Theoretical Computer Science 939 (2023) 140–160

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Using edge contractions to reduce the semitotal domination number ☆

Esther Galby [a], Paloma T. Lima [b,*], Felix Mann [c], Bernard Ries [c]

[a] CISPA Helmholtz Center for Information Security, Saarbrücken, Germany
[b] IT University of Copenhagen, Copenhagen, Denmark
[c] University of Fribourg, Fribourg, Switzerland

ARTICLE INFO

Article history:
Received 24 September 2021
Received in revised form 12 October 2022
Accepted 16 October 2022
Available online 21 October 2022
Communicated by M. Zehavi

Keywords:
Blocker problem
Edge contraction
Semitotal domination

ABSTRACT

In this paper, we consider the problem of reducing the semitotal domination number of a given graph by contracting k edges, for some fixed $k \geq 1$. We show that this can always be done with at most 3 edge contractions and further characterise those graphs requiring 1, 2 or 3 edge contractions, respectively, to decrease their semitotal domination number. We then study the complexity of the problem for $k = 1$ and obtain in particular a complete complexity dichotomy for monogenic classes.

1. Introduction

In the standard graph modification problem, one is interested in modifying a given graph, using a minimum number of graph operations from a prescribed set, such that the resulting graph belongs to some fixed graph class. The related family of so-called *blocker problems* considers not a graph class, but rather asks for a specific graph parameter π to decrease: given a graph G, a set $\mathcal{O}$ of one or more graph operations and an integer $k \geq 1$, the question is whether G can be transformed into a graph G' by using at most k operations from $\mathcal{O}$ such that $\pi(G') \leq \pi(G) - d$ for some *threshold* $d \geq 1$. These types of problems (as well as the variants where one wants to increase some parameter π) are related to other well-known graph problems like such as Hadwiger Number, Club Contraction and Graph Transversal (see [6,22]). While the set $\mathcal{O}$ generally consists of a single operation (namely vertex deletion, edge deletion, edge addition or edge contraction), a variety of parameters have been considered in the literature, including the chromatic number [1,6,7,22,24], the stability number [2,23], the clique number [18,20], the matching number [3,27], domination-like parameters [8,9,13,21] and others [4,14,17,16,25,28].

In this paper, we focus on one particular graph operation, namely *edge contraction*. Contracting an edge uv in a graph G corresponds to deleting both vertices u and v from G and adding a new vertex which is made adjacent to every neighbour of u or v in the original graph G. We denote by $ct_\pi(G)$ the smallest integer k such that there is a set of k edges in $E(G)$ whose contraction yields a graph for which the value of the parameter π is strictly smaller than $\pi(G)$. As a parameter, we consider the *semitotal domination number* (denoted by γ_{t2}), which was first introduced in [12]: a *semitotal dominating set* of a graph G is a set $D \subseteq V(G)$ such that every vertex in $V(G) \setminus D$ has a neighbour in D (that is, D is dominating) and every

☆ Research supported by the European Research Council (ERC) consolidator grant No. 725978 SYSTEMATICGRAPH.
* Corresponding author.
E-mail addresses: esther.galby@cispa.saarland (E. Galby), palt@itu.dk (P.T. Lima), felix.mann@unifr.ch (F. Mann), bernard.ries@unifr.ch (B. Ries).

https://doi.org/10.1016/j.tcs.2022.10.020
0304-3975/

vertex $x \in D$ is at distance at most two from a vertex in $D \setminus \{x\}$. Note that, by definition, every semitotal dominating set has at least two vertices. The *semitotal domination number* of G is the size of a minimum semitotal dominating set in G. We are more precisely interested in the following problem with $\pi = \gamma_{t2}$ and $k \geq 1$.

k-Edge Contraction(π)

Instance: A graph G.
Question: Is $ct_\pi(G) \leq k$?

We show that for every graph G such that $\gamma_{t2}(G) \geq 3$, we have $ct_{\gamma_{t2}}(G) \leq 3$. We further characterise for every fixed $k \in \{1, 2, 3\}$, those graphs for which $ct_{\gamma_{t2}}(G) = k$ in terms of the structure of their semitotal dominating sets (see Fig. 1). We then determine the computational complexity of 1-Edge Contraction(γ_{t2}) for several graph classes, such as bipartite graphs and chordal graphs, as well as for every monogenic graph class (that is, graph classes defined by excluding one graph as an induced subgraph). From these results, we deduce in particular the following theorem.

Theorem 1. *1-Edge Contraction(γ_{t2}) is polynomial-time solvable when restricted to H-free graphs if H is an induced subgraph of $P_5 + tK_1$ with $t \geq 0$ or H is an induced subgraph of $P_3 + pK_2 + tK_1$ with $p, t \geq 0$, and NP-hard or coNP-hard otherwise.*

Related work. In [13], Huang and Xu considered for π the *domination number* (denoted by γ) and the *total domination number* (denoted by γ_t). A *total dominating set* of a graph is a set D of vertices such that every vertex of the graph has a neighbour in D. The *total domination number* of a graph is the size of a minimum total dominating set. They showed that for $\pi \in \{\gamma, \gamma_t\}$, $ct_\pi(G)$ is never greater than 3 and further characterised for every fixed $k \in \{1, 2, 3\}$, the graphs for which $ct_\pi(G) = k$ in terms of the structure of their (total) dominating sets. More specifically, they showed the following (see Section 2 for missing definitions).

Theorem 2 *([13]). For any graph G, the following holds.*

(i) $ct_\gamma(G) = 1$ *if and only if there exists a minimum dominating set in G that is not independent.*
(ii) $ct_\gamma(G) = 2$ *if and only if every minimum dominating set in G is independent and there exists a dominating set D in G of size* $\gamma(G) + 1$ *such that G[D] contains at least two edges.*

Theorem 3 *([13]). For any graph G, the following holds.*

(i) $ct_{\gamma_t}(G) = 1$ *if and only if there exists a minimum total dominating set D in G such that G[D] contains a* P_3.
(ii) $ct_{\gamma_t}(G) = 2$ *if and only if every minimum total dominating set in G induces a graph that does not contain a* P_3 *and there exists a total dominating set D in G of size* $\gamma_t(G) + 1$ *such that G[D] contains a subgraph isomorphic to* P_4, $K_{1,3}$ *or* $2P_3$.

The paper is organised as follows. In Section 2, we present the necessary definitions and notations. Section 3 is devoted to the proofs of our structural results, which will be used later in the paper. In Section 4, we consider different graph classes and determine the complexity of 1-Edge Contraction(γ_{t2}) when restricted to these classes. We then combine these results in Section 4.3 to prove our main result, that is, Theorem 1.

2. Preliminaries

Unless specified otherwise, we only consider finite and simple graphs. Furthermore, since, for any $k \geq 1$ and any disconnected graph G, G is a Yes-instance for k-Edge Contraction(γ_{t2}) if and only if one connected component of G is a Yes-instance for k-Edge Contraction(γ_{t2}), we only consider connected graphs. We refer the reader to [5] for any terminology not defined here.

For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. For a set $S \subseteq V(G)$, we let $G[S]$ denote the graph *induced* by S, that is, the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. For an edge $xy \in E(G)$, we denote by G/xy the graph obtained from G by contracting the edge xy. We say that two vertices x and y are *adjacent*, or *neighbours*, if xy is an edge. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set $\{w \in V(G) : vw \in E(G)\}$ and the *closed neighbourhood* $N_G[v]$ of v is the set $N(v) \cup \{v\}$ (if it is clear from the context, we may omit the subscript G). For any two vertices $v, w \in V(G)$, a vertex $u \in N(v) \cap N(w)$ is called a *common neighbour* of v and w. Given two sets $S, S' \subseteq V(G)$, we say that S is *complete* to S' if every vertex in S is adjacent to every vertex in S'. We say a vertex $v \in V(G)$ is complete to a set $S \subseteq V(G)$ if $\{v\}$ is complete to S. For any two vertices $x, y \in V(G)$, the *distance between x and y* is the number of edges in a shortest path from x to y and is denoted $d_G(x, y)$ (if it is clear from the context, we may omit the subscript G). For a set $S \subseteq V(G)$ and a vertex $v \in V(G)$, we define $d_G(v, S) = \min_{w \in S} d_G(v, w)$. A set $D \subseteq V(G)$ is a *dominating set* if every vertex $v \in V(G) \setminus D$ has a neighbour in D.

Let D be a dominating set of G and $w \in V(G) \setminus D$. For any neighbour $v \in D \cap N(w)$, we say that v *dominates* w. If $N(w) \cap D = \{v\}$, we say that w is a *private neighbour* of v. The set of all private neighbours of a vertex $v \in D$ is called the *private neighbourhood* of v. For any two vertices $v, w \in D$ which are at distance at most two, we say that v *witnesses* w, or that v is a *witness* of w. This terminology allows us to characterise a semitotal dominating set as a dominating set in which every vertex is witnessed by another vertex in the dominating set.

We denote by K_n, P_n and C_n the complete graph, the path, and the cycle on n vertices, respectively. We may also call K_3 a *triangle*. For a path P with endpoints x and y, we call any vertex in $V(P) \setminus \{x, y\}$ an *internal vertex* of P. The *claw* is the complete bipartite graph with partition sizes one and three. For a graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph. For a family of graphs $\mathcal{H}$, we say that a graph G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. A graph is called *chordal* if it is C_k-free for every $k \geq 4$. A graph class is called *hereditary* if it is closed under vertex deletion. Any hereditary graph class can be characterised by a set of forbidden induced subgraphs [15]. A hereditary graph class which can be characterised by a single forbidden induced graph is called *monogenic*.

3. Structural results

In this section, we present our structural results which will then be used in Section 4. These results are comparable to those obtained by Huang and Xu [13] for the domination and the total domination numbers. Observe that, by definition, $\gamma_{t2}(G) \geq 2$ for any graph G, which justifies the lower bound on the semitotal domination number in the following.

Theorem 4. *For any graph G with $\gamma_{t2}(G) \geq 3$, $ct_{\gamma_{t2}}(G) \leq 3$.*

Proof. Let G be a graph with $\gamma_{t2}(G) \geq 3$ and let D be a minimum semitotal dominating set of G. Consider $u, v \in D$ such that $d_G(u, v) \leq 2$, and let $w \in D \setminus \{u, v\}$ be a closest vertex to $\{u, v\}$, that is, $d_G(w, \{u, v\}) = d_G(D \setminus \{u, v\}, \{u, v\}) = \min_{x \in D \setminus \{u,v\}} d_G(x, \{u, v\})$. We claim that $d_G(w, \{u, v\}) \leq 3$. Indeed, suppose that $d_G(w, \{u, v\}) > 3$ and let x be the vertex at distance two from w on a shortest path from w to $\{u, v\}$. Then since x is nonadjacent to w, u and v, there exists a vertex $y \in D \setminus \{w, u, v\}$ adjacent to x for otherwise, x would not be dominated. But then, $d_G(y, \{u, v\}) < d_G(w, \{u, v\})$, a contradiction to the choice of w. Thus, $d_G(w, \{u, v\}) \leq 3$. Now assume, without loss of generality, that $d_G(w, \{u, v\}) = d_G(w, u)$ and let P be a shortest path from w to u. We claim that the graph G' obtained by contracting the edges of P has a semitotal domination number strictly smaller than that of G. Indeed, denote by v_P the vertex resulting from the contraction of the edges of P and let $D' = (D \setminus \{u, w\}) \cup \{v_P\}$. Then D' is a semitotal dominating set of G': indeed, every vertex $x \in V(G) \setminus V(P)$ adjacent to a vertex of P in G is adjacent to v_P in G', and $d_{G'}(v_P, v) \leq d_G(u, v) \leq 2$. Thus, $\gamma_{t2}(G') \leq |D'| = \gamma_{t2}(G) - 1$ and since P has length at most three, the lemma follows. $\square$

Next, we give necessary and sufficient conditions for $ct_{\gamma_{t2}}$ to be equal to one or two. Given a graph G, a *friendly triple* is a subset of three vertices x, y and z such that $xy \in E(G)$ and $d_G(y, z) \leq 2$. The *ST-configurations* correspond to the set of configurations depicted in Fig. 1.

Theorem 5. *For any graph G, the following holds.*

(i) *$ct_{\gamma_{t2}}(G) = 1$ if and only if there exists a minimum semitotal dominating set D of G such that D contains a friendly triple.*
(ii) *$ct_{\gamma_{t2}}(G) = 2$ if and only if no minimum semitotal dominating set of G contains a friendly triple and there exists a semitotal dominating set of size $\gamma_{t2}(G) + 1$ that contains an ST-configuration.*

Proof. Let G be a graph. To prove (i), let D be a minimum semitotal dominating set of G containing a friendly triple, that is, there is a subset of three vertices $x, y, z \in D$ such that $xy \in E(G)$ and $d_G(y, z) \leq 2$. Let G' be the graph obtained from

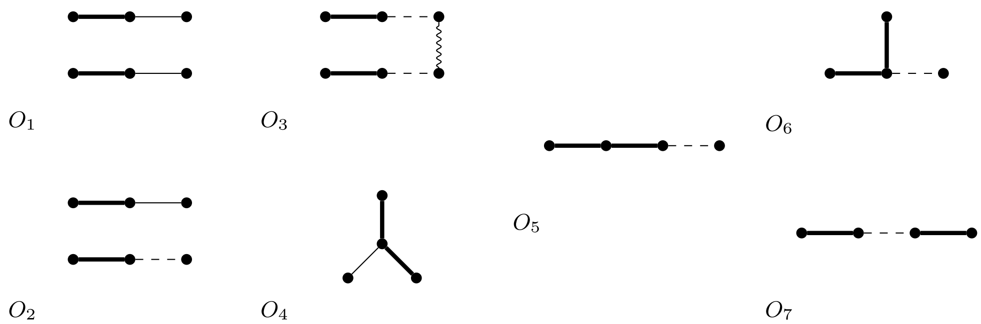

Fig. 1. The ST-configurations (the dashed lines indicate that the corresponding vertices are at distance 2 and the serpentine line indicates that the corresponding vertices may be the same vertex). The thick edges correspond to the edges to contract in the proof of Theorem 5(ii).

G by the contraction of the edge xy, and let v_{xy} be the vertex resulting from this contraction (note that $d_{G'}(z, v_{xy}) \leq 2$). Then it is easy to see that $(D \setminus \{x, y\}) \cup \{v_{xy}\}$ is a semitotal dominating set of G' of size $\gamma_{t2}(G) - 1$. Conversely, assume that G has an edge xy whose contraction decreases the semitotal domination number of G. Let G' and v_{xy} be the graph and the vertex obtained from this contraction, respectively. Let D' be a minimum semitotal dominating set of G' (note that $|D'| \leq \gamma_{t2}(G) - 1$). If $v_{xy} \in D'$, then there exists $z \in D'$ such that $d_{G'}(z, v_{xy}) \leq 2$; in particular, at least one vertex of $\{x, y\}$ is at distance at most two from z in G. It follows that $D = (D' \setminus \{v_{xy}\}) \cup \{x, y\}$ is a semitotal dominating set of G containing a friendly triple, namely x, y and z. Moreover, D is minimum since $|D'| \leq \gamma_{t2}(G) - 1$ and $|D| = |D'| + 1$. Now assume that $v_{xy} \notin D'$. Since v_{xy} is dominated in D', at least one vertex of $\{x, y\}$ is dominated by a vertex of D' in G, say x without loss of generality. Consider the set $D = D' \cup \{x\}$ in G (note that since $|D'| \leq \gamma_{t2}(G) - 1$, $|D| \leq \gamma_{t2}(G)$). Let us show that D is a semitotal dominating set of G. It is easy to see that D dominates every vertex of G and $|D| = \gamma_{t2}(G)$. It remains to show that every vertex of D has a witness. This holds for x: a witness for x is any vertex $z \in D'$ (thus, $z \in D$) that dominates v_{xy} in G' and is adjacent to x in G (such a vertex exists by the assumption that x is dominated in D'). Now consider a vertex $p \in D \setminus \{x\}$ (note that $p \in D'$) and let p' be a witness for p in D'. If $pp' \in E(G)$ or there exists a path pup' in G' with $u \neq v_{xy}$, then p' is still a witness for p in D. If a path of length at most two between p and p' in G' contains v_{xy} as an internal vertex, then $d_G(x, p) \leq 2$ and thus, x is a witness for p in D. Hence, every vertex in $D \setminus \{x\}$ has a witness and, thus, D is a semitotal dominating set of G. Finally, observe that D contains a friendly triple: indeed, denoting by w a witness for z in D', we have that $w \in D$ and, since $xz \in E(G)$, we conclude that $\{x, z, w\}$ is a friendly triple in D, which completes the proof of (i).

We now proceed to the proof of (ii). If no minimum semitotal dominating set of G contains a friendly triple then by (i), $ct_{\gamma_{t2}}(G) > 1$. Suppose that G has a semitotal dominating set S of size $\gamma_{t2}(G) + 1$ such that S contains an ST-configuration. It is straightforward to see that, for each configuration, the contraction of the two thick edges in Fig. 1 reduces the size of S by two. Moreover, after these contractions, S remains a semitotal dominating set of the resulting graph. Thus, we conclude that the contraction of two edges reduces the semitotal domination number of G and, so, $ct_{\gamma_{t2}}(G) = 2$.

For the other direction, let e and e' be two edges whose contraction decreases the semitotal domination number of G. In the remainder of this proof, we denote by G' the graph obtained from G by the contraction of the edges e and e' and by D' a minimum semitotal dominating set of G'. Note that $|D'| = \gamma_{t2}(G) - 1$ as $ct_{\gamma_{t2}}(G) > 1$ and the contraction of a single edge decreases the semitotal domination number of a graph by at most one. We start with the following observation that will be useful throughout the proof.

Observation 6. Let D be a semitotal dominating set of G. If D contains a (not necessarily induced) P_4, then D contains Configuration O_4 or Configuration O_6.

Indeed, let D be a semitotal dominating set of G containing a (not necessarily induced) P_4 on vertex set $\{a, b, c, d\}$ with $ab, bc, cd \in E(G)$. If $ac \in E(G)$, then $\{a, b, c, d\}$ contains O_4 in D since $ac, bc, cd \in E(G)$. Otherwise, $ac \notin E(G)$ in which case $d_G(a, c) = 2$ as b is a common neighbour of a and c. But then, $\{a, b, c, d\}$ forms an O_6 in D as $bc, cd \in E(G)$.

We now consider the following cases.

Case 1. *e and e′ share a vertex.* Let $e = xy$ and $e' = yz$ and let v_{xyz} be the vertex of G' resulting from the contraction of e and e'.

Case 1.1. $v_{xyz} \notin D'$. First note that, in this case, $D = D' \cup \{x, y\}$ is a semitotal dominating set of G (of size $\gamma_{t2}(G) + 1$). Indeed, D is a dominating set since D' is a dominating set of G' and y dominates z. Moreover, x is a witness for y (and vice versa). Now if there is a vertex p with witness p' in D' such that the unique path of length two connecting p to p' in G' contained v_{xyz}, then $d_G(p, y) \leq 2$ and thus, y is now a witness for p. Using similar arguments, we can show that $D' \cup \{y, z\}$ is also a semitotal dominating set of G.

Now since D' is a dominating set of G', at least one vertex of $\{x, y, z\}$ is dominated by D' in G. Suppose first that D' dominates x in G and consider the set $D = D' \cup \{x, y\}$. We next show that D contains an ST-configuration. Let $w_1 \in D'$ be a vertex that dominates x and let w_1' be a witness for w_1 in D'. If $d_G(w_1, w_1') = 2$, then $\{x, y, w_1, w_1'\}$ forms an O_5 in D. Otherwise, $d_G(w_1, w_1') = 1$ in which case D contains a P_4 on vertex set $\{y, x, w_1, w_1'\}$ and, so, by Observation 6, D contains an O_4 or an O_6. We conclude similarly in the case where D' dominates y (respectively z) by considering the semitotal dominating set $D = D' \cup \{x, y\}$ (respectively $D = D' \cup \{y, z\}$).

Case 1.2. $v_{xyz} \in D'$. We first show that $D = (D' \setminus \{v_{xyz}\}) \cup \{x, y, z\}$ is a semitotal dominating set of G (note that $|D| = \gamma_{t2}(G) + 1$). It is easy to see that D is a dominating set. Furthermore, if v_{xyz} was a witness for a vertex p in D', then in G, p is at distance at most two to a vertex of $\{x, y, z\}$ and thus, p has a witness in D.

We next show that D contains an ST-configuration. Let $w \in D'$ be a witness for v_{xyz} in D'. Suppose first that $d_{G'}(w, v_{xyz}) = 1$. If $wy \in E(G)$, then $\{x, y, z, w\}$ forms an O_4 in D. Otherwise, w is adjacent to x or z, in which case D contains a P_4 on vertex set $\{x, y, z, w\}$ and, so, by Observation 6, D contains an O_4 or an O_6. Now if $d_{G'}(w, v_{xyz}) = 2$, then $wx, wy, wz \notin E(G)$ and w is at distance two to a vertex of $\{x, y, z\}$. Then, either $d_G(w, y) = 2$, in which case $\{x, y, z, w\}$ forms an O_6 in D, or the same set forms an O_5 in D.

Case 2. *e and e′ do not share a vertex.* Let $e = xy$ and $e' = zw$ and let v_{xy} and v_{zw} be the vertices of G' resulting from the contraction of e and e', respectively.

Case 2.1. $D' \cap \{v_{xy}, v_{zw}\} = \varnothing$. Since D' dominates v_{xy} and v_{zw}, at least one of $\{x, y\}$ is dominated by D' (the same holds for $\{z, w\}$). Assume, without loss of generality, that x and z are dominated by D' and let $D = D' \cup \{x, z\}$. Note that D is a semitotal dominating set of G of size $\gamma_{t2}(G)+1$. We next show that D contains an ST-configuration. Let w_1 (respectively w_2) be a vertex of D that dominates x (respectively z). If $w_1 = w_2$, let w' be a witness of w_1 in D'. Then $\{x, z, w_1, w'\}$ forms an O_4 (if $d_G(w', w_1) = 1$) or an O_6 (if $d_G(w', w_1) = 2$) in D. Suppose next that $w_1 \neq w_2$ and let w'_1 (respectively w'_2) be a witness for w_1 (respectively w_2) in D'. Assume first that $w'_1 = w'_2$. If $d_G(w_2, w'_1) = 1$ and $d_G(w_1, w'_1) = 1$, then D contains a P_4 on vertex set $\{x, w_1, w'_1, w_2\}$ and, so, by Observation 6, D contains an O_4 or an O_6. If $d_G(w_2, w'_1) = 1$ and $d_G(w_1, w'_1) = 2$, then $\{w_1, w'_1, w_2, z\}$ forms an O_5 in D. Finally, if both $d_G(w_2, w'_1) = 2$ and $d_G(w_1, w'_1) = 2$, then $\{x, w_1, w'_1, w_2, z\}$ forms an O_3 in D. Assume henceforth that $w'_1 \neq w'_2$. If $w'_1 = w_2$ and $w_1w_2 \in E(G)$, then D contains a P_4 on vertex set $\{x, w_1, w_2, z\}$ and, so, by Observation 6, D contains an O_4 or an O_6. If $w'_1 = w_2$ and $w_1w_2 \notin E(G)$, then $\{x, w_1, w_2, z\}$ forms an O_7 in D. Finally, assume that w_1, w'_1, w_2, w'_2 are four distinct vertices in G. If $d_G(w_1, w'_1) = 1$ and $d_G(w_2, w'_2) = 1$, then $\{x, z, w_1, w'_1, w_2, w'_2\}$ forms an O_1 in D. If $d_G(w_1, w'_1) = 1$ and $d_G(w_2, w'_2) = 2$, then $\{x, z, w_1, w'_1, w_2, w'_2\}$ forms an O_2 in D. Finally, if both $d_G(w_1, w'_1) = 2$ and $d_G(w_2, w'_2) = 2$, then $\{x, z, w_1, w'_1, w_2, w'_2\}$ forms an O_3 in D.

Case 2.2. $D' \cap \{v_{xy}, v_{zw}\} \neq \varnothing$. If $|D' \cap \{v_{xy}, v_{zw}\}| = 1$ then assume, without loss of generality, that $v_{xy} \in D'$. Since $v_{zw} \notin D'$, there exists $z' \in D'$ such that z' is adjacent to z or w, say $zz' \in E(G)$ without loss of generality. Consider the set $D = (D' \setminus \{v_{xy}\}) \cup \{x, y, z\}$ (note that $|D| = \gamma_{t2}(G)+1$). Let us show that D contains an ST-configuration. Let p be a witness of v_{xy} in D'. Assume without loss of generality that $d_G(p, y) \leq 2$. Suppose first that $z' = v_{xy}$. If $d_G(p, y) = 1$ and $zy \in E(G)$, then $\{x, y, z, p\}$ forms an O_4 in D. If $d_G(p, y) = 1$ and $zy \notin E(G)$, then $d_G(y, z) = 2$ and therefore $\{x, y, z, p\}$ forms an O_6 in D. If $d_G(p, y) = 2$ and $zy \in E(G)$, then $\{x, y, z, p\}$ forms an O_6 in D. Finally, if $d_G(p, y) = 2$ and $zx \in E(G)$, then $\{z, x, y, p\}$ forms an O_5 in D. Second, suppose that $z' \neq v_{xy}$. If $p = z'$, we have two possibilities. Either $d_G(p, y) = 1$ in which case D contains a P_4 on vertex set $\{x, y, p, z\}$ and, so, by Observation 6, D contains an O_4 or an O_6. Or $d_G(p, y) = 2$ in which case $\{x, y, p, z\}$ forms an O_7 in D. Assume henceforth that $p \neq z'$ and let z'' be a witness of z' in D'. If $z'' = v_{xy}$, then either y or x is a witness of z' in D. By symmetry, we can assume that $d_G(y, z') \leq 2$. If $d_G(y, z') = 1$ then, by Observation 6, $\{z, z', y, x\}$ forms an O_4 or an O_6 in D. If $d_G(y, z') = 2$, then the same set forms an O_7 in D. Hence, we can safely assume that $z'' \neq v_{xy}$. Now note that $\{z, z', z''\}$ and $\{x, y, p\}$ form friendly triples in D (recall that $d_G(p, y) \leq 2$) and it may still be the case that $p = z''$. Suppose first that $p = z''$. If $d_G(z', z'') = 1$ and $d_G(z'', y) = 1$, then by Observation 6, we have either an O_4 or an O_6 in D. If $d_G(z', z'') = 1$ and $d_G(z'', y) = 2$, then $\{z, z', z'', y\}$ forms an O_5 in D. Finally, if $d_G(z', z'') = 2$ and $d_G(z'', y) = 1$ (respectively $d_G(z'', y) = 2$), then $\{z, z', z'', y\}$ (respectively $\{z, z', z'', y, x\}$) forms an O_7 (respectively O_3) in D. Thus we may assume that $p \neq z''$. Then $\{z, z', z''\}$ and $\{x, y, p\}$ are two disjoint friendly triples in D and thus, $\{z, z', z'', x, y, p\}$ forms either an O_1 (if $d_G(y, p) = d_G(z', z'') = 1$), an O_2 (if exactly one of $d_G(y, p)$ or $d_G(z', z'')$ equals two) or an O_3 (if $d_G(y, p) = d_G(z', z'') = 2$).

We conclude the proof by considering the case where $\{v_{xy}, v_{zw}\} \subseteq D'$ to which a similar case analysis applies. Consider the set $D = (D' \setminus \{v_{xy}, v_{zw}\}) \cup \{x, y, z, w\}$ (note that D is a semitotal dominating set of G of size $\gamma_{t2}(G)+1$). Let us show that D contains an ST-configuration. If a vertex of $\{x, y\}$ is adjacent to a vertex of $\{z, w\}$, then D contains a P_4 and so, by Observation 6, $\{x, y, z, w\}$ forms an O_4 or an O_6 in D. If a vertex of $\{x, y\}$ is at distance exactly two from a vertex in $\{z, w\}$, then D contains an O_7. If neither of these conditions hold, that is, if $d_{G'}(v_{xy}, v_{zw}) \geq 3$, then let p (resp. p') be a witness for v_{xy} (resp. v_{zw}) in D'. Note that $p \neq v_{zw}$ and $p' \neq v_{xy}$ since $d_{G'}(v_{xy}, v_{zw}) \geq 3$. Hence, if $p = p'$ then D contains an O_3 or an O_5. Otherwise, $\{x, y, p\}$ and $\{z, w, p'\}$ are two disjoint friendly triples in D and thus, $\{x, y, z, w, p, p'\}$ forms either an O_1, an O_2 or an O_3 in D, which concludes the proof. □

4. The complexity of 1-Edge Contraction(γ_{t2})

In this section, we consider several graph classes and determine for each of them whether 1-Edge Contraction(γ_{t2}) is (co)NP-hard (Section 4.1) or polynomial-time solvable (Section 4.2). Putting these results together then leads to our main theorem (Section 4.3).

4.1. Hardness results

Similarly to the case of domination, we have the two following results.

Theorem 7. 1-Edge Contraction(γ_{t2}) *is* coNP-*hard when restricted to claw-free graphs.*

Proof. We reduce from the Positive Exactly 3-Bounded 1-In-3 3-Sat problem which is a variant of the 3-Sat problem where, given a formula Φ in which all literals are positive, every clause contains exactly three literals and every variable appears in exactly three clauses, the problem is to determine whether there exists a truth assignment such that each clause has exactly one true literal. This problem was shown to be NP-complete in [19].

We first introduce the following graph, called the *long paw*, which we will use in the reduction (see Fig. 2).

As mentioned above, we reduce from Positive Exactly 3-Bounded 1-In-3 3-Sat: given an instance Φ of this problem, with variable set X and clause set C, we construct an instance G of 1-Edge Contraction(γ_{t2}) such that Φ is a Yes-instance for Positive 1-In-3 3-Sat if and only if G is a No-instance for 1-Edge Contraction(γ_{t2}), as follows. For every variable $x \in X$ contained in clauses c, c' and c'', we introduce the gadget G_x depicted in Fig. 3 (where the rectangles indicate that the

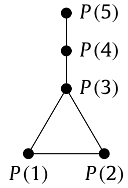

Fig. 2. The long paw P.

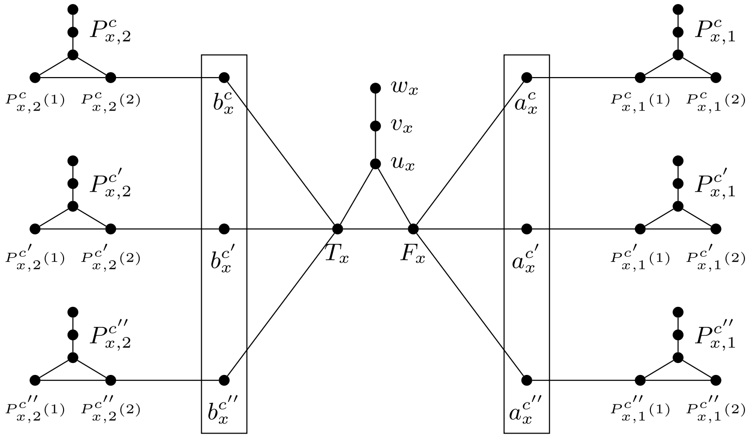

Fig. 3. The gadget G_x for a variable $x \in X$ contained in clauses c, c' and c'' (rectangles indicate that the corresponding set of vertices induces a clique).

(a) The graph G^T_c. **(b)** The graph G^F_c.

Fig. 4. The gadget G_c for a clause $c \in C$ containing variables x, y and z.

corresponding set of vertices is a clique). For every clause $c \in C$ containing variables x, y and z, we introduce the gadget G_c depicted in Fig. 4 consisting of the disjoint union of the graph G^T_c and the graph G^F_c. Finally, for every clause $c \in C$ containing variables x, y and z, we add edges between the corresponding gadgets as follows.

- For every $p \in \{x, y, z\}$, we connect $P^c_{p,1}(2)$ to f^{ab}_c if and only if $p \in \{a, b\}$.
- For every $p \in \{x, y, z\}$, we connect $P^c_{p,2}(1)$ to t^p_c and further connect $P^c_{p,2}(1)$ to w^{ab}_c if and only if $p \in \{a, b\}$.

We denote by G the resulting graph.

Observation 8. Let D be a semitotal dominating set of G. Then, for every variable $x \in X$, $|D \cap V(G_x)| \geq 14$.

Indeed, for every long paw P (see Fig. 2), the vertex $P(5)$ must be dominated and the vertex dominating $P(5)$ must have a witness. Since every variable gadget contains 7 long paws, the result follows.

Observation 9. Let D be a semitotal dominating set of G. If $|D \cap V(G_x)| = 14$ for some variable $x \in X$ contained in clauses c, c' and c'', then the following holds.

1. If $P^q_{x,2}(1) \in D$ for some $q \in \{c, c', c''\}$ then $T_x \in D$.
2. If $P^q_{x,1}(2) \in D$ for some $q \in \{c, c', c''\}$ then $F_x \in D$.

In particular, if $P^q_{x,2}(1) \in D$ for some $q \in \{c, c', c''\}$, then $D \cap \{P^c_{x,1}(2), P^{c'}_{x,1}(2), P^{c''}_{x,1}(2)\} = \varnothing$. Similarly, if $P^q_{x,1}(2) \in D$ for some $q \in \{c, c', c''\}$, then $D \cap \{P^c_{x,2}(1), P^{c'}_{x,2}(1), P^{c''}_{x,2}(1)\} = \varnothing$.

Indeed, suppose that $|D \cap V(G_x)| = 14$ for some variable $x \in X$ contained in clauses c, c' and c''. Observe first that, by Observation 8, $D \cap \{a^q_x, b^q_x \mid q \in \{c, c', c''\}\} = \varnothing$ and $|D \cap \{P^q_{x,j}(1), P^q_{x,j}(2)\}| \leq 1$ for any $j \in [2]$ and $q \in \{c, c', c''\}$ (similarly, $|D \cap \{T_x, F_x\}| \leq 1$). Thus, if $P^q_{x,2}(1) \in D$ for some clause $q \in \{c, c', c''\}$, then $T_x \in D$ as b^q_x must be dominated. Similarly, if $P^q_{x,1}(2) \in D$ for some $q \in \{c, c', c''\}$, then $F_x \in D$ as a^q_x must be dominated.

Observation 10. Let D be a semitotal dominating set of G. Then, for every clause $c \in C$, $D \cap V(G^T_c) \neq \varnothing$. Furthermore, if $P^c_{x,2}(1) \notin D$ for every variable x contained in c, then $|D \cap V(G^T_c)| \geq 2$.

Indeed, since u_c must be dominated, $D \cap V(G^T_c) \neq \varnothing$. Now if $P^c_{x,2}(1) \notin D$ for every variable x contained in c, then the result follows from the fact that $\gamma(G^T_c) = 2$.

Observation 11. Let D be a semitotal dominating set of G. Then, for every clause $c \in C$ containing variables x, y and z, if $|D \cap \{P^c_{x,1}(2), P^c_{y,1}(2), P^c_{z,1}(2)\}| < 2$ then $|D \cap V(G^F_c)| \geq 1$.

Indeed, if say $P^c_{x,1}(2), P^c_{y,1}(2) \notin D$ without loss of generality, then $N[f^{xy}_c] \setminus \{P^c_{x,1}(2), P^c_{y,1}(2)\} \cap D \neq \varnothing$ as f^{xy}_c should be dominated.

Claim 12. $\gamma_{t2}(G) = 14|X| + |C|$ if and only if Φ is a Yes-instance for Positive 1-In-3 3-Sat.

Proof. Assume first that Φ is a Yes-instance for Positive 1-In-3 3-Sat and consider a truth assignment satisfying Φ. We construct a semitotal dominating set D of G as follows. For every variable $x \in X$ contained in clauses c, c' and c'', if x is set to true, then we add $\{T_x, v_x\} \cup \{P^q_{x,j}(1), P^q_{x,j}(4) \mid j \in [2], q \in \{c, c', c''\}\}$; otherwise, we add $\{F_x, v_x\} \cup \{P^q_{x,j}(2), P^q_{x,j}(4) \mid j \in [2], q \in \{c, c', c''\}\}$. For every clause $c \in C$ containing variables x, y and z, exactly one variable is set to true, say x without loss of generality, in which case we add t^y_c to D. It is not difficult to see that the constructed set D is a semitotal dominating set of G of size $14|X| + |C|$. We then conclude by Observations 8 and 10 that D has minimum size.

Conversely, assume that $\gamma_{t2}(G) = 14|X| + |C|$ and consider a minimum semitotal dominating set D of G. Note that, by Observations 8 and 10, $|D \cap V(G_x)| = 14$ for every variable $x \in X$ and $|D \cap V(G^T_c)| = 1$ for every clause $c \in C$; in particular, $D \cap V(G^F_c) = \varnothing$ for every clause $c \in C$. Now consider a clause $c \in C$ containing variables x, y and z. Since $D \cap V(G^F_c) = \varnothing$, it follows from Observation 11 that $|D \cap \{P^c_{x,1}(2), P^c_{y,1}(2), P^c_{z,1}(2)\}| \geq 2$, say $P^c_{x,1}(2), P^c_{y,1}(2) \in D$ without loss of generality. Note that then, by Observation 9, $F_x, F_y \in D$ (and thus, $T_x, T_y \notin D$). Then, we claim that, $P^c_{z,2}(1) \in D$. Indeed, by Observation 9, we have that $P^c_{x,2}(1), P^c_{y,2}(1) \notin D$. Thus, if $P^c_{z,2}(1) \notin D$ then, by Observation 10, $|D \cap V(G^T_c)| \geq 2$ a contradiction. Thus, $P^c_{z,2}(1) \in D$ and so, by Observation 9, $T_z \in D$ (which implies that $F_z \notin D$). We thus construct a truth assignment satisfying Φ as follows: for every variable $x \in X$, if $T_x \in D$ then set x to true, otherwise set x to false. $\square$

Claim 13. $\gamma_{t2}(G) = 14|X| + |C|$ if and only if G is a No-instance for 1-Edge Contraction(γ_{t2}).

Proof. Assume first that $\gamma_{t2}(G) = 14|X| + |C|$ and consider a minimum semitotal dominating set D of G. Then, by Observations 8 and 10, $|D \cap V(G_x)| = 14$ for every variable $x \in X$ and $|D \cap V(G^T_c)| = 1$ for every clause $c \in C$; in particular, $D \cap V(G^F_c) = \varnothing$ for every clause $c \in C$. It follows that, for every variable $x \in X$, $D \cap V(G_x)$ contains no friendly triple: indeed, any two distinct long paws are at distance at least 2 from one another. Moreover, if some long paw P contains an edge $e \in E(D)$, then $P(4)$ is an endvertex of e and so, e is at distance at least three from any other vertex in $D \cap V(G_x)$. Now consider a clause $c \in C$ containing variables x, y and z and denote by u the vertex in $D \cap V(G^T_c)$. Since u cannot alone dominate every vertex in $V(G^T_c)$, there must exist $p \in \{x, y, z\}$ such that $P^c_{p,2}(1) \in D$, say $p = x$ without loss of generality. We claim that then, $P^c_{y,2}(1), P^c_{z,2}(1) \notin D$. Indeed, if say $P^c_{y,2}(1) \in D$ then, by Observation 9, $P^c_{x,1}(2), P^c_{y,1}(2) \notin D$. But then, by Observation 11, $D \cap V(G^F_c) \neq \varnothing$, a contradiction. Thus, $P^c_{y,2}(1) \notin D$ and we conclude similarly that $P^c_{z,2}(1) \notin D$. But then, $u \notin \{w^{xz}_c, w^{xy}_c, t^x_c\}$: indeed, if $u = t^x_c$ then w^{yz}_c is not dominated, and if $u \in \{w^{xz}_c, w^{xy}_c\}$ then u_c is not dominated. It follows that u is at distance at least two from $P^c_{x,2}(1)$; in particular, u cannot be part of a friendly triple. Hence, D contains no friendly triple and so, G is a No-instance for 1-Edge Contraction(γ_{t2}) by Theorem 5(i).

Conversely, assume that G is a No-instance for 1-Edge Contraction(γ_{t2}) and consider a minimum semitotal dominating set D of G. Observe first that if $|D \cap V(P)| \geq 3$ for some long paw P, then either $D \cap V(P)$ contains a friendly triple or $D \cap V(P) = \{P(1), P(2), P(5)\}$. In the latter case, we have that $D \setminus \{P(5)\} \cup \{P(4)\}$ is a minimum semitotal dominating set containing a friendly triple, a contradiction to the fact that G is a No-instance for 1-Edge Contraction(γ_{t2}). Thus for every variable $x \in X$ contained in clauses c, c' and c'', $|D \cap V(P^q_{x,j})| \leq 2$ for every $j \in [2]$ and $q \in \{c, c', c''\}$ (similarly, $|D \cap \{T_x, F_x, u_x, v_x, w_x\}| \leq 2$). By Observation 8, we conclude that in fact equality holds. We may further assume that

$P(5) \notin D$ for every long paw P of G_x (consider otherwise $(D \setminus \{P(5), P(4), P(3)\}) \cup \{P(3), P(4)\})$ which implies, in particular, that every vertex of a long paw P is dominated by some vertex in $D \cap V(P)$. It follows that, for any $q \in \{c, c', c''\}$, $b_x^q \notin D$: indeed, if $b_x^q \in D$ then $T_x \notin D$ (D would otherwise contain a friendly triple, namely b_x^q, T_x, v_x) and so, $D' = (D \setminus \{b_x^q\}) \cup \{T_x\}$ is a minimum semitotal dominating set of G containing a friendly triple, namely $D' \cap \{T_x, F_x, u_x, v_x, w_x\}$, a contradiction. By symmetry, we conclude that $a_x^q \notin D$ for any $q \in \{c, c', c''\}$. Hence, $|D \cap V(G_x)| = 14$. Now consider a clause $c \in C$ containing variables x, y and z. Suppose first that $|D \cap V(G_c^T)| \geq 2$. Then $D \cap \{P_{x,2}^c(1), P_{y,2}^c(1), P_{z,2}^c(1)\} = \varnothing$: indeed, if say $P_{x,2}^c(1) \in D$, then $(D \setminus V(G_c^T)) \cup \{t_c^x, t_c^y\}$ is a semitotal dominating set of G of size at most that of D containing a friendly triple, namely $t_c^x, t_c^y, P_{x,2}^c(1)$, a contradiction. Thus, $P_{x,2}^c(1) \notin D$ and we conclude similarly that $P_{y,2}^c(1), P_{z,2}^c(1) \notin D$. But then, $D' = (D \setminus V(G_c^T)) \cup \{t_c^y, P_{x,2}^c(1)\}$ is a semitotal dominating set of G of size at most that of D containing a friendly triple, namely $D' \cap V(P_{x,2}^c)$, a contradiction. Thus, $|D \cap V(G_c^T)| \leq 1$ and we conclude by Observation 10 that in fact equality holds. Second, observe that if $|D \cap V(G_c^F)| \geq 2$, say $f_c^{yz}, f_c^{xy} \in D$ without loss of generality, then D contains a friendly triple as $D \cap \{P_{x,1}^c(1), P_{x,1}^c(2), P_{x,1}^c(3)\} \neq \varnothing$ by the above, a contradiction. Thus, suppose that $|D \cap V(G_c^F)| = 1$, say $f_c^{xy} \in D$. Then $P_{x,1}^c(2), P_{y,1}^c(2) \notin D$: indeed, if $P_{p,1}^c(2) \in D$ for some $p \in \{x, y\}$, then $f_c^{xy} \cup (D \cap V(P_{p,1}^c))$ contains a friendly triple, a contradiction. It follows that $F_x \notin D$ for otherwise $(D \setminus V(P_{x,1}^c)) \cup \{P_{x,1}^c(2), P_{x,1}^c(4)\}$ is a minimum semitotal dominating set of G containing a friendly triple, namely $P_{x,1}^c(2), f_c^{xy}, P_{x,1}^c(4)$, a contradiction. We conclude similarly that $F_y \notin D$. But then, we may assume that $T_x, T_y \in D$ (consider otherwise $(D \setminus \{u_x, v_x, w_x, u_y, v_y, w_y\}) \cup \{T_x, v_x, T_y, v_y\}$) and that $P_{x,2}^c(1), P_{y,2}^c(1) \in D$ (consider otherwise $(D \setminus (V(P_{x,2}^c) \cup V(P_{y,2}^c))) \cup \{P_{p,2}(1), P_{p,2}(4) \mid p \in \{x, y\}\}$). But then, $(D \setminus V(G_c^T)) \cup \{t_c^x\}$ is a minimum semitotal dominating set of G containing a friendly triple, namely $P_{y,2}^c(1), t_c^x, P_{x,2}^c(1)$, a contradiction. Thus, $D \cap V(G_c^F) = \varnothing$ and so, $|D \cap V(G_c)| = 1$. Therefore, $|D| = 14|X| + |C|$, which concludes the proof. $\square$

By combining Claims 12 and 13, we obtain that G is a No-instance for 1-Edge Contraction(γ_{t2}) if and only if Φ is a Yes-instance for Positive 1-In-3 3-Sat. There remains to show that G is claw-free. To prove this, we show that, for every vertex $v \in V(G)$, the neighbourhood of v can be partitioned into (at most) two cliques. Consider first a clause $c \in C$ containing variables x, y and z. Clearly, $N(u_c) = \{t_c^x, t_c^y, t_c^z\}$ is a clique. For every $\ell \in \{x, y, z\}$, the neighbourhood of t_c^x consists of the clique $\{u_c, t_c^y, t_c^z\}$ and the clique $\{P_{x,2}^c(1), w_c^{xy}, w_c^{xz}\}$. We conclude by symmetry that the neighbourhood of t_c^y and that of t_c^z consist of two cliques as well. Similarly, the neighbourhood of w_c^{xy} consists of the clique $\{t_c^x, w_c^{xz}, P_{x,2}^c(1)\}$ and the clique $\{t_c^y, w_c^{yz}, P_{y,2}^c(1)\}$. We conclude by symmetry that the neighbourhood of w_c^{xz} and that of w_c^{yz} consist of two cliques as well. Finally, the neighbourhood of f_c^{xy} consists of the clique $\{f_c^{xz}, P_{x,1}^c(2)\}$ and the clique $\{f_c^{yz}, P_{y,1}^c(2)\}$. We conclude by symmetry that the neighbourhood of f_c^{xz} and that of f_c^{yz} consist of two cliques as well. Consider next a variable $x \in X$ contained in clauses c, c' and c''. Clearly, the neighbourhood of any vertex of degree at most two is partitioned into at most two cliques. For every long paw P of G_x, the neighbourhood of $P(3)$ consists of the clique $\{P(1), P(2)\}$ and the clique $\{P(4)\}$. The neighbourhood of T_x consists of the clique $\{F_x, u_x\}$ and the clique $\{b_x^c, b_x^{c'}, b_x^{c''}\}$. We conclude by symmetry that the neighbourhood of F_x consists of two cliques. For every $\ell \in \{c, c', c''\}$, the neighbourhood of b_x^ℓ consists of the clique $\{T_x\} \cup \{b_c^p \mid p \in \{c, c', c''\} \setminus \{\ell\}\}$ and the clique $\{P_{x,2}^\ell(2)\}$. We conclude by symmetry that, for every $\ell \in \{c, c', c''\}$, the neighbourhood of a_x^ℓ consists of two cliques. For every $\ell \in \{c, c', c''\}$, the neighbourhood of $P_{x,2}^\ell(2)$ consists of the cliques $\{P_{x,2}^\ell(1), P_{x,2}^\ell(3)\}$ and $\{b_x^\ell\}$. We conclude by symmetry that for every $\ell \in \{c, c', c''\}$, the neighbourhood of $P_{x,1}^\ell(1)$ consists of two cliques. Finally, consider a clause $\ell \in \{c,' c, c''\}$ and let $y, z \in X$ be the other two variables occurring in ℓ. Then the neighbourhood of $P_{x,2}^\ell(1)$ consists of the cliques $\{P_{x,2}^\ell(2), P_{x,2}^\ell(3)\}$ and $\{t_\ell^x, w_\ell^{xy}, w_\ell^{xz}\}$. Similarly, the neighbourhood of $P_{x,1}^\ell(2)$ consists of the clique $\{P_{x,1}^\ell(1), P_{x,1}^\ell(3)\}$ and the clique $\{f_\ell^{xy}, f_\ell^{xz}\}$. Therefore, G is claw-free. $\square$

Theorem 14. 1-Edge Contraction(γ_{t2}) *is* coNP*-hard when restricted to* $2P_3$*-free graphs.*

Proof. We introduce an auxiliary problem which will be helpful in showing the coNP-hardness of 1-Edge Contraction(γ_{t2}) when restricted to $2P_3$-free graphs.

> All Independent MSD
>
> **Instance:** A graph G.
> **Question:** Is every minimum semitotal dominating set of G independent?

In the following hardness proof, we reduce from the Positive 1-In-3 3-Sat problem which is a variant of the 3-Sat problem where, given a formula Φ in which all literals are positive, the problem is to determine whether there exists a truth assignment such that each clause has exactly one true literal. This problem was shown to be NP-complete in [26].

Lemma 15. All Independent MSD *is* NP*-hard when restricted to* $2P_3$*-free graphs.*

Proof. We reduce from Positive 1-In-3 3-Sat: given a instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance G_Φ of All Independent MSD as follows. For every variable $x \in X$, we introduce a triangle G_x which has two distinguished *truth vertices* T_x and F_x (we denote by u_x the third vertex of G_x). For every clause $c \in C$ containing variables x, y, z, we introduce a K_5, denoted by G_c, with vertex set $\{v_c^x, v_c^y, v_c^z, u_c^T, u_c^F\}$. The adjacencies between the gadgets are as follows.

- For every clause $c \in C$ containing variables x, y, z, we connect u_c^T to T_x, T_y, T_z and u_c^F to F_x, F_y, F_z. We further connect v_c^s to T_s and F_r for every $s \in \{x, y, z\}$ and every $r \in \{x, y, z\} \setminus \{s\}$.
- $\bigcup_{c\in C} V(G_c)$ induces a clique.

We denote by G_Φ the resulting graph.

Since u_x must be dominated in any semitotal dominating set, we trivially have the following.

Observation 16. Let D be a semitotal dominating set of G_Φ. Then $|D \cap V(G_x)| \geq 1$ for every variable $x \in X$.

Claim 17. $\gamma_{t2}(G_\Phi) = |X|$ if and only if Φ is satisfiable.

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a semitotal dominating set D of G_Φ as follows. For every variable $x \in X$, if x is set to true, then we add T_x to D; otherwise, we add F_x to D. Clearly, every variable gadget is dominated by some vertex in D. Now consider a clause c containing variables x, y, z. Then, exactly one variable is set to true, say x without loss of generality. Since $\{T_x, F_y, F_z\} \subset D$, v_c^x and u_c^T are dominated by T_x, v_c^y and u_c^F are dominated by F_z and v_c^z is dominated by F_y. Furthermore, T_x, F_y, F_z are pairwise at distance exactly two (v_c^x is a common neighbour). Thus, D is a semitotal dominating set of G_Φ and has minimum size by Observation 16.

Conversely, assume that $\gamma_{t2}(G_\Phi) = |X|$ and let D be a minimum semitotal dominating set of G_Φ. Then, by Observation 16, $|D \cap V(G_x)| = 1$ for every variable $x \in X$ which, in turn, implies that $D \cap \bigcup_{c\in C} V(G_c) = \varnothing$. It follows that, for any variable $x \in X$, $u_x \notin D$: indeed, if $u_x \in D$ for some variable $x \in X$, then u_x has no witness as $D \cap (\{T_x, F_x\} \cup \bigcup_{c\in C} V(G_c)) = \varnothing$, a contradiction. Now consider a clause $c \in C$ containing variables x, y, z. Suppose that there exist two variables $s, r \in \{x, y, z\}$ such that $\{T_s, T_r\} \subset D$. Then one of u_c^F and v_c^q, where $q \in \{x, y, z\} \setminus \{s, r\}$, is not dominated: indeed, either $T_q \in D$ in which case u_c^F is not dominated, or $F_q \in D$ in which case v_c^q is not dominated. Thus, there exists at most one variable $s \in \{x, y, z\}$ such that $T_s \in D$, and since u_c^T must be dominated, we conclude that such a variable exists. It follows that the truth assignment obtained by setting x to true if $T_x \in D$, and x to false if $F_x \in D$, satisfies Φ. $\square$

Claim 18. $\gamma_{t2}(G_\Phi) = |X|$ if and only if G_Φ is a Yes-instance for All Independent MSD.

Proof. Assume that $\gamma_{t2}(G_\Phi) = |X|$ and let D be a minimum semitotal dominating set of G_Φ. Then, by Observation 16, $|D \cap V(G_x)| = 1$ for any variable $x \in X$, which implies that $D \cap \bigcup_{c\in C} V(G_c) = \varnothing$. Thus, D is independent and so, G_Φ is a Yes-instance for All Independent MSD.

Conversely, assume that G_Φ is a Yes-instance for All Independent MSD and let D be a minimum semitotal dominating set of G_Φ. Since D is independent, $|D \cap V(G_x)| \leq 1$ for any variable $x \in X$, and we conclude by Observation 16 that, in fact, equality holds. Furthermore, we may assume that for any variable $x \in X$, $u_x \notin D$ as it suffices to consider $(D \setminus \{u_x\}) \cup \{T_x\}$ otherwise. It follows that if two variables x and y both occur in some clause c, and that $r \in D \cap V(G_x)$ and $s \in D \cap V(G_y)$, then r and s witness each other: indeed, $d(T_x, T_y) = d(F_x, F_y) = d(T_x, F_y) = d(T_y, F_x) = 2$. Now consider a clause $c \in C$ containing variables x, y, z, and suppose to the contrary that there exists $w \in V(G_c) \cap D$. Since D is independent, $(D \cap \bigcup_{c'\in C} V(G_{c'})) = \{w\}$ (recall that $\bigcup_{c'\in C} V(G_{c'})$ induces a clique). Furthermore, by the previous observation, any vertex $t \in D$ witnessed by w is also witnessed by a vertex in $D \setminus \{w\}$. Thus, we may replace w with either u_c^T or u_c^F, and obtain a semitotal dominating set of G_Φ which is not independent. Indeed, if $D \cap \{F_x, F_y, F_z\} \neq \varnothing$, then $(D \setminus \{w\}) \cup \{u_c^F\}$ is a minimum semitotal dominating set of G_Φ which is not independent. Otherwise, $(D \setminus \{w\}) \cup \{u_c^T\}$ is a minimum semitotal dominating set of G_Φ which is not independent. Since this contradicts the fact that G_Φ is a Yes-instance for All Independent MSD, we conclude that $D \cap \bigcup_{c\in C} V(G_c) = \varnothing$ and so, $\gamma_{t2}(G_\Phi) = |D| = |X|$. $\square$

By combining Claims 17 and 18, we obtain that Φ is satisfiable if and only if G_Φ is a Yes-instance for All Independent MSD. Since G_Φ is easily seen to be $2P_3$-free, the lemma follows. $\square$

Lemma 19. *Let G be a $2P_3$-free graph. Then G is a* Yes*-instance for* 1-Edge Contrac-tion(γ_{t2}) *if and only if G is a* No*-instance for* All Independent MSD.

Proof. If G is a Yes-instance for 1-Edge Contraction(γ_{t2}) then, by Theorem 5(i), G has a minimum semitotal dominating set containing a friendly triple which is, a fortiori, not independent. Thus, G is a No-instance for All Independent MSD.

Conversely, assume that G is a No-instance for All Independent MSD and let D be a minimum semitotal dominating set of G which is not independent. If D contains a friendly triple, then we conclude by Theorem 5(i) that G is a Yes-instance for 1-Edge Contraction(γ_{t2}). Thus, suppose that D contains no friendly triple. We show that we can modify D to obtain a minimum semitotal dominating set which contains a friendly triple. To this end, let $x, y \in D$ be two adjacent vertices. Consider a vertex $z \in D \setminus \{x, y\}$ such that $d(z, \{x, y\}) = \min_{u \in D\setminus\{x,y\}} d(u, \{x, y\})$ and assume without loss of generality that $d(z, \{x, y\}) = d(z, x)$. By assumption, $d(x, z) > 2$ and, since G is $2P_3$-free, $d(x, z) \leq 5$.

Suppose first that $d(x, z) = 3$ and let $P = xuvz$ be a shortest path from x to z. Let $w \in D$ be a closest witness for z. Suppose first that w is adjacent to z. If y has no private neighbour, then $(D \setminus \{y\}) \cup \{u\}$ is a minimum semitotal dominating set of G (indeed, since D contains no friendly triple by assumption, y is a witness for x only) containing a friendly triple, namely x, u, z. We conclude similarly if w has no private neighbour. Thus, we may assume that both y and w have at least one private neighbour, say p_y and p_w respectively. Then the private neighbourhood of w must be complete to the private neighbourhood of y: indeed, if y has a private neighbour a and w has a private neighbour b such that a and b are nonadjacent, then $\{a, y, x, b, w, z\}$ induces a $2P_3$, a contradiction. It follows that $(D \setminus \{y, w\}) \cup \{p_y, p_w\}$ is a minimum semitotal dominating set containing a friendly triple, namely p_y, p_w, x. Second, suppose that $d(z, w) = 2$. If w is adjacent to v, then every private neighbour of y is adjacent to v: indeed, if y has a private neighbour p_y which is nonadjacent to v, then $\{p_y, y, x, z, v, w\}$ induces a $2P_3$, a contradiction. But then, $(D \setminus \{y\}) \cup \{v\}$ is a minimum semitotal dominating set containing a friendly triple, namely z, v, w. Thus, assume that w is nonadjacent to v and let t be the internal vertex in a shortest path from z to w. If x has no private neighbour, then y is adjacent to u (u would otherwise be a private neighbour of x) and so, the minimum dominating set $(D \setminus \{x\}) \cup \{u\}$ contains a friendly triple, namely y, u, z. Thus, assume that x has at least one private neighbour. Then every private neighbour p_x of x must be adjacent to t, for otherwise $\{p_x, x, y, z, t, w\}$ induces a $2P_3$; in particular, x is at distance two from t. Similarly, we conclude that every private neighbour of y is adjacent to t. It then follows that $(D \setminus \{y\}) \cup \{t\}$ is a minimum semitotal dominating set of G containing a friendly triple, namely z, t, w.

Suppose next that $d(x, z) = 4$ and let $P = xuvtz$ be a shortest path from x to z. Let $w \in D$ be a witness for z. Suppose first that w is adjacent to z. We claim that either y has no private neighbour or w has no private neighbour. Indeed, if y has a private neighbour p_y and w has a private neighbour p_w, then p_y and p_w must be adjacent for otherwise, $\{p_y, y, x, p_w, w, z\}$ induces a $2P_3$, a contradiction. But then, $d(y, w) \leq 3 < d(x, z)$, a contradiction to our assumption. Thus, assume, without loss of generality, that y has no private neighbour. Then it suffices to consider $(D \setminus \{y\}) \cup \{u\}$ and go back to the previous case. Second, suppose that $d(z, w) = 2$ and let q be the internal vertex in a shortest path from z to w. Then y has no private neighbour: indeed, if y has a private neighbour a, then a is adjacent to q ($\{a, y, x, w, q, z\}$ would otherwise induce a $2P_3$), which implies that $d(y, z) \leq 3 < d(x, z)$, a contradiction to the choice of z. But then, it suffices to consider $(D \setminus \{y\}) \cup \{u\}$ and go back to the previous case.

Suppose, finally, that $d(x, z) = 5$ and let $P = u_1 \dots u_6$, where $u_1 = x$ and $u_6 = z$, be a shortest path from x to z. Then y has no private neighbour: indeed, if y has a private neighbour a, then a is adjacent to either u_4 or u_5 (since $\{a, y, x, u_4, u_5, z\}$ would otherwise induce a $2P_3$) and so, $d(y, z) \leq 4 < d(x, z)$, a contradiction to our assumption. But then, it suffices to consider $(D \setminus \{y\}) \cup \{u_2\}$ and go back to the previous case, which concludes the proof. $\square$

Theorem 14 now follows from Lemmas 15 and 19. $\square$

We next focus on $\mathcal{C}$-free graphs, where $\mathcal{C}$ is a (possibly infinite) family of cycles, and show a relation between 1-Edge Contraction(γ) and 1-Edge Contraction(γ_{t2}).

Lemma 20. *Let $\mathcal{C}$ be a (possibly infinite) family of cycles. If* 1-Edge Contraction(γ) *is* NP*-hard when restricted to $\mathcal{C}$-free graphs then* 1-Edge Contraction(γ_{t2}) *is* NP*-hard when restricted to $\mathcal{C}$-free graphs.*

Proof. Let G be a $\mathcal{C}$-free graph. We construct a $\mathcal{C}$-free graph $T(G)$ such that G is a Yes-instance for 1-Edge Contraction(γ) if and only if $T(G)$ is a Yes-instance for 1-Edge Contraction(γ_{t2}), as follows. For every vertex $v \in V(G)$, we attach a copy of the tree T_v depicted in Fig. 5 by connecting v to a_v. We let $T(G)$ be the resulting graph. Clearly, $T(G)$ is $\mathcal{C}$-free.

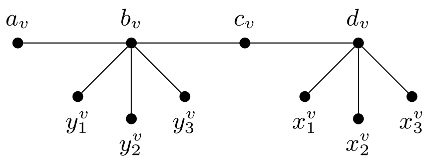

Fig. 5. The tree T_v.

Let us first show that $\gamma_{t2}(T(G)) = \gamma(G) + 2|V(G)|$. Clearly, if D is a minimum dominating set of G then $D \cup \{b_v, d_v : v \in V(G)\}$ is a semitotal dominating set of $T(G)$. Thus, $\gamma_{t2}(T(G)) \leq \gamma(G) + 2|V(G)|$. Conversely, let D be a minimum semitotal dominating set of $T(G)$. We claim that, for every $v \in V(G)$, $b_v \in D$. Indeed, suppose to the contrary that there exists $v \in V(G)$ such that $b_v \notin D$. Since for every $i \in [3]$, y_i^v must be dominated, it follows that $y_1^v, y_2^v, y_3^v \in D$. But then, $(D \setminus \{y_1^v, y_2^v\}) \cup \{b_v\}$ is a semitotal dominating set of $T(G)$ of size strictly less than $|D|$, a contradiction to the minimality of D.

Using similar arguments, we can show that, for every $v \in V(G)$, $d_v \in D$. This implies that, for every $v \in V(G)$ and $i \in [3]$, $c_v, x_i^v, y_i^v \notin D$. Moreover, if $a_v \in D$ for some $v \in V(G)$, then $(D \setminus \{a_v\}) \cup \{v\}$ is a semitotal dominating set of $T(G)$ of size at most $|D|$. Thus, $T(G)$ has a minimum semitotal dominating set D such that $D \cap \{a_v\colon\ v \in V(G)\} = \varnothing$, and, since $b_v, d_v \in D$ for every $v \in V(G)$, in fact $D \cap V(T_v) = \{b_v, d_v\}$ for every $v \in V(G)$. Let D be such a minimum semitotal dominating set. We claim that $D \setminus \{b_v, d_v\colon\ v \in V(G)\}$ is a dominating set of G. Indeed, since, for every $v \in V(G)$, $a_v \notin D$, necessarily $D \cap (N_{T(G)}[v] \setminus \{a_v\}) \neq \varnothing$ for, otherwise, v would not be dominated in D. Thus, $\gamma(G) \leq \gamma_{t2}(T(G)) - 2|V(G)|$ and, combined with the above inequality, we conclude that in fact equality holds.

Now assume that G is a Yes-instance for 1-Edge Contraction(γ) and let D be a minimum dominating set of G containing at least one edge $xy \in E(G)$ (see Theorem 2(i)). Then, clearly, $D \cup \{b_v, d_v\colon\ v \in V(G)\}$ is a minimum semitotal dominating set containing a friendly triple, namely x, y, b_y.

Conversely, assume that $T(G)$ is a Yes-instance for 1-Edge Contraction(γ_{t2}) and let D be a minimum semitotal dominating set containing a friendly triple (see Theorem 5), say x, y, z where $xy \in E(T(G))$ and $d_{T(G)}(y, z) \leq 2$. Now observe that either both x and y belong to $V(G)$, or there exists $v \in V(G)$ such that both x and y belong to $V(T_v)$. Indeed, if $x \in V(G)$ and $y \in V(T_v)$ for some $v \in V(G)$, then necessarily $v = x$ and $y = a_v$. But then, since $b_v \in D$ by the above, $D \setminus \{a_v\}$ is a semitotal dominating set of $T(G)$ of size strictly less that $|D|$, a contradiction to the minimality of D. Now if both x and y belong to $V(G)$ then, by the above, $(D \cap V(G)) \cup \{v : a_v \in D\}$ is a minimum dominating set of G containing an edge, namely xy. Next, assume that there exists $v \in V(G)$ such that $x, y \in V(T_v)$. As shown above, $\{x, y\} \cap \{y_1^v, y_2^v, y_3^v, x_1^v, x_2^v, x_3^v, c_v\} = \varnothing$ (it would otherwise contradict the minimality of D as $b_v, d_v \in D$) and so, $\{x, y\} = \{a_v, b_v\}$. But then, $v \notin D$ for otherwise, $D \setminus \{a_v\}$ would be a semitotal dominating set of $T(G)$ of size strictly less than $|D|$. Now consider a neighbour $w \in V(G)$ of v. Then $w \notin D$ for otherwise, $D \setminus \{a_v\}$ would be a semitotal dominating set of $T(G)$ of size strictly less than $|D|$, a contradiction to the minimality of D. But since w is dominated in D, w has a neighbour u in D. If $u = a_w$ then, by the above, $(D \cap V(G)) \cup \{t : a_t \in D\}$ is a minimum dominating set of G containing an edge, namely wv. Otherwise, $u \in V(G)$ and so, $(D \setminus \{a_v\}) \cup \{w\}$ is a minimum semitotal dominating set of $T(G)$ containing a friendly triple whose edge lies in $V(G)$, namely u, w, b_w, and we proceed as previously. Since, in any case, we can construct a minimum dominating set of G containing an edge, we conclude by Theorem 2(i) that G is a Yes-instance for 1-Edge Contraction(γ). □

In [8], the authors showed the following result for 1-Edge Contraction(γ).

Theorem 21 *([8]).* 1-Edge Contraction(γ) *is* NP*-hard when restricted to $\{C_3, \ldots, C_\ell\}$-free graphs for any $\ell \geq 3$, and when restricted to bipartite graphs.*

By combining Lemma 20 and Theorem 21, we obtain the following.

Theorem 22. 1-Edge Contraction(γ_{t2}) *is* NP*-hard when restricted to $\{C_3, \ldots, C_\ell\}$-free graphs for any $\ell \geq 3$, and when restricted to bipartite graphs.*

Finally, similar to the case of domination, we can show the following.

Theorem 23. 1-Edge Contraction(γ_{t2}) *is* NP*-hard when restricted to $\{P_6, P_4 + P_2\}$-free chordal graphs.*

Proof. We use the same construction as in [8, Theorem 3.1]: given an instance (G, ℓ) of Dominating Set, we construct an equivalent instance G' of 1-Edge Contraction(γ_{t2}) as follows. We denote by $\{v_1, \ldots, v_n\}$ the vertex set of G. The vertex set of the graph G' is given by $V(G') = V_0 \cup \ldots \cup V_\ell \cup \{x_0, \ldots, x_\ell, y\}$, where each V_i is a copy of the vertex set of G. We denote the vertices of V_i by $v_1^i, v_2^i, \ldots, v_n^i$. The adjacencies in G' are then defined as follows (see Fig. 6):

- $V_0 \cup \{x_0\}$ is a clique;
- $yx_0 \in E(G')$;

and for $1 \leq i \leq \ell$,

- V_i is an independent set;
- x_i is adjacent to all the vertices of $V_0 \cup V_i$;
- v_j^i is adjacent to $\{v_a^0 \mid v_a \in N_G[v_j]\}$ for any $1 \leq j \leq n$.

Claim 24. $\gamma_{t2}(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

Proof. It is clear that $\{x_0, x_1, \ldots, x_\ell\}$ is a semitotal dominating set of G' and so, $\gamma_{t2}(G') \leq \ell + 1$. Conversely, if $\gamma(G) \leq \ell$ and $\{v_{i_1}, \ldots, v_{i_p}\}$ is a minimum dominating set of G, it is easily seen that $\{v_{i_1}^0, \ldots, v_{i_p}^0, x_0\}$ is a semitotal dominating set of G'. Thus, $\gamma_{t2}(G') \leq \gamma(G) + 1$ and so, $\gamma_{t2}(G') \leq \min\{\gamma(G) + 1, \ell + 1\}$. Now, suppose to the contrary that $\gamma_{t2}(G') <$

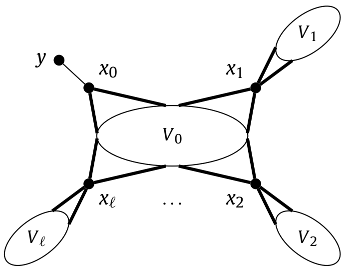

Fig. 6. The graph G' (thick lines indicate that the vertex x_i is adjacent to every vertex in V_0 and V_i, for $i = 0, \dots, \ell$).

$\min\{\gamma(G)+1, \ell+1\}$, and consider a minimum semitotal dominating set D' of G'. We first make the following simple observation.

Observation 25. For any semitotal dominating set D of G', $D \cap \{y, x_0\} \neq \varnothing$.

Since $\gamma_{t2}(G') < \ell + 1$, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$ (otherwise, $\{x_1, \dots, x_\ell\} \subset D'$ and, combined with Observation 25, D' would be of size at least $\ell + 1$). But then, $D'' = D' \cap (V_0 \cup V_i)$ must dominate every vertex in V_i, and so $|D''| \geq \gamma(G)$. Since $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \varnothing$), we then have $\gamma(G) \leq |D'| - 1$, a contradiction. Thus, $\gamma_{t2}(G') = \min\{\gamma(G)+1, \ell+1\}$. □

We now show that (G, ℓ) is a Yes-instance for Dominating Set with $\gamma(G) \geq 2$ if and only if G' is a Yes-instance for 1-Edge Contraction(γ_{t2}).

Assume first that $\gamma(G) \leq \ell$. Then $\gamma_{t2}(G') = \gamma(G) + 1$ by the previous claim, and, if $\{v_{i_1}, \dots, v_{i_p}\}$ is a minimum dominating set of G, then $\{v^0_{i_1}, \dots, v^0_{i_p}, x_0\}$ is a minimum semitotal dominating set of G' containing a friendly triple (recall that we assume that $\gamma(G) \geq 2$). Hence, by Theorem 5(i), G' is a Yes-instance for 1-Edge Contraction(γ_{t2}).

Conversely, assume that G' is a Yes-instance for 1-Edge Contraction(γ_{t2}), that is, there exists a minimum semitotal dominating set D' of G' containing a friendly triple (see Theorem 5(i)), say x, y, z where $xy \in E(G')$ and $d_{G'}(y, z) \leq 2$. Then Observation 25 implies that there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$: indeed, if this were not the case, then we would have, by Claim 24, that $\gamma_{t2}(G') = \ell + 1$. But then, D' would consist of $x_1, \dots, x_\ell$ and x_0, and so, D' would not contain a friendly triple, a contradiction. It follows that $D'' = D' \cap (V_0 \cup V_i)$ must dominate every vertex in V_i and, thus, $|D''| \geq \gamma(G)$. But $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \varnothing$) and so, by Claim 24, $\gamma(G) \leq |D'| - 1 \leq (\ell + 1) - 1$, that is, (G, ℓ) is a Yes-instance for Dominating Set.

Since it was shown in [8, Theorem 3.1] that G' is a $\{P_6, P_4 + P_2\}$-free chordal graph, the result follows. □

4.2. Polynomial cases

We now focus on graph classes for which 1-Edge Contraction(γ_{t2}) can be solved in polynomial time. A first simple approach to the problem, from which we obtain Proposition 26, is based on brute force.

Proposition 26. 1-Edge Contraction(γ_{t2}) *(respectively* 2-Edge Contraction(γ_{t2})*) can be solved in polynomial time when restricted to a graph class $\mathcal{C}$, if either*

(a) *$\mathcal{C}$ is closed under edge contractions and* Semitotal Dominating Set *can be solved in polynomial time on $\mathcal{C}$; or*
(b) *for every $G \in \mathcal{C}$, $\gamma_{t2}(G) \leq q$, where q is some fixed constant; or*
(c) *$\mathcal{C}$ is the class of $(H + K_1)$-free graphs, where $|V(H)| = q$ is a fixed constant and* 1-Edge Contraction(γ_{t2}) *(respectively* 2-Edge Contraction(γ_{t2})*) is polynomial-time solvable on H-free graphs.*

Proof. In order to prove item (a), it suffices to note that, if we can compute $\gamma_{t2}(G)$, $\gamma_{t2}(G/e)$ and $\gamma_{t2}(G/\{e, e'\})$, for any edges e, e' of G, in polynomial time, then we can determine whether a graph G is a Yes-instance for 1-Edge Contraction(γ_{t2}) or 2-Edge Contraction(γ_{t2}) in polynomial time.

For item (b), we proceed as follows. Given a graph G of $\mathcal{C}$, we consider every subset $S \subseteq V(G)$ with $|S| \leq q+1$ and check whether it is a semitotal dominating set of G. Since there are at most $O(n^{q+1})$ such possible subsets, we can determine the semitotal domination number of G and check whether the conditions given in Theorem 5 are satisfied in polynomial time.

Finally, so as to prove item (c), we provide the following algorithm. Let H and q be as stated and let G be a $(H + K_1)$-free graph. We first test whether G is H-free (note that this can be done in time $O(n^q)$). If this is the case, we use the polynomial-time algorithm for 1-Edge Contraction(γ_{t2}) (respectively 2-Edge Contraction(γ_{t2})) on H-free graphs. Otherwise, G has an induced subgraph isomorphic to H. But, since G is a $(H + K_1)$-free graph, $V(H)$ must then be a dominating

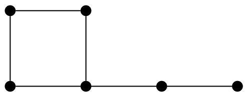

Fig. 7. The graph P.

set of G and so, $\gamma_{t2}(G) \leq 2q$. We then conclude by Proposition 26(b) that 1-Edge Contraction(γ_{t2}) (respectively 2-Edge Contraction(γ_{t2})) is also polynomial-time solvable in this case. □

We use below the following result by Galby et al. [8].

Lemma 27 *([8]). If G is a P_5-free graph and $\gamma(G) \geq 3$, then $ct_\gamma(G) = 1$.*

Lemma 28. *Let G be a P_5-free graph. If $\gamma_{t2}(G) = 2$ then G is a* No-*instance for* 1-Edge Contraction(γ_{t2})*, otherwise G is a* Yes-*instance for* 1-Edge Contraction(γ_{t2}).

Proof. Let G be a P_5-free graph. If $\gamma_{t2}(G) = 2$ then G is clearly a No-instance for 1-Edge Contraction(γ_{t2}). Assume henceforth that $\gamma_{t2}(G) \geq 3$. Since G is P_5-free, G is in particular (C_6, P_6, P)-free (see Fig. 7). It then follows from [11] that $\gamma(G) = \gamma_{t2}(G)$. Now, by Lemma 27, $ct_\gamma(G) = 1$, which implies that there exists a minimum dominating set of G which is not independent (see Theorem 2(i)). Amongst those non-independent minimum dominating sets, consider one D with the fewest unwitnessed vertices. Let us show that D is a semitotal dominating set.

Suppose to the contrary that there exists $w \in D$ such that w has no witness, and let $u \in D$ be a vertex such that $d_G(w, D \backslash \{w\}) = d_G(w, u)$. Since G is P_5-free, $d_G(u, w) \leq 3$, and, since $d_G(u, w) > 2$ by assumption, in fact $d_G(u, w) = 3$. Let x (respectively y) be the neighbour of u (respectively w) on a shortest path from u to w. We claim that $N_G(u) \cup N_G(w) \subseteq N_G(x) \cup N_G(y)$. Indeed, if a is a neighbour of u, then a is nonadjacent to w (otherwise $d_G(u, w) \leq 2$). But then, a is adjacent to either x or y for otherwise, $auxyw$ would induce a P_5. We conclude similarly if a is a neighbour of w. It follows that $(D \backslash \{u, w\}) \cup \{x, y\})$ is a dominating set which is not independent and contains fewer unwitnessed vertices than D, a contradiction to its minimality. Thus, D is a minimum semitotal dominating set.

Now consider $u, v \in D$ such that $uv \in E(G)$. If there exists $w \in D$ such that $d_G(w, \{u, v\}) \leq 2$, then u, v, w is a friendly triple contained in D, and we conclude by Theorem 5(i). Assume henceforth that no such vertex exists, and consider a vertex $w \in D$ closest to $\{u, v\}$. Since G is P_5-free, $d_G(w, \{u, v\}) \leq 3$, and, since $d_G(w, \{u, v\}) > 2$ by assumption, in fact $d_G(w, \{u, v\}) = 3$. Assume, without loss of generality, that $d_G(w, v) \geq d_G(w, u) = 3$ and denote by x (respectively y) the neighbour of u (respectively w) on a shortest path from u to w. Then, as previously, we have that $N_G(w) \cup N_G(u) \subseteq N_G(x) \cup N_G(y)$. It follows that $D' = (D \backslash \{u, w\}) \cup \{x, y\}$ is a minimum semitotal dominating set: indeed, by assumption, no vertex in $D \setminus \{v\}$ has u as a witness, and, since $N_G(w) \subseteq N_G(x) \cup N_G(y)$, any vertex in D witnessed by w is witnessed by x or y in D'. But D' contains a friendly triple, namely x, y, v, and, thus, $ct_{\gamma_{t2}}(G) = 1$ by Theorem 5(i). □

By combining Lemma 28 and Proposition 26(c), we obtain the following.

Theorem 29. *For any fixed $t \geq 0$,* 1-Edge Contraction(γ_{t2}) *is polynomial-time solvable when restricted to $(P_5 + tK_1)$-free graphs.*

Let us now present the last result of this section which concerns $P_3 + kP_2$-free graphs.

Theorem 30. *For any $k \geq 0$,* 1-Edge Contraction(γ_{t2}) *is polynomial-time solvable when restricted to $P_3 + kP_2$-free graphs.*

Proof. First observe that, if G does not contain an induced P_3, then G is a clique (recall that, by assumption, G is connected) and, thus, a No-instance for 1-Edge Contraction(γ_{t2}). Assume henceforth that $k \geq 1$ and let G be a $P_3 + kP_2$-free graph containing an induced $P_3 + (k-1)P_2$. The following proof is similar to that of [10, Theorem 2]. Let $A \subseteq V(G)$ be a set of vertices which induces a $P_3 + (k-1)P_2$, let $B \subset V(G)$ be the set of vertices at distance one from A and let $C \subset V(G)$ be the set of vertices at distance two from A. Note that, since G is $P_3 + kP_2$-free, the sets A, B and C partition $V(G)$ and C is an independent set. We call a vertex $v_1 \in C$ a *regular vertex* if there exist k vertices $v_2, \ldots, v_{k+1} \in C$ such that $v_1, \ldots, v_{k+1}$ are pairwise at distance at least four and, for every $i \in [k+1]$, $N(v_i)$ is a clique. We denote by $\mathcal{R}$ the set of regular vertices.

Claim 31. If $\mathcal{R} \neq \varnothing$ then the following holds.

(i) $\gamma(G) = \gamma_{t2}(G)$.
(ii) G is a Yes-instance for 1-Edge Contraction(γ) if and only if G is a Yes-instance for 1-Edge Contraction(γ_{t2}).

Proof. Let $V_1 \subseteq V(G) \setminus N[\mathcal{R}]$ be the set of vertices at distance one from $N[\mathcal{R}]$ and let $V_2 = V(G) \setminus (N[\mathcal{R}] \cup V_1)$. Note that, since $G[N[\mathcal{R}]]$ contains an induced $(k+1)P_2$, V_2 is P_3-free and, thus, $G[V_2]$ is a disjoint union of cliques.

The following was shown in the proof of [10, Claim 6].

Claim 32 *([10]).* If a vertex $v \in V_1$ is adjacent to a vertex in $N(c)$, for some regular vertex $c \in \mathcal{R}$, then there exists $c' \in \mathcal{R} \setminus \{c\}$ such that v is complete to $N(c')$.

Let D be a minimum dominating set of G. We first show how to modify D in order to obtain a minimum dominating set $S(D)$ of G satisfying the following.

(1) For every regular vertex c, $S(D) \cap N[c] = S(D) \cap N(c) = \{b\}$ for some vertex b with at least one neighbour in $V(G) \setminus N[c]$.
(2) For every $u \in S(D) \cap V_2$, u has at least one neighbour in V_1.
(3) If D contains an edge e then $S(D)$ also contains e.

First, it was shown in [10, Claim 7] that $|D \cap N[c]| = 1$ for every regular vertex c. Now suppose that D does not already satisfy (1), that is, there exists a regular vertex c such that $D \cap N[c] = \{v\}$ for some vertex v with no neighbour in $V(G) \setminus N[c]$ (possibly $v = c$). Since G is connected, c must have a neighbour b with at least one neighbour in $V(G) \setminus N[c]$. Since c is regular, any neighbour of c contains $N[c]$ in its neighbourhood and so $N[b] \supset N[v] = N[c]$.

It follows that $(D \setminus \{v\}) \cup \{b\}$ is a dominating set which contains fewer regular vertices violating (1) than D. Further note that, if D contains an edge e, then v is not an endpoint of e since $N[v] = N[c]$ and $|N[c] \cap D| = 1$. Thus, no edge is destroyed by replacing v with b in D. By reiterating this process if necessary, we obtain a minimum dominating set satisfying (1). Suppose next that D does not already satisfy (2), that is, there exists a vertex $v \in D \cap V_2$ such that v has no neighbour in V_1. Denote by C_v the clique of V_2 containing v. Note that, by minimality of D, $D \cap C_v = \{v\}$. Furthermore, since G is connected, $C_v \setminus \{v\} \neq \varnothing$ and $N_v = \{w \in C_v \setminus \{v\} \mid N(w) \cap V_1 \neq \varnothing\} \neq \varnothing$. Thus, the dominating set $(D \setminus \{v\}) \cup \{u\}$, where $u \in N_v$, contains fewer vertices in V_2 violating (2) than D. Additionally, if D contains an edge e, then v is not an endpoint of e and so, no edge is destroyed by replacing v with u in D. By reiterating this process if necessary, we obtain a minimum dominating set satisfying (2), and still satisfying (1). Furthermore, no edge, if it exists, is destroyed in a replacement process and thus, (3) holds true as well.

We now claim that $S(D)$ is in fact a semitotal dominating set. Indeed, let us show that every vertex in $S(D)$ is within distance at most two from another vertex in $S(D)$. Consider first a regular vertex $c \in \mathcal{R}$. By (1), $S(D) \cap N[c] = S(D) \cap N(c) = \{b\}$ for some vertex b with at least one neighbour outside of $N[c]$, that is, with at least one neighbour in $V_1 \cup N[\mathcal{R} \setminus \{c\}]$. Suppose first that b has a neighbour in $N[\mathcal{R} \setminus \{c\}]$, and let $c' \in \mathcal{R} \setminus \{c\}$ be a regular vertex such that b is adjacent to a vertex in $N[c']$. Let us show that $b \notin N(c')$ (note that $b \neq c'$ since C is an independent set). Suppose for a contradiction that b is adjacent to c'. Since c is a regular vertex, there exist k regular vertices $c_1, \dots, c_k \in \mathcal{R} \setminus \{c, c'\}$ pairwise at distance at least four, such that c is at distance at least four from each c_i (which is the reason why $c' \neq c_i$ for each i). Now for every $i \in [k]$, let $v_i \in N(c_i)$ be a neighbour of c_i. Then, for every $i \in [k]$, v_i is nonadjacent to c': indeed, if there exists $i \in [k]$ such that $v_i \in N(c')$ then v_i is adjacent to b since $N(c')$ is a clique (recall that c' is a regular vertex); but, then, $c_i v_i bc$ is a path of length three from c_i to c, a contradiction to the fact that $d(c_i, c) \geq 4$. It follows that the set $\{c, b, c'\} \cup \{c_i, v_i \mid i \in [k]\}$ induces a $P_3 + kP_2$, a contradiction. Therefore, $b \notin N(c')$; but, by (1), $N(c') \cap S(D) \neq \varnothing$, and $N(c')$ is a clique, and, so, b is at distance at most two from the vertex in $N(c') \cap S(D)$. Second, suppose that b has no neighbour in $N[\mathcal{R} \setminus \{c\}]$. Then, by assumption, b must have a neighbour $v \in V_1$. By Claim 32, there then exists $c' \in \mathcal{R} \setminus \{c\}$ such that v is complete to $N(c')$. But, by (1), $N(c') \cap S(D) \neq \varnothing$, and $b \notin N(c')$ by assumption, and, so, b is at distance two from the vertex in $N(c') \cap S(D)$. Thus, in both cases, we conclude that b is within distance at most two from another vertex in $S(D)$. Now, every vertex $v \in S(D) \cap V_1$ has a witness in $S(D) \cap N(\mathcal{R})$ as, by Claim 32, v is complete to $N(c)$ for some regular vertex c, and $N(c) \cap S(D) \neq \varnothing$ by (1). Similarly, every vertex in $S(D) \cap V_2$ is within distance at most two from a vertex in $S(D) \cap N(\mathcal{R})$. Indeed, every vertex $v \in S(D) \cap V_2$ has at least one neighbour $u \in V_1$ by (2). However, by Claim 32, u is complete to $N(c)$ for some regular vertex c, and $S(D) \cap N(c) \neq \varnothing$ by (1). It follows that every vertex in $S(D)$ has a witness and, thus, $S(D)$ is a semitotal dominating set of G as claimed. Since $\gamma(H) \leq \gamma_{t2}(H)$ for any graph H, we conclude that $\gamma(G) = \gamma_{t2}(G)$.

Now suppose that D initially contained an edge $e = uv$, that is, G is a Yes-instance for 1-Edge Contraction(γ). Then, by (3), $S(D)$ also contains the edge e. Suppose first that $u \in N(c)$ and $v \in N(c')$ for some $c, c' \in \mathcal{R}$ (note that $c \neq c'$ as, by (1), $|S(D) \cap N(v)| = 1$ for every regular vertex v). Since c is a regular vertex, there exist $c_1, \dots, c_k \in \mathcal{R}$ such that $c, c_1, \dots, c_k$ are pairwise at distance at least four (note that, since u and v are adjacent, $d(c, c') \leq 3$ and, so, $c' \neq c_i$ for every $i \in [k]$). For every $i \in [k]$, denote by v_i the vertex in $S(D) \cap N(c_i)$ (such vertices exist by (1)). Note that, since, for every $i \in [k]$, c is at distance at least four from c_i, u is nonadjacent to v_i. Similarly, for every $i \in [k]$, v is nonadjacent to c_i: indeed, if there exists $i \in [k]$ such that v is adjacent to c_i, then the path $cuvc_i$ has length $3 < d(c, c_i)$ (recall that, by assumption, c is at distance at least four from c_i), a contradiction. It follows that there exists $j \in [k]$ such that v is adjacent to v_j for, otherwise, $v, u, c, c_1, v_1, \dots, c_k, v_k$ induce a $P_3 + kP_2$. But, then, u, v, v_j is a friendly triple. Now suppose that one of u and v belongs to V_1, say $u \in V_1$ without loss of generality. By definition of V_1, there exists $w \in \mathcal{R}$ such that u is adjacent to a vertex of $N(w)$. By Claim 32, there then exists a regular vertex $c \in \mathcal{R} \setminus \{w\}$ such that u is complete to $N(c)$; and, by applying Claim 32 a second time with c, we conclude that there exists a regular vertex $c' \in \mathcal{R} \setminus \{c\}$ such that u is complete to $N(c')$. Note that c' could be w. Now, by (1), there are vertices x and y such that $S(D) \cap N(c) = \{x\}$ and $S(D) \cap N(c') = \{y\}$. Since u is complete to both $N(c)$ and $N(c')$ it follows that u is adjacent to both x and y. We claim that $x \neq y$. Indeed, suppose for a contradiction that $x = y$. Since c is a regular vertex, there exist $c_1, \dots, c_k \in \mathcal{R}$ such that $c, c_1, \dots, c_k$ are pairwise at distance at least four (note that, since $d(c,c') \leq 2$, $c' \neq c_i$ for every $i \in [k]$). Now, for every $i \in [k]$, let $v_i \in N(c_i)$ be a neighbour of c_i. Then, for every $i \in [k]$, v_i is nonadjacent to x: indeed, if there exists $i \in [k]$ such that $v_i \in N(x)$ then cxv_ic_i is a path of length three from c to c_i, a contradiction to the fact that $d(c,c_i) \geq 4$. Similarly, for every $i \in [k]$, v_i is nonadjacent to c': indeed, if there exists $i \in [k]$ such that $v_i \in N(c')$ then v_i is adjacent to $y = x$ since $N(c')$ is a clique (recall that c' is a regular vertex); but, then, c_iv_ixc is a path of length three from c_i to c, a contradiction to the fact that $d(c,c_i) \geq 4$. It follows that the set $\{c,x,c'\} \cup \{c_i, v_i \mid i \in [k]\}$ induces a $P_3 + kP_2$, a contradiction. Therefore, $x \neq y$. Assuming without loss of generality that $v \neq y$, we then have that u, v, y is a friendly triple. Finally, if $u, v \in V_2$ then, by (2), u is adjacent to some vertex $w \in V_1$. However, by the above, w is then complete to $N(c)$ for some regular vertex c, and since, by (1), $S(D) \cap N(c) = \{b\}$, it follows that u, v, b is a friendly triple. Since, in every case, we can find a friendly triple, we conclude by Theorem 5(i) that G is a Yes-instance for 1-Edge Contraction(γ_{t2}). Conversely, if there exists a minimum semitotal dominating set D of G containing a friendly triple, then D is, a fortiori, a minimum dominating set of G containing an edge and, thus, G is a Yes-instance for 1-Edge Contraction(γ). □

Proposition 33. *If $\mathcal{R} = \varnothing$ and G is a* No-*instance for* 1-Edge Contraction (γ_{t2}) *then* $\gamma_{t2}(G) \leq (k+1)(|A| + 2k + 4) + k + 6|A| - 4$.

Proof. To prove Proposition 33, we first prove the following claims. Assume henceforth that $\mathcal{R} = \varnothing$.

Claim 34. If G is a No-instance for 1-Edge Contraction(γ_{t2}) and D is a minimum semitotal dominating set of G, then every connected component of $G[D]$ has cardinality at most two and there are at most $|A|$ components of cardinality 2.

Proof. The first claim follows from Theorem 5(i), since any connected component of size at least three in $G[D]$ would contain a friendly triple. For the second claim, since any component of size two has to be at distance at least three to every other component (D would otherwise contain a friendly triple), every vertex of A can be adjacent to at most one component of size two. On the other hand, every size-two-component C_0 of $G[D]$ has to be adjacent to at least one vertex in A for otherwise $A \cup C_0$ would induce a $P_3 + kP_2$. □

Assume henceforth that G is a No-instance. In the following, given a minimum semitotal dominating set D of G, we denote by $D' \subseteq D$ the set of size-one components in D (note that D' is an independent set).

Claim 35. Let D be a minimum semitotal dominating set of G. If there exists a vertex $b \in B \cap D'$ such that b has more than one private neighbour in C, then $|B \cap D'| \leq k|A|$.

Proof. Assume that there exists a vertex $b \in B \cap D'$ which has at least two private neighbours in C, say x and y. Suppose for a contradiction that there are at least $k|A|$ further vertices in $B \cap D'$ besides b, say $b_1, \ldots, b_{k|A|}$. For every $i \in [k]$, there has to be a vertex $c_i \in C$ such that $N(c_i) \cap D \subsetneq \{b_{(i-1)|A|+1}, \ldots, b_{i|A|}\}$. Indeed, if, for some $i \in [k]$, no such vertex in C exists, then $(D \setminus \{b_{(i-1)|A|+1}, \ldots, b_{i|A|}\}) \cup A$ is a minimum semitotal dominating set containing a friendly triple P_3 in A (note indeed that A dominates all of $A \cup B$ and every vertex is within distance at most two of a vertex in A), a contradiction to Theorem 5. Thus, assume, without loss of generality, that $b_{i|A|}$ is adjacent to c_i for every $i \in [k]$. Then, the vertices $x, b, y, c_1, \ldots, c_k, b_{|A|}, b_{2|A|}, \ldots, b_{k|A|}$ induce a $P_3 + kP_2$, a contradiction. □

Claim 36. Let D be a minimum semitotal dominating set of G. If there exists a vertex $c \in C$ such that $|N(c) \cap D'| \geq 2$, then c is adjacent to all vertices in $B \cap D'$ except for at most $k|A| - 1$.

Proof. Assume that $c \in C$ has at least two neighbours $x, y \in B \cap D'$. Suppose for a contradiction that there are at least $k|A|$ vertices in $B \cap D'$ which are not adjacent to c, say $b_1, \ldots, b_{k|A|}$. As shown in the proof of Claim 35, there then has to be, for every $i \in [k]$, a vertex $c_i \in C$ such that $N(c_i) \cap D \subsetneq \{b_{(i-1)|A|+1}, \ldots, b_{i|A|}\}$. Assume, without loss of generality, that $b_{i|A|}$ is adjacent to c_i for every $i \in [k]$. Then the vertices $x, c, y, c_1, \ldots, c_k, b_{|A|}, b_{2|A|}, \ldots, b_{k|A|}$ induce a $P_3 + kP_2$, a contradiction. □

Claim 37. Let D be a minimum semitotal dominating set of G. If there are $|A|$ vertices in $B \cap D'$ which do not have a private neighbour in C, then $|B \cap D'| \leq (k+1)|A| - 1$.

Proof. Assume that $|B \cap D'| \geq (k+1)|A|$. Suppose for a contradiction that there are $|A|$ vertices, say $b_1, \ldots, b_{|A|} \in B \cap D'$, which have no private neighbours in C. Then, for every $i \in [|A|]$, any vertex $c \in N(b_i) \cap C$ has to be adjacent to at least two vertices in $B \cap D'$ and thus, by Claim 36, c has to be adjacent to at least $|A| + 1$ vertices in $B \cap D'$. But then, $(D \setminus \{b_1, \ldots, b_{|A|}\}) \cup A$ is a minimum semitotal dominating set containing a friendly triple, a contradiction to Theorem 5(i). □

Claim 38. Let D be a minimum semitotal dominating set of G. If there exists a vertex $v \in B \cap D'$ which has a private neighbour $c \in N(v) \cap C$ and a private neighbour $b \in N(v)$ such that c is not adjacent to b, then $|B \cap D'| \leq (k+1)|A|$.

Proof. If there exists a vertex in $B \cap D'$ with two private neighbours in C, then we conclude by Claim 35. Thus, from now on, we may assume that no vertex in $B \cap D'$ has more than one private neighbour in C. Now assume that there exists a vertex $v \in B \cap D'$ which has exactly one private neighbour $c \in C$ and another private neighbour $b \in B \cup A$ such that b and c are not adjacent. Suppose for a contradiction that $|B \cap D'| \geq (k+1)|A| + 1$. Then, by Claim 37, there are at most $|A| - 1$ vertices in $B \cap D'$ which do not have a private neighbour in C. Hence, besides v, there are at least $k|A| + 1$ further vertices in $B \cap D'$ which do have private neighbours in C. Let $b_1, \dots, b_{|A|+k} \in B \cap D'$ be $|A| + k$ of them (notice that $k|A| + 1 \geq |A| + k$, since $k \geq 1$) and let $c_1, \dots, c_{|A|+k} \in C$ be their private neighbours, respectively (recall that we assumed that no vertex in $B \cap D'$ has more than one private neighbour in C, and hence each b_i has in fact a unique private neighbour in C). By the pigeonhole principle, there are either k indices $i \in [|A| + k]$ such that c_i is nonadjacent to b, or $|A| + 1$ indices $i \in [|A| + k]$ such that c_i is adjacent to b. In the first case, assume, without loss of generality, that $c_1, \dots, c_k$ are nonadjacent to b. Then $c, v, b, b_1, \dots, b_k, c_1, \dots, c_k$ induce a $P_3 + kP_2$, a contradiction. In the second case, assume, without loss of generality, that b is complete to $\{c_1, \dots, c_{|A|+1}\}$. Claim 36 then implies that any vertex $c \in C$ with $|N(c) \cap D'| \geq 2$ is adjacent to at least $(k+1)|A| + 1 - (k|A| - 1) \geq |A| + 2$ vertices in $B \cap D'$. We then conclude that every vertex in C which is adjacent to a vertex in $\{b_1, \dots, b_{|A|+1}\}$ is adjacent to a vertex in $((B \cap D) \setminus \{b_1, \dots, b_{|A|+1}\}) \cup \{b\}$ as well, and, so, $(D \setminus \{b_1, \dots, b_{|A|+1}\}) \cup \{b\} \cup A$ is a minimum semitotal dominating set containing a friendly triple, a contradiction to Theorem 5(i). □

Claim 39. Let D be a minimum semitotal dominating set of G. If there exists a vertex $v \in V(G) \setminus D$ such that v has exactly two neighbours in D', then $|B \cap D'| \leq (k+1)(|A| + 1) - 1$.

Proof. Assume that there exists a vertex $v \in V(G) \setminus D$ such that v has exactly two neighbours in D', say b and b'. Suppose to the contrary that $|B \cap D'| \geq (k+1)(|A|+1)$. Then, by Claim 35, every vertex in $B \cap D'$ has at most one private neighbour in C. Moreover, Claim 37 ensures that there are at least $k(|A| + 1) + 2$ vertices $b_1, \dots, b_{k(|A|+1)+2} \in B \cap D'$ which do have a private neighbour in C, say $c_1, \dots, c_{k(|A|+1)+2} \in C$ respectively. Assume without loss of generality that b and b' are distinct from $b_1, \dots, b_{k(|A|+1)}$. Then there exist at least $k|A| + 1$ indices $i \in [k(|A| + 1)]$ such that c_i is adjacent to v: indeed, if there are at most $k|A|$ such indices, then there are at least k indices $i \in [k(|A|+1)]$ such that c_i is nonadjacent to v, say indices 1 through k without loss of generality. But, then, the vertices $b, v, b', b_1, \dots, b_k, c_1, \dots, c_k$ induce a $P_3 + kP_2$, a contradiction. Thus, assume, without loss of generality, that c_i is adjacent to v for every $i \in [k|A| + 1]$. Then $S = (D \setminus \{b_1, \dots, b_{|A|+1}\}) \cup \{v\} \cup A$ is a minimum semitotal dominating set of G: indeed, if some vertex $c \in C$ is adjacent to both b_i and b_j for two distinct $i, j \in [k(|A| + 1) + 2]$ then, by Claim 36, c is adjacent to at least one vertex in $S \cap \{b_\ell \mid \ell \in [k(|A| + 1) + 2]\}$. However, S contains a friendly triple, a contradiction to Theorem 5(i). □

Claim 40. There exists a minimum semitotal dominating set D of G with a maximum number of size-two components amongst all minimum semitotal dominating sets of G, such that the number of size-one components outside of C is at most $(k+1)(|A| + 2) + 2|A|$.

Proof. Let D be a minimum semitotal dominating set of G with the maximum number of size-two components amongst all minimum semitotal dominating sets of G, such that $|B \cap D'|$ has minimum size amongst all minimum semitotal dominating sets with the maximum number of size-two components. First note that $|D' \cap A| \leq |A|$. Now suppose for a contradiction that $|B \cap D'| \geq (k+1)(|A| + 2) + |A| + 1$. Then, by Claim 35, every vertex in $B \cap D'$ has at most one private neighbour in C. Moreover, Claim 37 ensures that there are at least $(k+1)(|A| + 2) + 2$ vertices $b_1, \dots, b_{(k+1)(|A|+2)+2} \in B \cap D'$ which do have a private neighbour in C, say $c_1, \dots, c_{(k+1)(|A|+2)+2} \in C$ respectively. Now observe that the set $S = (D \setminus \{b_1\}) \cup \{c_1\}$ is a dominating set of G of size $\gamma_{t2}(G)$ for, otherwise, b_1 has a private neighbour $p \in N(b_1)$ which is not adjacent to c_1, a contradiction to Claim 38. However, S cannot be a semitotal dominating set of G as it would contradict the fact that $|B \cap D'|$ has minimum size amongst all minimum dominating sets with the maximum number of size-two components. Thus, in S, either c_1 has no witness, or there exists a vertex $w \in D'$ which is at distance two from b_1 but at distance at least three from c_1 and every other vertex in D'. In the latter case, let v be a common neighbour of b_1 and w. Then, by assumption, v has exactly two neighbours in D', namely b_1 and w, a contradiction to Claim 39 as $|B \cap D'| \geq (k+1)(|A| + 2) + |A| + 1$ by assumption. Thus, assume that c_1 has no witness in S. Suppose that $N(c_1)$ contains two nonadjacent vertices, say x and y. Notice that neither x nor y is adjacent to some b_j, as otherwise b_j would be a witness for c_1. Then, since, for any $i \in [|A| + 2]$, the set $\{x, c_1, y, b_{(i-1)k+2}, c_{(i-1)k+2}, \dots, b_{ik+1}, c_{ik+1}\}$ cannot induce a $P_3 + kP_2$, it follows that x or y has to be adjacent to $c_{(i-1)k+j+1}$ for some $j \in [k]$, say, for every $i \in [|A| + 2]$, c_{ik} is adjacent to x or y without loss of generality. Then $S = (D \setminus \{b_{ik} \mid i \in [|A| + 2]\}) \cup \{x, y\} \cup A$ is a minimum semitotal dominating set of G: indeed, if some vertex $c \in C$ is adjacent to both b_i and b_j for two distinct $i, j \in [(k+1)(|A|+2)+2]$ then, by Claim 36, c is adjacent to at least one vertex in $S \cap \{b_\ell \mid \ell \in [(k+1)(|A|+2)+2]\}$. However, S contains a friendly triple, a contradiction to Theorem 5(i). It follows that $N(c_1)$ is a clique, and a similar reasoning shows that, in fact, $N(c_i)$ is a clique for every $i \in [(k+1)(|A|+2)+2]$. Now suppose that there exist two indices $i, j \in [(k+1)(|A|+2)+2]$ such that c_i and c_j are at distance two, and let b be a common neighbour of c_i and c_j. Then, by Claim 38, every private neighbour of b_i, besides c_i, has to be adjacent to c_i and, thus, to b as well, since $N(c_i)$ is a clique. But, then, $(D \setminus \{b_i\}) \cup \{b\}$ is a minimum semitotal dominating set containing more size-two components than D: indeed, b_i belongs to no size-two components of D by assumption (recall that $\{b_1, \dots, b_{(k+1)(|A|+2)+2}\} \subseteq D'$) and b is adjacent to b_j since $N(c_j)$ is a clique. This, however, contradicts the choice of D. Now if there exist two indices

$i, j \in [(k+1)(|A|+2)+2]$ such that c_i and c_j are at distance three, then there are two adjacent vertices $v_i \in N(c_i)$ and $v_j \in N(c_j)$. Then, for any $p \in \{i, j\}$, any private neighbour of b_p, besides c_p, has to be adjacent to c_p by Claim 38, and, thus, to v_p, since $N(c_p)$ is a clique. Moreover, any common neighbour of b_i and b_j must have at least one other neighbour in D' by Claim 39. It follows that $S' = (D \setminus \{b_i, b_j\}) \cup \{v_i, v_j\}$ is a dominating set of G of size γ_{t2}. We further claim that S' is a semitotal dominating set of G. Indeed, suppose to the contrary that there exists a vertex $w \in D'$ such that w has no witness in S'. Then w must be at distance two from b_i or b_j, say $d_G(w, b_i) = 2$ without loss of generality. Let v be a common neighbour of w and b_i. Then there must exist at least two indices $p, q \in [(k+1)(|A|+2)+2] \setminus \{i\}$ such that v is adjacent to c_p and c_q for, otherwise, $\{w, v, b_i\} \cup \{b_\ell, c_\ell \mid \ell : v \notin N(c_\ell)\}$ would contain an induced $P_3 + kP_2$, a contradiction. But this implies, in particular, that c_p and c_q are at distance two, a contradiction by the previous case. Thus, S' is a minimum semitotal dominating set of G. However, S' contains strictly more size-two components than D, a contradiction. It follows that, for any $i, j \in [(k+1)(|A|+2)+2]$, c_i and c_j are at distance at least four and, so, for any $i \in [(k+1)(|A|+2)+2]$, c_i is a regular vertex, a contradiction to the fact that $\mathcal{R}$ is empty. This implies that $|B \cap D'| \leq (k+1)(|A|+2) + |A|$ and, since $|D' \cap A| \leq |A|$ as observed above, the claim follows. □

Claim 41. Let D be a minimum semitotal dominating set of G with a maximum number of size-two components amongst all minimum semitotal dominating set of G. If $S \subseteq C \cap D'$ is a subset of vertices which are pairwise at distance at least three, and every vertex in S has two nonadjacent neighbours, then $|S| \leq (k+1)^2 - 1$.

Proof. Assume that $S' = \{c_1, \ldots, c_{k+1}\} \subseteq C \cap D'$ is a set of $k+1$ vertices which are pairwise at distance at least three such that, for every $i \in [k+1]$, there are two nonadjacent vertices $b_i, b'_i \in N(c_i)$. If, for every $i, j \in [k+1]$, the vertices c_i and c_j are at distance at least four, then $b_1, b'_1, c_1, \ldots, b_{k+1}, b'_{k+1}, c_{k+1}$ induce a $(k+1)P_3$, a contradiction. Hence, there exist two indices $i, j \in [k+1]$ such that c_i and c_j are at distance exactly three. In other words, there exist $i, j \in [k+1]$, $b_i \in N(c_i)$ and $b_j \in N(c_j)$ such that b_i and b_j are adjacent. Therefore, we have proven the following.

Observation 42. Let $S' \subseteq C \cap D'$ be a set of at least $k+1$ vertices which are pairwise at distance at least three. If every vertex in S' has two non-adjacent neighbours, then $N(S')$ contains an edge.

Suppose for a contradiction that there is a set $S \subseteq C \cap D'$ of at least $(k+1)^2$ vertices which are pairwise at distance at least three and such that, for every vertex $v \in S$, there are two nonadjacent vertices in $N(v)$. By the above remark, there exist two vertices at distance exactly three in S, which implies in particular that $N(S)$ contains an edge. Let $S_1 \subseteq N(S)$ be a maximum subset of $N(S)$ such that $G[S_1]$ contains exactly one edge and no two vertices in S_1 share a common neighbour in S. Observe that $|N(S_1) \cap S| = |S_1|$ since $d_G(u, v) \geq 3$ for each u and v in S, and that $S_1 \cup (N(S_1) \cap S)$ induces a $P_4 + (|S_1| - 2)P_2$. In particular, $|S_1| \leq k+1$. We construct a sequence of sets of vertices according to the following procedure.

1. Initialize $i = 1$. Set $C_1 = N(S_1) \cap S$ and $B_1 = N(C_1)$.
2. Increase i by one.
3. If there exists a set $L \subset N(S) \setminus B_{i-1}$ such that $G[L]$ contains exactly one edge and no two vertices in L share a common neighbour in S, then let S_i be such a set which has maximum size amongst all such sets. Otherwise, set $S_i = \varnothing$. Set $C_i = C_{i-1} \cup (N(S_i) \cap S)$ and $B_i = B_{i-1} \cup N(C_i)$.
4. If $|S_i| = |S_{i-1}|$, stop the procedure. Otherwise, return to Step 2.

Consider the value of i at the end of the procedure (note that $i \geq 2$). Observe that, for each $j > 1$, B_{j-1} is included in B_j by definition, and hence $N(S) \setminus B_j$ is included in $N(S) \setminus B_{j-1}$; in particular, for each $j > 1$, $S_j \subseteq N(S) \setminus B_{j-1}$. It follows that for every $j \in [i-1] \setminus \{1\}$, $|S_j|$ is not strictly larger than $|S_{j-1}|$ for, otherwise, S_{j-1} would not have been picked in Step 3, since it does not have maximum size (the procedure could have indeed picked S_j, for instance). Since, additionally, for any $j \in [i-1] \setminus \{1\}$, $|S_j| \neq |S_{j-1}|$ (the procedure would have otherwise stopped in j), it follows that, for any $j \in [i-1] \setminus \{1\}$, $|S_j| < |S_{j-1}|$. Since $|S_1| \leq k+1$, this implies in particular that, for any $j \in [i-1]$, $|S_j| \leq k+2-j$.

Let us now show that $S_j \neq \varnothing$ for any $j \in [i]$. Note that it is enough to show that $S_i \neq \varnothing$. Observe first that $|C_j| = \sum_{p \in [j]} |S_p|$ for every $j \in [i-1]$. Indeed, since $C_1 = N(S_1) \cap S$ and no two vertices in S_1 have a common neighbour in S, we have that $|C_1| = |S_1|$. Now $B_i = N(C_1) \cup \ldots \cup N(C_i)$, which implies in particular that, for each $j > 1$, S_j is a subset of $N(S) \setminus B_{j-1} = N(S) \setminus (N(C_1) \cup \ldots \cup N(C_{j-1}))$. Therefore, there is no edge with one endpoint in S_j and one endpoint in C_{j-1}. In other words, $N(S_j) \cap C_{j-1}$ is empty. It follows that, for any $j > 1$, $|C_j| = |C_{j-1} \cup (N(S_j) \cap S)| = |C_{j-1}| + |N(S_j) \cap S|$. Since no two vertices in S_j have a common neighbour in S, and since $d_G(u, v) \geq 3$ for each u and v in S, it follows that $|N(S_j) \cap S| = |S_j|$. Thus, since $|C_1| = |S_1|$, we conclude by induction that, for each $j > 1$:

$$|C_j| = |C_{j-1}| + |S_j| = \sum_{p \in [j]} |S_p|.$$

Since $|S| \geq (k+1)^2$ and $|S_j| \leq k+1$ for any $j \in [i-1]$, it follows that

$$|S \setminus C_j| = |S| - \sum_{p=1}^{j} |S_p| \geq (k+1)^2 - j(k+1) = (k+1-j)(k+1).$$

Thus, for any $j \in [\min\{i-1,k\}]$, we have that $|S \setminus C_j| \geq k+1$. It now follows from Observation 42 that, for any $j \in [\min\{i-1,k\}]$, the set $N(S \setminus C_j)$ contains an edge. Since the vertices in S are pairwise at distance at least three, it follows that $N(S \setminus C_j) = N(S) \setminus N(C_j)$ and it follows from the construction that $B_j = N(C_1) \cup \ldots \cup N(C_j) = N(C_1 \cup \ldots \cup C_j) = N(C_j)$ (the last equality follows from the fact that $C_1 \subseteq \ldots \subseteq C_j$, by definition of the sets $C_1, \ldots, C_j$). The two facts above imply that the set $N(S) \setminus B_j$ contains an edge and, in particular, the set S_{j+1} is not empty for any $j \in [\min\{i-1,k\}]$ since there exist, in Step 3, such sets L. We now claim that i cannot be larger than $k+1$. Indeed, if $i > k+1$ then, for any $j \in [k+1] \setminus 1$, $|S_j| < |S_{j-1}|$, with $S_{k+1} \neq \varnothing$ as shown above; in particular, $|S_{k+1}| \geq 2$, since S_{k+1} contains an edge by definition. However, $|S_j| \leq k+2-j$ for any $j \in [i-1]$, which implies that $|S_{k+1}| \leq 1$, a contradiction. Thus, $i \leq k+1$ and, so, $S_i \neq \varnothing$ by the above. Let us now prove the two following observations.

Observation 43. For any vertex $c \in N(S_i) \cap S$, every vertex $v \in N(c)$ is adjacent to a vertex in S_{i-1}.

Indeed, suppose for a contradiction that there exist $c \in N(S_i) \cap S$ and $v \in N(c)$ such that v has no neighbour in S_{i-1}. Let us show that, in this case, the procedure could have output $S_{i-1} \cup \{v\}$ in place of S_{i-1}, which, if true, would contradict the maximality of $|S_{i-1}|$. To this end, let us first show that no two vertices in $S_{i-1} \cup \{v\}$ have a common neighbour in S. Suppose to the contrary that this does not hold. Since, by definition, no two vertices in S_{i-1} have a common neighbour in S, there must exist $u \in S_{i-1}$ such that u and v have a common neighbour $c' \in S$. If $c' \neq c$ then $d(c,c') = 2$, a contradiction to the definition of S. Suppose therefore that $c = c'$ and let $c'' \in N(S_{i-1}) \cap S$ be a neighbour of u. Then $c \neq c''$ since $c \in N(S_i) \cap S$ and $N(S_i) \cap N(S_{i-1}) \cap S = \emptyset$ (indeed, recall that $N(S_i) \cap C_{i-1} = \emptyset$, as shown above, and that $N(S_{i-1}) \cap S \subseteq C_{i-1}$ by definition). It follows that $d(c,c'') = 2$, a contradiction to the definition of S. Therefore, no two vertices in $S_{i-1} \cup \{v\}$ have a common neighbour in S. Now, by construction, $c \in N(S_i) \cap S = C_i \setminus C_{i-1}$ and $v \in N(c)$, which implies that $v \in B_i \setminus B_{i-1}$; in particular, $v \notin B_{i-2} \subseteq B_{i-1}$. Since $S_{i-1} \subseteq N(S) \setminus B_{i-2}$, it follows that $S_{i-1} \cup \{v\} \subseteq N(S) \setminus B_{i-2}$. Finally observe that, by assumption, v is nonadjacent to every vertex in S_{i-1}, and, since S_{i-1} contains exactly one edge, it follows that $S_{i-1} \cup \{v\}$ contains exactly one edge. Therefore, by combining the above, we conclude that the procedure could have output $S_{i-1} \cup \{v\}$ in place of S_{i-1}, a contradiction which proves Observation 43.

Observation 44. For any vertex $c \in N(S_{i-1}) \cap S$, every vertex $v \in N(c)$ is adjacent to a vertex in S_i.

Indeed, suppose for a contradiction that there exists $c \in N(S_{i-1}) \cap S$ and $v \in N(c)$ such that v has no neighbour in S_i. Let us show that, in this case, the procedure could have output $S_i \cup \{v\}$ in place of S_{i-1}, which, if true, would contradict the maximality of S_{i-1} (recall, indeed, that $|S_i| = |S_{i-1}|$ by construction). To this end, let us first show that no two vertices in $S_i \cup \{v\}$ have a common neighbour in S. Suppose to the contrary that this does not hold. Since, by construction, no two vertices in S_i have a common neighbour in S, there must exist $u \in S_i$ such that u and v have a common neighbour $c' \in S$. If $c \neq c'$ then $d(c,c') = 2$, a contradiction to the definition of S. Suppose therefore that $c = c'$ and let $c'' \in N(S_i) \cap S$ be a neighbour of u. Then $c \neq c''$, since $c \in N(S_{i-1}) \cap S$ and $N(S_i) \cap N(S_{i-1}) \cap S = \emptyset$, and, so, $d(c,c'') = 2$, a contradiction to the definition of S. Now, by construction, $S_i \subseteq N(S) \setminus B_{i-1} \subseteq N(S) \setminus B_{i-2}$ since $B_{i-2} \subseteq B_{i-1}$. Since c has a neighbour in S_{i-1} it follows that c cannot be contained in C_{i-2}. Since v cannot have two neighbours in S (recall that the vertices in S are pairwise at distance at least three), it follows that v cannot have a neighbour in C_{i-2} and thus cannot be contained in B_{i-2}. Furthermore, by assumption, v has no neighbour in S_i, and since, by construction, S_i contains exactly one edge, it follows that $S_i \cup \{v\}$ contains exactly one edge. Therefore, by combining the above, we conclude that the procedure could have output $S_i \cup \{v\}$ in place of S_{i-1}, a contradiction which proves Observation 44.

To conclude, let us show that the set $T = (D \setminus (N(S_i \cup S_{i-1}) \cap S)) \cup (S_i \cup S_{i-1})$ is a minimum semitotal dominating set of G. Observe first that, for any $j \in \{i-1, i\}$, since no two vertices in S_j have a common neighbour in S, and since $d_G(u,v) \geq 3$ for each u and v in S, it follows that $|N(S_j) \cap S| = |S_j|$. Furthermore, by construction, $S_{i-1} \cap S_i = \emptyset$ (since $S_i \subseteq N(S) \setminus B_{i-1}$ and $S_{i-1} \subseteq B_{i-1}$) and $N(S_{i-1}) \cap N(S_i) \cap S = \emptyset$ (this implies in particular that $N(S_{i-1}) \cap S$ and $N(S_i) \cap S$ are disjoint sets), which implies that

$$\begin{aligned}|N(S_i \cup S_{i-1}) \cap S| &= |N(S_i) \cap S \cup N(S_{i-1}) \cap S| = |N(S_i) \cap S| + |N(S_{i-1}) \cap S| \\ &= |S_i| + |S_{i-1}| = |S_i \cup S_{i-1}|.\end{aligned}$$

Since, further, we have that S is a subset of D, $S_i \cup S_{i-1} \subset N(S)$ and $N(S) \cap S = \varnothing$, we can conclude that $|T| = |D|$. Furthermore, by Observations 43 and 44, any vertex which is dominated (or witnessed) by a vertex in $N(S_i) \cap S$ or $N(S_{i-1}) \cap S$ has to be dominated (or witnessed) by a vertex in S_{i-1} or S_i, respectively. However, T contains strictly more size-two components than D: indeed, no vertex of S belongs to a size-two component of D by assumption (recall that $S \subseteq D'$), while S_i and S_{i-1} both contain an edge. This, however, contradicts the choice of D, which concludes the proof. $\square$

Claim 45. Let D be a minimum semitotal dominating set of G with a maximum number of size-two components amongst all minimum semitotal dominating set of G. Then the number of vertices in $C \cap D'$ which are at distance two from another vertex in $C \cap D'$ is at most $2|A| + (k+1)^2 - 3$.

Proof. If every two vertices in $C \cap D'$ are at distance at least three from one another, then the claim trivially holds. Thus, assume that there are two vertices in $C \cap D'$ which are at distance two from one another. Let $\mathcal{S} = \arg\max_{S \subseteq B} |N(S) \cap C \cap D'| - |S|$ and let $S \in \mathcal{S}$ be a set of minimum size in $\mathcal{S}$. As there are two vertices in $C \cap D'$ which have a common neighbour, $\mathcal{S}$ is non-empty. If $|N(S) \cap C \cap D'| \geq |A| + |S|$, then $D \setminus (N(S) \cap C \cap D') \cup S \cup A$ is a semitotal dominating set of G which has cardinality at most $|D|$ and which contains a P_3, a contradiction to Theorem 5(i). Hence, $|N(S) \cap C \cap D'| < |S| + |A|$. We now claim that every vertex in S is adjacent to two vertices in $C \cap D'$ which are adjacent to no other vertex in S. Indeed, if there exists a vertex $s \in S$ such that every one of its neighbours in $C \cap D'$ is adjacent to another vertex in S, then we could remove s from S without changing the cardinality of $|N(S) \cap C \cap D'|$, thereby contradicting the fact that $S \in \mathcal{S}$. If a vertex $s \in S$ has only one neighbour c in $C \cap D'$ which is adjacent to no other vertex in S, then removing s from S would only remove c from $N(S) \cap C \cap D'$, thus leaving the value of $|N(S) \cap C \cap D| - |S|$ unchanged while decreasing the cardinality of S, a contradiction to the minimality of $|S|$. This implies that $|N(S) \cap C \cap D'| \geq 2|S|$. Combined with the inequality above, it follows that $|S| < |A|$ and $|N(S) \cap C \cap D'| \leq 2|A| - 2$. Now denote by $C' = (C \cap D') \setminus N(S)$ the set of vertices in $C \cap D'$ which are not adjacent to a vertex in S. Note that every pair of vertices $c, c' \in C'$ does not have a common neighbour b for, otherwise, $S' = S \cup \{b\}$ would be such that $|S'| = |S| + 1$ and $|N(S') \cap C \cap D'| \geq |N(S) \cap C \cap D'| + 2$ and, thus, $|N(S') \cap C \cap D'| - |S'| > |N(S) \cap C \cap D'| - |S|$, a contradiction to the choice of S. Hence, C' is a set of vertices which are pairwise at distance at least three. By Claim 41, it follows that at most $(k+1)^2 - 1$ vertices in C' do not have cliques as neighbourhoods. Denote $C'' \subset C'$ the set of vertices whose neighbourhoods are cliques. Note that no vertex c in C'' can be at distance two to any other vertex c' in $C \cap D'$ for, otherwise, we could remove c from D' and replace it with a common neighbour of c and c', thus yielding a minimum semitotal dominating set containing strictly more size-two components than D, a contradiction to the choice of D. Thus, every vertex in $C \cap D'$ which has a common neighbour with another vertex in $C \cap D'$ must be contained in $N(S) \cap C \cap D'$ or in $C' \setminus C''$, which together have cardinality at most $2|A| + (k+1)^2 - 3$. □

Claim 46. There exists a minimum semitotal dominating set D of G such that $|D'| \leq (k+1)(|A| + 2k + 4) + k + 4(|A| - 1)$.

Proof. It follows from Claim 40 that there exists a minimum semitotal dominating set D with the maximum number of size-two components amongst all minimum semitotal dominating sets of G such that $|D' \setminus C| \leq (k+1)(|A| + 2) + 2|A|$. Let $C_1 \subset C \cap D'$ be the set of vertices in $C \cap D'$ which are at distance at least three to every other vertex in $C \cap D'$. Let $C_2 \subseteq C_1$ be the set of vertices in C_1 whose neighbourhood is a clique. Suppose for a contradiction that there are two vertices $c, c' \in C_2$ which are at distance three. Let $b \in N(c)$ and $b' \in N(c')$ be two adjacent vertices. Then $(D \setminus \{c, c'\}) \cup \{b, b'\}$ is a minimum semitotal dominating set containing strictly more size-two components than D, a contradiction to the choice of D. Thus, the vertices in C_2 are pairwise at distance at least four from one another and, so, $|C_2| \leq k$ as $\mathcal{R} = \varnothing$. It now follows from Claim 45 that $|(C \cap D') \setminus C_1| \leq 2|A| + (k+1)^2 - 3$ and from Claim 41 that $|C_1 \setminus C_2| \leq (k+1)^2 - 1$, which implies the claim. □

Proposition 33 now follows from Claims 34 and 46. □

Consider now the following algorithm whose correctness is guaranteed by Claim 31 and Proposition 33.

1. Compute A, B, C and $\mathcal{R}$.
2. If $\mathcal{R} \neq \varnothing$ then check whether G is a Yes-instance for 1-Edge Contraction(γ).
   - 2.1 If the answer is yes then output Yes.
   - 2.2 Otherwise output No.
3. If $\mathcal{R} = \varnothing$ then check whether there exists a semitotal dominating set of size at most $(k+1)(|A| + 2k + 4) + k + 6|A| - 4$.
   - 3.1 If the answer is no then output Yes.
   - 3.2 Otherwise, determine whether there exists a minimum semitotal dominating set containing a friendly triple or not using brute force (see Theorem 5(i)).

Regarding its complexity, first note that the sets in Step 1 can be computed in time $n^{O(k)}$ by simple brute force (recall that, by definition, $|A| = 3 + 2(k-1)$). Furthermore, it is shown in [10] that checking whether G is a Yes-instance for 1-Edge Contraction(γ) can be done in polynomial time. Thus, Step 2 can be done in polynomial time. Checking whether there exists a minimum semitotal dominating set of size at most $(k+1)(|A| + 2k + 4) + k + 6|A| - 4$ containing a friendly triple can also be done in polynomial time (by simple brute force). We conclude that the algorithm above runs in polynomial time. □

4.3. Proof of Theorem 1

We finally prove Theorem 1. Let H be a graph. If H contains a cycle then 1-Edge Contraction(γ_{t2}) is NP-hard when restricted to H-free graphs by Theorem 22. Thus, we may assume that H is a forest. If H contains a vertex of degree at least three, then H contains an induced claw and, so, 1-Edge Contraction(γ_{t2}) is coNP-hard when restricted to H-free graphs by Theorem 7. Assume henceforth that H is a linear forest. If H contains a path on at least six vertices, then 1-Edge Contraction(γ_{t2}) is NP-hard when restricted to H-free graphs by Theorem 23. Thus, we may assume that every connected component of H induces a path on at most five vertices. Now suppose that H contains a path on at least four vertices. If H has another connected component on more than one vertex, then 1-Edge Contraction(γ_{t2}) is NP-hard when restricted to H-free graphs by Theorem 23. Otherwise, every other connected component of H (if any) contains exactly one vertex, in which case 1-Edge Contraction(γ_{t2}) is polynomial-time solvable by Theorem 29. Now suppose that the longest path in H has length three. If H has another connected component on three vertices, then 1-Edge Contraction(γ_{t2}) is coNP-hard by Theorem 14. Otherwise, every other connected component of H (if any) has at most two vertices, in which case 1-Edge Contraction(γ_{t2}) is polynomial-time solvable when restricted to H-free graphs by Theorem 30. Finally, if every connected component of H has at most two vertices, then 1-Edge Contraction(γ_{t2}) is polynomial-time solvable when restricted H-free graphs by Theorem 30, which concludes the proof.

5. Conclusion

It has been shown in [11] that the complexities of the Dominating Set problem (that is, given a graph G and an integer $k \geq 0$, does there exist a dominating set of size at most k?), the Total Dominating Set problem (given a graph G and an integer $k \geq 0$, does there exist a total dominating set of size at most k?) and the Semitotal Dominating Set problem (given a graph G and an integer $k \geq 0$, does there exist a semitotal dominating set of size at most k?) agree on all monogenic graph classes. Interestingly, this is no longer the case when we consider blocker problems with respect to these parameters together with edge contractions: combining our results with the complexity dichotomies for 1-Edge Contraction(γ) and 1-Edge Contraction(γ_t) obtained in [8,10] and [9], respectively, we can observe that the complexities of 1-Edge Contraction(γ_{t2}) and 1-Edge Contraction(γ_t) disagree on some monogenic graph classes. Whether there is a hereditary graph class on which 1-Edge Contraction(γ) and 1-Edge Contraction(γ_{t2}) differ remains an open question. We note however that, if such a class exists, its characterising set of forbidden induced subgraphs has to contain at least two graphs. In light of Lemma 20, we conjecture that such a graph class, if it exists, has to have a graph non-isomorphic to a cycle as a forbidden induced subgraph.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

References

[1] Cristina Bazgan, Cédric Bentz, Christophe Picouleau, Bernard Ries, Blockers for the stability number and the chromatic number, Graphs Comb. 31 (1) (2015) 73–90.
[2] Cristina Bazgan, Sonia Toubaline, Zsolt Tuza, The most vital nodes with respect to independent set and vertex cover, Discrete Appl. Math. 159 (2011) 1933.
[3] Cédric Bentz, Costa Marie-Christine, Dominique de Werra, Christophe Picouleau, Bernard Ries, Rico Zenklusen, Blockers and transversals in some subclasses of bipartite graphs: when caterpillars are dancing on a grid, Discrete Math. 310 (2010) 132.
[4] Guantao Chen, Julia Ehrenmüller, Cristina G. Fernandes, Carl Georg Heise, Songling Shan, Ping Yang, Amy N. Yates, Nonempty intersection of longest paths in series–parallel graphs, Discrete Math. 340 (3) (2017) 287–304.
[5] Reinhard Diestel, Graph Theory, Springer-Verlag, Heidelberg, 2016, p. 37.
[6] Öznur Yaşar Diner, Daniël Paulusma, Christophe Picouleau, Bernard Ries, Contraction blockers for graphs with forbidden induced paths, in: Algorithms and Complexity, Springer International Publishing, 2015, pp. 194–207.
[7] Öznur Yaşar Diner, Daniël Paulusma, Christophe Picouleau, Bernard Ries, Contraction and deletion blockers for perfect graphs and H-free graphs, Theor. Comput. Sci. 746 (2018) 49–72.
[8] Esther Galby, Paloma T. Lima, Bernard Ries, Reducing the domination number of graphs via edge contractions and vertex deletions, Discrete Math. 344 (1) (2021) 112–169.
[9] Esther Galby, Felix Mann, Bernard Ries, Blocking total dominating sets via edge contractions, Theor. Comput. Sci. 877 (2021) 18–35.
[10] Esther Galby, Felix Mann, Bernard Ries, Reducing the domination number of $P_3 + kP_2$-free graphs via one edge contraction, Discrete Appl. Math. 305 (2021) 205–210.
[11] Esther Galby, Andrea Munaro, Bernard Ries, Semitotal domination: new hardness results and a polynomial-time algorithm for graphs of bounded mim-width, Theor. Comput. Sci. 814 (2020) 28–48.
[12] W. Goddard, M.A. Henning, C.A. McPillan, Semitotal domination in graphs, Util. Math. 94 (2014) 67–81.
[13] Jia Huang, Jun-Ming Xu, Domination and total domination contraction numbers of graphs, Ars Comb. 94 (2010) 431–443.
[14] Chaya Keller, Micha A. Perles, Blockers for simple hamiltonian paths in convex geometric graphs of even order, Discrete Comput. Geom. 60 (1) (2018) 1–8.
[15] John M. Lewis, Mihalis Yannakakis, The node-deletion problem for hereditary properties is NP-complete, J. Comput. Syst. Sci. 20 (2) (1980) 219–230.
[16] Paloma T. Lima, Vinicius F. dos Santos, Ignasi Sau, Uéverton S. Souza, Reducing graph transversals via edge contractions, J. Comput. Syst. Sci. 120 (2021) 62–74.

[17] Paloma T. Lima, Vinicius F. dos Santos, Ignasi Sau, Uéverton S. Souza, Prafullkumar Tale, Reducing the vertex cover number via edge contractions, in: 47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022), in: Leibniz International Proceedings in Informatics (LIPIcs), vol. 241, 2022, pp. 69:1–69:14.

[18] Foad Mahdavi Pajouh, Vladimir Boginski, Eduardo Pasiliao, Minimum vertex blocker clique problem, Networks 64 (2014) 48.

[19] C. Moore, J.M. Robson, Hard tiling problems with simple tiles, Discrete Comput. Geom. 26 (4) (2001) 573–590.

[20] Farzaneh Nasirian, Foad Mahdavi Pajouh, Josephine Namayanja, Exact algorithms for the minimum cost vertex blocker clique problem, Comput. Oper. Res. 103 (2019) 296–309.

[21] Foad Mahdavi Pajouh, Jose L. Walteros, Vladimir Boginski, Eduardo L. Pasiliao, Minimum edge blocker dominating set problem, Eur. J. Oper. Res. 247 (1) (2015) 16–26.

[22] Daniël Paulusma, Christophe Picouleau, Bernard Ries, Reducing the clique and chromatic number via edge contractions and vertex deletions, in: ISCO 2016, in: LNCS, vol. 9849, 2016, pp. 38–49.

[23] Daniël Paulusma, Christophe Picouleau, Bernard Ries, Blocking independent sets for H-free graphs via edge contractions and vertex deletions, in: TAMC 2017, in: LNCS, vol. 10185, 2017, pp. 470–483.

[24] Daniël Paulusma, Christophe Picouleau, Bernard Ries, Critical vertices and edges in H-free graphs, Discrete Appl. Math. 257 (2019) 361–367.

[25] Dieter Rautenbach, Jean-Sébastien Sereni, Transversals of longest paths and cycles, SIAM J. Discrete Math. 28 (1) (2014) 335–341.

[26] Thomas J. Schaefer, The complexity of satisfiability problems, in: Proceedings of the Tenth Annual ACM Symposium on Theory of Computing, STOC'78, ACM, 1978, pp. 216–226.

[27] Rico Zenklusen, Matching interdiction, Discrete Appl. Math. 158 (15) (2010) 1676–1690.

[28] Rico Zenklusen, Network flow interdiction on planar graphs, Discrete Appl. Math. 158 (13) (2010) 1441–1455.