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In each dimension $4k+1 \geq 9$, we exhibit infinite families of closed manifolds with fundamental group $\mathbb{Z}_2$ for which the moduli space of metrics of nonnegative sectional curvature has infinitely many path components. Examples of closed manifolds with finite fundamental group with this property were known before only in dimension 5 and dimensions $4k+3 \geq 7$.

53C20, 58D27, 58J28; 19K56, 53C27, 57R55

1 Introduction

We give examples of closed manifolds of dimension $4k+1$ with $k \geq 2$ for which the moduli spaces of metrics of nonnegative sectional curvature and positive Ricci curvature have infinitely many path components.

For a closed manifold M, let $\mathcal{R}_{\sec\geq 0}(M)$ denote the space of Riemannian metrics of nonnegative sectional curvature on M endowed with the smooth topology. The diffeomorphism group $\mathrm{Diff}(M)$ acts on $\mathcal{R}_{\sec\geq 0}(M)$ by pulling back metrics. The orbit space $\mathcal{M}_{\sec\geq 0}(M) := \mathcal{R}_{\sec\geq 0}(M)/\mathrm{Diff}(M)$ equipped with the quotient topology is called the *moduli space* of metrics of nonnegative sectional curvature on M. The corresponding notation will be used for the moduli space of metrics satisfying other curvature bounds.

A basic problem in Riemannian geometry is to determine whether a given manifold admits a metric with prescribed curvature properties. If this is the case, one may ask whether the respective moduli space carries some interesting topology. In contrast to scalar curvature, where surgery techniques are available, little is known about the topology of moduli spaces of metrics satisfying lower bounds (nonnegative or positive) on sectional or Ricci curvature.

The first results in this direction are due to Kreck and Stolz [22], who introduced an invariant for certain $(4k+3)$–dimensional spin manifolds which is constant on path

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components of the moduli space of metrics of positive scalar curvature. Kreck and Stolz used this invariant to show that there exists an Aloff–Wallach space for which the moduli space of metrics of positive sectional curvature is disconnected. They also exhibited an infinite family of 7–dimensional Witten manifolds for which the moduli spaces of metrics of positive Ricci curvature have infinitely many path components [22]. Another more basic invariant to distinguish path components is the relative index of Gromov and Lawson [20, page 327]. Using these invariants manifolds in dimension $4k+3\geq 7$ have been found — see Kapovitch, Petrunin and Tuschmann [21], Dessai, Klaus and Tuschmann [12], Dessai [10] and Goodman [16] — for which the moduli space $\mathcal{M}_{\mathrm{sec}\geq 0}$ has infinitely many path components (see also Wraith [35], Tuschmann and Wiemeler [28] and Goodman [15] as well as Tuschmann and Wraith [29] for related results).

Dessai and González-Álvaro [11] used relative η–invariants to show that for every homotopy $\mathbb{R}P^5$ the moduli space $\mathcal{M}_{\mathrm{sec}\geq 0}$ has infinitely many path components (see also Wermelinger [32]). Here we apply η–invariants to prove that manifolds with this property also exist in all dimensions $4k+1$ with $k\geq 2$.

Main theorem *In each dimension $4k+1\geq 9$, there are infinitely many closed manifolds M_i, $i\in\mathbb{N}$, with pairwise nonisomorphic integral cohomology for which the moduli space $\mathcal{M}_{\mathrm{sec}\geq 0}(M_i)$ of metrics of nonnegative sectional curvature has infinitely many path components. The same holds true for the moduli space $\mathcal{M}_{\mathrm{Ric}>0}(M_i)$ of metrics of positive Ricci curvature on M_i.*

It follows that the corresponding spaces of metrics, $\mathcal{R}_{\mathrm{sec}\geq 0}(M_i)$ and $\mathcal{R}_{\mathrm{Ric}>0}(M_i)$, also have infinitely many path components.

In combination with [6, Proposition 2.8] of Belegradek, Kwasik and Schultz, the theorem implies that for every such manifold the moduli space of complete metrics of nonnegative sectional curvature on the total space of a real line bundle over M_i has infinitely many path components.

The manifolds in the theorem above may be described as total spaces of two-stage iterated fiber bundles over $\mathbb{C}P^1$ with fibers $\mathbb{C}P^{2k-1}$ and S^1 (see the next section for definitions and details) and are closely related to the manifolds considered by Witten [34], Wang and Ziller [31], Kreck and Stolz [22], Kapovitch, Petrunin and Tuschmann [21] and Dessai, Klaus and Tuschmann [12]. They can also be described as quotients of the product of round spheres $S^3\times S^{4k-1}$ by a free isometric action of $S^1\times\mathbb{Z}_2$. The metrics which represent distinct path components in the respective moduli space are obtained as submersion metrics and have nonnegative sectional and positive Ricci curvature. To distinguish components, we compute relative η–invariants for these metrics. The construction can also be carried out for $k = 1$, in which case one obtains a finite number of 5–dimensional $\mathbb{Z}_2$–quotients of $S^2 \times S^3$. Their moduli spaces of metrics of nonnegative sectional curvature and positive Ricci curvature also have infinitely many path components (see [17; 32] and Remark 7.1 for related results).

This paper is structured as follows. In the next section we introduce a family of $(4k+1)$–dimensional manifolds with fundamental group $\mathbb{Z}_2$ which are total spaces of two-stage iterated fiber bundles and collect some of their topological properties. In Section 3, we give a rough diffeomorphism classification for these manifolds. More precisely, we first study their homotopy type via Postnikov towers and then apply the exact surgery sequence to show that certain infinite subfamilies belong to only finitely many oriented diffeomorphism types.

The manifolds come with a submersion metric of nonnegative sectional and positive Ricci curvature, which, when lifted to the universal cover, extends in a nice way to an associated disk bundle. They also carry a $\mathrm{Spin}^{\mathbf{c}}$–structure and a flat line bundle for which the relative η–invariant of the corresponding Dirac operator is nontrivial. This is explained in Sections 4 and 5. Computations for the relative η–invariant via equivariant index theory are detailed in Section 6. These computations are then used in the final section to prove the main theorem.

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2 A family of $(4k+1)$–dimensional manifolds

In this section we describe a family of simply connected manifolds of nonnegative sectional curvature which will be used in the proof of the main theorem. A manifold in this family is given as the total space of an S^1–bundle over the total space of a projective bundle over $\mathbb{C}P^1$, where the bundles depend on three parameters, $s, t, c \in \mathbb{Z}$ (see Definition 2.3). The case of a trivial projective bundle (the untwisted case) includes Witten manifolds and was considered in [34; 31; 22].

The manifolds we are interested in are obtained from certain twisted (ie nontrivial) projective bundles and are of dimension $4k+1$. The twisting is necessary to obtain nontrivial relative η–invariants (see Section 6), which will be used to distinguish components in the moduli space (see Section 7).

Although the construction involves the parameters s, t and c, the integral cohomology ring of these manifolds depends up to isomorphism only on s (see Remark 2.5 and [26]). As in the untwisted case, the manifolds can be described as quotients of $S^3 \times S^{4k-1}$ by a free action of S^1 (see Lemma 2.6).

We now come to the construction of the aforementioned manifolds. Let γ_1 denote the canonical complex line bundle over $\mathbb{C}P^1$ and let $\mathrm{pr}\colon S^3 \to S^3/S^1$ be the Hopf fibration, where S^3 is the sphere of unit quaternions. We will always identify $\mathbb{C}P^1$ with S^3/S^1 and identify γ_1 with the complex line bundle $S^3 \times_{\rho_1} \mathbb{C} \to S^3/S^1 = \mathbb{C}P^1$ associated to the Hopf fibration and the standard 1–dimensional representation ρ_1 of S^1. Similarly, we will identify the n^{th} tensor power γ_1^n with $S^3 \times_{\rho_n} \mathbb{C} \to \mathbb{C}P^1$ for $n \in \mathbb{Z}$, where $\rho_n(\lambda)$ for $\lambda \in S^1$ acts on $\mathbb{C}$ via multiplication with λ^n. Let ϵ_l denote the trivial complex vector bundle of rank l over $\mathbb{C}P^1$. We consider the standard inner product on $\mathbb{C}$ and equip the line bundles above with the induced inner products. For $k > 0$ fixed and $c \in \mathbb{Z}$, let $E_c \to \mathbb{C}P^1$ denote the direct sum of γ_1^c and ϵ_{2k-1}.

Next we consider the pullback of the bundles above via the projection $\mathrm{pr}\colon S^3 \to S^3/S^1$. Note that for every $c \in \mathbb{Z}$ there is a canonical trivialization of the complex line bundle $\mathrm{pr}^*(\gamma_1^c)$. Hence, $\mathrm{pr}^*(E_c) \to S^3$ and its associated sphere bundle $S(\mathrm{pr}^*(E_c)) \to S^3$ have a canonical trivialization. In the following, we will identify $S(\mathrm{pr}^*(E_c))$ with $S^3 \times S^{4k-1}$ via the corresponding diffeomorphism.

Let B_c be the total space of the projective bundle associated to $E_c \to \mathbb{C}P^1$ and let $q\colon B_c \to \mathbb{C}P^1$ denote the projection. Under the identification above $\mathrm{pr}^*(B_c)$ corresponds to the quotient of $S^3 \times S^{4k-1}$ by S^1, where S^1 acts trivially on S^3 and acts by complex multiplication on $S^{4k-1} \subset \mathbb{C}^{2k}$. The following lemma follows directly from the description above. The proof is left to the reader.

Lemma 2.1 *B_c is diffeomorphic to the quotient of $S^3 \times S^{4k-1}$ by a 2–dimensional torus T^2, where T^2 acts freely on $S^3 \times S^{4k-1}$ by*

$$(\lambda, \mu)(x, y) := (x \cdot \lambda^{-1}, (\lambda^c \cdot y_1 \cdot \mu, y_2 \cdot \mu, \dots, y_{2k} \cdot \mu))$$

for $(\lambda, \mu) \in T^2$, $x \in S^3$, $y_1, \dots, y_{2k} \in \mathbb{C}$ and $y = (y_1, \dots, y_{2k}) \in S^{4k-1} \subset \mathbb{C}^{2k}$. □

Let $u \in H^2(B_c;\mathbb{Z})$ be the negative of the first Chern class of the canonical complex line bundle over the projective bundle $B_c \to \mathbb{C}P^1$ and let v be the generator of $H^2(\mathbb{C}P^1;\mathbb{Z})$ defined by $v := -c_1(\gamma_1)$.

The tangent bundle along the fibers of q, denoted by $T^{\triangle}$, is a complex vector bundle of rank $2k-1$ over B_c and the tangent bundle TB_c is isomorphic to the complex vector bundle $q^*(T\mathbb{C}P^1)\oplus T^{\triangle}$ of rank $2k$ (see [8]). We equip B_c with the induced orientation.

Lemma 2.2 *B_c is a simply connected closed oriented $4k$–dimensional manifold. The integral cohomology of B_c as an $H^*(\mathbb{C}P^1;\mathbb{Z})$–module is given by*

$$H^*(B_c;\mathbb{Z}) \cong \mathbb{Z}[u,v]/(v^2, u^{2k} - c\cdot u^{2k-1}\cdot v).$$

In particular, $H^2(B_c;\mathbb{Z}) \cong \mathbb{Z}\langle u,v\rangle$. Under this identification

$$c(TB_c) = (1+2v)\cdot((1+u)^{2k} - c\cdot v\cdot(1+u)^{2k-1})$$

and $c_1(TB_c) = (-c+2)\cdot v + 2k\cdot u$.

Proof Using the homotopy long exact sequence, it follows directly that B_c is simply connected.

By the Leray–Hirsch theorem, $H^*(B_c;\mathbb{Z})$ is generated as a $H^*(\mathbb{C}P^1;\mathbb{Z})$–module by u subject to the relation $u^{2k} + c_1(E_c)\cdot u^{2k-1} + \dots + c_{2k}(E_c) = 0$. Since $c(E_c) = c(\gamma_1^c) = 1 - c\cdot v$, this gives the statement on the cohomology of B_c.

The total Chern class of $T^{\triangle}$ satisfies $c(T^{\triangle}) = \sum_{i=0}^{2k}(1+u)^{2k-i}\cdot c_i(E_c)$ (see [8, page 514]). Since $TB_c \cong q^*(T\mathbb{C}P^1)\oplus T^{\triangle}$, the total Chern class $c(B_c)$ is as stated. □

Definition 2.3 Let $\overline{M}_{s,t,c}$ be the total space of the principal S^1–bundle over B_c with Euler class equal to $e := su + tv$. Let $\pi\colon \overline{M}_{s,t,c} \to B_c$ denote the projection.

From now on we will assume that c is odd, $k \geq 2$, s and t are nonzero coprime integers, and s is even.

Lemma 2.4 (1) *$\overline{M}_{s,t,c}$ is simply connected.*

(2) *$H^2(\overline{M}_{s,t,c};\mathbb{Z}) \cong \mathbb{Z}$, $H^{2i}(\overline{M}_{s,t,c};\mathbb{Z}) \cong \mathbb{Z}_{s^2}$ is generated by $\pi^*(u)^i$ for $4 \leq 2i \leq 4k-2$ and $H^{2i+1}(\overline{M}_{s,t,c};\mathbb{Z}) = 0$ for $1 \leq 2i+1 \leq 4k-3$.*

(3) *$H^{4k-1}(\overline{M}_{s,t,c};\mathbb{Z}) \cong \mathbb{Z}$, $H^{4k}(\overline{M}_{s,t,c};\mathbb{Z}) = 0$ and $H^{4k+1}(\overline{M}_{s,t,c};\mathbb{Z}) \cong \mathbb{Z}$.*

(4) *$H^*(\overline{M}_{s,t,c};\mathbb{Q}) \cong H^*(\mathbb{C}P^1 \times S^{4k-1};\mathbb{Q})$.*

Proof First note that the Euler class e is part of a basis of $H^2(B_c;\mathbb{Z})\cong\mathbb{Z}^2$ since s and t are coprime. Using the Gysin sequence for $\overline{M}_{s,t,c}\to B_c$, one finds that $H^1(\overline{M}_{s,t,c};\mathbb{Z})=0$ and $H^2(\overline{M}_{s,t,c};\mathbb{Z})\cong\mathbb{Z}$. Hence, $\pi_1(\overline{M}_{s,t,c})$ vanishes by the Hurewicz theorem and the universal coefficient theorem.

Next note that the cokernel of $\mathbb{Z}\langle v\cdot u^{l-1},u^l\rangle\xrightarrow{e\cup}\mathbb{Z}\langle v\cdot u^l,u^{l+1}\rangle$ is cyclic of order s^2 and generated by u^{l+1} for $1\le l\le 2k-2$. The remaining statements now follow from Lemma 2.2 and the Gysin sequence. □

Remark 2.5 Using Poincaré duality one finds that the isomorphism type of the ring $H^*(\overline{M}_{s,t,c};\mathbb{Z})$ depends up to finite ambiguity only on s. A closer look shows that $\pi^*(f)^i$ generates $H^{2i}(\overline{M}_{s,t,c};\mathbb{Z})$, $2\le 2i\le 4k-2$, where f is chosen so that (e,f) is a basis of $H^2(B_c;\mathbb{Z})$. It follows that the isomorphism type of the integral cohomology ring of $\overline{M}_{s,t,c}$ is uniquely determined by s.

Next we consider the smooth 2–connected cover of B_c. Since $H^2(B_c;\mathbb{Z})\cong\mathbb{Z}^2$, it can be described as the total space of a principal T^2–bundle over B_c. We note that the 2–connected cover is unique up to diffeomorphism

Lemma 2.6 (1) *The 2–connected cover of B_c is diffeomorphic to $S^3\times S^{4k-1}$.*

(2) *$\overline{M}_{s,t,c}$ is diffeomorphic to a quotient of $S^3\times S^{4k-1}$ by a free action of a subgroup $S^1\subset T^2$.*

Proof The first statement follows directly from Lemma 2.1. For the second statement recall that s and t are coprime. Hence, there is a principal S^1–bundle $S'\to B_c$ such that the Euler classes of $\overline{M}_{s,t,c}\to B_c$ and $S'\to B_c$ generate $H^2(B_c;\mathbb{Z})$. The two bundles define a principal T^2–bundle over B_c with 2–connected total space. Hence, the latter can be identified with $S^3\times S^{4k-1}$ and $\overline{M}_{s,t,c}$ is diffeomorphic to a quotient of $S^3\times S^{4k-1}$ by a free action of S^1. □

Remarks 2.7 (1) From the homotopy long exact sequence for the fibration

$$S^3\times S^{4k-1}\to\overline{M}_{s,t,c},$$

one gets $\pi_i(\overline{M}_{s,t,c})\cong\pi_i(S^3\times S^{4k-1})$ for $i\ge 3$.

(2) The principal T^2–action on the 2–connected cover is not equivalent to the standard T^2–action on $S^3\times S^{4k-1}$ given by componentwise multiplication since B_c is not diffeomorphic to $\mathbb{C}P^1\times\mathbb{C}P^{2k-1}$.

(3) In [26], the cohomology rings of certain S^1–quotients of a product of spheres, including $\overline{M}_{s,t,c}$, have been computed. The results there can also be used to prove Lemma 2.4 and Remark 2.5.

3 Diffeomorphism finiteness of $\mathbb{Z}_2$–quotients

In this section we show that certain infinite families of $\mathbb{Z}_2$–quotients of the manifolds $\overline{M}_{s,t,c}$ fall into finitely many oriented diffeomorphism types. *Throughout this section, s will be a fixed nonzero even integer.*

As before, let $\overline{M}_{s,t,c}$ be the total space of the principal S^1–bundle over B_c with Euler class equal to $su+tv$, where B_c is the total space of the projective bundle associated to $\gamma_1^c \oplus \epsilon_{2k-1}$ and assume that $k \geq 2$, c is odd, and s and t are coprime. Let $L \to B_c$ denote the complex line bundle which is associated to the principal S^1–bundle.

Consider the total space $M_{s,t,c}$ of the principal S^1–bundle over B_c associated to $L \otimes L \to B_c$. Note that the Euler class of $M_{s,t,c} \to B_c$ is equal to $2(su+tv)$.

By construction, S^1 acts (fiberwise) on L, $\overline{M}_{s,t,c}$, $L \otimes L$ and $M_{s,t,c}$. Let τ denote multiplication by $-1 \in S^1$ on the fibers of $L \to B_c$ and on the fibers of $\overline{M}_{s,t,c} \to B_c$. Note that the map $L \to L \otimes L$, $v \mapsto v \otimes v$, is equivariant with respect to the $\mathbb{Z}_2$–action via τ on L and the trivial $\mathbb{Z}_2$–action on $L \otimes L$. By passing to the associated principal S^1–bundles, it follows that $M_{s,t,c}$ can be identified with $\overline{M}_{s,t,c}/\tau$ and that the quotient map $p\colon \overline{M}_{s,t,c} \to M_{s,t,c}$ is a universal covering map.

Since the action of τ on $\overline{M}_{s,t,c}$ extends to an action of S^1, the fundamental group $\pi_1(M_{s,t,c}) = \mathbb{Z}_2$ acts trivially on $\pi_*(\overline{M}_{s,t,c})$. Hence, $M_{s,t,c}$ is a simple space. In addition, $H^*(M_{s,t,c};\mathbb{Q}) \cong H^*(\overline{M}_{s,t,c};\mathbb{Q})^{\pi_1(M_{s,t,c})} \cong H^*(\overline{M}_{s,t,c};\mathbb{Q})$, which is isomorphic to $H^*(\mathbb{C}P^1 \times S^{4k-1};\mathbb{Q})$ by Lemma 2.4.

We equip $\overline{M}_{s,t,c}$ and $M_{s,t,c}$ with the orientation induced from the orientation of B_c (see Section 2) and the complex structure of the complex line bundles.

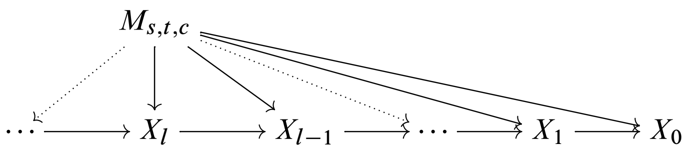
Our aim is to show diffeomorphism finiteness for the family of $(4k+1)$–dimensional oriented manifolds $\mathcal{F}_s := \{M_{s,t,c} \mid c \text{ and } t \text{ odd and } t \text{ coprime to } s\}$.

Theorem 3.1 *The family $\mathcal{F}_s$ contains only finitely many oriented diffeomorphism types.*

Proof We first show homotopy finiteness and then diffeomorphism finiteness.

Homotopy finiteness claim We claim that the family $\mathcal{F}_s$ belongs to only finitely many simple homotopy types. Note that this is equivalent to showing finiteness of homotopy types since $\pi_1(M_{s,t,c}) = \mathbb{Z}_2$ and the Whitehead group of $\mathbb{Z}_2$ is trivial.

Since the members of $\mathcal{F}_s$ are simple spaces, they can be described by Postnikov towers which are classified by their respective k–invariants (see for example [33, Theorem 4.11]). To show the claim it suffices to prove that there are up to homotopy only finitely many Postnikov towers for the manifolds in this family. Let



be the Postnikov tower of $M_{s,t,c}$. Recall that each $X_l \to X_{l-1}$ is a principal fibration (with fiber an Eilenberg–Mac Lane space) which can be described as the pullback of the path fibration

$$K(\pi_l(M_{s,t,c}), l) \hookrightarrow \Gamma K(\pi_l(M_{s,t,c}), l+1) \to K(\pi_l(M_{s,t,c}), l+1)$$

via a map $\kappa_{l+1}\colon X_{l-1} \to K(\pi_l(M_{s,t,c}), l+1)$. Up to homotopy, the fibration is classified by the homotopy class of κ_{l+1}, which corresponds to a class $k_{l+1} \in H^{l+1}(X_{l-1}; \pi_l(M_{s,t,c}))$. As noted before, the Postnikov tower is determined by its k–invariants k_l for $l \geq 1$. For showing homotopy finiteness, it therefore suffices to show finiteness of the possible k–invariants.

By Lemma 2.6, $\overline{M}_{s,t,c}$ is a quotient of $S^3 \times S^{4k-1}$ by a free action of S^1. Since $\overline{M}_{s,t,c}$ is simply connected and $M_{s,t,c}$ is the quotient of a free $\mathbb{Z}_2$–action on $\overline{M}_{s,t,c}$, we have $\pi_1(M_{s,t,c}) = \mathbb{Z}_2$, $\pi_2(M_{s,t,c}) \cong \pi_2(\overline{M}_{s,t,c}) \cong \mathbb{Z}$ and $\pi_l(M_{s,t,c}) \cong \pi_l(S^3 \times S^{4k-1})$ for $l \geq 3$.

It follows that the stages $X_{\leq 2}$ of the Postnikov tower of $M_{s,t,c}$ do not depend, up to homotopy, on the choice of the parameters. In fact, one has $X_0 = \{\mathrm{pt}\}$,

$$\kappa_2\colon X_0 \to K(\mathbb{Z}_2, 2), \qquad k_2 = 0, \quad X_1 \simeq K(\mathbb{Z}_2, 1) \simeq \mathbb{R}P^\infty,$$
$$\kappa_3\colon X_1 \to K(\mathbb{Z}, 3), \quad k_3 \in H^3(\mathbb{R}P^\infty; \mathbb{Z}) = 0, \quad X_2 \simeq X_1 \times K(\mathbb{Z}, 2) \simeq \mathbb{R}P^\infty \times \mathbb{C}P^\infty.$$

In the following we will consider the stages X_l and partial Postnikov towers up to homotopy without explicit mention.

The next stage X_3 in the Postnikov tower is determined by the invariant

$$k_4 \in H^4(X_2; \pi_3(S^3 \times S^{4k-1})) \cong H^4(\mathbb{R}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(recall that $k \geq 2$). Using the Gysin sequence, one finds that $|H^4(M_{s,t,c};\mathbb{Z})| = 4s^2$. Since X_3 is obtained from $M_{s,t,c}$ by attaching cells of dimension ≥ 5, the homomorphism $H^4(X_3;\mathbb{Z}) \to H^4(M_{s,t,c};\mathbb{Z})$ is injective. Hence, the cohomology group $H^4(X_3;\mathbb{Z})$ is finite and determined up to finite ambiguity by s. The invariant k_4 can be identified with the transgression of the fundamental class of the fiber in the Leray–Serre spectral sequence for the fibration $X_3 \to X_2$. It follows that k_4 is determined up to finite ambiguity by s. Hence, X_3 is determined up to finite ambiguity by s as well. For later reference we note that $H^{\geq 3}(X_3;\mathbb{Q}) = 0$ since $s \neq 0$ (again by applying the Leray–Serre spectral sequence).

Since $\pi_l(S^3 \times S^{4k-1}) \otimes \mathbb{Q} = 0$ for $3 < l < 4k-1$, it follows by induction that, for $l < 4k-1$, the invariants k_{l+1} for $M_{s,t,c}$ and its stages X_l are determined, up to finite ambiguity, by s. Hence, the same holds for the partial Postnikov tower $(X_{4k-2} \to X_{4k-3} \to \cdots \to X_1 \to X_0)$ of $M_{s,t,c}$. Again by induction, or by using the minimal model, one finds that $H^{\geq 3}(X_l;\mathbb{Q}) = 0$ for $2 < l < 4k-1$.

By the above, the invariant $k_{4k} \in H^{4k}(X_{4k-2};\pi_{4k-1}(M_{s,t,c}))$ is also determined up to finite ambiguity by s. Since $\pi_{>4k-1}(M_{s,t,c}) \otimes \mathbb{Q} = 0$ we can argue as before to see that, for every $l \geq 4k-1$, the partial Postnikov tower $(X_l \to X_{l-1} \to \cdots \to X_1 \to X_0)$ of $M_{s,t,c}$ is also determined up to finite ambiguity by s. Moreover, the construction of the infinitely many stages X_l for $l > 4k+1$, of the Postnikov tower is formal, ie it depends only on X_{4k+1} (see [19, page 72]). Hence, the entire tower is determined up to finite ambiguity by s, and the claim follows.

Diffeomorphism finiteness claim We claim that after restricting to a (simple) homotopy type there are only finitely many oriented diffeomorphism types among the manifolds $M_{s,t,c}$. Let us fix a homotopy type represented by $M \in \mathcal{F}_s$ and consider the subfamily $\mathcal{F}_s' := \{M_{s,t,c} \in \mathcal{F}_s \mid M_{s,t,c} \simeq M\}$ of manifolds homotopy equivalent to $M_{s,t,c}$. Recall that each $M_{s,t,c}$ comes with an orientation. To show that the family $\mathcal{F}_s'$ contains only finitely many oriented diffeomorphism types, we apply the surgery exact sequence [30]

$$\cdots \to L_{4k+2}(\mathbb{Z}_2) \to \mathcal{S}(M) \to [M, G/O] \to \cdots .$$

Note that $H^*(M;\mathbb{Q}) \cong H^*(\mathbb{C}P^1 \times S^{4k-1};\mathbb{Q})$ and the homotopy groups $\pi_i(G/O)$ of the H–space G/O are finite for $i \not\equiv 0 \bmod 4$. Hence, $[M, G/O]$ is finite. Since $L_{4k+2}(\mathbb{Z}_2) = \mathbb{Z}_2$, the smooth structure set $\mathcal{S}(M)$ is also finite, and the claim follows.

Combining the two claims above, we conclude that for fixed s there are, up to orientation-preserving diffeomorphism, only finitely many $(4k+1)$–dimensional manifolds in the family $\mathcal{F}_s$. □

4 Nonnegative sectional and positive Ricci curvature

In this section we consider submersion metrics of nonnegative sectional and positive Ricci curvature on $M_{s,t,c}$ and $\overline{M}_{s,t,c}$ and extend the latter to the associated disk bundle.

Let (S^l, h_{S^l}) denote the round sphere of radius 1 and let $h_{S^3} \times h_{S^{4k-1}}$ denote the product metric on $S^3 \times S^{4k-1}$. Recall from Lemma 2.1 that T^2 acts freely and isometrically on $(S^3 \times S^{4k-1}, h_{S^3} \times h_{S^{4k-1}})$ with quotient diffeomorphic to B_c. By Lemma 2.6, $\overline{M}_{s,t,c}$ is diffeomorphic to a quotient of $S^3 \times S^{4k-1}$ by an S^1–subaction of T^2. Let $\bar{g}_{s,t,c}$ denote the submersion metric on $\overline{M}_{s,t,c}$, ie $(S^3 \times S^{4k-1}, h_{S^3} \times h_{S^{4k-1}}) \to (\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ is a Riemannian submersion. We note that $M_{s,t,c}$ can be identified with the quotient of $S^3 \times S^{4k-1}$ by a subgroup of T^2 which is isomorphic to $S^1 \times \mathbb{Z}_2$. Let $g_{s,t,c}$ denote the submersion metric on $M_{s,t,c}$. By construction, $p\colon (\overline{M}_{s,t,c}, \bar{g}_{s,t,c}) \to (M_{s,t,c}, g_{s,t,c})$ is a Riemannian universal covering.

Lemma 4.1 $(M_{s,t,c}, g_{s,t,c})$ *and* $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ *both have nonnegative sectional and positive Ricci curvature.*

Proof Note that the sectional curvature of $(S^3 \times S^{4k-1}, h_{S^3} \times h_{S^{4k-1}})$ is always nonnegative and vanishes only on mixed planes. It is easy to see that there is, for any horizontal vector of the Riemannian submersion $S^3 \times S^{4k-1} \to M_{s,t,c}$ (resp. $S^3 \times S^{4k-1} \to \overline{M}_{s,t,c}$), a horizontal plane of positive sectional curvature which contains this vector. Hence, the statements follow from the Gray–O'Neill formula [18; 25]. □

Recall that $\overline{M}_{s,t,c}$ (resp. $M_{s,t,c}$) is a quotient of $S^3 \times S^{4k-1}$ by a subgroup $H \subset T^2$ which is isomorphic to S^1 (resp. $S^1 \times \mathbb{Z}_2$). We remark that the normalizer N of H in the isometry group of $(S^3 \times S^{4k-1}, h_{S^3} \times h_{S^{4k-1}})$ acts with cohomogeneity 1 on $\overline{M}_{s,t,c}$ (resp. $M_{s,t,c}$) and the metric $\bar{g}_{s,t,c}$ (resp. $g_{s,t,c}$) is N–invariant.

For the computation of η–invariants in the next sections, we will also need to put a suitable metric on the disk bundle associated to the principal S^1–bundle $\overline{M}_{s,t,c} \to B_c$. Let $W_{s,t,c} := \overline{M}_{s,t,c} \times_{S^1} D^2$, where $D^2 \subset \mathbb{R}^2$ is the disk of radius 1. We equip D^2 with a metric g_{D^2} (a torpedo metric) such that g_{D^2} is S^1–invariant, is of product type on the annulus $\{x \in D^2 \mid |x| \geq 1-\epsilon\}$ for a fixed small positive ϵ, and such that g_{D^2} is of positive curvature outside of the annulus. Next we consider the product metric $\bar{g}_{s,t,c} \times g_{D^2}$ on $\overline{M}_{s,t,c} \times D^2$ and denote by $h_{s,t,c}$ the submersion metric on $W_{s,t,c}$ with respect to the quotient map $\overline{M}_{s,t,c} \times D^2 \to W_{s,t,c}$. The next lemma follows directly from the construction and the Gray–O'Neill formula [18; 25].

Lemma 4.2 *The metric $h_{s,t,c}$ extends $\bar{g}_{s,t,c}$ to an S^1–invariant metric on $W_{s,t,c}$ of nonnegative sectional and positive scalar curvature which is of product type near the boundary.* □

5 Spin$^\mathbf{c}$–structures and Dirac operators

In this section we introduce suitable Spin$^\mathbf{c}$–structures and corresponding Dirac operators on $(M_{s,t,c}, g_{s,t,c})$, on its universal cover and on the associated disk bundle. These will be used to compute η–invariants in the next section. For background information on and references for Spin$^\mathbf{c}$–manifolds and Dirac operators, we refer to [2; 3; 23; 11].

We begin by defining the relevant Spin$^\mathbf{c}$–structures. Recall that π denotes the projection $\overline{M}_{s,t,c} \to B_c$. In the following we will also denote the projections $W_{s,t,c} \to B_c$ and $M_{s,t,c} \to B_c$ by π. Also we will suppress the parameters s, t and c in the notation for Spin$^\mathbf{c}$–structures and Dirac operators.

Recall that τ acts freely by multiplication with $-1 \in S^1$ on the fibers of $\overline{M}_{s,t,c} \to B_c$ and that the quotient can be identified with $M_{s,t,c}$. Let τ also denote the action by -1 on the fibers of the disk bundle $W_{s,t,c} \to B_c$.

The action of $\mathbb{Z}_2 = \{\mathrm{id}, \tau\}$ on $\overline{M}_{s,t,c}$ and $W_{s,t,c}$ lifts via differentials to the respective oriented orthonormal frame bundles.

Lemma 5.1 $(W_{s,t,c}, h_{s,t,c})$ *has a unique* Spin*–structure.*

Proof Recall that c and t are odd and s is even. Since $TW_{s,t,c} \cong \pi^*(TB_c \oplus L)$, $c_1(TB_c) = (-c+2)\cdot v + 2k \cdot u$ and $c_1(L) = su + tv$ (see Section 2), the manifold $W_{s,t,c}$ is spin. Moreover, the Spin–structure on $(W_{s,t,c}, h_{s,t,c})$ is unique since $H^1(W_{s,t,c}; \mathbb{Z}_2) = 0$. □

We note that the induced structure on the boundary is the unique Spin–structure on $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ since $\pi_1(\overline{M}_{s,t,c}) = 0$. Note, however, that $M_{s,t,c}$ is not spin but does admit a $\mathrm{Spin}^{\mathbf{c}}$–structure, as will be explained below. It also carries a twisted Spin–structure in the sense of [9].

Let $P_{\mathrm{SO}}(W) \to W_{s,t,c}$ be the principal bundle of oriented orthonormal frames and let $P_{\mathrm{Spin}}(W) \to P_{\mathrm{SO}}(W)$ be the covering map defining the Spin–structure. Its restriction to a fiber of $P_{\mathrm{Spin}}(W) \to W_{s,t,c}$ can be identified (noncanonically) with the nontrivial covering $\rho\colon \mathrm{Spin}(4k+2) \to \mathrm{SO}(4k+2)$.

The fixed-point manifold of the τ–action on $W_{s,t,c}$ is the zero section B_c, which is of codimension 2. Hence, the involution τ is of odd type and the $\mathbb{Z}_2$–action on $P_{\mathrm{SO}}(W)$ does not lift to the Spin–structure (see [1, page 487]). However, as we will see below, the $\mathbb{Z}_2$–action does lift to a suitable $\mathrm{Spin}^{\mathbf{c}}$–structure.

Let $P_{U(1)}(W) \to W_{s,t,c}$ be the trivial principal $U(1)$–bundle and consider the two-fold covering map $P_{U(1)}(W) \to P_{U(1)}(W)$ for which the restriction to a fiber is given by the nontrivial two-fold covering $(\cdot)^2\colon U(1) \to U(1)$, $\lambda \mapsto \lambda^2$.

Let $\mathbb{Z}_2$ act by multiplication with ± 1 on $U(1)$. The $\mathbb{Z}_2$–actions on $W_{s,t,c}$ and $U(1)$ define a $\mathbb{Z}_2$–action on $P_{U(1)}(W)$. Note that this $\mathbb{Z}_2$–action does not lift in the two-fold covering $P_{U(1)}(W) \to P_{U(1)}(W)$.

Let $P_{\mathrm{Spin}^{\mathrm{c}}}(W) \to W_{s,t,c}$ denote the $\mathrm{Spin}^{\mathbf{c}}$–structure associated to the Spin–structure on $W_{s,t,c}$.

Lemma 5.2 *The $\mathbb{Z}_2$–actions on $P_{\mathrm{SO}}(W)$ and $P_{U(1)}(W)$ lift to a $\mathbb{Z}_2$–action on $P_{\mathrm{Spin}^{\mathrm{c}}}(W)$.*

Proof By definition, $P_{\mathrm{Spin}^{\mathrm{c}}}(W)$ is the extension of $P_{\mathrm{Spin}}(W)$ with respect to the inclusion

$$\mathrm{Spin}(4k+2) \hookrightarrow (\mathrm{Spin}(4k+2) \times U(1))/\{\pm(1,1)\} = \mathrm{Spin}^{\mathbf{c}}(4k+2).$$

Moreover, there is a $\mathrm{Spin}^{\mathbf{c}}(4k+2)$–equivariant bundle map

$$P_{\mathrm{Spin}^{\mathrm{c}}}(W) \to P_{\mathrm{SO}(n)}(W) \times P_{U(1)}(W)$$

with respect to the homomorphism $\mathrm{Spin}^{\mathbf{c}}(4k+2) \xrightarrow{\rho\times(\cdot)^2} \mathrm{SO}(4k+2) \times U(1)$ (here $P_{\mathrm{SO}(n)}(W) \times P_{U(1)}(W)$ denotes the fiberwise product of $P_{\mathrm{SO}(n)}(W)$ and $P_{U(1)}(W)$).

Recall that the $\mathbb{Z}_2$–actions on $P_{\mathrm{SO}}(W)$ and $P_{U(1)}(W)$ do not lift as $\mathbb{Z}_2$–actions in the coverings $P_{\mathrm{Spin}}(W) \to P_{\mathrm{SO}}(W)$ and $P_{U(1)}(W) \to P_{U(1)}(W)$. In both cases the induced action on the total spaces is by an effective action of $\mathbb{Z}_4$. Note, however, that the diagonal action of $\mathbb{Z}_4$ on $P_{\mathrm{Spin}^c}(W)$ has $\mathbb{Z}_2 \subset \mathbb{Z}_4$ as ineffective kernel. Hence, the $\mathbb{Z}_2$–action on $P_{\mathrm{SO}}(W) \times P_{U(1)}(W)$ lifts as a $\mathbb{Z}_2$–action to the $\mathrm{Spin}^{\mathbf{c}}$–structure $P_{\mathrm{Spin}^c}(W) \to W_{s,t,c}$. □

Recall that the $\mathbb{Z}_2$–actions on $(W_{s,t,c}, h_{s,t,c})$ and on the trivial principal $U(1)$–bundle $P_{U(1)}(W) \to W_{s,t,c}$ are of product form near the boundary of $W_{s,t,c}$. We fix a flat unitary $\mathbb{Z}_2$–equivariant connection $\nabla^{\mathbf{c}}(W)$ on $P_{U(1)}(W) \to W_{s,t,c}$ which is constant in the normal direction near the boundary of $W_{s,t,c}$.

Next we describe the relevant Dirac operators on $W_{s,t,c}$ and its boundary. Let $S(W_{s,t,c})$ denote the spinor bundle for the $\mathrm{Spin}^{\mathbf{c}}$–structure on $W_{s,t,c}$ defined before. The Levi-Civita connection of $(W_{s,t,c}, h_{s,t,c})$ together with the connection $\nabla^{\mathbf{c}}(W)$ determine a connection $\nabla(W)$ on $S(W_{s,t,c})$. Let D_W be the associated $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator, ie D_W is the composition

$$\Gamma(S(W_{s,t,c})) \to \Gamma(S(W_{s,t,c}) \otimes T^* W_{s,t,c}) \to \Gamma(S(W_{s,t,c}) \otimes T W_{s,t,c}) \to \Gamma(S(W_{s,t,c})),$$

where the first map is the connection $\nabla(W)$, the second map uses the isomorphism given by the metric $h_{s,t,c}$ and the last map is induced from Clifford multiplication (see [23, D.9]).

Since $W_{s,t,c}$ is of even dimension, the spinor bundle $S(W_{s,t,c})$ splits as a direct sum $S^+(W_{s,t,c}) \oplus S^-(W_{s,t,c})$ and the operator D_W restricts to an operator

$$\mathfrak{D}_W{}^+ \colon \Gamma(S^+(W_{s,t,c})) \to \Gamma(S^-(W_{s,t,c})).$$

The $\mathrm{Spin}^{\mathbf{c}}$–structure on $W_{s,t,c}$ induces a $\mathrm{Spin}^{\mathbf{c}}$–structure on the boundary. Let $\overline{P} \to \overline{M}_{s,t,c}$ denote the corresponding principal $\mathrm{Spin}^{\mathbf{c}}$–bundle. The restriction of $S^+(W_{s,t,c})$ and $\mathfrak{D}_W{}^+$ to the boundary can be identified with the spinor bundle $S(\overline{M}_{s,t,c})$ and the $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator

$$D_{\overline{M}} \colon \Gamma(S(\overline{M}_{s,t,c})) \to \Gamma(S(\overline{M}_{s,t,c}))$$

on $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$, which is defined with respect to $\overline{P} \to \overline{M}_{s,t,c}$ and the restriction $\overline{\nabla}^{\mathbf{c}}$ of the connection $\nabla^{\mathbf{c}}(W)$ to the principal $U(1)$–bundle (see [3])

$$\overline{P}_{U(1)} := P_{U(1)}(W)|_{\overline{M}_{s,t,c}} \to \overline{M}_{s,t,c}.$$

Consider the orthogonal projection of $\Gamma(S^+(W_{s,t,c})|_{\overline{M}_{s,t,c}}) = \Gamma(S(\overline{M}_{s,t,c}))$ onto the space spanned by the eigenfunctions of $D_{\overline{M}}$ for nonnegative eigenvalues. Following Atiyah, Patodi and Singer, we impose the *APS boundary condition*, ie we restrict to sections $\phi \in \Gamma(S^+(W_{s,t,c}))$ for which $\phi|_{\overline{M}_{s,t,c}}$ is in the kernel of the projection. After imposing this condition, the operator $\mathfrak{D}_W{}^+$ has finite-dimensional kernel and will be denoted by D_W^+. Similarly, the formal adjoint of $\mathfrak{D}_W{}^+$ (defined via bundle metrics) subject to the adjoint APS boundary condition has finite-dimensional kernel and will be denoted by $(D_W^+)^*$. The index of D_W^+ is defined as $\operatorname{ind} D_W^+ := \dim\ker D_W^+ - \dim\ker(D_W^+)^* \in \mathbb{Z}$ (see [3] for details). Note that by construction the operators D_W^+, $(D_W^+)^*$ and $D_{\overline{M}}$ are $\mathbb{Z}_2$–equivariant. For later reference we point out the following crucial lemma:

Lemma 5.3 *The operators D_W^+, $(D_W^+)^*$ and $D_{\overline{M}}$ are injective. In particular, we have* $\operatorname{ind} D_W^+ = 0$.

Proof Since $h_{s,t,c}$ and $\bar{g}_{s,t,c}$ are of positive scalar curvature and all relevant connections are flat, the statements follow from the argument of Schrödinger and Lichnerowicz [27; 24; 23]. □

Note that the objects on $W_{s,t,c}$ considered above, when restricted to the boundary $\overline{M}_{s,t,c}$, induce corresponding objects on $M_{s,t,c}$ by passing to the quotient with respect to the $\mathbb{Z}_2$–action. For example, the quotient of $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ by the free isometric $\mathbb{Z}_2$–action can be identified with $(M_{s,t,c}, g_{s,t,c})$ and the same is true for the respective principal bundles of oriented orthonormal frames and the Levi-Civita connections.

Similarly, the $\mathbb{Z}_2$–quotient of the principal $U(1)$–bundle $\overline{P}_{U(1)} \to \overline{M}_{s,t,c}$ with its flat connection $\overline{\nabla}^{\mathbf{c}}$ and the quotient of the Spin$^{\mathbf{c}}$–structure $\overline{P} \to \overline{M}_{s,t,c}$ can be identified with a principal $U(1)$–bundle $P_{U(1)} \to M_{s,t,c}$ with flat connection $\nabla^{\mathbf{c}}$ and a Spin$^{\mathbf{c}}$–structure $P \to M_{s,t,c}$ on $M_{s,t,c}$, respectively.

Since the generator of $\mathbb{Z}_2$ acts by $(\tau, -1)$ on $\overline{P}_{U(1)} = \overline{M}_{s,t,c} \times U(1)$, the bundle $P_{U(1)} \to M_{s,t,c}$ can be identified with $\overline{M}_{s,t,c} \times_{\mathbb{Z}_2} U(1) \to M_{s,t,c}$. This bundle is nontrivial. In fact, its first Chern class is of order 2 and generates the kernel of $p^*\colon H^2(M_{s,t,c};\mathbb{Z}) \to H^2(\overline{M}_{s,t,c};\mathbb{Z})$.

Let $S(M_{s,t,c})$ denote the spinor bundle associated to the Spin$^{\mathbf{c}}$–structure on $M_{s,t,c}$ and let

$$D_M\colon \Gamma(S(M_{s,t,c})) \to \Gamma(S(M_{s,t,c}))$$

denote the associated Spin$^{\mathbf{c}}$ Dirac operator. It follows from the construction that D_M lifts to the $\mathbb{Z}_2$–equivariant Spin$^{\mathbf{c}}$ Dirac operator $D_{\overline{M}}$ with respect to the covering map $p\colon \overline{M}_{s,t,c} \to M_{s,t,c}$.

6 Computation of η–invariants

In this section we will compute relative η–invariants for the $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator D_M on $M_{s,t,c}$ twisted with the nontrivial complex 1–dimensional representation of $\pi_1(M_{s,t,c})$. These computations will be used in the next section to prove the main theorem. Alternatively, one could use a twisted Spin–structure of $M_{s,t,c}$ and compute η–invariants of associated Dirac operators along the lines of [9, Section 2].

The idea to use relative η–invariants to distinguish components of moduli spaces goes back to Atiyah, Patodi and Singer, who explained this for positive scalar curvature metrics on spin manifolds in [4]. They also pointed out the possibility to extend this idea to certain $\mathrm{Spin}^{\mathbf{c}}$–manifolds. For background information on η–invariants of $\mathrm{Spin}^{\mathbf{c}}$–manifolds, we also refer to [11].

Recall that $\pi_1(M_{s,t,c}) = \mathbb{Z}_2$ and $p\colon \overline{M}_{s,t,c} \to M_{s,t,c}$ is a universal covering. Let $\alpha\colon \pi_1(M_{s,t,c}) \to U(1)$ denote the nontrivial homomorphism and let α also denote the associated complex line bundle $\overline{M}_{s,t,c} \times_\alpha \mathbb{C} \to M_{s,t,c}$. We fix a flat unitary connection on α. Let $D_{M,\alpha}$ denote the $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator D_M twisted with α.

Next consider the η–invariants $\eta(M_{s,t,c})$ and $\eta_\alpha(M_{s,t,c})$ of D_M and $D_{M,\alpha}$, respectively. Recall that $\eta(M_{s,t,c})$ (resp. $\eta_\alpha(M_{s,t,c})$) is given by the value at $z=0$ of the meromorphic extension of the series $\sum_\lambda \mathrm{sign}(\lambda)/|\lambda|^z$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \gg 0$ to the complex plane, where the sum is taken over all nonzero eigenvalues λ of D_M (resp. $D_{M,\alpha}$) (see [3] for background information on η–invariants).

Definition 6.1 The relative η–invariant $\tilde{\eta}_\alpha(M_{s,t,c})$ is defined by

$$\tilde{\eta}_\alpha(M_{s,t,c}) := \eta_\alpha(M_{s,t,c}) - \eta(M_{s,t,c}).$$

Recall from Section 5 that D_M lifts to the $\mathbb{Z}_2$–equivariant $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator $D_{\overline{M}}$. The η–invariant of $D_{\overline{M}}$ refines to a $\mathbb{Z}_2$–equivariant invariant with values denoted by $\eta(M_{s,t,c})_g$ for $g \in \mathbb{Z}_2 = \{1,\tau\}$. As pointed out in [13, Theorem 3.4], the η–invariants for $M_{s,t,c}$ can be computed from equivariant η–invariants for $\overline{M}_{s,t,c}$. In our situation this relation is given by

$$\eta_\alpha(M_{s,t,c}) = \tfrac{1}{2}(\eta(\overline{M}_{s,t,c})_1 \cdot \chi_\alpha(1) + \eta(\overline{M}_{s,t,c})_\tau \cdot \chi_\alpha(\tau)) = \tfrac{1}{2}(\eta(\overline{M}_{s,t,c}) - \eta(\overline{M}_{s,t,c})_\tau)$$

and

$$\eta(M_{s,t,c}) = \eta_e(M_{s,t,c}) = \tfrac{1}{2}(\eta(\overline{M}_{s,t,c})_1 \cdot 1 + \eta(\overline{M}_{s,t,c})_\tau \cdot 1) = \tfrac{1}{2}(\eta(\overline{M}_{s,t,c}) + \eta(\overline{M}_{s,t,c})_\tau),$$

where χ_α is the character of α and $e\colon \pi_1(M_{s,t,c}) \to U(1)$ denotes the trivial representation. This gives, for the relative η–invariant,

$$\widetilde{\eta}_\alpha(M_{s,t,c}) = \eta_\alpha(M_{s,t,c}) - \eta(M_{s,t,c}) = -\eta(\overline{M}_{s,t,c})_\tau.$$

Next we consider the $\mathbb{Z}_2$–action on the disk bundle $W_{s,t,c}$ over B_c and the equivariant $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator D_W^+ which was defined in Section 5. Since τ acts by -1 on the fibers of $W_{s,t,c}$, the fixed-point manifold can be identified with B_c. Let $a(B_c)(\tau)$ be the local datum of the Lefschetz fixed-point formula for the $\mathbb{Z}_2$–equivariant operator D_W^+ at B_c evaluated at $\tau \in \mathbb{Z}_2$, as described in [5] (see also [11]).

The index formula for manifolds with boundary [3] refines in the presence of symmetries and gives a relation between equivariant η–invariants, local data and certain representations attached to the index of D_W^+ and the kernel of $D_{\overline{M}}$ (see [13] for details). In our situation one obtains:

Proposition 6.2 $$\widetilde{\eta}_\alpha(M_{s,t,c}) = -2a(B_c)(\tau).$$

Proof As a warm-up we first consider the nonequivariant APS index formula for D_W^+, which takes the form (see [3, Theorem 3.10 and Section 4])

$$\operatorname{ind} D_W^+ = \left(\int_{W_{s,t,c}} e^{c_1/2}\hat{\mathcal{A}}(W_{s,t,c}, h_{s,t,c})\right) - \tfrac{1}{2}(\dim h(\overline{M}_{s,t,c}, \bar{g}_{s,t,c}) + \eta(\overline{M}_{s,t,c})),$$

where c_1 denotes the first Chern form of $\nabla^{\mathbf{c}}(W)$, $\hat{\mathcal{A}}(W_{s,t,c}, h_{s,t,c})$ represents the $\hat{\mathcal{A}}$–series evaluated on the Pontryagin forms $p_i(W_{s,t,c}, h_{s,t,c})$ and $h(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ is the kernel of $D_{\overline{M}}$. Since $\nabla^{\mathbf{c}}(W)$ is flat, c_1 vanishes. Since $(W_{s,t,c}, h_{s,t,c})$ and $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ are of positive scalar curvature, $\operatorname{ind} D_W^+$ and $h(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ both vanish (see Lemma 5.3).

Next we consider the index of the $\mathbb{Z}_2$–equivariant operator D_W^+. The index evaluated at τ can be expressed by the formula above after making the following replacements (see [13, Theorem 1.2] for details): First, $\dim h(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ is replaced by the character of the $\mathbb{Z}_2$–representation given by the kernel of $D_{\overline{M}}$ evaluated at τ. We denote this value by h_τ. Next, $\eta(\overline{M}_{s,t,c})$ is replaced by $\eta(\overline{M}_{s,t,c})_\tau$. Finally, the integral is replaced by the local datum $a(B_c)(\tau)$. Hence, one has

$$\operatorname{ind} D_W^+(\tau) = a(B_c)(\tau) - \tfrac{1}{2}(h_\tau + \eta(\overline{M}_{s,t,c})_\tau).$$

Since $(W_{s,t,c}, h_{s,t,c})$ and $(\overline{M}_{s,t,c}, \bar{g}_{s,t,c})$ are of positive scalar curvature, the representations which are used to define $\operatorname{ind} D_W^+(\tau)$ and h_τ are all 0–dimensional and trivial (see Lemma 5.3). Hence, ind $D^+_W(\tau)$ and h_τ both vanish and

$$\widetilde{\eta}_\alpha(M_{s,t,c}) = -\eta(\overline{M}_{s,t,c})_\tau = -2a(B_c)(\tau). \qquad \square$$

We proceed to describe the local datum $a(B_c)(\tau)$ (see [5, Section 3] for the general discussion). Let $\{\pm x_1, \dots, \pm x_{2k}\}$ denote the formal roots of TB_c and let y be the Euler class of the oriented normal bundle ν_{B_c} of $B_c \subset W_{s,t,c}$. Let c_1 denote now the first Chern class of the $\mathrm{Spin}^{\mathbf{c}}$ Dirac operator D^+_W. Then the local datum evaluated at τ is given by

$$a(B_c)(\tau) = \epsilon \cdot \int_{B_c} e^{c_1/2} \cdot \widehat{A}(B_c) \cdot \frac{1}{i \cdot e^{y/2} + i \cdot e^{-y/2}},$$

where $\widehat{A}(B_c) = \prod_{j=1}^{2k} x_j/(e^{x_j/2} - e^{-x_j/2})$ and $\epsilon \in \{\pm i\}$ depends on the lift of the $\mathbb{Z}_2$–action to the $\mathrm{Spin}^{\mathbf{c}}$–structure. We will not discuss this ambiguity further since it will not affect the results on moduli spaces stated in the main theorem. The class c_1 vanishes since the bundle $P_{U(1)}(W)$ is trivial. Note that $y = su + tv$ since ν_{B_c} is isomorphic to the complex line bundle associated to the principal S^1–bundle $\pi\colon \overline{M}_{s,t,c} \to B_c$ (see Definition 2.3). Hence,

$$a(B_c)(\tau) = \pm \int_{B_c} \widehat{A}(B_c) \cdot \frac{1}{e^{(su+tv)/2} + e^{-(su+tv)/2}}.$$

Next recall from Lemma 2.2 that TB_c has a complex structure and the total Chern class of TB_c is given by

$$c(TB_c) = (1+2v)\cdot((1+u)^{2k} - c\cdot v\cdot(1+u)^{2k-1}) = (1+2v)\cdot(1+u)^{2k-1}\cdot(1+u-c\cdot v).$$

Hence, one obtains the local term $a(B_c)(\tau)$ up to sign by integrating

$$(1)\qquad \frac{2v}{e^v - e^{-v}} \cdot \left(\frac{u}{e^{u/2} - e^{-u/2}}\right)^{2k-1} \cdot \frac{u - cv}{e^{(u-cv)/2} - e^{-(u-cv)/2}} \cdot \frac{1}{e^{(su+tv)/2} + e^{-(su+tv)/2}}$$

over B_c. Note that the integral is given by evaluating the cohomological expression on the fundamental class of B_c, which, by Lemma 2.2, amounts to computing the coefficient of $u^{2k-1} \cdot v$ in (1). In the following, k and c will be fixed.

Proposition 6.3 *For almost all $s \neq 0$ with s even, $a(B_c)(\tau)$ is a nonzero polynomial in t of degree 1.*

The proposition as stated is sufficient for our purposes. It is likely that the statement is true for all $s \neq 0$. We leave it to the interested reader to prove the more general statement.

Proof For a fixed odd integer c, let $A \in \mathbb{Q}[s,t]$ denote the polynomial obtained by integrating the expression in (1) over B_c. To prove the proposition we first note that the factor in (1) involving t is equal to

$$\frac{1}{e^{su/2}+e^{-su/2}}\cdot\left(1-\frac{tv}{2}\cdot\frac{e^{su/2}-e^{-su/2}}{e^{su/2}+e^{-su/2}}\right)$$

since $v^2=0$ by Lemma 2.2.

Hence, A is a polynomial in t of degree ≤ 1, say $A = A_0 - A_1\cdot t$ with $A_i \in \mathbb{Q}[s]$. Moreover, by looking at the other factors of (1), we see that A_1 is given by integrating

$$\frac{2v}{e^v-e^{-v}}\cdot\left(\frac{u}{e^{u/2}-e^{-u/2}}\right)^{2k-1}\cdot\frac{u-cv}{e^{(u-cv)/2}-e^{-(u-cv)/2}}\cdot\frac{1}{e^{su/2}+e^{-su/2}}\cdot\left(\frac{v}{2}\cdot\frac{e^{su/2}-e^{-su/2}}{e^{su/2}+e^{-su/2}}\right)$$

over B_c. Since $v^2=0$, we get

$$A_1=\int_{B_c}\left(\frac{u}{e^{u/2}-e^{-u/2}}\right)^{2k-1}\cdot\frac{u}{e^{u/2}-e^{-u/2}}\cdot\frac{1}{e^{su/2}+e^{-su/2}}\cdot\frac{e^{su/2}-e^{-su/2}}{e^{su/2}+e^{-su/2}}\cdot\frac{v}{2}.$$

Using Lemma 2.2 again, it follows that A_1 is equal to the coefficient of u^{2k-1} in the formal power series

$$\left(\frac{u}{e^{u/2}-e^{-u/2}}\right)^{2k}\cdot\frac{e^{su/2}-e^{-su/2}}{2(e^{su/2}+e^{-su/2})^2}\in\mathbb{Q}[s][\![u]\!].$$

Note that A_1 is an odd polynomial in s of degree $\le 2k-1$, which can be written as a residue,

$$A_1=\operatorname{Res}_{u=0}\left(\left(\frac{1}{e^{u/2}-e^{-u/2}}\right)^{2k}\cdot\frac{e^{su/2}-e^{-su/2}}{2(e^{su/2}+e^{-su/2})^2}\right).$$

Using the substitution $w := 2\cdot\sinh\left(\frac{1}{2}u\right)=e^{u/2}-e^{-u/2}=u+\cdots$, one finds that

$$A_1=\operatorname{Res}_{w=0}\frac{1}{w^{2k}}\cdot\frac{\sinh\left(\frac{1}{2}su\right)}{(2\cosh\left(\frac{1}{2}su\right))^2}\cdot\frac{1}{\cosh\left(\frac{1}{2}u\right)}.$$

To show that the polynomial $A_1\in\mathbb{Q}[s]$ is nonzero, we will compute its value for $s=2$ with the help of the addition theorems for sinh and cosh:

$$\begin{aligned}
\operatorname{Res}_{w=0}\frac{1}{w^{2k}}\cdot\frac{\sinh\bigl(2\cdot\frac12 u\bigr)}{(2\cosh\bigl(2\cdot\frac12 u\bigr))^2}\cdot\frac{1}{\cosh\bigl(\frac12 u\bigr)}
&= \operatorname{Res}_{w=0}\frac{1}{w^{2k}}\cdot\frac{2\sinh\bigl(\frac12 u\bigr)\cdot\cosh\bigl(\frac12 u\bigr)}{4(1+2\sinh^2\bigl(\frac12 u\bigr))^2}\cdot\frac{1}{\cosh\bigl(\frac12 u\bigr)}\\
&= \operatorname{Res}_{w=0}\frac{1}{w^{2k}}\cdot\frac{w}{4\bigl(1+\frac12 w^2\bigr)^2}\\
&= \text{coefficient of } w^{2k-2} \text{ in } \frac{\frac14}{\bigl(1+\frac12 w^2\bigr)^2}\\
&\neq 0.
\end{aligned}$$

Hence, A_1 is a nonzero polynomial in s. This shows that A_1 does not vanish for almost all even integers s. It follows that $a(B_c)(\tau)=\pm A$ is a nonzero polynomial in t of degree 1 for almost all even integers s. □

7 Proof of the main theorem

The proof of the main theorem follows from the previous results by an argument similar to the one in [11]. We will focus on the statement on $\mathcal{M}_{\mathrm{sec}\geq 0}$; the statement for $\mathcal{M}_{\mathrm{Ric}>0}$ is analogous and easier. The main steps are the following (see [11, Section 6] for more details).

As before we will assume that c is odd, $k\geq 2$, s and t are nonzero coprime integers, and s is even. In the following we will fix c and k and choose $s>0$ such that the local datum $a(B_c)(\tau)$ is a nonzero polynomial in t of degree 1. By Proposition 6.3, there are infinitely many choices for such s and, by Lemma 2.4, different choices for s lead to different manifolds $M_{s,t,c}$, which can be distinguished by their integral cohomology. We will fix a choice for s.

By Theorem 3.1, the family $\mathcal{F}_{c,s}=\{M_{s,t,c}\mid t \text{ coprime to } s\}$ of $(4k+1)$–dimensional oriented manifolds belongs to finitely many oriented diffeomorphism types. Let us choose a sequence $t_0<t_1<t_2<\cdots$ such that each $M_{s,t_l,c}$ for $l\geq 0$ is diffeomorphic to $M_{s,t_0,c}$ as an oriented manifold and such that the relative η–invariants $\widetilde{\eta}_\alpha(M_{s,t_l,c})$ for $l\in\mathbb{N}$ are pairwise distinct (see Propositions 6.2 and 6.3). An orientation-preserving diffeomorphism $M_{s,t_0,c}\to M_{s,t_l,c}$ may not preserve the topological Spin^c–structures. However, since $M_{s,t_l,c}$ has only finitely many (namely two) topological Spin^c–structures with trivial first Chern class, we may assume, after passing to a subsequence, again denoted by $M_{s,t_l,c}$, that all manifolds in this sequence are diffeomorphic by diffeomorphisms preserving the topological Spin^c–structures. Let $M := M_{s,t_0,c}$ and let $F_l : M \to M_{s,t_l,c}$ be such a diffeomorphism.

Let $g_l := F_l^*(g_{s,t_l,c})$, where $g_{s,t_l,c}$ is the submersion metric of nonnegative sectional and positive Ricci curvature on $M_{s,t,c}$ from Section 4. Since η–invariants are preserved under pullback, we conclude that the relative η–invariants of the Spin^c–manifold M with respect to g_l for $l \in \mathbb{N}$ are pairwise distinct.

Let $\mathcal{D}$ denote the subgroup of diffeomorphisms of M which preserve its topological Spin^c–structure. Note that $\mathcal{D}$ has finite index in the full diffeomorphism group $\mathrm{Diff}(M)$. Hence, it suffices to show that the elements $[g_l] \in \mathcal{R}_{\sec\geq 0}(M)/\mathcal{D}$ for $l \in \mathbb{N}$ defined by g_l represent infinitely many path components.

We argue by contradiction. Suppose there is a path $\tilde{\gamma}\colon [0,1] \to \mathcal{R}_{\sec\geq 0}(M)/\mathcal{D}$ connecting $[g_l]$ to $[g_{l'}]$ with $l \neq l'$. By Ebin's slice theorem [14], this path can be lifted to a continuous path γ in $\mathcal{R}_{\sec\geq 0}(M)$ with $\gamma(0) = g_l$ and $\gamma(1) = \Phi^*(g_{l'})$ for some $\Phi \in \mathcal{D}$. Since η–invariants are preserved under pullback, it follows that the relative η–invariants of the Spin^c–manifold M with respect to $\gamma(0) = g_l$ and $\gamma(1) = \Phi^*(g_{l'})$ are distinct.

The path γ may be deformed inside of $\mathcal{R}_{\mathrm{scal}>0}(M)$ to a path $\hat{\gamma}$ with the same endpoints as γ and whose interior points lie in $\mathcal{R}_{\mathrm{Ric}>0}(M)$ (this can be done via Ricci flow using [7]). Since the relative η–invariant is constant on path components of $\mathcal{R}_{\mathrm{scal}>0}(M)$ (see [4, page 417; 11, Proposition 3.3]), we get a contradiction.

Hence, the classes $[g_l]$ for $l \in \mathbb{N}$ represent infinitely many pairwise distinct path components of $\mathcal{R}_{\sec\geq 0}(M)/\mathcal{D}$. Since $\mathcal{D}$ has finite index in $\mathrm{Diff}(M)$, the same holds for the moduli space $\mathcal{M}_{\sec\geq 0}(M)$. As explained in the beginning, we can argue in this way for infinitely many choices of s. Hence, we obtain infinitely many manifolds $M_i := M_{s_i,t_i,c}$, indexed by $i \in \mathbb{N}$, which can be distinguished by their integral cohomology, such that for each $i \in \mathbb{N}$ the moduli space $\mathcal{M}_{\sec\geq 0}(M_i)$ has infinitely many path components. This completes the proof of the first statement of the main theorem. An analogous argument gives the statement for $\mathcal{M}_{\mathrm{Ric}>0}$. □

Remark 7.1 For $k = 1$, the manifolds $\overline{M}_{s,t,c}$ given in Definition 2.3 can be shown to be diffeomorphic to $S^2 \times S^3$ (for c odd, s even, and s and t coprime). The $\mathbb{Z}_2$–quotients $M_{s,t,c}$ can be described as total spaces of S^1–principal bundles over $B_c \cong \mathbb{C}P^2 \sharp -\mathbb{C}P^2$ and fall into finitely many diffeomorphism types. Their moduli spaces of metrics of nonnegative sectional curvature and positive Ricci curvature also have infinitely many path components (see also the recent work of Goodman and Wermelinger [17; 32] on such moduli spaces for the class of all orientable nonspin $\mathbb{Z}_2$–quotients of $S^2 \times S^3$).

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