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# **Valuation of Derivatives Based on CKLS Interest Rate Models**

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## **Based on CKLS Interest Rate Models**

### **Abstract**

The CKLS (1992) short-term risk-free interest rate process leads to valuation model for both default free bonds and contingent claims that can only be solved numerically for the general case. Valuation equations of this nature in the past have been solved using the Crank Nicholson scheme. In this paper, we introduce a new numerical scheme – the Box method, and compare it with the traditional Crank Nicholson scheme. We find that in specific cases of the CKLS process where analytical prices are available, the new scheme leads to more accurate results than the Crank Nicholson scheme.

**Key Words:** CKLS Interest rate model, Box method, Crank Nicholson scheme.

## 1. Introduction

A feature distinguishing interest rate models from equity models is the need for interest rate models to exhibit mean reversion and for the volatility to be dependent on the interest rate. Due to these complexities and to the fact that interest rates cannot be traded like stock options, two main pricing methodologies have arisen for fixed income securities.

Both pricing methodologies price the option and the underlying bond as functions of the term structure of interest rates. The first pricing methodology prices the option as a function of an exogenously given term structure of interest rates. The second methodology restricts the behavior of the term structure to ensure that the observed market prices of zero coupon bonds are respected by the model. The first group comprises of models proposed by Vasicek (1977) and Dothan (1978) amongst many others. The second group comprises of models proposed by Cox, Ingersoll and Ross (1985), Ho and Lee (1986), HJM (1992) amongst others. For security valuation purposes, the lattice approach is generally used with the second group of models; with the notable exception of the Cox, Ingersoll and Ross model. The partial differential equation is used with the first approach. The partial differential equation approach leads to analytical solutions for specific models such as the Vasicek and Cox, Ingersoll and Ross. The existence of analytical solutions leads to quick valuation of the bond, option prices, and the hedge parameters such as delta for risk management purposes.

The first group of models in addition to the Cox, Ingersoll and Ross model can be enclosed in the most general Chan, Karolyi, Longstaff and Sanders (CKLS, 1992) model. Unlike many of the specific short-term interest rate model, no analytical solution is available for the bond and option prices based on the CKLS model. Thus in order to value bonds, options and the relevant hedge parameters a numerical approach is required. In option pricing literature, the standard numerical approach to value bonds and options on bonds is the Crank Nicholson scheme (see for example Courtadon (1982)). The basic idea behind the Crank Nicholson scheme is that the numerical solution should converge

to the true solution. However, under the Crank Nicholson scheme, although convergence may be guaranteed, the convergence to the true solution is not guaranteed. Furthermore, to arrive at the numerical solution based on the Crank Nicholson scheme we need to solve a system of finite difference equations expressed as a matrix; which may be solved iteratively or by direct elimination.

The contribution of the present paper is to introduce a new numerical method to finance from engineering and the physical sciences called the Box method. The objective of this paper is to compare the bond and option prices derived using the Crank Nicholson scheme and the Box method. We find that the Box method is superior. Its performance is dramatically better for the valuation of options on bonds with high volatility and low mean reversion. Our result is of practical relevance in the light of the recent findings of Bliss and Smith (1998).

Section II discusses the CKLS model in depth. In Section III we develop the Crank Nicholson scheme and the Box method. Section IV compares bond and option prices calculated using both the Crank Nicholson scheme and the Box method. Section V contains summary and conclusion.

## **II. CKLS Model**

CKLS used the following stochastic differential equation to specify the general form of the short- term interest rate, nesting a range of different term structure models.

$$dr = k(\theta - r) + \sigma r^\gamma dZ \quad (1)$$

where  $k, \theta, \sigma, \gamma$  are unknown parameters. As in the CKLS paper, we can obtain the alternative term structure models given below by imposing restrictions on the parameters  $k, \theta, \sigma, \gamma$ :

1. Merton (1973)  $dr = k\theta + \sigma dZ$
2. Vasicek (1977)  $dr = k(\theta - r) + \sigma dZ$
3. Cox, Ingersoll, and Ross (1985)  $dr = k(\theta - r) + \sigma r^{\frac{1}{2}} dZ$
4. Dothan (1978)  $dr = \sigma r dZ$
5. Geometric Brownian Motion  $dr = -kr + \sigma dZ$
6. Brennan and Schwartz (1980)  $dr = k(\theta - r) + \sigma r dZ$
7. Cox, Ingersoll, and Ross (1980)  $dr = \sigma r^{\frac{3}{2}} dZ$
8. Constant Elasticity of Variance  $dr = kr dt + \sigma r^{\gamma} dZ$

CKLS found that the value of  $\gamma$  is the most important feature distinguishing interest rate models. In particular they find that for the U.S.  $\gamma \geq 1$  captures the dynamics of the short-term interest rate because the volatility of the process is highly sensitive to the value of  $r$ .

Based on standard arbitrage arguments, we can derive valuation equations for default free bonds and options based on the CKLS model. The valuation equation for the default free bond will be the same irrespective of the type of option, which is based on it. However, the valuation equation for different types of options will be different, due to the differing boundary conditions associated with each type of options. In this paper we concentrate

on the valuation of zero coupon default free bonds and the valuation of <sup>1</sup>call options based on the zero coupon bonds.

As in Courtadon (1982), we take the time expiry of the option as  $\tau$ , and the time to maturity of the bond as  $\tau' = \tau + T$ , where  $T$  is the time to maturity remaining on the bond when the option expires.

Letting  $u(r, \tau')$  be the value of the default-free bond; the valuation equation based on the CKLS process is:

$$\frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 u}{\partial r^2} + k[\theta - r] \frac{\partial u}{\partial r} - ru = \frac{\partial u}{\partial \tau'} \quad (2)$$

Subject to:  $u(r, 0) = 1$

$$u(\infty, \tau') = 0$$

Similarly the valuation equation for a call option  $w(r, \tau)$  based on the CKLS process is:

$$\frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 w}{\partial r^2} + k[\theta - r] \frac{\partial w}{\partial r} - rw = \frac{\partial w}{\partial \tau} \quad (3)$$

Subject to:  $w(r, 0) = \text{Max}[u(r, T) - E, 0]$

$$w(\infty, \tau) = 0$$

where E is the exercise price of the option.

### III Numerical Solution for the Valuation of Default Free Bonds and Options

In this section we discuss alternative discretization schemes for the numerical valuation of both default-free bond and options. We start with the widely used Crank Nicholson scheme, and then we introduce the Box method discretization scheme.

#### Crank Nicholson Method

The starting point with the Crank Nicholson scheme is to transform the interest rate

variable  $r$  such that  $s = \frac{\eta r}{1 + \eta r}$  with  $0 \leq s \leq 1$ . Based on this transformation equations (2)

and (3) become respectively:

$$\begin{aligned} & \left\{ \frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^4 \right\} \frac{\partial^2 U}{\partial s^2} \\ & + \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^3 + \left[ k\theta - \frac{s}{\eta(1-s)} k \right] \eta (1-s)^2 \right\} \frac{\partial U}{\partial s} \\ & - \frac{s}{\eta(1-s)} U = \frac{\partial U}{\partial \tau'} \end{aligned} \quad (4)$$

for:  $U(s, \tau') = u(r, \tau')$



$$\begin{aligned}
& \left\{ \frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^4 \right\} \frac{\partial^2 W}{\partial s^2} \\
& + \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^3 + \left[ k\theta - \frac{s}{\eta(1-s)} k \right] \eta (1-s)^2 \right\} \frac{\partial W}{\partial s}
\end{aligned} \tag{5}$$

$$-\frac{s}{\eta(1-s)} W = \frac{\partial W}{\partial \tau}$$

for:  $W(s, \tau) = w(r, \tau)$

We can represent both equations (4) and (5) as a general equation:

$$\begin{aligned}
& \left\{ \frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^4 \right\} \frac{\partial^2 v}{\partial s^2} \\
& + \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^3 + \left[ k\theta - \frac{s}{\eta(1-s)} k \right] \eta (1-s)^2 \right\} \frac{\partial v}{\partial s}
\end{aligned} \tag{6}$$

$$-\frac{s}{\eta(1-s)} v = \frac{\partial v}{\partial t}$$

where  $v$  may represent either  $U(s, \tau')$  or  $W(s, \tau)$  and  $t$  may represent either  $\tau'$  or  $\tau$ .

As in Courtadon (1982) we let  $s$  take value on the interval  $\Gamma = [0, S]$  and  $t$  take value on the interval  $T = [0, T']$ . To solve equation (6) we need to fit the space  $\Gamma \times T$  with a grid.

We let  $\Delta s$  represent the grid spacing in the  $s$  dimension and  $\Delta t$  represent the grid spacing in the  $t$  direction, such that:

$s_n = n\Delta s$ ,  $t_m = m\Delta t$  with  $0 \leq n \leq N, 0 \leq m \leq M$  such that  $N\Delta s = S$  and  $M\Delta t = T'$ .

The value of  $v$  is approximated by  $V_n^m$  at the grid points  $s_n$  and  $t_m$ . Based on the finite difference approximations give in the appendix, the following finite difference equation is derived:

$$\alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m \quad (7)$$

where:

$$\alpha_n = -A_n [V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}] - B_n [V_{n+1}^{m-1} - V_{n-1}^{m-1}] - [1 + C_n] V_n^{m-1}$$

$$\chi_n = A_n - B_n$$

$$\eta_n = C_n - 2A_n - 1$$

$$\beta_n = A_n + B_n$$

$$A_n = \frac{\sigma^2 \Delta t}{2} \left[ \frac{n\Delta s}{\eta(1-n\Delta s)} \right]^{2\gamma} \frac{\eta^2 (1-n\Delta s)}{2(\Delta s)^2}$$

$$B_n = \frac{\Delta t}{4\Delta s} \left\{ -\sigma^2 \left[ \frac{n\Delta s}{\eta(1-n\Delta s)} \right]^{2\gamma} \eta^2 (1-n\Delta s)^3 + \left[ k\theta - \frac{n\Delta s}{\eta(1-n\Delta s)} k \right] \eta (1-n\Delta s)^2 \right\}$$

$$C_n = -\frac{n\Delta s \Delta t}{2\eta(1-n\Delta s)}$$

## Box Method

To derive the algorithm for the Box method, we focus on equation (2), as exactly the same analysis holds for equation (3). We start by dividing equation (2) by  $\frac{\sigma^2 r^{2\gamma}}{2}$  and further we let,  $a = \frac{2k\theta}{\sigma^2}$ ,  $b = \frac{2k}{\sigma^2}$ ,  $c = \frac{2}{\sigma^2}$ . This results in the following equation:

$$\frac{\partial^2 u}{\partial r^2} + [ar^{-2\gamma} - br^{1-2\gamma}] \frac{\partial u}{\partial r} - cr^{1-2\gamma} u = cr^{-2\gamma} \frac{\partial u}{\partial \tau'} \quad (8)$$

We combine the first term and the second term on the left hand side of the above equation by choosing a function  $\Psi(a, b, r, \gamma)$  or  $\Psi(r)$  abbreviated such that:

$$\frac{1}{\Psi(r)} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial r^2} + [ar^{-2\gamma} - br^{1-2\gamma}] \frac{\partial u}{\partial r} \quad (9)$$

Expansion and simplification of the above formula leads to the following expression.

$$\frac{1}{\Psi(r)} \frac{\partial \Psi}{\partial r} = ar^{-2\gamma} - br^{1-2\gamma} \quad (10)$$

Integrating the above equation with respect to  $r$  gives:

$$\Psi(r) = \exp \left[ \frac{ar^{1-2\gamma}}{1-2\gamma} - \frac{br^{2-2\gamma}}{2-2\gamma} \right] \quad (11)$$

Note that with the above expression for  $\Psi(r)$  there is singularity at  $\gamma = \frac{1}{2}$  and  $\gamma = 1$ .

Thus the above expression for  $\Psi(r)$  is not valid at these two specific points. Further if

$\gamma \neq 1$  or  $\gamma \neq \frac{1}{2}$  but  $\gamma$  is very close to  $\gamma = 1$  or  $\gamma = \frac{1}{2}$ , then the value of  $\Psi(r)$  may be

excessively large because of the nature of the denominators in equation (11). In such

cases we need to use a more complex approach or simply switch to the expression for

$\Psi(r)$  when  $\gamma = 1$  or  $\gamma = \frac{1}{2}$ . To derive expression for  $\Psi(r)$  when  $\gamma = 1$  or  $\gamma = \frac{1}{2}$ , we

substitute, these two values of  $\gamma$  directly into equation (10) and integrate to give:

$$\Psi(r) = \exp\left(\frac{-a}{r}\right) r^{-b} \quad \text{for } \gamma = 1$$

$$\Psi(r) = \exp(-br) r^a \quad \text{for } \gamma = \frac{1}{2}$$

With this choice of  $\Psi(r)$ , our original equation becomes

$$\frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial u}{\partial r} \right) - \Psi(r) r^{1-2\gamma} c u = c \Psi(r) r^{-2\gamma} \frac{\partial u}{\partial \tau} \quad (12)$$

Similar analysis of the option valuation equation yields:

$$\frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial w}{\partial r} \right) - \Psi(r) r^{1-2\gamma} c w = c \Psi(r) r^{-2\gamma} \frac{\partial w}{\partial \tau} \quad (13)$$

We can represent, equations, (10) and (11) as a general equation:

$$\frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) - \Psi(r) r^{1-2\gamma} c v = c \Psi(r) r^{-2\gamma} \frac{\partial v}{\partial t} \quad (14)$$

where  $v$  may represent either  $u(r, \tau')$  or  $w(r, \tau)$  and  $t$  may represent either  $\tau'$  or  $\tau$ . As

with the Crank Nicholson scheme, we let  $r$  take value on the interval  $H = [0, R]$  and  $t$  take

value on the interval  $T = [0, T']$ . To solve equation (14) we need to fit the space  $H \times T$

with a grid. We let  $\Delta r$  represent the grid spacing in the  $r$  dimension and  $\Delta t$  represent the grid spacing in the  $t$  direction, such that:

$$r_n = n\Delta r, \quad t_m = m\Delta t \quad \text{with } 0 \leq n \leq N, 0 \leq m \leq M \quad \text{such that } N\Delta r = R \quad \text{and } M\Delta t = T'.$$

To derive the Box method scheme, we integrate equation (14) from  $r_a = \frac{r_n + r_{n-1}}{2}$  to

$r_b = \frac{r_{n+1} + r_n}{2}$  yielding the following equation:

$$\int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) dr - \int_{r_a}^{r_b} (\Psi(r) r^{1-2\gamma} c v) dr = \int_{r_a}^{r_b} \left( c \Psi(r) r^{-2\gamma} \frac{\partial v}{\partial \tau} \right) dr \quad (15)$$

Equation (15) is solved by numerical integration (full details in the Appendix). As with the Crank Nicholson scheme, the value of  $v$  is approximated by  $V_n^m$  at the grid points  $r_n$  and  $t_m$ . The resulting difference equation is:

$$\alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m \quad (16)$$

$$\begin{aligned} \alpha_n &= \frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) V_n^{m-1} && \text{if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1 \\ &= -\frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) V_n^{m-1} && \text{for } \gamma = \frac{1}{2} \\ &= \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) V_n^{m-1} && \text{for } \gamma = 1 \end{aligned}$$

$$\chi_n = -\frac{1}{\Delta r} \frac{\Psi(r_a)}{\Psi(r_n)}$$

$$\beta_n = -\frac{1}{\Delta r} \frac{\Psi(r_b)}{\Psi(r_n)}$$

$$\eta_n = \frac{1}{\Delta r} \left( \frac{\Psi(r_b)}{\Psi(r_n)} + \frac{\Psi(r_a)}{\Psi(r_n)} \right) + X$$

where:

$$\begin{aligned}
X &= \frac{cr_b^{2-2\gamma}}{2-2\gamma} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) + \frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) \quad \text{provided } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1 \\
&= c(r_b - r_a) - \frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \quad \text{for } \gamma = \frac{1}{2} \\
&= -c \ln \left( \frac{r_a}{r_b} \right) + \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \quad \text{for } \gamma = 1
\end{aligned}$$

### Valuation of Finite Difference System of Equations

Equations (7) and (16) as a system of equations covering the whole grid can be represented by the following matrix equation:

$$\begin{pmatrix} \alpha_1 - \chi_1 V_0^m \\ \alpha_2 V_2^{m-1} \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{N-1} - \beta_{N-1} V_N^m \end{pmatrix} = \begin{pmatrix} \eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \chi_{N-3} & \eta_{N-3} & \beta_{N-3} & 0 \\ \vdots & \ddots & \ddots & 0 & \chi_{N-1} & \eta_{N-2} & \beta_{N-2} \\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1} \end{pmatrix} \begin{pmatrix} V_1^m \\ V_2^m \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ V_{N-1}^m \end{pmatrix} \quad (17)$$

There exists two separate approaches to solving the above matrix equation. The elimination approach, and the iterative approach. An example of the former is the gaussian elimination approach widely used in option pricing literature (Courtadon (1982) develops this approach in depth). An example of the latter approach is the Successive Over Relaxation or SOR approach, which we further discuss in depth below.

The first step of the SOR process involves forming an intermediate quantity  $z_n^m$  for a point  $n$  on the grid. Based on this intermediate quantity, a trial solution  $V_n^m$  is formed. This trial solution is iterated until, a certain <sup>3</sup>accuracy is achieved between successive iterations. Having achieved this accuracy we move onto  $n+1$  point on the grid at a particular time step.

$$z_n^m = \frac{1}{\eta_n} (\alpha_n V_n^{m-1} - \chi_n V_{n-1}^m - \beta_n V_{n+1}^{m-1}) \quad (18)$$

$$V_n^m = \omega z_n^m + (1 - \omega) V_n^{m-1} \quad (19)$$

for  $n=1, \dots, N-1$ , and  $\omega \in (1, 2]$

#### IV. Analysis of Results

In this section, we investigate the two numerical methods. Each method is implemented to value zero coupon bond and call option prices when the short-term interest rate follows the CKLS stochastic process..

Initially we check each of the numerical methods using assumed parameter values. We then use the parameters for the U.S. as estimated in the CKLS paper. As in Tian (1994), we define a quantity  $\alpha_1 = (4k\theta - \sigma^2)/8$  for the assumed parameter values.

$\alpha_1 > 0$  corresponds to low volatility and high mean reversion rate. For  $\alpha_1 < 0$  the



converse condition holds. We consider the specific case  $\gamma = 0.5$ , because of the results in Barone-Adesi, Dinenis and Sorwar (1997) and Bliss and Smith (1998). The former authors find that instability appears in binomial trees for low values of  $\gamma$  and  $\alpha_1 < 0$ , the latter authors find evidence of low values of  $\gamma$  in the market in recent times.

The maturities of our bonds are 5 and 15 years. The face value of the zero coupon bond is \$100. Short-term interest rates of 5% and 11% are considered. For

$\alpha_1 > 0, k = 0.5, \sigma = 0.1, \theta = 0.08$ , and for  $\alpha_1 < 0, k = 0.1, \sigma = 0.5, \theta = 0.08$ . Table 1 – Table 2, and Table 3 – Table 4 contain the bond and call prices respectively calculated using each of the suggested numerical methods for different combinations of  $\alpha_1$ . Table 5

and Table 6; contain the bond and call option prices based on the CKLS parameters respectively. For the sake of brevity, following notation will be used in all of the tables:

BMS: prices calculated using the Box method, which uses Successive-Over-Relaxation.

BMG: prices calculated using the Box method, which uses gaussian elimination.

CNS: prices calculated using the Crank Nicholson method, which uses Successive-Over-Relaxation.

CNG: prices calculated using the Crank Nicholson method, which uses gaussian elimination.

Table 1 – Table 2 both have the same format and contain zero coupon bond prices. In each of these tables, we alter the annual number of time steps from 20 to 1000. This variation serves as a check as to the convergence of each of the numerical schemes.

Examination of Table 1 – Table 2 leads to the following observations:

From Table 1 we see that all four combinations converge to produce bond prices which are in agreement with the analytical prices. Furthermore, we find that SOR and gaussian elimination yield almost identical prices with each of the two methods. As an example consider a 5-year bond at 5% interest rate and 50 annual time steps; we find that the Box price using both SOR and gaussian elimination is identical at \$71.0754. Whilst the Crank Nicholson prices are \$71.6853 and \$71.6958, using SOR and gaussian elimination respectively. We also find that Box bond prices are always lower than Crank Nicholson, and that Box bond prices are closer to the analytical prices than Crank Nicholson prices. For example, we see that a 5-year bond at 5% interest is valued at 71.0379. The same bond at twenty annual time step is valued at \$71.1006 using Box (SOR) and \$71.6853 using Crank Nicholson (SOR).

In Table 2, only the prices using SOR are stated, as gaussian elimination in this case did not lead to prices, which agreed with the analytical prices. The bond prices calculated show the same traits as in Table 1.

Table 3 – Table 4 all have the same format and comprise of call options based on zero coupon bond prices for various expiry dates and exercise prices. In Table 3 – Table 4 the first column indicates the range of exercise prices and the first row indicates the different expiry dates of the option ranging from 1 year to 5 years. All the call options are based on a 10 year zero coupon bond, the call options are written during the last 5 years of the bond's maturity date. Further the third column entitled, "Bond Price", indicates the price of a 10 year zero coupon bond based on each of the possible combinations. For example,

turning to Table 3's, third column, we find that the price of a 10 year zero coupon bond calculated using the Box method (SOR) is \$45.5000, whereas the same bond is priced at \$45.8809 using the Crank Nicholson method (SOR). Examination of Table 3 – Table 4 leads to the following observations:

Box prices are closer to the analytical prices than Crank Nicholson call prices. For example, from Table 3, consider a 5-year call option, exercised at \$35. The analytical call price is \$21.8802; Box pricing using SOR is \$21.9445 and the Crank Nicholson price again using SOR is \$22.1132.

As with bonds, Box prices are always lower than the corresponding call prices calculated using Crank Nicholson. However, unlike bonds, the differences are significant when  $\alpha_1 < 0$ . To illustrate the differences in call prices between the Box and the Crank Nicholson methods consider an example from Table 4. In particular, consider a 5-year option, exercise at \$60, the analytical call price is \$23.9008, the Box price is \$23.9476, and the Crank Nicholson price is \$32.2997. The large error of Crank Nicholson mirrors the instability of binomial trees for similar parameter values.

Again, as with bonds, when all four combinations yield sensible prices, we again find that SOR and gaussian elimination lead to almost identical calls prices. For examples, from Table 3, consider a 4 year call option, exercised at \$35, we find that the Box price using SOR or gaussian elimination are identical at \$20.0181. Whilst the Crank Nicholson prices are \$20.1790 and \$20.1846 using SOR and gaussian respectively.

Table 5 contains bond prices based on the parameter values calculated by CKLS. We find that all four combinations converge to produce similar prices. As with Table 1 and Table 2, we find that Box prices are slightly lower than the corresponding Crank Nicholson prices.

Table 6 contains call option prices based on the parameter values calculated by CKLS. We again observe the same the same trend as with Table 3 and Table 4.

### **3. Conclusion**

We have introduced a new numerical method from engineering and the physical sciences to finance – the Box method. We have compared it with the existing scheme in finance – the Crank Nicholson using both Successive Over Relaxation and gaussian elimination. By, first assuming parameter values for the CKLS process, and then using historical parameters calculated by CKLS; we were able to test each of the numerical schemes both for the case of, extreme parameters values and historical parameter values.

We found that for assumed parameter values, where there was high mean reversion rate and low volatility parameter both the numerical schemes yielded bond and call option prices which were close to the analytical prices. However, where the mean reversion rate was low and volatility parameter was high, we found that gaussian elimination did not produce values that agreed with the analytical prices irrespective of whether the Box

method or the Crank Nicholson scheme was used. Further more, when using the SOR iterative process, we found that although the bond prices based on the Crank Nicholson and the Box method were close to the analytical bond price, the call option prices were not. In fact, in this case we found that Box prices were in excellent agreement with the analytical prices, whereas the Crank Nicholson prices were too high. We also found that both Box bond and call option prices were lower than Crank Nicholson bond and call option prices; and that Box prices are always closer to the analytical bond and call option prices than the Crank Nicholson bond and call option prices.

Finally, we find that when using the historical parameter values from the CKLS parameter values both numerical schemes yield bond and call option prices, which are close to each other, with Box prices being lower again than the corresponding prices calculated using the Crank Nicholson scheme.

Table 1. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$\alpha_1 = (4k\theta - \sigma^2) / 8 > 0$								
$k = 0.5, \theta = 0.08, \sigma = 0.1, \Delta r = 0.5\%, \gamma = 0.5$								
Maturity (years)	Model	r(%)	Annual number of time steps (n)					
			20	50	100	300	500	1000
5	CIR	5	71.0379	71.0379	71.0379	71.0379	71.0379	71.0379
	BMS		71.1006	71.0754	71.0670	71.0614	71.0603	71.0595
	BMG		71.1006	71.0754	71.0670	71.0614	71.0603	71.0595
	CNS		71.6853	71.6853	71.6858	71.6914	71.6853	71.6854
	CNG		71.6937	71.6958	71.6966	71.6971	71.6973	71.6973
5	CIR	11	63.7161	63.7161	63.7161	63.7161	63.7161	63.7161
	BMS		63.7850	63.7475	63.7349	63.7266	63.7249	63.7237
	BMG		63.7850	63.7475	63.7349	63.7266	63.7249	63.7237
	CNS		64.3129	64.3130	64.3134	64.3188	64.3130	64.3131
	CNG		64.3143	64.3147	64.3148	64.3149	64.3150	64.3150
15	CIR	5	32.5442	32.5442	32.5442	32.5442	32.5442	32.5442
	BMS		32.6428	32.5979	32.5829	32.5728	32.5711	32.5689
	BMG		32.6428	32.5979	32.5289	32.5729	32.5709	32.5694
	CNS		32.8647	32.8648	32.8648	32.8648	32.8646	32.8657
	CNG		32.8745	32.8770	32.8779	32.8785	32.8786	32.8787
15	CIR	11	28.9322	28.9322	28.9322	28.9322	28.9322	28.9322
	BMS		29.0135	28.9735	28.9601	28.9511	28.9496	28.9476
	BMG		29.0135	28.9735	28.9601	28.9512	28.9494	28.9481
	CNS		29.2251	29.2251	29.2251	29.2251	29.2250	29.2259
	CNG		29.2304	29.2317	29.2322	29.2326	29.2326	29.2327

Table 2. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$\alpha_1 = (4k\theta - \sigma^2) / 8 < 0$								
$k = 0.1, \theta = 0.08, \sigma = 0.5, \Delta r = 0.5\%, \gamma = 0.5$								
Maturity (years)	Model	r(%)	Annual number of time steps (n)					
			20	50	100	300	500	1000
5	CIR	5	83.4832	83.4832	83.4832	83.4832	83.4832	83.4832
	BMS		83.6040	83.5409	83.5244	83.5145	83.5115	83.5098
	CNS		84.3837	84.3614	84.3554	84.3538	84.3516	84.3503
5	CIR	11	72.5572	72.5572	72.5572	72.5572	72.5572	72.5572
	BMS		72.6956	72.6166	72.5961	72.5836	72.5802	72.5781
	CNS		73.2609	73.2389	73.2338	73.2319	73.2305	73.2293
15	CIR	5	68.2741	68.2741	68.2741	68.2741	68.2741	68.2741
	BMS		68.4127	68.3836	68.3730	68.3668	68.3657	68.3631
	CNS		69.0981	69.0851	69.0807	69.0802	69.0801	69.0801
15	CIR	11	58.9177	58.9177	58.9177	58.9177	58.9177	58.9177
	BM		59.0348	59.0095	59.0002	58.9947	58.9940	58.9913

Table 3. Call Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$\alpha_1 = (4k\theta - \sigma^2)/8 > 0$ $\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$							
Exercise Price	Model	Bond Price	Maturity (years)				
			5	4	3	2	1
35	CIR	45.1561 <sup>2</sup>	21.8802	19.9509	17.8585	15.5863	13.1552
	BMS	45.5000 <sup>1</sup>	21.9445	20.0181	17.9293	15.6615	13.1957
	BMG	45.5140	21.9445	20.0181	17.9293	15.6615	13.1957
	CNS	45.8809	22.1132	20.1790	18.0921	15.8450	13.4362
	CNG	45.8866	22.1177	20.1846	18.0987	15.8524	13.4438
40	CIR		18.5163	16.3114	13.9201	11.3233	8.4993
	BMS		18.5836	16.3605	13.9887	11.3968	8.5789
	BMG		18.5774	16.3759	13.9887	11.3968	8.5789
	CNS		18.7181	16.5076	14.0513	11.5545	8.8015
	CNG		18.7226	16.5132	14.1291	11.5618	8.8092
45	CIR		15.1524	12.6719	9.9819	7.0636	3.9137
	BMS		15.2104	12.7336	10.0482	7.1352	3.9896
	BMG		15.2104	12.7336	10.0482	7.1351	3.9896
	CNS		15.3230	12.8362	10.1531	7.2662	4.1834
	CNG		15.3275	12.8418	10.1597	7.2735	4.1910
50	CIR		11.7886	9.0330	6.0560	2.9514	0.4535
	BMS		11.8433	9.0919	6.1191	3.0126	0.4788
	BMG		11.8433	9.0919	6.1191	3.0126	0.4788
	CNS		11.9820	9.1653	6.1943	3.1020	0.5267
	CNG		11.9324	9.1709	6.2008	3.1090	0.5317
55	CIR		8.4257	5.4156	2.3804	0.3118	0.0001
	BMS		8.4772	5.4705	2.4305	0.3307	0.0001
	BMG		8.4772	5.4705	2.4305	0.3308	0.0001
	CNS		8.5338	5.5143	2.4679	0.3443	0.0000



CNG                      8.5382      5.5200      2.4746      0.3486      0.0001

*Table 4.* Call Prices calculated analytically (CIR), using the Box Method and the Crank Nicholson methods.

$\alpha_1 = (4k\theta - \sigma^2) / 8 < 0$							
$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$							
Exercise	Model	Bond	Maturity (years)				
			5	4	3	2	1
Price		Price					
60	CIR	63.4557	23.9008	22.8564	20.2596	19.8902	16.9798
	BMS	69.9969	23.9476	22.9006	21.6375	19.9112	16.9769
	CNS	70.8166	32.2997	32.0170	31.4356	30.1946	27.3805
65	CIR		20.1770	19.0843	17.7967	16.0922	13.2470
	BM		20.2200	19.1255	17.8313	16.1109	13.2320
	CNS		28.2373	27.9676	27.4063	26.1936	23.3519
70	CIR		16.4887	15.3565	14.0532	12.3971	9.7260
	BMS		16.5281	15.3950	14.0865	12.4102	9.7061
	CNS		24.1833	23.9313	23.4043	22.2519	19.4636
75	CIR		12.8444	11.6829	10.3819	8.8038	6.4487
	BM		12.8803	11.7194	10.4151	8.8246	6.4317
	CNS		20.1358	19.4299	19.4299	18.3732	15.7371
80	CIR		9.2570	8.0789	6.8019	5.3528	3.4558
	BM		9.2895	8.1135	6.8352	5.3787	3.4527
	CNS		16.0962	15.8990	15.4794	14.5314	12.0601

Table 5. Bond Prices calculated using the Box and the Crank Nicholson methods based on the original CKLS parameters.

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$$k = 0.2213, \theta = 0.0786, \sigma = 1.1767, \gamma = 1.4808$$

$$\Delta r = 0.5\%, \Delta t = 0.05$$


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<b>Maturity</b> <b>(years)</b>	<b>r(%)</b>	<b>BMS</b>	<b>BMG</b>	<b>CNS</b>	<b>CNG</b>
5	5	73.9710	73.9709	74.9387	74.7724
	8	68.0032	68.0029	68.8242	68.8009
	11	62.8360	62.8355	63.4843	63.4766
10	5	52.4782	52.4781	53.2131	52.7969
	8	47.4292	47.4290	48.0326	47.8550
	11	43.2494	43.2493	43.7180	43.6028
15	5	36.9671	36.9670	37.4302	36.9492
	8	33.3173	33.3172	33.6941	33.4163
	11	30.3189	30.3188	30.6063	30.3946

---

Table 6. Call Prices calculated using the Box and the Crank Nicholson methods based on the original CKLS parameters.

$k = 0.2213, \theta = 0.0786, \sigma = 1.1767, \gamma = 1.4808$							
$\Delta t = 0.05, \Delta r = 0.5\%$							
Exercise Price	Model	Bond Price	Maturity (years)				
			5	4	3	2	1
40	BMS	47.4292	20.2416	18.1490	15.8549	13.3260	10.5193
	BMG	47.4290	20.2414	18.1488	15.8548	13.3258	10.5191
	CNS	48.0326	20.5324	18.4391	16.1612	13.6756	10.9583
	CNG	47.8550	20.4290	18.2968	15.9905	13.4890	10.7663
45	BMS		16.8573	14.5175	11.9626	9.1602	6.0511
	BMG		16.8571	14.5174	11.9624	9.1600	6.0509
	CNS		17.1065	14.7647	12.2240	9.4592	6.4189
	CNG		17.0037	14.6225	12.0534	9.2732	6.2295
50	BMS		13.4905	10.9259	8.1616	5.2122	2.1787
	BMG		13.4903	10.9257	8.1615	5.2121	2.1786
	CNS		13.6978	11.1282	8.3707	5.4372	2.3901
	CNG		13.5953	10.9860	8.2006	5.2547	2.2242
55	BMS		10.1617	7.4289	4.6026	1.9276	0.1891
	BMG		10.1615	7.4287	4.6025	1.9275	0.1890
	CNS		10.3262	7.5820	4.7445	2.0392	0.2051
	CNG		10.2238	7.4401	4.5774	1.8737	0.1455
60	BMS		6.9194	4.1675	1.6821	0.1986	0.0001
	BMG		6.9198	4.1674	1.6825	0.1986	0.0001
	CNS		7.0386	4.2640	1.7460	0.2078	0.0003
	CNG		6.9375	4.1232	1.5912	0.1285	0.0000

## APPENDIX

### Crank-Nicholson Method

For the time derivative we use the Euler backward difference:

$$\frac{\partial v}{\partial t} = \frac{V_n^m - V_n^{m-1}}{\Delta t}$$

$$v = \frac{1}{2} V_n^m + \frac{1}{2} V_n^{m-1}$$

$$\frac{\partial v}{\partial s} = \frac{V_{n+1}^m - V_{n-1}^m}{4\Delta s} + \frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{4\Delta s}$$

$$\frac{\partial^2 v}{\partial s^2} = \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{2(\Delta s)^2} + \frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{2(\Delta s)^2}$$

Substituting the above discretization leads to the following discrete equation.

$$\begin{aligned}
& \frac{\sigma^2 \Delta t}{2} \left[ \frac{n \Delta s}{\eta(1-n \Delta s)} \right]^{2\gamma} \frac{\eta^2 (1-n \Delta s)^4}{2(\Delta s)^2} \\
& \times \left\{ V_{n+1}^m - 2V_n^m + V_{n-1}^m + V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1} \right\} \\
& + \frac{\Delta t}{4\Delta s} \left\{ -\sigma^2 \left[ \frac{n \Delta s}{\eta(1-n \Delta s)} \right]^{2\gamma} \eta^2 (1-n \Delta s)^3 \right. \\
& \quad \left. + \left[ k\theta - \frac{n \Delta s}{\eta(1-n \Delta s)} (k + \lambda) \right] \eta(1-n \Delta s)^2 \right\} \\
& \times \left\{ V_{n+1}^m - V_{n-1}^m + V_{n+1}^{m-1} - V_{n-1}^{m-1} \right\} \\
& - \frac{n \Delta s \Delta t}{2\eta(1-n \Delta s)} V_n^m - \frac{n \Delta s \Delta t}{2\eta(1-n \Delta s)} V_n^{m-1} = V_n^m - V_n^{m-1}
\end{aligned} \tag{A.1}$$

We can further simplify the above equation as:

$$\begin{aligned}
& A_n [V_{n+1}^m - 2V_n^m + V_{n-1}^m] + A_n [V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}] \\
& + B_n [V_{n+1}^m - V_{n-1}^m] + B_n [V_{n+1}^{m-1} - V_{n-1}^{m-1}] \\
& + C_n V_n^m + C_n V_n^{m-1} = V_n^m - V_n^{m-1}
\end{aligned} \tag{A.2}$$

where:

$$A_n = \frac{\sigma^2 \Delta t}{2} \left[ \frac{n \Delta s}{\eta(1 - n \Delta s)} \right]^{2\gamma} \frac{\eta^2 (1 - n \Delta s)}{2(\Delta s)^2}$$

$$B_n = \frac{\Delta t}{4 \Delta s} \left\{ -\sigma^2 \left[ \frac{n \Delta s}{\eta(1 - n \Delta s)} \right]^{2\gamma} \eta^2 (1 - n \Delta s)^3 + \left[ k\theta - \frac{n \Delta s}{\eta(1 - n \Delta s)} k \right] \eta (1 - n \Delta s)^2 \right\}$$

$$C_n = -\frac{n \Delta s \Delta t}{2 \eta (1 - n \Delta s)}$$

Further rearrangement leads to:

$$\alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m \quad (\text{A.3})$$

where:

$$\alpha_n = -A_n [V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}] - B_n [V_{n+1}^{m-1} - V_{n-1}^{m-1}] - [1 + C_n] V_n^{m-1}$$

$$\chi_n = A_n - B_n$$

$$\eta_n = C_n - 2A_n - 1$$

$$\beta_n = A_n + B_n$$

### Box Method

To derive the Box method scheme, our starting position is to integrate equation (15)

$$\text{where } r_a = \frac{r_n + r_{n-1}}{2} \text{ to } r_b = \frac{r_{n+1} + r_n}{2} :$$

$$\int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) dr - \int_{r_a}^{r_b} (\Psi(r) r^{1-2\gamma} c v) dr = \int_{r_a}^{r_b} \left( c \Psi(r) r^{-2\gamma} \frac{\partial v}{\partial \tau} \right) dr \quad (\text{A.4})$$

For  $\frac{\partial u}{\partial t}$  we use the backward Euler approximation as with the Crank Nicholson to obtain

the following equation.

$$\frac{\partial v}{\partial \tau} = \frac{v - v_0}{\Delta t}$$

Such that equation (A.4) becomes:

$$\int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) dr - \int_{r_a}^{r_b} (\Psi(r) r^{1-2\gamma} c v) dr = \int_{r_a}^{r_b} \left( c \Psi(r) r^{-2\gamma} \left( \frac{v - v_0}{\Delta t} \right) \right) dr$$

Further rearrangement leads to the following expression

$$-\int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) dr + \int_{r_a}^{r_b} \left( c \Psi(r) r^{1-2\gamma} v + \frac{\Psi(r) r^{-2\gamma}}{\Delta t} v \right) dr = \int_{r_a}^{r_b} \left( \frac{\Psi(r) r^{-2\gamma}}{\Delta t} v_0 \right) dr$$

Approximating each of the integrals, we have:

$$-\int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) dr = -\Psi(r_b) \left( \frac{V_{n+1}^m - V_n^m}{\Delta r} \right) + \Psi(r_a) \left( \frac{V_n^m - V_{n-1}^m}{\Delta r} \right)$$

$$\int_{r_a}^{r_b} \left( c \Psi(r) r^{1-2\gamma} v + \frac{\Psi(r) dr^{-2\gamma}}{\Delta t} v \right) dr$$

$$= \Psi(r_n) \left[ \frac{c r_b^{2-2\gamma}}{2-2\gamma} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) + \frac{c r_b^{1-2\gamma}}{\Delta t (1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) \right] V_n^m \text{ if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1$$

$$= \Psi(r_n) \left[ c(r_b - r_a) - \frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \right] V_n^m \quad \text{for } \gamma = \frac{1}{2}$$

$$= \Psi(r_n) \left[ -c \ln \left( \frac{r_a}{r_b} \right) + \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \right] V_n^m \quad \text{for } \gamma = 1$$

$$\int_{r_a}^{r_b} \left( \frac{\Psi(r) dr^{-2\gamma}}{\Delta t} V_0 \right) dr$$

$$= \Psi(r_n) \left[ \frac{c r_b^{1-2\gamma}}{\Delta t (1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) \right] V_n^{m-1} \quad \text{if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1$$

$$= \Psi(r_n) \left[ -\frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \right] V_n^{m-1} \quad \text{for } \gamma = \frac{1}{2}$$

$$= \Psi(r_n) \left[ \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \right] V_n^{m-1} \quad \text{for } \gamma = 1$$

Substituting the above approximations into the original equation yields

$$\alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m$$



## Notes

1. We concentrate on zero coupon call options for illustrative purposes. In the case of zero coupon bonds, the American call option prices are identical to European call option prices as it is not optimal to exercise the option until the expiry date. This allows us to check the calculated values of call options with the analytical call options for  $\gamma = 0.5$ .
2. We take  $\eta = 0.2$  for all our calculations.
3. We ensure that the accuracy is  $10^{-6}$
4.  $\omega$  - with few exceptions (see Ames (1977)) is estimated by numerical experimentation. For the purposes of our calculations, for  $\alpha_1 > 0$ ,  $\omega = 1.5$  for the historical CKLS parameters and  $\alpha_1 < 0$ ,  $\omega = 1.955$ .
5. For this approximation, strictly  $v$  should be  $V_n^m$  and  $v_0$  should be  $V_n^{m-1}$ . We are using this notation to simplify the algebraic manipulation.

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