, V_p)$ satisfies $|V_i| = 2$ for all $i \in \{1, \dots, p\}$.*

Proof. We use a reduction from Vertex Cover which is known to be $\mathcal{NP}$-hard even in cubic graphs (see [15]). Recall that Vertex Cover consists in finding in a graph $G = (V, E)$, a subset $V' \subseteq V$ with minimum size which covers the edges of G (i.e., $\forall uv \in E$, $u \in V'$ or $v \in V'$).

Consider an instance $\mathcal{I}$ of Vertex Cover in a cubic graph $H = (V_H, E_H)$ with $|V_H| = q$. We construct a complete q-partite graph $G = (L_1, \dots, L_q, E)$ such that $|L_j| = 3$ for all $j \in \{1, \dots, q\}$ as follows: with each vertex $v_i \in V_H$, we associate a set $L_i = \{v_{i_1}, v_{i_2}, v_{i_3}\}$, for $i = 1, \dots, n$; we add all possible edges between any two sets L_i, L_j for $i, j \in \{1, \dots, q\}$ with $i \neq j$. Now we define a partition $\mathcal{V}$ of V: with every edge $v_iv_j \in E_H$ we associate a cluster $V_{ij} = \{v_{i_\ell}, v_{j_q}\}$ for $\ell, q \in \{1, 2, 3\}$. Thus we obtain an instance $\mathcal{I}'$ of Sel-Col.

Now suppose that $\mathcal{I}$ has a feasible solution of value $s \le q$ and let V' be a vertex cover of size s. Then, for every $v_i \in V'$, $i \in \{1, \dots, s\}$, we add the vertices of L_i to V^*. Thus we obtain a set V^* containing at least one vertex from each cluster. If necessary, we delete some vertices from V^* such that it contains exactly one vertex from each cluster. Since $|V'| = s$, there are at most s sets L_i such that $V^* \cap L_i \neq \emptyset$ for $i \in \{1, \dots, q\}$. Thus $G[V^*]$ is s-colorable.

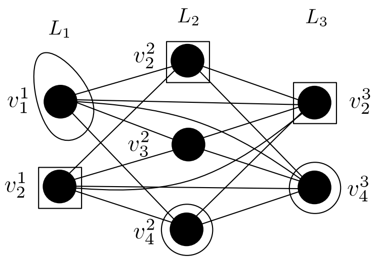

Fig. 3. The construction of $G = (L_1, \ldots, L_3, E)$ for $S_1 = \{x_1, x_2\}, S_2 = \{x_2, x_3, x_4\}, S_3 = \{x_2, x_4\}$. The clusters are $V_{x_1} = \{v_1^1\}, V_{x_2} = \{v_2^1, v_2^2, v_2^3\}, V_{x_3} = \{v_3^2\}$, $V_{x_4} = \{v_4^2\}$.

Conversely, suppose that $\mathcal{I}'$ has a feasible solution of value $s \leq q$. We construct a vertex cover V' of H with $|V'| \leq s$ as follows: for every set L_i such that $V^* \cap L_i \neq \emptyset, i \in \{1, \ldots, p\}$, we add v_i to V'. Since V^* intersects at most s sets L_i (recall that $G[V^*]$ is s-colorable), $i \in \{1, \ldots, p\}$, we obtain that $|V'| \leq s$. Furthermore, since V^* intersects every cluster exactly once, it follows that for each edge in E_H at least one endvertex belongs to V'. Thus V' is a vertex cover with $|V'| \leq s$. □

Notice that, as previously, the result given in Theorem 4.3 is best possible with respect to the maximum size of the clusters. Next we will focus on a polynomial-time approximation algorithm for Sel-Col in complete q-partite graphs. First we obtain the following.

Theorem 4.4. *From an approximation point of view,* Sel-Col *in complete q-partite graphs is equivalent to* Set-Cover*.*

Proof. We will use two approximation preserving reductions from and to Set Cover which is defined as follows: we are given a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of a finite set $X = \{x_1, \ldots, x_m\}$; we want to find a subset $\mathcal{S}' \subseteq \mathcal{S}$ of minimum size such that every element of X belongs to at least one member of $\mathcal{S}'$. This problem is known to be $\mathcal{NP}$-hard even if $|S_i| \leq 3$ for all $i \in \{1, \ldots, n\}$ (see [14]). Furthermore it is $H(\max_{i=1,\ldots,n} |S_i|)$-approximable (see [6]) where $H(r)$ is the r-th harmonic number and it is not $(1-\varepsilon)\log m$-approximable for any $\varepsilon > 0$ unless $\mathcal{NP} \subset TIME(n^{O(\log\ \log\ n)})$ (see [10]).

The reduction from Set Cover is defined as follows: given an instance $\mathcal{I} = (\mathcal{S}, X)$ of Set Cover, where $\mathcal{S} = \{S_1, \ldots, S_n\}$, $X = \{x_1, \ldots, x_m\}$ and $|S_i| \leq 3$ for all $i \in \{1, \ldots, n\}$, we construct a complete q-partite graph $G = (L_1, \ldots, L_q, E)$ where $q = n$ as follows: for each occurrence of x_j in a set S_i, we create a vertex v_j^i; we set $L_i = \{v_j^i : x_j \in S_i\}$ for all $i \in \{1, \ldots, n\}$. Finally, we define m clusters each of which corresponds to an element of X: for $j = 1, \ldots, m$, $V_{x_j} = \{v_j^i : x_j \in S_i\}$ and $\mathcal{V} = (V_{x_1}, \ldots, V_{x_m})$. This clearly gives us an instance of Sel-Col in a complete q-partite graph G (see Fig. 3 for an example).

Let $\mathcal{S}^* \subseteq \mathcal{S}$ be an optimal solution for instance $\mathcal{I}$ of value $OPT(\mathcal{I})$. For each $x_j \in X, j = 1, \ldots, m$, let $f(x_j) \in \{1, \ldots, n\}$ such that $x_j \in S_{f(x_j)} \in \mathcal{S}^*$. Let $V^* = \{v_j^{f(x_j)} : x_j \in X\}$. Obviously, V^* is such that $|V^* \cap V_{x_j}| \geq 1$ for all $j \in \{1, \ldots, m\}$. If necessary, we delete some vertices in V^* such that $|V^* \cap V_{x_j}| = 1$ for all $j \in \{1, \ldots, m\}$. Thus V^* is a feasible solution of Sel-Col in G and $G[V^*]$ is $|\mathcal{S}^*| = OPT(\mathcal{I})$-colorable since G is a complete partite graph. Hence,

$$\chi_{SEL}(G, \mathcal{V}) \leq \chi(G[V^*]) = |\mathcal{S}^*| = OPT(\mathcal{I}) \tag{1}$$

Conversely, let V^* be a feasible solution for Sel-Col in G and set $\mathcal{S}' = \{S_i : V^* \cap L_i \neq \emptyset, i = 1, \ldots, n\}$. We claim that $\mathcal{S}'$ is a set cover of X and thus a feasible solution of Set Cover for $\mathcal{I}$. Indeed, for every $x_j \in X, j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, n\}$ such that $\{v_j^i\} = V^* \cap V_{x_j}$ since $|V^* \cap V_{x_\ell}| = 1$ for all $\ell \in \{1, \ldots, m\}$. It follows that $x_j \in S_i \in \mathcal{S}'$ and hence $\mathcal{S}'$ is a set cover of X. Finally, notice that $G[V^*]$ contains a clique K of size $|\mathcal{S}'|$ because G is a complete partite graph. Thus,

$$|\mathcal{S}'| = |K| \leq \chi(G[V^*]) \tag{2}$$

Since $|\mathcal{S}'| \geq OPT(\mathcal{I})$ and since inequalities (1) and (2) hold for any feasible solution V^* of Sel-Col, we may apply them to an optimal solution $V^{*'}$ of Sel-Col and obtain that $\chi_{SEL}(G, \mathcal{V}) = OPT(\mathcal{I})$. Furthermore, from any ρ-approximation for Sel-Col on G, we polynomially get a ρ-approximation for SET-COVER on $\mathcal{I}$.

The reduction to Set Cover is defined as follows: consider an instance of SEL-COL in a complete q-partite graph $G = (L_1, \ldots, L_q, E)$ with partition $V_1, \ldots, V_p$ of its vertex set; we will construct an instance $\mathcal{I}$ of Set Cover by setting $X = \{x_1, \ldots, x_p\}$ and defining the subsets $S_i, i = 1, \ldots, q$, as follows: $x_j \in S_i$ if and only if $V_j \cap L_i \neq \emptyset$ for $j = 1, \ldots, p$.

Consider an optimal solution V^* of Sel-Col for G with value $\chi_{SEL}(G)$. Then we clearly obtain a feasible solution of Set Cover for instance $\mathcal{I}$ by taking $\mathcal{S}' = \{S_i : L_i \cap V^* \neq \emptyset, i = 1, \ldots, q\}$. Thus

$$OPT(\mathcal{I}) \leq |\mathcal{S}'| = \chi_{SEL}(G, \mathcal{V}) \tag{3}$$

Now consider any set cover $\mathcal{S}' = \{S_{i_1}, \ldots, S_{i_r}\}$ of $\mathcal{I}$ with size $r = |\mathcal{S}'|$. Then we obtain a feasible solution of Sel-Col for G as follows: for $j = 1, \ldots, r$, we add the vertices of L_{i_j} to V^*; if necessary we delete some vertices of V^* such that $|V^* \cap V_i| = 1$ for $i = 1, \ldots, p$. This gives us a feasible solution of Sel-Col such that

$$\chi(G[V^*]) \leq |\mathcal{S}'| \tag{4}$$

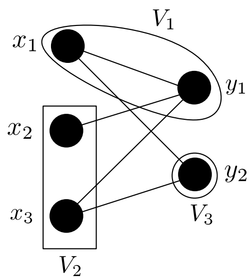

Fig. 4. An instance of Sel-Col for which we obtain $C_1^1 = x_1 \vee y_1$, $C_1^2 = \bar{x}_1 \vee \bar{x}_2$, $C_2^1 = x_2 \vee x_3$, $C_2^2 = \bar{x}_2 \vee \bar{x}_3$, $C_3 = y_2$, $C = \bar{x}_1 \vee \bar{y}_2$, $C' = \bar{x}_2 \vee \bar{y}_1$, $C'' = \bar{x}_3 \vee \bar{y}_2$, $C''' = \bar{x}_3 \vee \bar{y}_1$.

Using inequalities (3) and (4), we deduce that $\chi_{SEL}(G, \mathcal{V}) = OPT(\mathcal{I})$. Furthermore, from any ρ-approximation for Set Cover on $\mathcal{I}$, we polynomially get a ρ-approximation for Sel-Col in G. $\square$

Corollary 4.5. *Let $G = (L_1, \ldots, L_q, E)$ be a complete q-partite graph and let $V_1, \ldots, V_p$ be a partition of its vertex set. Then there exists a polynomial-time $H(\alpha(G))$-approximation algorithm for* Sel-Col*, where $H(r) = \sum_{i=1}^{r} \frac{1}{i}$, and there exists no $(1-\varepsilon)\log p$-approximation of* Sel-Col *for any $\varepsilon > 0$ unless $\mathcal{NP} \subset TIME(n^{O(\log\ \log\ n)})$.*

Proof. First, using the $H(\max_{i=1,\ldots,n}|S_i|)$-approximation of [6] for Set Cover, it follows from Theorem 4.4 that there exists a polynomial-time $H(\alpha(G))$-approximation for Sel-Col in complete q-partite graphs since $\alpha(G) = \max_{i=1,\ldots,n}|S_i|$ in the above construction. Furthermore, using the negative result of [10] for Set Cover, we conclude that there exists no $(1-\varepsilon)\log p$-approximation for Sel-Col in complete q-partite graphs since in the construction given in Theorem 4.4, we have $|X| = p$. $\square$

5. Bipartite graphs

In this section, we consider the class of bipartite graphs. Since for a bipartite graph $G = (V, E)$ we have $\chi(G) \leq 2$, it follows that the only interesting case for k-DSel-Col is when $k = 1$. Furthermore, it follows that if 1-DSel-Col is polynomial-time solvable, then the selective chromatic number can be determined in polynomial time.

First we obtain the following result for general bipartite graphs.

Theorem 5.1. Sel-Col *is polynomial-time solvable in bipartite graphs if the partition $\mathcal{V} = (V_1, \ldots, V_p)$ satisfies $|V_i| \leq 2$ for all $i \in \{1, \ldots, p\}$.*

Proof. We first check whether $\chi_{SEL}(G, \mathcal{V}) = 1$ by using a reduction to 2Sat which is known to be polynomial-time solvable (see [14]). Consider an instance $\mathcal{I}$ of Sel-Col, i.e, a bipartite graph $G = (V, E)$ and a partition $V_1, \ldots, V_p$ of V such that for all $i \in \{1, \ldots, p\}$ we have $|V_i| \leq 2$. We define an instance of 2Sat as follows: (i) with each vertex x we associate a variable x; (ii) with each cluster V_i, $i \in \{1, \ldots, p\}$, such that $V_i = \{x\}$, we associate a clause $C_i = x$; (iii) with each cluster V_i, $i \in \{1, \ldots, p\}$, such that $V_i = \{x, y\}$, we associate two clauses $C_i^1 = x \vee y$ and $C_i^2 = \bar{x} \vee \bar{y}$; (iv) with each edge $xy \in E$ such that x, y belong to different clusters, we associate a clause $C = \bar{x} \vee \bar{y}$. This clearly defines an instance $\mathcal{I}'$ of 2Sat (see Fig. 4 for an example).

Now suppose that $\mathcal{I}$ has a feasible solution of value 1. For all vertices that are in V^*, we set the corresponding variables to true. Thus all clauses associated with clusters are satisfied. Furthermore, since V^* is a stable set, it follows that all clauses associated with edges of G are satisfied as well. Thus $\mathcal{I}'$ is a yes-instance.

Conversely, suppose now that $\mathcal{I}'$ is a yes-instance. For all variables that are true, we add the corresponding vertices to V^*. Due to the definition of the clauses associated with the clusters and the edges, this clearly gives us a stable set V^* such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. Thus $\mathcal{I}'$ has a feasible solution of value 1.

Now, suppose that by applying the above reduction we conclude that $\chi_{SEL}(G, \mathcal{V}) > 1$. Then we arbitrarily choose one vertex in every cluster and add it to V^*. Clearly $G[V^*]$ is bipartite and thus it is 2-colorable. Hence $\chi_{SEL}(G, \mathcal{V}) = 2$. $\square$

Next we consider graphs which are the disjoint union of C_4's or the disjoint union of P_3's. We obtain the following.

Theorem 5.2. 1-DSel-Col *is $\mathcal{NP}$-complete for the disjoint union of C_4's even if the partition $\mathcal{V} = (V_1, \ldots, V_p)$ satisfies $|V_i| = 3$ for all $i \in \{1, \ldots, p\}$.*

Proof. We use a reduction from $(3, B2)$-Sat which was shown to be $\mathcal{NP}$-complete in [4]. This problem is defined as follows: we are given a set of clauses each of which contains exactly three literals and every literal appears exactly two times; then we want to decide whether there exists a truth assignment such that each clause contains at least one true literal.

Consider an instance $\mathcal{I}$ of $(3, B2)$-Sat consisting of p clauses and n variables. With each variable x, we associate a C_4 with edge set $\{x_1\bar{x}_1, \bar{x}_1x_2, x_2\bar{x}_2, \bar{x}_2x_1\}$. This clearly gives us a graph $G = (V, E)$ isomorphic to nC_4. Then we define a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V as follows: for each variable x, if it appears as a positive literal in a clause C_i, we add x_1 or x_2 to V_i;

if it appears as a negative literal in a clause C_i, we add $\bar{x}_1$ or $\bar{x}_2$ to V_i. Since each clause contains exactly three literals and since each literal appears exactly two times, this clearly gives us a partition $\mathcal{V} = (V_1, \ldots, V_p)$ of V such that $|V_i| = 3$ for all $i \in \{1, \ldots, p\}$. We set $k = 1$. Thus we obtain an instance $\mathcal{I}'$ of 1-DSEL-COL.

Now suppose that $\mathcal{I}$ is a yes-instance. For every variable x, if x is true, then we add x_1 and x_2 into V^*; if x is false we add $\bar{x}_1$ and $\bar{x}_2$ into V^*. Since every clause contains at least one true literal, it follows that $|V^* \cap V_i| \geq 1$ for all $i \in \{1, \ldots, p\}$. If necessary we delete some vertices from V^* such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. Clearly V^* is a stable set. Thus $\mathcal{I}'$ is a yes-instance.

Conversely, suppose that $\mathcal{I}'$ is a yes-instance. For every set V_i, $i \in \{1, \ldots, p\}$, if $V^* \cap V_i = \{x_j\}$, for $j \in \{1, 2\}$, we set the corresponding variable x to true; if $V^* \cap V_i = \{\bar{x}_j\}$, for $j \in \{1, 2\}$, we set the corresponding variable x to false. Now recall that each vertex in a set V_i corresponds to a literal appearing in clause C_i. Thus by setting to true each literal corresponding to a vertex of $V_i \cap V^*$, for $i = 1, 2, \ldots, p$, we obtain that clause C_i contains one true literal and hence $\mathcal{I}$ is a yes-instance. □

It follows from Theorem 5.2 that deciding whether $\chi_{SEL}(G, \mathcal{V}) = 1$ or $\chi_{SEL}(G, \mathcal{V}) = 2$ is $\mathcal{NP}$-complete if G is the disjoint union of C_4's and its vertex partition $\mathcal{V}$ satisfies $|V_i| = 3$ for all $i \in \{1, \ldots, p\}$. The next result shows that if the clusters of a partition $\mathcal{V}$ in such a graph satisfy $|V_i| \geq 4$ for all $i \in \{1, \ldots, p\}$, then we always have $\chi_{SEL}(G, \mathcal{V}) = 1$ and thus SEL-COL becomes polynomial-time solvable.

Theorem 5.3. *Let $G = (V, E)$ be the disjoint union of C_4's and let $\mathcal{V} = (V_1, \ldots, V_p)$ be a partition of V satisfying $|V_i| \geq 4$ for all $i \in \{1, \ldots, p\}$. Then $\chi_{SEL}(G, \mathcal{V}) = 1$.*

Proof. Consider an instance of SEL-COL in a disjoint union of n cycles C_4 and let $\mathcal{V} = (V_1, \ldots, V_p)$ be a partition of its vertex set satisfying $|V_i| \geq 4$ for all $i \in \{1, \ldots, p\}$. Denote the cycles by $C_4^1, \ldots, C_4^n$. We will construct the following auxiliary bipartite multigraph $H = (X, Y, E)$: we associate with every cluster V_i, for $i \in \{1, \ldots, p\}$, a vertex x_i ($\rightarrow$ set X); we associate with every cycle C_4^j, $j \in \{1, \ldots, n\}$, a vertex y_j ($\rightarrow$ set Y); finally for every vertex $u \in V_i \cap V(C_4^j)$, $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, n\}$, we add an edge between x_i and y_j.

Notice that since $|V_i| \geq 4$ for all $i \in \{1, \ldots, p\}$, it follows that $p \leq n$. Thus $|X| \leq |Y|$. Furthermore, notice that $d(x_i) \geq 4$ and $d(y_j) = 4$, for $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, n\}$. Thus $\min_{x_i \in X} d(x_i) \geq \max_{y_j \in Y} d(y_j)$. Using a result of [3], we conclude that there exists a matching M in H which saturates X.

To finish the proof, we will show that a matching M in H which saturates X corresponds to a set $V^* \subseteq V$ satisfying $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$ and such that $\chi(G[V^*]) = 1$.

Consider a matching M in H which saturates X. For every edge $x_i y_j \in M$, $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$, we add a vertex of $V_i \cap V(C_4^j)$ to V^*. Since M is a matching which saturates X, it follows that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. Furthermore, since M is a matching, we have that $|V(C_4^j) \cap V^*| \leq 1$. Hence $\chi(G[V^*]) = 1$. □

Theorem 5.4. 1-DSEL-COL *is $\mathcal{NP}$-complete for the disjoint union of P_3's even if the partition $\mathcal{V} = (V_1, \ldots, V_p)$ satisfies $2 \leq |V_i| \leq 3$ for all $i \in \{1, \ldots, p\}$.*

Proof. Consider the problem (2, 1)-3SAT which is defined as follows: we are given a set U of variables as well as a set C of clauses over U such that each clause contains either two or three literals; furthermore each variable occurs exactly three times, once as a negative literal and twice as a positive literal; we want to decide whether there exists a truth assignment such that each clause contains at least one true literal. (2, 1)-3SAT was shown to be $\mathcal{NP}$-complete in [9].

Now we use a reduction from (2, 1)-3SAT. Consider an instance $\mathcal{I}$ of (2, 1)-3SAT containing n variables and p clauses. We construct the following graph $G = (V, E)$: with each variable x we associate a path P_3 with edge set $\{x\bar{x}, \bar{x}x'\}$. Thus G is isomorphic to nP_3. Now consider the following partition $\mathcal{V}$ of V: for each clause C_i, if the variable x appears in C_i as a positive literal, then we add x or x' to V_i; if the variable x appears a negative literal in C_i, then we add $\bar{x}$ to V_i. This clearly gives us a partition $V_1, \ldots, V_p$ with $2 \leq |V_i| \leq 3$ for all $i \in \{1, \ldots, p\}$. We set $k = 1$. Thus we obtain an instance $\mathcal{I}'$ of 1-DSEL-COL.

Now suppose that $\mathcal{I}$ is a yes-instance. For each variable x which is true, we add x, x' to V^*. For each variable x which is false, we add $\bar{x}$ to V^*. Thus we obtain a stable set V^* such that $|V^* \cap V_i| \geq 1$ for all $i \in \{1, \ldots, p\}$. If necessary we delete some vertices in V^* such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. Thus $\mathcal{I}'$ is a yes-instance.

Conversely, suppose now that $\mathcal{I}'$ is a yes-instance. We proceed as follows: for each path P_3, if $x \in V^*$ or $x' \in V^*$ we set the variable x to true; if $\bar{x} \in V^*$, then we set x to false. Since each cluster V_i, $i \in \{1, \ldots, p\}$, corresponds to a clause C_i, it follows that $\mathcal{I}$ is a yes-instance. □

Notice that the result given in Theorem 5.4 is best possible in the sense that if G is the disjoint union of P_2's, then SEL-COL is polynomial-time solvable (see Theorem 7.2).

Now using similar arguments as in the proof of Theorem 5.3, we obtain the following result.

Theorem 5.5. *Let $G = (V, E)$ be the disjoint union of P_3's and let $\mathcal{V} = (V_1, \ldots, V_p)$ be a partition of V satisfying $|V_i| \geq 3$ for all $i \in \{1, \ldots, p\}$. Then $\chi_{SEL}(G, \mathcal{V}) = 1$.*

From Theorem 5.4 we obtain the following.

Corollary 5.6. 1-DSEL-COL *is* $\mathcal{NP}$*-complete for paths even if the partition* $\mathcal{V} = (V_1, \dots, V_p)$ *satisfies* $2 \le |V_i| \le 3$ *for all* $i \in \{1, \dots, p\}$.

Proof. We use a reduction from 1-DSEL-COL for the union of P_3's which we previously showed to be $\mathcal{NP}$-complete even if the partition $V_1, \dots, V_p$ satisfies $2 \le |V_i| \le 3$ for all $i \in \{1, \dots, p\}$.

Consider the following instance $\mathcal{I}$ of 1-DSEL-COL. Let $G = (V, E)$ be isomorphic to nP_3 and let $\mathcal{V} = (V_1, \dots, V_p)$ be a partition of V satisfying $2 \le |V_i| \le 3$ for all $i \in \{1, \dots, p\}$. We denote by $P_3^1 = \{x_{11}x_{12}, x_{12}x_{13}\}, \dots, P_3^n = \{x_{n1}x_{n2}, x_{n2}x_{n3}\}$ the P_3's of G. We construct a path $P = (V', E')$ as follows. For $j = 1, \dots, n-1$, we add a path $\{y_{j1}y_{j2}, y_{j2}y_{j3}\}$ as well as the edges $x_{j3}y_{j1}, y_{j3}x_{(j+1)1}$. We obtain a partition $\mathcal{V}'$ of V' by using the sets $V_1, \dots, V_p$ as well as the sets $V_{p+1}, \dots, V_{p+n-1}$, where $V_{p+j} = \{y_{j1}, y_{j2}, y_{j3}\}$. This gives us an instance $\mathcal{I}'$ of 1-DSEL-COL.

Clearly if $\mathcal{I}'$ is a yes-instance, then $\mathcal{I}$ is a yes-instance.

Conversely, suppose now that $\mathcal{I}$ is a yes-instance. Let V^* be the stable set in a solution of $\mathcal{I}$. Then we clearly obtain a solution $V^{*'}$ of $\mathcal{I}'$ by adding to V^* the vertices $y_{12}, \dots, y_{(n-1)2}$. Thus $\mathcal{I}'$ is a yes-instance. $\square$

Applying similar arguments, we obtain the following result.

Corollary 5.7. 1-DSEL-COL *is* $\mathcal{NP}$*-complete for cycles even if the partition* $\mathcal{V} = (V_1, \dots, V_p)$ *satisfies* $2 \le |V_i| \le 3$ *for all* $i \in \{1, \dots, p\}$.

It follows from Corollaries 5.6 and 5.7, that SEL-COL cannot be approximated within a factor less than 2 in paths or cycles with clusters of size 2 or 3, unless $\mathcal{P} = \mathcal{NP}$.

6. Compact clustering

In this section, we consider the special case when every cluster V_i, $i \in \{1, \dots, p\}$, of the partition $\mathcal{V}$ induces a clique. We will say that $\mathcal{V}$ is a *compact clustering*. It can be immediately deduced from the definition of SEL-COL that the solution does not change if one changes edges between vertices of the same cluster and in particular if we replace each cluster by a clique. However this same argument cannot be applied if we consider specific graph classes not stable under edge adding operation. Therefore if we do not consider only general graphs the problem is not necessary equivalent in the case of compact clustering.

We are interested in this particular case for two main reasons, one theoretical and one dealing with potential applications. From the theoretical point of view, most of reductions pointing out $\mathcal{NP}$-hard cases involve instances of SEL-COL for which clusters are stable sets. The case when clusters are cliques then becomes natural. Moreover, in several applications, compact clustering corresponds to natural situations. Note first that, for the multicolored clique problem (MCC) (see Section 1), the instance of SEL-COL associated to any instance of MCC corresponds to a compact clustering. Considering now the scheduling problem mentioned in Section 1, a compact clustering corresponds to the case when, for each task t_j, all the possible intervals $I_1(j), \dots, I_{n_j}(j)$ share a common point; this is natural when a preferred starting time is determined with some flexibility represented by a collection of admissible intervals around this preferred time or when a specific event scheduled at a fixed date must occur during the execution of the task.

Theorem 6.1. *Let* $G = (V, E)$ *be a graph and let* $\mathcal{V} = (V_1, \dots, V_p)$ *be a compact clustering. Then* $\chi_{SEL}(G, \mathcal{V}) = 1$ *if and only if* $\alpha(G) = p$.

Proof. Suppose that $\alpha(G) = p$ and let S be a stable set in G of size p. Since $G[V_i]$ is a clique, for $i = 1, 2, \dots, p$, it follows that $|S \cap V_i| = 1$. It follows that S is a solution of SEL-COL in G with respect to $\mathcal{V}$ and thus $\chi_{SEL}(G, \mathcal{V}) = 1$.

Conversely, suppose that $\chi_{SEL}(G, \mathcal{V}) = 1$. Thus there exists a stable set S in G such that $|S \cap V_i| = 1$ for all $i \in \{1, \dots, p\}$. It follows that $|S| = p$. Since $G[V_i]$ is a clique, for $i = 1, 2, \dots, p$, there exists no stable set S' in G such that $|S \cap V_i| \ge 2$ for some $i \in \{1, \dots, p\}$. Hence $\alpha(G) = p$. $\square$

Let us denote by $\mathcal{STAB}$ the class of graphs G for which the stability number $\alpha(G)$ can be determined in polynomial time. Then the following is an immediate consequence of Theorem 6.1.

Corollary 6.2. *Let* $G \in \mathcal{STAB}$ *and let* $\mathcal{V} = (V_1, \dots, V_p)$ *be a compact clustering. Then* 1-DSEL-COL *is polynomial-time solvable.*

Theorem 6.3. 1-DSEL-COL *is* $\mathcal{NP}$*-complete in planar graphs of maximum degree* 3 *when* $\mathcal{V} = (V_1, \dots, V_p)$ *is a compact clustering and* $|V_i| \le 3$ *for all* $i \in \{1, \dots, p\}$.

Proof. We use a reduction from RESTRICTED PLANAR 3-SAT which is defined as follows: we are given a set U of variables as well as a set C of clauses over U such that each clause contains either two or three literals; furthermore each variable occurs exactly three times, once as a negative literal and twice as a positive literal; finally the bipartite graph $H = (U \cup V, E)$, where $uc \in E$ if the variable corresponding to u appears (as positive or negative literal) in the clause corresponding to c, is

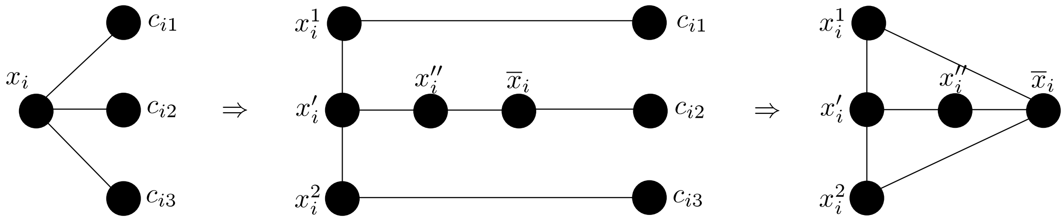

Fig. 5. Replacement of x_i in H.

planar; we want to decide whether there exists a truth assignment such that each clause contains at least one true literal. Restricted Planar 3-Sat was shown to be $\mathcal{NP}$-complete in [9].

Let $\mathcal{I}$ be an instance of Restricted Planar 3-Sat with variables $x_1, \ldots, x_n$ and clauses $c_1, \ldots, c_m$. Consider the associated planar bipartite graph $H = (U \cup C, E)$. Notice that every vertex in U has degree exactly three. Consider a vertex $x_i \in U$ (corresponding to variable x_i) as well as its neighbors $c_{i1}, c_{i2}, c_{i3} \in C$ (corresponding to the clauses in which x_i appears). Suppose that x_i appears as a negative literal in c_{i2} (and hence it appears as positive literal in c_{i1} and in c_{i3}). We delete x_i and replace it by the graph H_i with vertex set $\{x_i^1, x_i', x_i'', \overline{x_i}, x_i^2\}$ and edge set $\{x_i^1x_i', x_i'x_i^2, x_i'x_i'', x_i''\overline{x_i}\}$; then we make c_{i1} adjacent to x_i^1, c_{i2} adjacent to $\overline{x_i}$ and c_{i3} adjacent to x_i^2 (see Fig. 5). We do this for every vertex $x_i \in U$. Clearly the resulting graph is still planar and has maximum degree 3. Finally, we delete every vertex $c_j \in C$ and make its three neighbors pairwise adjacent. This can clearly be done in such a way that the resulting graph $G = (V, E')$ is still planar. Notice that G has still maximum degree 3. We define a partition $\mathcal{V} = (V_1, \ldots, V_{m+n})$ of V by adding to V_i, $i = 1, \ldots, m$, the vertices in V representing the literals occurring in clause c_i, furthermore for every variable x_i we define a cluster $V_{m+i} = \{x_i', x_i''\}$. Thus every cluster V_i induces a clique (of size 2 or 3), $i = 1, \ldots, m+n$, and hence we obtain an instance $\mathcal{I}'$ of 1-DSel-Col in a planar graph with maximum degree 3 with a compact clustering $\mathcal{V}$ such that $|V_i| \leq 3$ for all $i \in \{1, \ldots, m+n\}$.

Now suppose that $\mathcal{I}$ is a yes-instance. For every variable x_i which is set to true, we add the corresponding vertices x_i^1, x_i^2 to V^*; similarly for every variable x_i which is set to false, we add the corresponding vertex $\overline{x_i}$ to V^*. Furthermore, if x_i is set to true, we add x_i'' to V^*; otherwise, if x_i is set to false we add x_i' to V^*. From the construction of G, it follows that $|V^* \cap V_i| \geq 1$ for $i = 1, \ldots, m+n$. If necessary we may delete some vertices from V^* to obtain $|V^* \cap V_i| = 1$ for $i = 1, \ldots, m+n$. This implies that V^* is necessarily a stable set. Thus $\mathcal{I}'$ is a yes-instance.

Conversely, suppose now that $\mathcal{I}'$ is a yes-instance. Let V^* be a solution of $\mathcal{I}'$, i.e., V^* is a stable set such that $|V^* \cap V_i| = 1$ for all $i \in \{1, \ldots, m+n\}$. We obtain a solution of $\mathcal{I}$ as follows: for $i = 1, \ldots, m$, if $V^* \cap V_i = \{x_{i1}\}$ (or $V^* \cap V_i = \{x_{i2}\}$) then we set the corresponding variable x_i to true; otherwise, if $V^* \cap V_i = \{\overline{x_i}\}$, then we set the corresponding variable x_i to false. All remaining variables are arbitrarily set to true or false. From the construction of G and the fact that V^* is a stable set, it follows that this gives us a feasible truth assignment and furthermore since every V_i corresponds to a clause c_i, for $i = 1, \ldots, m$, it follows that every clause contains at least one true literal. Thus $\mathcal{I}$ is a yes-instance. □

Theorem 6.4. *Let $G = (V, E)$ be a graph with maximum degree 3 and a compact clustering $\mathcal{V} = (V_1, \ldots, V_p)$ such that $|V_i| \leq 3$. Then there exists a polynomial-time 2-approximation algorithm for* Sel-Col *in G.*

Proof. Consider an instance $\mathcal{I}$ of Sel-Col consisting of a graph $G = (V, E)$ with maximum degree 3 and a compact clustering $\mathcal{V} = (V_1, \ldots, V_p)$ such that $|V_i| \leq 3$. Consider the graph G' induced by all vertices belonging to a cluster V_i with $|V_i| \leq 2$, for $i \in \{1, \ldots, p\}$. We check whether there exists a stable set S in G' which contains exactly one vertex of each cluster in G'. This can be done in polynomial time (see Corollary 7.1).

If such a stable set does not exist, then the optimal solution of Sel-Col for $\mathcal{I}$, $OPT(\mathcal{I})$ must satisfy $OPT(\mathcal{I}) \geq 2$. Since G has maximum degree 3, it follows that we can color the vertices of G with at most 4 colors in polynomial time by assigning greedily the first available color to each vertex. Thus we can obtain a set V^* in polynomial time such that $\chi(G[V^*]) \leq 4 \leq 2 \cdot OPT(\mathcal{I})$.

If such a stable set S exists, then we add all vertices of S to V^* and proceed as follows: for each cluster V_i in G such that $|V_i| = 3$, we arbitrarily choose one vertex in V_i and add it to V^*. We claim that every connected component of $G[V^*]$ is isomorphic to one of the following graphs: $K_1, K_2, P_3, K_{1,3}$. Indeed, suppose that a connected component of $G[V^*]$ contains a triangle on vertices a, b, c. Since S is a stable set it follows that at least two of the vertices a, b, c belong to a cluster of size 3. Without loss of generality, we may assume that a and b belong both to such a cluster. Since we chose exactly one vertex in each such cluster, it follows that $a \in V_i, b \in V_j$ with $i \neq j$ and $|V_i| = |V_j| = 3$. Furthermore, for the same reason, $c \notin V_i, V_j$. But this implies that a and b must have degree at least 4, a contradiction. We conclude that no connected component contains a triangle. Finally, suppose that a connected component of $G[V^*]$ contains a path (not necessarily induced) on 4 vertices, say a, b, c, d with edges ab, bc, cd. First notice that b cannot belong to a cluster V_i with $|V_i| = 3$. Indeed, since G has maximum degree 3, this would imply that either a or c belong to V_i as well, a contradiction because we chose exactly one vertex in each cluster of size 3. By symmetry the same holds for c. But this implies each of b, c belongs to a cluster of size at most two. This contradicts the fact that S exists. Thus no connected component of $G[V^*]$ contains a path on 4 vertices. Hence every connected component of $G[V^*]$ must be isomorphic to one of the following graphs: $K_1, K_2, P_3, K_{1,3}$. This implies that $\chi(G[V^*]) \leq 2 \leq 2 \cdot OPT(\mathcal{I})$. □

7. Further results

Using a similar approach as for Theorem 5.1, we obtain the following.

Corollary 7.1. 1-DSEL-COL *is polynomial-time solvable if the partition* $\mathcal{V} = (V_1, \dots, V_p)$ *satisfies* $|V_i| \leq 2$ *for all* $i \in \{1, \dots, p\}$.

Next we will consider the disjoint union of cliques. Let G be the disjoint union of n cliques $K^1, \dots, K^q$ and let $\mathcal{V} = (V_1, \dots, V_p)$ be a partition of its vertex set. Notice that since for every two vertices u, v belonging to a same clique we have $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, we may assume that $|V_i \cap K^j| \leq 1$ for all $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Furthermore notice that we have $1 \leq \chi_{SEL}(G, \mathcal{V}) \leq \max_{j=1,\dots,q}\{|K^j|\}$. We obtain the following.

Theorem 7.2. SEL-COL *is polynomial-time solvable for the disjoint union of cliques.*

Proof. Consider an instance $\mathcal{I}$ of SEL-COL in a graph $G = (V, E)$ which is the disjoint union of n cliques $K^1, \dots, K^q$ and let $\mathcal{V} = (V_1, \dots, V_p)$ be a partition of V. As mentioned above, we may assume that $|V_i \cap K^j| \leq 1$ for all $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, n\}$.

We use a reduction to the MAXIMUM FLOW problem which can be solved in polynomial time (see [1]). We construct the following network N: with every cluster $V_i, i \in \{1, \dots, p\}$, we associate a vertex x_i; with every clique $K^j, j \in \{1, \dots, q\}$, we associate a vertex y_j; we add a vertex s and arcs (s, x_i) for $i \in \{1, \dots, p\}$ as well as a vertex t and arcs (y_j, t) for $j \in \{1, \dots, q\}$; for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ we add an arc (x_i, y_j) if and only if $V_i \cap K^j \neq \emptyset$; finally we assign a capacity of one to all arcs $(s, x_i), i \in \{1, \dots, p\}$, and to all arcs (x_i, y_j) for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, and a capacity of $k \geq 1$ to all arcs (y_j, t) for $j \in \{1 \dots, q\}$. This clearly gives us an instance $\mathcal{I}'$ of the MAXIMUM FLOW problem.

Now suppose that $\mathcal{I}$ has a solution of value $s \leq k$ and let V^* be a set of vertices in G such that $|V^* \cap V_i| = 1$ for $i \in \{1, \dots, p\}$ and such that $G[V^*]$ is s-colorable. We will show that there exists a flow of value p in N. For every vertex $v \in V^*$ we proceed as follows: if $v \in V_i \cap K^j, i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, then we add a unit of flow on the path $\{sx_i, x_iy_j, y_jt\}$. Since $G[V^*]$ is s-colorable, with $s \leq k$, it follows that $|V^* \cap K^j| \leq k$ for every $j \in \{1, \dots, q\}$. Thus we obtain a flow of value p.

Conversely, assume that there exists a flow in N of value p. Thus every arc (s, x_i) is used by one flow unit for $i \in \{1, \dots, p\}$. We construct a solution of $\mathcal{I}$ as follows: for every arc (x_i, y_j) used by one flow unit, we add the vertex $V_i \cap K^j$ to V^*. Since the arcs (y_j, t), for $j \in \{1, \dots, q\}$, all have capacity k, it follows that $|V^* \cap K^j| \leq k$. Thus we obtain a set V^* such that $|V^* \cap V_i| = 1$ for $i \in \{1, \dots, p\}$ and such that $|V^* \cap K^j| \leq k$ for $j \in \{1 \dots, q\}$. It follows that $G[V^*]$ is k-colorable.

The above shows that SEL-COL has a solution of value $s \leq k$ if and only if there exists a maximum flow of value p in N. Since $1 \leq \chi_{SEL}(G, \mathcal{V}) \leq \max_{j=1\dots,q}\{|K^j|\}$ for the disjoint union of cliques, it follows that by taking $k = 1, \dots, \max_{j=1\dots,q}\{|K^j|\}$, we can determine the selective chromatic number of G in polynomial time. □

Next, we consider graphs which have stability number at most 2. Clearly for such graphs $G = (V, E)$ we have $\lceil \frac{p}{2} \rceil \leq \chi_{SEL}(G, \mathcal{V}) \leq p$, for any partition $\mathcal{V}$ of V. We obtain the following.

Theorem 7.3. SEL-COL *is polynomial-time solvable for graphs with stability number at most* 2.

Proof. Consider an instance $\mathcal{I}$ of SEL-COL in a graph $G = (V, E)$ with stability number at most 2. If G is a clique, then clearly $\chi_{SEL}(G, \mathcal{V}) = p$. Thus we may assume that G is not a clique and hence has stability number exactly 2.

We use a reduction to the MAXIMUM MATCHING problem which is polynomial-time solvable (see for instance [18]). We will build the following auxiliary graph $H = (V_H, E_H)$: with every set $V_i, i \in \{1, \dots, p\}$ we associate a vertex v_i; we add an edge between two vertices $v_i, v_j, i, j \in \{1, \dots, p\}$, if there exists two nonadjacent vertices $u \in V_i$ and $w \in V_j$. This gives us an instance $\mathcal{I}'$ of MAXIMUM MATCHING.

First assume that $\mathcal{I}$ has a feasible solution of value $p - k_1$, for $0 \leq k_1 \leq \lfloor \frac{p}{2} \rfloor$ and let c be a selective $(p - k_1)$-coloring of G. Notice that since G has stability number two, every color class has size at most 2. Thus k_1 is the number of color classes having size exactly 2. We build a matching M in H as follows. For every pair $u, w \in V^*$ such that $c(u) = c(w), u \in V_i, w \in V_j$, for $i, j \in \{1, \dots, p\}$ with $i \neq j$, add the edge v_iv_j to M. This gives us a feasible solution of size k_1 for instance $\mathcal{I}'$.

Conversely, suppose that $\mathcal{I}'$ has a feasible solution M of size k_1. Then we obtain a feasible solution of value $p - k_1$ for $\mathcal{I}$ as follows. For every edge $v_iv_j \in M$, we color the corresponding nonadjacent vertices $u \in V_i$ and $w \in V_j$ with a same color c_{ij}. Thus there remain $p - k_1$ sets of the partition not having any colored vertex yet. We arbitrarily choose one vertex in each of these sets and color it with a new color. Thus we obtain a feasible selective $(p - k_1)$-coloring of G. □

Let us finally conclude with a *Log-APX* result for SEL-COL in the case when all clusters have the same size. We denote by MAX k-STABLE the problem consisting in finding a k-colorable induced subgraph of maximum size in a given graph G. In the following, we suppose that there exists an approximation algorithm for this problem, where k is supposed to be part of the input. A solution of this algorithm will be referred to as an approximated k-Stable.

Theorem 7.4. *Consider an hereditary class of graphs $\mathcal{H}$ for which* Max k-Stable *can be approximated within ρ. Let $G = (V, E) \in \mathcal{H}$ and let $\mathcal{V}$ a partition of V such that each cluster has the same size $\upsilon \geq 2$. Then* Sel-Col *can be approximated within $(-\frac{\log(n)}{\log(1-\rho/\upsilon)})$, where n is the number of vertices in G.*

Algorithm 1

Require: $G = (V, E)$ is a graph in $\mathcal{H}$ and $\mathcal{V} = V_1, \dots, V_p$ a partition of V s.t. $|V_i| = \upsilon \geq 2, i = 1, \dots, p$

for $k = 1, \dots, p$ **do**
  $N_k \leftarrow 0$; $V^k \leftarrow \emptyset$
  repeat
    compute an approximated k-Stable $V' = S_1 \cup \cdots \cup S_k$
    for each cluster V_i such that $V_i \cap V' \neq \emptyset$, select one vertex in $V_i \cap V'$ and add it to V^k
    for each $i = 1, \dots, k$ such that $V^k \cap S_i \neq \emptyset$, let $N_k \leftarrow N_k + 1$ and keep $V^k \cap S_i$ as a new color
    remove from V all clusters intersecting V'
  until $V = \emptyset$
  let C_k be the resulting selective coloring of $G[V^k]$
end for
Select the solution (V^{k_0}, C_{k_0}) where $k_0 \in \text{argmin}_{k=1,\dots,p}(N_k)$

Proof. Consider Algorithm 1 shown above. Its complexity is $pnC(n)$ where $C(n)$ is the complexity of the approximation algorithm for Max k-Stable.

For each $k = 1, \dots, p$, the algorithm computes successively k stable sets and removes the clusters that are covered by at least one of these sets. Consequently, it computes $N_k \leq kR_k$ stable sets covering all clusters, where R_k is the number of iterations in the **Repeat**-loop corresponding to k. Then it selects the solution that minimizes N_k, $k = 1, \dots, p$. Let us suppose $k_0 = \chi_{SEL}(G, \mathcal{V})$. Of course $k_0 \leq p$ and consequently the solution computed is at least as good as the solution computed during the **Repeat**-loop associated to k_0. Consequently, the related ratio is not more than R_{k_0}.

Let us consider the **Repeat**-loop associated to k_0. We have $n = |V| = p\upsilon$ and a maximum k_0-Stable of G is of size at least p. Consequently the approximated k_0-Stable contains at least ρp vertices and since all the covered clusters are removed, the remaining graph contains at most $p\upsilon(1 - \rho/\upsilon)$ vertices distributed among the $p(1 - \rho/\upsilon)$ remaining clusters. Since $\mathcal{H}$ is a hereditary class, the approximation algorithm stays valid in the remaining graph and consequently the same argument can be repeated to justify that, after ℓ iterations of the **Repeat**-loop, the number of remaining vertices is $p\upsilon(1 - \rho/\upsilon)^\ell = n(1 - \rho/\upsilon)^\ell$. After $R_{k_0} - 1$ iterations, there are at least υ remaining vertices and consequently we have:

$$n(1 - \rho/\upsilon)^{R_{k_0}-1} \geq \upsilon$$

implying

$$R_{k_0} \leq 1 - \frac{\log(n/\upsilon)}{\log(1 - \rho/\upsilon)} \leq -\frac{\log(n)}{\log(1 - \rho/\upsilon)}$$

where the last inequality holds because $\upsilon \geq 1 + \rho$. This completes the proof. □

Notice that we assumed that $\upsilon \geq 2$ since if $\upsilon = 1$, the problem corresponds to the usual coloring problem and in this case the ratio is $1 - \frac{\log(n)}{\log(1-\rho)}$.

Consider now the case when $\mathcal{H}$ is the class of interval graphs for which Max k-Stable is known to be polynomial [29] or the class of chordal graphs for which there exists a $\frac{1}{2}$-approximation algorithm [5].

Corollary 7.5. *Let $G = (V, E)$ be an interval graph with a partition $\mathcal{V}$ of V in which each cluster has the same size $\upsilon \geq 2$. Then* Sel-Col *can be approximated within*

$$-\frac{\log(n)}{\log(1 - 1/\upsilon)}.$$

Let $G = (V, E)$ be a chordal graph with a partition $\mathcal{V}$ of V in which each cluster has the same size υ. Then Sel-Col *can be approximated within*

$$-\frac{\log(n)}{\log(1 - 1/(2\upsilon))}.$$

8. Conclusion

In this paper, we considered the selective graph coloring problem and analyzed its computational complexity in various classes of graphs. Our results are summarized in Fig. 6.

We also proposed some first approximation results. It would be interesting to consider the selective graph coloring problem in graphs where the partition satisfies some specific constraints. In Section 6 we started such an approach but many other configurations are still to be analyzed.

Graph class	$\lvert V_i \rvert$ $i = 1, \dots, p$	SEL-COL	k-DSEL-COL	
split graphs	≤ 2	$\mathcal{NP}$-hard	$\mathcal{P}$	Theorem 3.1 Corollary 3.3
threshold graphs		$\mathcal{P}$	$\mathcal{P}$	Remark 3.1
complete q-partite graphs			$\mathcal{P}$	Theorem 4.2
complete q-partite graphs q fixed		$\mathcal{P}$		Theorem 4.1
complete q-partite graphs $\lvert L_j \rvert = 3, j = 1, \dots, n$	$= 2$	$\mathcal{NP}$-hard		Theorem 4.3
bipartite graphs	≤ 2	$\mathcal{P}$	$\mathcal{P}$	Theorem 5.1
nC_4	$= 3$	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete, $k = 1$	Theorem 5.2
nC_4	≥ 4	$\mathcal{P}$	$\mathcal{P}$	Theorem 5.3
nP_3	$2 \leq \ \leq 3$	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete, $k = 1$	Theorem 5.4
nP_3	≥ 3	$\mathcal{P}$	$\mathcal{P}$	Theorem 5.5
paths	$2 \leq \ \leq 3$	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete	Corollary 5.6
cycles	$2 \leq \ \leq 3$	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete	Corollary 5.7
	≤ 2		$\mathcal{P}, k = 1$	Corollary 7.1
disjoint union of cliques		$\mathcal{P}$	$\mathcal{P}$	Theorem 7.2
$\alpha(G) \leq 2$		$\mathcal{P}$	$\mathcal{P}$	Theorem 7.3
$\mathcal{STAB}$	compact		$\mathcal{P}$ for $k = 1$	Corollary 6.2
planar, $\Delta(G) \leq 3$	≤ 3 compact	$\mathcal{NP}$-hard	$\mathcal{NP}$-complete, $k = 1$	Theorem 6.3

Fig. 6. Complexity results for SEL-COL and k-DSEL-COL.

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