Discrete Applied Mathematics 180 (2015) 101–110

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Coloring graphs characterized by a forbidden subgraph[☆]

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ARTICLE INFO

Article history:
Received 8 June 2012
Received in revised form 13 November 2013
Accepted 5 August 2014
Available online 27 August 2014

Keywords:
Complexity
Algorithms
Graph coloring
Forbidden subgraphs

ABSTRACT

The Coloring problem is to test whether a given graph can be colored with at most k colors for some given k, such that no two adjacent vertices receive the same color. The complexity of this problem on graphs that do not contain some graph H as an induced subgraph is known for each fixed graph H. A natural variant is to forbid a graph H only as a subgraph. We call such graphs strongly H-free and initiate a complexity classification of Coloring for strongly H-free graphs. We show that Coloring is NP-complete for strongly H-free graphs, even for $k = 3$, when H contains a cycle, has maximum degree at least 5, or contains a connected component with two vertices of degree 4. We also give three conditions on a forest H of maximum degree at most 4 and with at most one vertex of degree 4 in each of its connected components, such that Coloring is NP-complete for strongly H-free graphs even for $k = 3$. Finally, we classify the computational complexity of Coloring on strongly H-free graphs for all fixed graphs H up to seven vertices. In particular, we show that Coloring is polynomial-time solvable when H is a forest that has at most seven vertices and maximum degree at most 4.

1. Introduction

Graph coloring involves the labeling of the vertices of some given graph by integers called colors such that no two adjacent vertices receive the same color. The corresponding Coloring problem is to decide whether a graph can be colored with at most k colors for some given integer k. Due to the fact that Coloring is NP-complete for any fixed $k \geq 3$ [15], there has been considerable interest in studying its complexity when restricted to certain graph classes. One of the most well-known results in this respect is due to Grötschel, Lovász, and Schrijver [9] who show that Coloring is polynomial-time solvable on perfect graphs. A well-known structural result that is useful for the design of algorithms for special graph classes is Brooks' Theorem (Theorem 5.2.4 in [6]), which states that any connected graph G that is neither complete nor an odd cycle can be colored with at most $\Delta(G)$ colors where $\Delta(G)$ is the maximum degree of G. General motivation, background and related work on coloring problems restricted to special graph classes can be found in several surveys [17,18].

We study the complexity of the Coloring problem restricted to graph classes defined by forbidding a graph H as a (not necessarily induced) subgraph. So far, Coloring has not been studied in the literature as regards to such graph classes. Before we summarize some related results and present our results, we first state the necessary terminology and notations.

☆ This paper was supported by EPSRC (EP/G043434/1) and the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ERC Grant Agreement n. 267959. The results of this paper have appeared in the proceedings of MFCS 2012.

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http://dx.doi.org/10.1016/j.dam.2014.08.008

1.1. Terminology

We consider finite undirected graphs without loops and multiple edges. We refer to the textbook of Diestel [6] for any undefined graph terminology. Let $G = (V, E)$ be a graph. The subgraph of G induced by a subset $U \subseteq V$ is denoted by $G[U]$. The graph $G - u$ is obtained from G by removing vertex u. For a vertex u of G, its *open neighborhood* is $N(u) = \{v \mid uv \in E\}$, its *closed neighborhood* is $N[u] = N(u) \cup \{u\}$, and its degree is $d(u) = |N(u)|$. The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$.

The *length* of a path or a cycle is the number of its edges. The *distance* $\mathrm{dist}(u, v)$ between two vertices u and v of G is the length of a shortest path between them. The *girth* $g(G)$ is the length of a shortest cycle in G.

For two graphs F and G, we may write $G \supseteq F$ if G contains F as a subgraph. We say that G is *(strongly) H-free* for some graph H if G has no subgraph isomorphic to H; note that this is more restrictive than forbidding H as an induced subgraph.

A *subdivision* of an edge $uv \in E$ is the operation that removes uv and adds a new vertex adjacent to u and v. A graph H is a *subdivision* of G if H is obtained from G by a sequence of edge subdivisions.

A *coloring* of G is a mapping $c : V \to \{1, 2, \ldots\}$, such that $c(u) \neq c(v)$ if $uv \in E$. We call $c(u)$ the *color* of u. A *k-coloring* of G is a coloring c of G with $1 \leq c(u) \leq k$ for all $u \in V$. If G has a k-coloring, then G is called *k-colorable*. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. The k-Coloring problem is to test whether a graph admits a k-coloring for some fixed integer k. If k is in the input, then we call this problem Coloring.

The graphs C_n, K_n, and P_n denote the cycle, complete graph and path on n vertices, respectively.

1.2. Related work

Král', Kratochvíl, Tuza and Woeginger [13] completely determined the computational complexity of Coloring for graph classes characterized by a forbidden *induced* subgraph and achieved the following dichotomy. Here, $P_1 + P_3$ denotes the disjoint union of P_1 and P_3.

Theorem 1 (*[13]*). *If some fixed graph H is a (not necessarily proper) induced subgraph of P_4 or of $P_1 + P_3$, then* Coloring *is polynomial-time solvable on graphs with no induced subgraph isomorphic to H; otherwise it is* NP*-complete on this graph class.*

The complexity classification of the k-Coloring problem for graphs with no induced subgraphs isomorphic to some fixed graph H is still open. For $k = 3$, it has been classified for graphs H up to six vertices [3], and for $k = 4$ for graphs H up to five vertices [8]. We refer to the latter paper for a survey on the complexity status of k-Coloring for graph classes characterized by a forbidden induced subgraph and to a recent paper of Huang [10], who showed that 5-Coloring is NP-complete for P_6-free graphs and that 4-Coloring is NP-complete for P_7-free graphs.

1.3. Our results

Recall that a strongly H-free graph denotes a graph with no subgraph isomorphic to some fixed graph H. Forbidding a graph H as an induced subgraph is equivalent to forbidding H as a subgraph if and only if H is a complete graph (a graph with an edge between any two distinct vertices). Hence, Theorem 1 tells us that Coloring is NP-complete for strongly H-free graphs if H is a complete graph on at least three vertices. We extend this result by proving the following two theorems in Sections 2 and 3, respectively; note that the case when H is a complete graph is covered by condition (a) of Theorem 2. The trees $T_1, \ldots, T_6$ are displayed in Fig. 1. For an integer $p \geq 0$, the graph T_2^p is the graph obtained from T_2 after subdividing the edge st p times; note that $T_2^0 = T_2$.

Theorem 2. 3-Coloring *(and hence* Coloring*) is* NP*-complete for strongly H-free graphs if*

(a) *H contains a cycle, or*
(b) *$\Delta(H) \geq 5$, or*
(c) *H has a connected component with at least two vertices of degree 4, or*
(d) *H contains a subdivision of the tree T_1 as a subgraph, or*
(e) *H contains the tree T_2^p as a subgraph for some $0 \leq p \leq 9$, or*
(f) *H contains one of the trees T_3, T_4, T_5, T_6 as a subgraph.*

Theorem 3. Coloring *is polynomial-time solvable for strongly H-free graphs if*

(a) *H is a forest with $\Delta(H) \leq 3$, such that each connected component has at most one vertex of degree 3, or*
(b) *H is a forest with $\Delta(H) \leq 4$ and $|V_H| \leq 7$.*

Theorems 1–3 tell us that the Coloring problem behaves differently on graphs characterized by forbidding H as an induced subgraph or as a subgraph. As a consequence of Theorems 2 and 3(b) we can classify the Coloring problem on strongly H-free graphs for graphs H up to 7 vertices. The problem is NP-complete if H is not a forest or $\Delta(H) \geq 5$, and polynomial-time solvable otherwise.

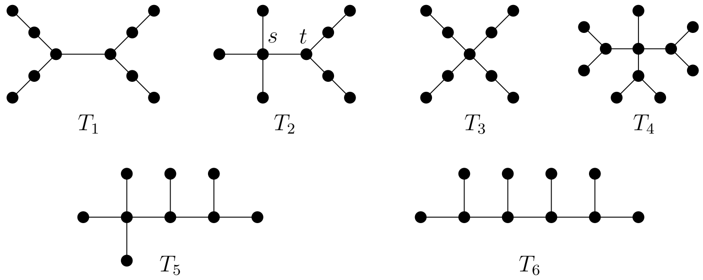

Fig. 1. The trees $T_1, \ldots, T_6$.

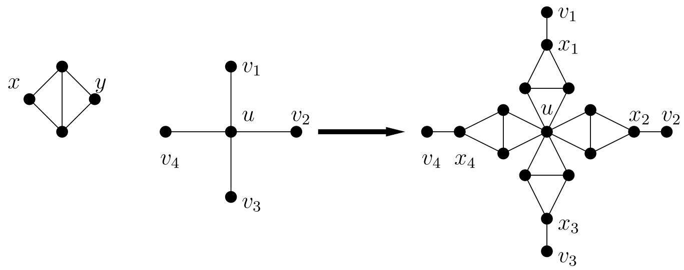

Fig. 2. A diamond with poles x, y and the vertex-diamond operation.

2. The proof of Theorem 2

In the remainder of the paper we write H-free instead of strongly H-free as a shorthand notation. Here is the proof of Theorem 2.

(a) Maffray and Preissmann [16] showed that 3-Coloring is NP-complete for triangle-free graphs. This result has been extended by Kamiński and Lozin [11], who proved that k-Coloring is NP-complete for the class of graphs of girth at least p for any fixed $k \geq 3$ and $p \geq 3$. Suppose that H contains a cycle. Then $g(H)$ is finite. Let $p = g(H) + 1$. It remains to observe that any graph of girth at least p does not contain H as a subgraph, and (a) follows.

For the remaining cases, namely cases (b)–(f), we reduce from the 3-Coloring problem restricted to graphs of maximum degree at most 4. It is well known that 3-Coloring is NP-complete for this graph class [7]. It will be readily seen that all our reductions can be carried out in polynomial time.

(b) Let G be a graph with $\Delta(G) \leq 4$. Then G does not contain a graph H with $\Delta(H) \geq 5$ as a subgraph. Hence (b) holds.

(c) Let $G = (V, E)$ be a graph of maximum degree at most 4. We define a useful graph operation. In order to do this, we need the graph displayed in Fig. 2. It has vertex set $\{x, y, z, t\}$ and edge set $\{xz, xt, yz, yt, zt\}$ and is called a *diamond* with *poles* x, y. We observe that in any 3-coloring of a diamond with poles x, y, the vertices x and y are colored alike.

The graph operation that we use is displayed in Fig. 2. For a vertex $u \in V$ with four neighbors $v_1, \ldots, v_4$, we do as follows. We delete the edges uv_i for $i = 1, \ldots, 4$. We then add 4 diamonds with poles x_i, y_i for $i = 1, \ldots, 4$ and identify u with each y_i. Finally, we add the edges v_ix_i for $i = 1, \ldots, 4$. We call this operation the *vertex-diamond* operation. Note that this operation is only defined on vertices of degree 4. Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if G is 3-colorable. We also observe that this operation when applied on a vertex u increases the distance between u and any other vertex of G by 2. Moreover, the new vertices added have degree 3.

To complete the proof of (c), let H be a graph that has a connected component D with at least two vertices of degree 4. Let α denote the maximum distance between two such vertices in D. Then we apply α vertex-diamond operations on each vertex of degree 4 in G. By our previous observations, the resulting graph G^* is D-free, and consequently, H-free, and in addition, G^* is 3-colorable if and only if G is 3-colorable. Hence (c) holds.

(d) Let $G = (V, E)$ be a graph of maximum degree at most 4. We define the following graph operation displayed in Fig. 3. For an edge $x_0y_0 \in E$, we do as follows. We delete the edge x_0y_0 (but we keep the vertices x_0 and y_0) and add vertices $x_1, y_1, \ldots, x_\ell, y_\ell$. We then construct diamonds with poles x_{i-1}, x_i and y_{i-1}, y_i respectively, for $i = 1, \ldots, \ell$. Finally, we add the edge $x_\ell y_\ell$. We call this operation the *edge-diamond* operation *of type* ℓ. We let G_ℓ be the graph obtained from G after applying an edge-diamond operation of type ℓ on each of its edges. Because any 3-coloring gives the poles of a diamond the same color, G_ℓ is 3-colorable for any $\ell \geq 1$ if and only if G is 3-colorable.

To complete the proof of (d), let H be a graph that contains a subdivision of T_1, which we will denote by T'. Let u, v be the vertices of degree 3 in T'. We choose $\ell = \mathrm{dist}_{T'}(u, v)$. Then G_ℓ is H-free, and (d) holds.

Subcases $p = 0$ and $p = 1$ of (e) *and subcase $H \supseteq T_5$ of* (f). Let $G = (V, E)$ be a graph of maximum degree at most 4. We apply one vertex-diamond operation on each vertex of degree 4 in G. This results in a graph G^*. We observe that G^* is T_2^0-free,

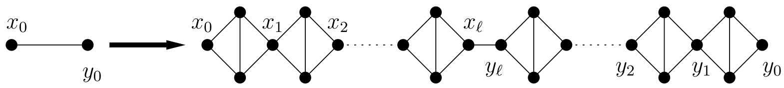

Fig. 3. The edge-diamond operation.

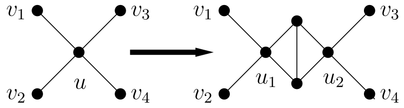

Fig. 4. The balanced-diamond operation.

T_2^1-free and T_5-free, because every vertex of degree at least 4 in G^* is obtained by identifying pole vertices of diamonds. Recall that G^* is 3-colorable if and only if G is 3-colorable. Hence, the subcases $p = 0$ and $p = 1$ of (e) and the subcase $H \supseteq T_5$ of (f) hold.

Remaining eight subcases of (e) *and subcase $H \supseteq T_6$ of* (f). Let $G = (V, E)$ be a graph of maximum degree at most 4. To complete the proof of (e), let H be a graph that contains T_2^p as a subgraph for some $2 \leq p \leq 9$. Recall that the graph G_ℓ defined in case (d) is 3-colorable if and only if G is 3-colorable. We choose $\ell = \lceil \frac{p-1}{2} \rceil$. Then G_ℓ is H-free, and the remaining subcases of (e) hold. As an aside, note that for $p \geq 10$, there exists no ℓ such that G_ℓ is T_2^p-free, because for all $\ell \geq 1$ we can "map" the degree-3 vertex t of T_2^p on a degree-4 vertex in G_ℓ that corresponds to an original degree-4 vertex of G. Then we will either find in G_ℓ a suitable vertex u that is in a diamond or that is a degree-4 vertex that corresponds to an original degree-4 vertex of G, such that we can "map" the degree-4 vertex s of T_2^p to u in order to obtain a subgraph in G_ℓ that is isomorphic to T_2^p. Hence, the case $p \geq 10$ is still open.

Now let H be a graph that contains T_6 as a subgraph. We choose $\ell = 1$. Then G_2 is H-free, and the corresponding subcase of (f) holds.

Remaining two subcases of (f). Let $G = (V, E)$ be a graph of maximum degree at most 4. The last graph operation that we use is displayed in Fig. 4. For a vertex $u \in V$ with four neighbors $v_1, \ldots, v_4$, we do as follows. We remove u and add two new vertices u_1 and u_2. We make u_1 adjacent to v_1 and v_2, whereas we make u_2 adjacent to v_3 and v_4. Finally, we add two more vertices that together with u_1 and u_2 form a diamond, in which u_1 and u_2 are the poles. We call this operation the *balanced-diamond* operation. Note that we only define this operation on vertices of degree 4 (we refer to the paper of Kamiński and Lozin [12] for a more general variant called diamond implementation). Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if G is 3-colorable.

To complete the proof of (f), let H be a graph that contains T_3 or T_4 as a subgraph. We apply the balanced-diamond operation on each vertex of degree 4 in G. The resulting graph G' is H-free. Moreover, by our observation, G' is 3-colorable if and only if G is 3-colorable. This concludes the proof of Theorem 2. □

3. The proof of Theorem 3

Let G be a graph. A graph H is a *minor* of G if H can be obtained from a subgraph of G by a sequence of edge contractions, or equivalently, if H can be obtained from G by a sequence of edge deletions, vertex deletions and edge contractions.

We start by proving the following theorem.

Theorem 3(a). *Let H be a fixed forest with $\Delta(H) \leq 3$, such that each connected component of H has at most one vertex of degree 3. Then* Coloring *can be solved in polynomial time for H-free graphs.*

Proof. Let $H_1, \ldots, H_p$ be the connected components of H. By our assumption on H, each H_i is either a path or a subdivided star, in which the center vertex has degree 3. As such, H_i is a subgraph of a graph G if and only if H_i is a minor of G. Consequently, H is a subgraph of a graph G if and only if H is a minor of G. By a result of Bienstock et al. [2], every graph that does not contain H as a minor has path-width, and consequently treewidth, at most $|V_H| - 2$. Because Coloring can be solved in linear time on graphs of bounded treewidth as shown by Arnborg and Proskurowski [1], the result follows. □

Theorem 3(a) limits the remaining cases of Theorem 3(b) to those graphs H that are a forest on at most seven vertices and that contain a vertex of degree 4 or two vertices of degree at least 3. Moreover, our goal is to show polynomial-time solvability for such cases, and a graph is H-free if it is H'-free for any subgraph H' of H. This narrows down our case analysis to the trees $H_1, \ldots, H_5$ shown in Fig. 5. We consider each such tree, but we first give some auxiliary results.

Observation 1. *Let G be a graph with $|V_G| \geq 2$. Let $u \in V_G$ with $d_G(u) < k$ for some integer $k \geq 1$. Then G is k-colorable if and only if $G - u$ is k-colorable.*

We say that a vertex u of a graph G is *universal* if $G = G[N_G[u]]$, that is, if u is adjacent to all other vertices of G.

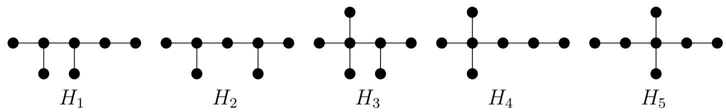

Fig. 5. The trees $H_1, \ldots, H_5$.

Observation 2. *Let u be a universal vertex of a graph G with $|V_G| \geq 2$. Let $k \geq 2$ be an integer. Then G is k-colorable if and only if $G - u$ is $(k-1)$-colorable.*

A vertex u of a connected graph G with at least two vertices is a *cut-vertex* if $G - u$ is disconnected. A maximal connected subgraph of G with no cut-vertices is called a *block* of G.

Observation 3. *Let G be a connected graph, and let k be a positive integer. Then G is k-colorable if and only if each block of G is k-colorable.*

Let (G, k) be an instance of Coloring. We apply the following preprocessing rules *exhaustively*, which in our context means recursively and as long as possible; in particular, if after the application of some rule we can apply some other rule with a smaller index, then we will do this.

Rule 1. Find all connected components of G and consider each of them.
Rule 2. Check if G is 1-colorable or 2-colorable. If so, then stop considering G.
Rule 3. If $|V_G| \geq 2$, $k \geq 3$, and G has a vertex u with $d_G(u) \leq 2$, take $(G - u, k)$.
Rule 4. If $|V_G| \geq 2$, $k \geq 3$, and G has a universal vertex u, take $(G - u, k - 1)$.
Rule 5. If G is connected, then find all blocks of G and consider each of them.

We obtain the following lemma.

Lemma 1. *Let (G, k) be an instance of* Coloring *with $k \geq 3$. Exhaustively applying Rules* 1–5 *takes polynomial time and yields a set I of at most $|V_G|$ instances, such that (G, k) is a yes-instance if and only if every instance of I is a yes-instance. Moreover, each $(G', k') \in I$ has the following properties:*

(i) $|V_{G'}| \leq |V_G|$;
(ii) *if* $k' \geq 3$, *then* $\delta(G') \geq 3$;
(iii) *if* $k' \geq 3$, *then* G' *has no universal vertices;*
(iv) G' *is* 2*-connected;*
(v) $k' \leq k$;
(vi) *if G is H-free for some graph H, then G' is H-free as well.*

Proof. Let (G, k) be an instance of Coloring with $k \geq 3$. We denote the number of vertices of G by n.

We first show that applying Rules 1–5 exhaustively takes polynomial time. Rule 1 takes linear time, because we only have to find the connected components of G. Rule 2 takes linear time, because G is 1-colorable if and only if G has no edges, and G is 2-colorable if and only if G is bipartite. Rules 3 and 4 take linear time, because we only need to check the degree of each vertex. Rule 5 takes linear time, because we only need to find the set of blocks of G. Because the size of G decreases after applying Rule 3 or Rule 4, our procedure terminates.

We now show that Rules 1–5 are *correct*, that is, applying them yields a set of one or more new instances such that the original instance is a yes-instance of Coloring if and only if each newly created instance is a yes-instance. It is readily seen that G is k-colorable if and only if each connected component of G is k-colorable. Hence, Rule 1 is correct. Clearly, Rule 2 is correct as well. Rule 3 is correct due to Observation 1. Rule 4 is correct due to Observation 2. Rule 5 is correct due to Observation 3. Hence, our procedure creates a set I of instances, such that (G, k) is a yes-instance if and only if each instance of I is a yes-instance. In particular, we note that (G, k) is a yes-instance if $I = \emptyset$, as in that case G is 2-colorable, and consequently, k-colorable, due to one or more applications of Rule 2.

The number of instances created only increases after applying Rule 1 or Rule 5. Because the total number of blocks of all connected components is at most n, the set I has size at most n.

Let (G', k') be an instance of I. Then $|V_{G'}| \leq |V_G|$ because we only decreased the size of G. This proves (i). By Rule 3, G' has minimum degree at least 3 if $k' \geq 3$. This proves (ii). By Rule 4, G' has no universal vertices if $k' \geq 3$. This proves (iii). By Rule 5, G' is 2-connected. This proves (iv). By our assumption, $k \geq 3$. We have $k' \leq k$, because we do not increase k when applying Rules 1–5. This proves (v). Because we only removed vertices from G, we find that G' is a subgraph of G. Hence, if G is H-free for some graph H, then G' is H-free. This proves (vi). $\square$

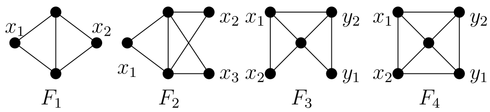

Fig. 6. The graphs F_1, F_2, F_3, F_4.

3.1. The cases $H = H_1$ and $H = H_2$

We first give some extra terminology. Let $G = (V, E)$ be a graph. We let $\omega(G)$ denote the size of a maximum clique in G. The *complement* of G is the graph $\overline{G}$ with vertex set V, such that any two distinct vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If $\chi(F) = \omega(F)$ for any induced subgraph F of G, then G is called *perfect*. We will use the Strong Perfect Graph Theorem proved by Chudnovsky et al. [5]. This theorem tells us that a graph is perfect if and only if it does not contain C_r or $\overline{C}_r$ as an induced subgraph for any odd integer $r \geq 5$.

Lemma 2. *Let G be a 2-connected graph with $\delta(G) \geq 3$ that has no universal vertices. If G is H_1-free or H_2-free, then G is perfect.*

Proof. Note that H_1 and H_2 are both subgraphs of $\overline{C}_r$ for any $r \geq 7$. Moreover, $C_5 = \overline{C_5}$. Then, by the Strong Perfect Graph Theorem [5], we are left to prove that G contains no induced cycle C_r for any odd integer $r \geq 5$. To obtain a contradiction, assume that G does contain an induced cycle $C = v_0v_1 \cdots v_{r-1}v_{r-1}v_0$ for some odd integer $r \geq 5$.

First suppose that G is H_1-free. Let $0 \leq i \leq r-1$ and consider the path $v_iv_{i+1} \cdots v_{i+3}v_{i+4}$, where the indices are taken modulo r. Since $\delta(G) \geq 3$, v_{i+1} and v_{i+2} each have at least one neighbor in $V' = V \setminus \{v_0, \ldots, v_{r-1}\}$, say v_{i+1} is adjacent to some vertex u and v_{i+2} is adjacent to some vertex v. Because G is H_1-free, $u = v$, and moreover, $|N(v_{i+1}) \cap V'| = |N(v_{i+2}) \cap V'| = 1$. Because $0 \leq i \leq r-1$ was taken arbitrarily, we deduce that the vertices $v_0, \ldots, v_{r-1}$ are all adjacent to the same vertex $u \in V'$ and to no other vertices in V'. Because G is 2-connected, u is not a cut-vertex. Hence, $V' = \{u\}$. However, then u is a universal vertex. This is a contradiction.

Now suppose that G is H_2-free. By the same arguments and the fact that r is odd, we conclude again that there exists a universal vertex $u \in V'$. This is a contradiction. □

We are now ready to prove that Coloring is polynomial-time solvable for H_1-free and for H_2-free graphs. Let G be a graph, and let $k \geq 1$ be an integer. If $k \leq 2$, then Coloring is even polynomial-time solvable for general graphs. Suppose that $k \geq 3$. Then, by Lemma 1, we may assume without loss of generality that G is 2-connected, has $\delta(G) \geq 3$ and does not contain any universal vertices. Lemma 2 then tells us that G is perfect. Because Grötschel et al. [9] showed that Coloring is polynomial-time solvable for perfect graphs, our result follows.

3.2. The case $H = H_3$

We start with showing the following useful lemma that gives an upper bound on the maximum degree of connected H_3-free graphs with no universal vertices and with minimum degree at least 3. We may impose the latter two conditions, because our polynomial-time algorithm for solving Coloring on H_3-free graphs will apply Rules 1–5 exhaustively.

Lemma 3. *Let G be a connected H_3-free graph with no universal vertices. If $\delta(G) \geq 3$, then $\Delta(G) \leq 4$.*

Proof. Let $G = (V, E)$ be an H_3-free graph with no universal vertices. Suppose that $\delta(G) \geq 3$. To obtain a contradiction assume that $d_G(u) \geq 5$ for some vertex $u \in V$. Because G has no universal vertices and because G is connected, there is a vertex $v \in N_G(u)$ such that v has a neighbor $x \in V \setminus N_G[u]$. Because $d_G(v) \geq \delta(G) \geq 3$, we deduce that v has another neighbor $y \notin \{u, x\}$. Because $d_G(u) \geq 5$, we also deduce that u has three neighbors z_1, z_2, z_3 neither equal to v nor to y. However, the subgraph of G with vertices $u, v, x, y, z_1, z_2, z_3$ and edges $uz_1, uz_2, uz_3, uv, vx, vy$ is isomorphic to H_3. This is a contradiction, because G is H_3-free. □

We now state some additional terminology. We say that we *identify* two distinct vertices $u, v \in V_G$ if we first remove u, v and then add a new vertex w by making it (only) adjacent to the vertices of $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$.

Consider the graphs $F_1, \ldots, F_4$ shown in Fig. 6. Vertices x_1, x_2 of F_1, vertices x_1, x_2, x_3 of F_2 and vertices x_1, x_2, y_1, y_2 of F_3 and F_4 are called the *pole vertices* of the corresponding graph F_i, whereas the other vertices of F_i are called *center vertices*. We say that a graph G *properly contains* F_i for some $1 \leq i \leq 4$ if G contains F_i as an induced subgraph, in such a way that center vertices of F_i are only adjacent to vertices of F_i, that is, the subgraph F_i is connected to other vertices of G only via its poles.

Our polynomial-time algorithm for solving Coloring on H_3-free graphs will try to apply Rules 1–5 and one additional rule.

Rule 6. If G properly contains F_i for some $1 \leq i \leq 4$, then remove the center vertices of F_i from G and identify the pole vertices of F_i as follows:

- if $i = 1$, then identify x_1 and x_2;
- if $i = 2$, then identify x_1, x_2, and x_3;
- if $i = 3$ or $i = 4$, then identify x_1 and y_1, and also identify x_2 and y_2.

The next lemma shows that we may safely apply Rule 6 on an H_3-free graph G with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$.

Lemma 4. *Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$. Let G' be the graph obtained from G after one application of Rule 6. Then G' is 3-colorable if and only if G is 3-colorable. Moreover, G' is H_3-free.*

Proof. Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that properly contains a graph F_i for some $1 \leq i \leq 4$. Let G' be the graph obtained from G after applying Rule 6 with respect to F_i.

We first prove that G' is 3-colorable if and only if G is 3-colorable. First suppose that G' is 3-colorable. Consider a 3-coloring of G'. We color all vertices in $V \setminus V_{F_i}$ by the same colors as in G', the pole vertices of F_i are colored by the same color as the vertex obtained from them by the identification. It remains to observe that if $i = 1$ or $i = 2$, then the neighbors of the two center vertices are colored by one color, and if $i = 3$ or $i = 4$, then the neighborhood of the unique center vertex is colored by two colors. Hence, we can safely color the center vertices of F_i. Now suppose that G is 3-colorable. Because in any 3-coloring of F_i the identified vertices are necessarily colored with the same color, G' is 3-colorable as well.

Now we show that G' is H_3-free. To obtain a contradiction, assume that G' has a subgraph H isomorphic to H_3. Let u be the vertex of degree 4 in H, and let v be the vertex of degree 3. Because G is H_3-free, at least one of u, v must be obtained by identifying pole vertices of F_i.

First suppose that u is not obtained by identifying pole vertices of F_i. Then v must be obtained by identifying pole vertices of F_i. Then, in G, we find that u is adjacent to a vertex v' that is a pole vertex of F_i and that corresponds to v in G' by the identification of pole vertices. Moreover, because u has degree 4 in G', we find that u has three other neighbors z_1, z_2, z_3 not equal to v' in G that are not identified with each other or with v' after applying Rule 6; one of them may still be a pole vertex in the case that $i = 3$ or $i = 4$, but then such z_i is identified with some vertex of G not in $\{v', z_1, z_2, z_3\} \setminus \{z_i\}$. Also, z_1, z_2, z_3 cannot be center vertices of F_i, as center vertices are removed by Rule 6.

Because u is in G' and Rule 6 removes center vertices of F_i, we find that u is not a center vertex of F_i. Because u is not a pole vertex of F_i either, this means that $u \in V \setminus V_{F_i}$. If $i = 1$ or $i = 2$, then let w_1 and w_2 be the two center vertices of F_i. Then the subgraph of G with vertices $u, v', w_1, w_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w_1, v'w_2$ is isomorphic to H_3. This is a contradiction. Hence, $i = 3$ or $i = 4$.

Let w be the unique center vertex of F_i and assume that $v' \in \{x_1, x_2\}$. Let v'' denote the other vertex of $\{x_1, x_2\}$. If none of the vertices z_1, z_2, z_3 is in V_{F_i}, then the subgraph of G that has vertices $u, v', v'', w, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'v''$ is isomorphic to H_3. This is a contradiction. Therefore, one of the vertices z_1, z_2, z_3, say z_1, is a pole vertex of F_i. Note that z_2 and z_3 are not in F_i, as we already deduced. We also deduced that z_1 is not identified with v'. Suppose that $z_1 \in \{y_1, y_2\}$. Then again the subgraph of G that has vertices $u, v', v'', w, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'v''$ is isomorphic to H_3, which is a contradiction. Hence, $z_1 \in \{x_1, x_2\}$. If $z_1 = x_1$, then $v' = x_2$. Then the subgraph of G with vertices $u, v', w, y_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, z_1w, z_1y_2$ is isomorphic to H_3. If $z_1 = x_2$, then $v' = x_1$. Then the subgraph of G with vertices $u, v', w, y_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'y_2$ is isomorphic to H_3. Both cases are not possible. We conclude that u must be obtained by identifying pole vertices, namely x_1 and x_2 if $i = 1$, x_1, x_2, x_3 if $i = 2$, and we may assume without loss of generality that u is obtained by identifying x_1 and y_1 if $i = 3$ or $i = 4$.

First suppose that $i = 1$. Because $\Delta(G) \leq 4$ and $d_{G'}(u) = 4$, each pole x_j must have two neighbors s_1^j and s_2^j in G that are not in F_1 for $j = 1, 2$. Because G' contains H_3, one of the vertices $s_1^1, s_2^1, s_1^2, s_2^2$, say s_1^1, has two neighbors t_1 and t_2 in G that are not in $V_{F_1} \cup \{s_1^1, s_2^1, s_1^2, s_2^2\}$. Let w_1 and w_2 denote the two center vertices of F_1. We find that the subgraph of G with vertices $s_1^1, s_2^1, t_1, t_2, w_1, w_2, x_1$ and edges $x_1s_1^1, x_1s_2^1, x_1w_1, x_1w_2, s_1^1t_1, s_1^1t_2$ is isomorphic to H_3. This is a contradiction.

Now suppose that $i = 2$. Because $\Delta(G) \leq 4$ and $d_{G'}(u) = 4$, one pole, say x_1, has two neighbors s_1 and s_2 in G that are not in F_2. Let w_1 and w_2 denote the two center vertices of F_2. We find that the subgraph of G with vertices $s_1, s_2, w_1, w_2, x_1, x_2, x_3$ and edges $x_1s_1, x_1s_2, x_1w_1, x_1w_2, w_1x_2, w_1x_3$ is isomorphic to H_3. This is a contradiction.

Finally suppose that $i = 3$ or $i = 4$. Recall that we assume that $u \in V_H$ was obtained by identifying x_1 and y_1. Then, because $d_{G'}(u) = 4$ and $\Delta(G) \leq 4$, we find that $i = 3$ and that y_1 has two neighbors s_1 and s_2 in G that are not in F_3. Let w denote the center vertex of F_3. We find that the subgraph of G with vertices $s_1, s_2, w, x_1, x_2, y_1, y_2$ and edges $y_1s_1, y_1s_2, y_1y_2, y_1w, wx_1, wx_2$ is isomorphic to H_3. This is a contradiction. We conclude that u cannot be obtained by identifying pole vertices. This completes the proof of Lemma 4. □

Before we can present our polynomial-time algorithm that solves Coloring for H_3-free graphs, we prove one final lemma.

Lemma 5. *Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that does not properly contain any of the graphs $F_1, \ldots, F_4$. Then G is 3-colorable if and only if G is K_4-free.*

Proof. Let $G = (V, E)$ be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that does not properly contain any of the graphs $F_1, \ldots, F_4$. First suppose that G is 3-colorable. This immediately implies that G is K_4-free.

Now suppose that G is K_4-free. If $\Delta(G) \leq 3$, then Brooks' Theorem (cf. [6]) tells us that G is 3-colorable unless $G = K_4$, which is not the case. Hence, we may assume that G contains at least one vertex of degree 4. To obtain a contradiction, assume that G is a minimal counter-example, that is, $\chi(G) \geq 4$ and the graph obtained from $G - v$ by removing vertices of degree at most 2 as long as possible is 3-colorable for all $v \in V$; note that this graph may be empty.

Let u be a vertex of degree 4 in G, and let $N_G(u) = \{v_1, v_2, v_3, v_4\}$. We first show the following four claims.

(a) $G[N_G(u)]$ is C_3-free;
(b) $G[N_G(u)]$ contains no vertex of degree 3;
(c) $G[N_G(u)]$ is not isomorphic to P_4;
(d) $G[N_G(u)]$ is not isomorphic to C_4.

Claims (a)–(d) can be seen as follows. If $G[N_G(u)]$ contains C_3 as a subgraph, then $G[N_G[u]]$, and consequently, G contains K_4 as a subgraph of G. This proves (a). If $G[N_G(u)]$ contains a vertex of degree 3, then G properly contains F_2, as $G[N_G(u)]$ is C_3-free due to (a). This proves (b). If $G[N_G(u)]$ is isomorphic to P_4, then G properly contains F_3. This proves (c). If $G[N_G(u)]$ is isomorphic to C_4, then G properly contains F_4. This proves (d).

Because G is H_3-free, each v_j has at most one neighbor in $V \setminus N_G[u]$. Because $\delta(G) \geq 3$, this means that $G[N_G(u)]$ contains no isolated vertices. Then, by claims (a)–(d), we find that $G[N_G(u)]$ contains exactly two edges. Moreover, $d_G(v_j) = 3$ for $j \in \{1, \ldots, 4\}$ as $\delta(G) \geq 3$.

We assume without loss of generality that v_1v_2 and v_3v_4 are edges in G. Let w_j be the neighbor of v_j in $V \setminus N_G[u]$ for $j = 1, \ldots, 4$. We note that $w_1 \neq w_2$ and $w_3 \neq w_4$, as otherwise G properly contains F_1.

Because G is a minimal counterexample, we find that the graph obtained from $G - u$ by removing vertices of degree at most 2 as long as possible is 3-colorable. Hence, $G - u$ is 3-colorable. Let c be an arbitrary 3-coloring of $G - u$. We show that the following two claims are valid for c up to a permutation of the colors 1, 2, 3.

(1) $c(v_1) = c(v_3) = 1$, $c(v_2) = 2$ and $c(v_4) = 3$;
(2) $c(w_1) = c(w_2) = 3$ and $c(w_3) = c(w_4) = 2$.

Claims (1) and (2) can be seen as follows. If c uses at most two different colors on $v_1, \ldots, v_4$, then we can extend c to a 3-coloring of G, which is not possible as $\chi(G) \geq 4$. Hence, c uses three different colors on $v_1, \ldots, v_4$. Then we may assume without loss of generality that $c(v_1) = c(v_3) = 1$, $c(v_2) = 2$ and $c(v_4) = 3$. This proves (1). We now prove (2). In order to obtain a contradiction, assume that $c(w_1) \neq c(w_2)$. Because $c(v_2) = 2$, we find that $c(w_2) = 1$ or $c(w_2) = 3$. If $c(w_2) = 1$, then we change the color of v_2 into 3, contradicting (1). Hence, $c(w_2) = 3$. Then, as $c(v_1) = 1$, we obtain $c(w_1) = 2$. However, we can now change the colors of v_1 and v_2 into 3 and 1, respectively, again contradicting (1). We conclude that $c(w_1) = c(w_2)$. Hence, $c(w_1) = c(w_2) = 3$. By the same arguments, we find that $c(w_3) = c(w_4)$. Hence, $c(w_3) = c(w_4) = 2$. This proves (2).

The facts that $w_1 \neq w_2$ and $w_3 \neq w_4$ together with Claim (2) imply that w_1, w_2, w_3, w_4 are four distinct vertices. We observe that $d_G(w_j) = 3$ for $j = 1, \ldots, 4$, as otherwise H_3 is a subgraph of G. See Fig. 7 for an illustration. In this figure we also indicate that w_1, w_2 have neighbors colored with colors 1 and 2, and that w_3, w_4 have neighbors colored with colors 1 and 3, as otherwise we could recolor $w_1, \ldots, w_4$ such that $c(w_1) \neq c(w_2)$ or $c(w_3) \neq c(w_4)$, and hence we would contradict Claim (2). We may also assume without loss of generality that c is chosen in such a way that the set of vertices with color 1 is maximal, that is, each vertex with color 2 or 3 has a neighbor with color 1.

Consider the subgraph Q of $G - u$ induced by the vertices colored with colors 2 and 3. We claim that the vertices w_1 and v_2 are in the same connected component of Q. To show this, suppose that there is a connected component Q' of Q that contains w_1 but not v_2. Then we recolor all vertices of Q' colored 2 with color 3 and all vertices of Q' colored 3 with color 2. We obtain a 3-coloring of $G - u$ such that w_1 and w_2 are colored by distinct colors, contradicting Claim (2). Using the same arguments, we conclude that w_3 and v_4 are in the same connected component of Q. Now we show that all the vertices w_1, v_2, w_3, v_4 are in the same connected component of Q. Suppose that there is a connected component Q' of Q that contains w_1, v_2 but not w_3, v_4. Then we recolor all vertices of Q' colored 2 with color 3 and all vertices colored 3 with color 2. We obtain a 3-coloring of $G - u$ such that w_1, w_2, w_3, w_4 are colored with the same color, contradicting Claim (2).

We observe that $d_Q(w_1) = d_Q(v_2) = d_Q(w_3) = d_Q(v_4) = 1$. Then, because w_1, v_2, w_3, v_4 belong to the same connected component of Q, we find that Q contains a vertex x with $d_Q(x) \geq 3$.

Let $y_1, \ldots, y_r$ denote the neighbors of x in Q for some $r \geq 3$. Because $y_1, \ldots, y_r$ are colored with the same color, they are pairwise non-adjacent. Because $\Delta(G) \leq 4$, we find that $r \leq 4$. First suppose that $r = 4$. Because $d_G(y_1) \geq 3$ as $\delta(G) \geq 3$ and $y_1, \ldots, y_4$ are pairwise non-adjacent, y_1 has at least two neighbors in $V \setminus N_G[x]$. However, then G contains H_3 as a subgraph. This is a contradiction. Now suppose that $r = 3$. Recall that the set of vertices with color 1 is maximal. Hence x is adjacent to a vertex z with color 1. Because G is H_3-free and $d_G(y_i) \geq 3$ for $i = 1, 2, 3$, we find that z is adjacent to y_1, y_2, y_3. However, since $\Delta(G) \leq 4$, this means that $G[N_G[z]]$ is isomorphic to F_2. Consequently, G properly contains F_2. This contradiction completes the proof of Lemma 5. □

We are now ready to prove that Coloring can be solved in polynomial time for H_3-free graphs. Let G be an H_3-free graph on n vertices, and let $k \geq 1$ be an integer.

Case 1. $k \leq 2$.

Then Coloring can be solved in polynomial time even for general graphs.

Case 2. $k \geq 3$.

By Lemma 1, we may assume without loss of generality that $\delta(G) \geq 3$ and that G contains no universal vertices. By Lemma 3 we find that $\Delta(G) \leq 4$. Because G has no universal vertices, $G \neq K_5$. Then applying Brooks' Theorem (cf. [6]) yields that G is 4-colorable.

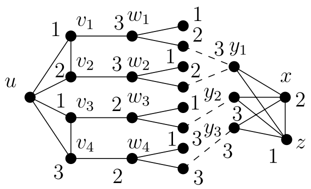

Fig. 7. The structure of the graph G. We note that neighbors of $w_1, \ldots, w_4$ not equal to $v_1, \ldots, v_4$ may not be distinct.

Case 2a. $k \geq 4$.

Then (G, k) is a yes-answer.

Case 2b. $k = 3$.

We apply Rule 6 exhaustively. This takes polynomial time, because each application of Rule 6 takes linear time and reduces the size of G. In order to maintain the properties of having minimum degree at least 3 and containing no universal vertices, we first apply Rules 1–5 exhaustively before another application of Rule 6. Afterward, by Lemmas 1 and 4, we have found in polynomial time a (possibly empty) set $\mathcal{G}$ of at most n graphs, such that G is 3-colorable if and only if each graph in $\mathcal{G}$ is 3-colorable. Moreover, each $G' \in \mathcal{G}$ is H_3-free, has minimum degree at least 3, contains no universal vertices, and in addition, does not properly contain any of the graphs $F_1, \ldots, F_4$. Then, by Lemma 3, each $G' \in \mathcal{G}$ has $\Delta(G') \leq 4$. As a consequence, we may apply Lemma 5. This lemma tells us that a graph $G' \in \mathcal{G}$ is 3-colorable if and only if it does not contain K_4 as a subgraph. As we can check the latter condition in polynomial time and $|\mathcal{G}| \leq n$, that is, we have at most n graphs to check, also the last step of our algorithm runs in polynomial time.

3.3. The cases $H = H_4$ and $H = H_5$

For these cases we replace Rule 4 by a new rule. Let $G = (V, E)$ be a graph and k be an integer.

Rule 4*. If $k \geq 3$ and $V \setminus N_G[u]$ is an independent set for some $u \in V$, take $(G[N_G(u)], k - 1)$.

The next lemma shows that Rule 4 is correct.

Lemma 6. *Let $k \geq 2$ be an integer, and let u be a vertex of a graph $G = (V, E)$ such that $V \setminus N_G[u]$ is an independent set. Then G is k-colorable if and only if $G[N_G(u)]$ is $(k-1)$-colorable.*

Proof. First suppose that G is k-colorable. Let c be a k-coloring of G. Then the vertices of $N_G(u)$ are colored with at most $k-1$ colors, which are different from $c(u)$. Hence, $G[N_G(u)]$ is $(k-1)$-colorable. Now suppose that $G[N_G(u)]$ is $(k-1)$-colorable. Then we extend this coloring to a k-coloring of G by coloring $V \setminus N_G(u)$ with a new color. □

We will also need the following lemma.

Lemma 7. *Let $G = (V, E)$ be a 2-connected graph with $\delta(G) \geq 3$ such that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. If G is H_4-free or H_5-free, then $\Delta(G) \leq 3$.*

Proof. Let $G = (V, E)$ be a 2-connected graph with $\delta(G) \geq 3$ such that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. Assume that G has a vertex u with $d_G(u) \geq 4$. We will show that G contains a subgraph isomorphic to H_4 and a subgraph isomorphic to H_5.

By our assumption, $V \setminus N_G[u]$ contains two adjacent vertices v and w. We choose v and w so that at least one of them, say v, is adjacent to a vertex $z_1 \in N_G(u)$. Because $d_G(u) \geq 4$, we find that $N_G(u)$ contains at least three other vertices, which we denote by z_2, z_3 and z_4. Then the subgraph of G with vertices $u, v, w, z_1, z_2, z_3, z_4$ and edges $uz_1, uz_2, uz_3, uz_4, z_1v, vw$ is isomorphic to H_4. Because G is 2-connected, G contains a path P from w to u that neither uses v nor z_1. Let v' be the vertex of P that is in $V \setminus N_G[u]$ and that is adjacent to a neighbor of u, say to z_2. Then the subgraph of G with vertices $u, v, v', z_1, z_2, z_3, z_4$ and edges $uz_1, uz_2, uz_3, uz_4, z_1v, z_2v'$ is isomorphic to H_5. □

We are now ready to prove that Coloring can be solved in polynomial time for H_4-free graphs and for H_5-free graphs. Let $G = (V, E)$ be a graph, and let $k \geq 1$ be an integer. If $k \leq 2$, then Coloring can be solved in polynomial time even for general graphs. Now suppose that $k \geq 3$. Lemma 6 shows that Rule 4* is correct. Moreover, an application of Rule 4* takes linear time and reduces the number of vertices of G by at least one. Hence, we can replace Rule 4 by Rule 4* in Lemma 1. Due to this, we may assume without loss of generality that G is 2-connected and has $\delta(G) \geq 3$, and moreover, that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. Then Lemma 7 tells us that $\Delta(G) \leq 3$. By using Brooks' Theorem (cf. [6]) we find that G is 3-colorable, unless $G = K_4$. Hence, (G, k) is a yes-answer when $k \geq 4$, whereas (G, k) is a yes-answer when $k = 3$ if and only if $G \neq K_4$.

4. Conclusions

We classified the complexity of Coloring restricted to strongly H-free graphs for all graphs H up to seven vertices. We also identified an infinite number of polynomial-time solvable and NP-complete cases. The only open cases left are when

H is a forest on at least eight vertices that does not satisfy the conditions of Theorem 2 (for instance, we may assume that each connected component of H has at most one vertex of degree 4). However, the exact borderline between tractability and hardness is not clear. Even determining the computational complexity of Coloring restricted to strongly H-free graphs for some graphs H on eight vertices, such as the 8-vertex trees that contain the graph H_3, seems to be a difficult task.

As our current proof techniques are rather diverse, a more unifying approach may be required in order to complete the computational complexity classification of Coloring for strongly H-free graphs. Also the fact that Coloring (and even the more general problem Precoloring Extension [4]) is polynomial-time solvable for graphs of maximum degree at most 3 makes the problem harder to classify for strongly H-free graphs than some other decision problems that are NP-complete for graphs of maximum degree at most 3. To illustrate this, we consider the Independent Set problem, which is the problem of deciding whether a graph has an independent set of at least k vertices for some given integer k. It is well known that Independent Set is already NP-complete for graphs of maximum degree at most 3 [7]. This allows us to use a well-known and simple edge-replacing gadget in order to prove that Independent Set is NP-complete on strongly H-free graphs for almost all graphs H.

Proposition 1. *Let H be a graph. Then* Independent Set *is polynomial-time solvable for strongly H-free graph if H is a forest with $\Delta(H) \leq 3$, each connected component of which contains at most one vertex of degree* 3*. In all other cases,* Independent Set *is* NP*-complete for strongly H-free graphs.*

Proof. First suppose that H is a forest with $\Delta(H) \leq 3$, each connected component of which contains at most one vertex of degree 3. We apply exactly the same arguments as we used in the proof of Theorem 3(a) in order to show that Independent Set is polynomial-time solvable on strongly H-free graphs.

Now suppose that H contains at least one connected component that contains either a vertex of degree at least 4 or two vertices of degree 3 or a cycle. Recall that Independent Set is NP-complete on graphs of maximum degree at most 3 [7]. Hence, Independent Set is NP-complete on strongly H-free graphs if H contains a vertex of degree at least 4. Due to this, we are left with the case when H is a graph with $\Delta(H) \leq 3$ that contains either two vertices of degree 3 or a cycle.

If Independent Set is NP-complete for a graph class $\mathcal{G}$, then it remains NP-complete on the graph class obtained by subdividing each edge of each graph of $\mathcal{G}$ exactly twice (the subdivision of an edge uv in a graph replaces uv by two new edges uw and wv for some new vertex w). Hence, Independent Set is NP-complete on graphs of maximum degree at most 3 that have girth at least g for any fixed $g \geq 3$ (the girth of a graph is the length of a shortest induced cycle in the graph) such that any two vertices of degree 3 are of distance at least h for any fixed $h \geq 1$. As a consequence, Independent Set is NP-complete for strongly H-free graphs. □

We note that, just as the complexity classification of k-Coloring (see Section 1.2), also the complexity classification of Independent Set is wide open when H is forbidden as an induced subgraph, and that so far only partial results have obtained; very recently, Lokshtanov, Vatshelle, and Villanger [14] solved a long-standing open problem by showing that Independent Set is polynomial-time solvable on P_5-free graphs.

Acknowledgments

We thank the two anonymous referees for their useful comments that helped us to improve the presentation of our paper.

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