



Maximal metric surfaces and the Sobolev-to-Lipschitz property

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Abstract

We find maximal representatives within equivalence classes of metric spheres. For Ahlfors regular spheres these are uniquely characterized by satisfying the seemingly unrelated notions of Sobolev-to-Lipschitz property, or volume rigidity. We also apply our construction to solutions of the Plateau problem in metric spaces and obtain a variant of the associated intrinsic disc studied by Lytchak–Wenger, which satisfies a related maximality condition.

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1 Introduction

1.1 Main result

A celebrated result due to Bonk and Kleiner [3] states that an Ahlfors 2-regular metric sphere Z is quasimetrically equivalent to the standard sphere if and only if it is linearly locally connected. Recently, it was shown in [35] that the quasimetric homeomorphism $u_Z : S^2 \rightarrow Z$ may be chosen to be of minimal energy $E_+^2(u_Z)$, and in this case is unique up to a conformal diffeomorphism of S^2 .

The map u_Z gives rise to a measurable Finsler structure on S^2 , defined by the approximate metric differential $\text{apmd } u_Z$; cf. [33] and Sect. 6 below. When Z is a smooth Finsler surface, the approximate metric differential carries all the metric information of Z . In the present

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generality, however, $\text{apmd } u_Z$ is defined only almost everywhere and thus does not determine the length of every curve.

Definition 1.1 Let Y and Z be Ahlfors 2-regular, linearly locally connected metric spheres. We say that Y and Z are *analytically equivalent* if there exist energy minimizing parametrizations u_Y and u_Z such that

$$\text{apmd } u_Y = \text{apmd } u_Z \quad (1.1)$$

almost everywhere.

By the aforementioned uniqueness result, analytic equivalence defines an equivalence relation on the class of linearly locally connected, Ahlfors 2-regular spheres. The main result of this paper states that the equivalence class of such a sphere contains a maximal representative, unique up to an isometry.

Theorem 1.2 *Let Z be a linearly locally connected, Ahlfors 2-regular sphere. Then there is linearly locally connected, Ahlfors 2-regular sphere \widehat{Z} which is analytically and bi-Lipschitz equivalent to Z and satisfies the following properties.*

- (1) **Sobolev-to-Lipschitz property:** *If $f \in N^{1,2}(\widehat{Z})$ has weak upper gradient 1, then f has a 1-Lipschitz representative.*
- (2) **Thick geodecity:** *For arbitrary measurable subsets $E, F \subset \widehat{Z}$ of positive measure and $C > 1$, one has $\text{Mod}_2 \Gamma(E, F; C) > 0$.*
- (3) **Maximality:** *If Y is analytically equivalent to \widehat{Z} , there exists a 1-Lipschitz homeomorphism $f : \widehat{Z} \rightarrow Y$.*
- (4) **Volume rigidity:** *If Y is a linearly locally connected, Ahlfors 2-regular sphere, and $f : Y \rightarrow \widehat{Z}$ is a 1-Lipschitz area preserving map which is moreover cell-like, then f is an isometry.*

Moreover \widehat{Z} is characterized uniquely, up to isometry, by any of the listed properties.

In the setting of Ahlfors regular metric spheres, Theorem 1.2 links the essential metric investigated in [1,7,8], the concept of volume rigidity studied under curvature bounds in [27–29], the Sobolev-to-Lipschitz property arising in the context of RCD spaces in [15,16], and the notion of thick quasiconvexity whose connection to Poincaré inequalities is investigated in [10–12].

More generally, an Ahlfors 2-regular disc Z with finite boundary length admits a quasimetric parametrization by the standard disc \mathbb{D} if and only if it is linearly locally connected. The parametrization may again be chosen to be of minimal Reshetynak energy and is then unique up to a conformal diffeomorphism, see [35]. A suitable variant of Theorem 1.2 also holds in this setting. In this case in the formulation of volume rigidity one has to assume that f preserves the boundary curves and the restriction $f : \partial Y \rightarrow \partial Z$ is monotone, instead of assuming that the entire map f is cell-like.

Other parametrization results for metric surfaces have been obtained for example in [23, 36,39,40,43]. These results however do not include a uniqueness part. Thus it seems unclear how to define a suitable notion of analytic equivalence in the settings therein.

1.2 Construction of essential metrics

Our construction of essential metrics relies on the p -essential infimum over curve families and extends the construction of the essential metric in [1] to the case $p < \infty$; cf. (1.3) and

Definition 5.1. For a metric measure space X , $p \in [1, \infty]$, and a family of curves Γ in X , we define its p -essential length $\text{ess}\ell_p(\Gamma) \in [0, \infty]$ as the essential infimum of the length function on Γ with respect to p -modulus. The p -essential distance $d'_p : X \times X \rightarrow [0, \infty]$ is defined by

$$d'_p(x, y) := \lim_{\delta \rightarrow 0} \text{ess}\ell_p \Gamma(B(x, \delta), B(y, \delta)), \quad x, y \in X, \quad (1.2)$$

where $\Gamma(E, F)$ denotes the family of curves in X joining two given measurable subsets $E, F \subset X$. This quantity does not automatically satisfy the triangle inequality, and we consider instead the largest metric $d_p \leq d'_p$; see the discussion after Definition 5.3. In general, d_p might take infinite values, and its finiteness is related to an abundance of curves of uniformly bounded length connecting given disjoint sets. The condition of *thick quasiconvexity*, introduced in [12], quantifies the existence of an abundance of quasiconvex curves.

Definition 1.3 Let (X, d, μ) be a metric measure space and $p \in [1, \infty]$. We say that X is **p -thick quasiconvex** with constant $C \geq 1$ if, for all measurable subsets $E, F \subset X$ of positive measure, we have

$$\text{Mod}_p \Gamma(E, F; C) > 0.$$

We say that X is **p -thick geodesic** if X is p -thick quasiconvex with constant C for every $C > 1$.

Here $\Gamma(E, F; C)$ denotes the family of curves $\gamma : [0, 1] \rightarrow X$ joining E and F such that $\ell(\gamma) \leq C \cdot d(\gamma(0), \gamma(1))$. Note that this is equivalent to the original definition in [12] when X is infinitesimally doubling.

Theorem 1.4 Let (X, d, μ) be an infinitesimally doubling metric measure space which is p -thick quasiconvex with constant C and $1 \leq p \leq \infty$. Then there exists a metric d_p on X , for which $X_p := (X, d_p, \mu)$ is p -thick geodesic and $d \leq d_p \leq Cd$. Moreover d_p is minimal among metrics $\rho \geq d$ for which $X_\rho := (X, \rho, \mu)$ is p -thick geodesic.

It is known that doubling p -Poincaré spaces are p -thick quasiconvex for some constant depending only on the data of X , though the converse need not be true unless $p = \infty$; see [11, 12]. For $p < q$, the assumption of p -thick geodesicity is strictly stronger than that of q -thick geodesicity, see Example 3.3 below. However, under the assumption of a suitable Poincaré inequality the particular value of q is immaterial. Indeed, for Poincaré spaces the essential metrics of all indices coincide with the essential metric d_{ess} introduced in [1]. The essential metric d_{ess} is given by

$$d_{\text{ess}}(x, y) := \text{ess}\ell_\infty \Gamma(\{x\}, \{y\}), \quad x, y \in X. \quad (1.3)$$

Proposition 1.5 Let X be a doubling metric measure space satisfying a p -Poincaré inequality. Then $d_q = d_{\text{ess}}$ for every $q \in [p, \infty]$.

In the proof of Theorem 1.2 we will set $\widehat{Z} := (Z, d_2)$. A posteriori, the chosen metric agrees with the essential metric (1.3). However from this characterization it is unclear that the disc obtained in Theorem 1.2 is unique. This follows from the use of 2-modulus in the construction of the metric, and its quasi-invariance under quasisymmetric maps.

1.3 The Sobolev-to-Lipschitz property, thick geodesicity, and volume rigidity

The Sobolev-to-Lipschitz property and thick geodesicity are defined and equivalent in a much broader framework.

Definition 1.6 A metric measure space X is said to have the p -Sobolev-to-Lipschitz property, if every $f \in N^{1,p}(X)$ for which the minimal p -weak upper gradient g_f satisfies $g_f \leq 1$ almost everywhere has a 1-Lipschitz representative.

To illustrate this equivalence, and contrast the difference of the Sobolev-to-Lipschitz property with merely being geodesic, recall that a proper metric space is geodesic if and only if, whenever a function f has (genuine) upper gradient 1, it is 1-Lipschitz. The p -Sobolev-to-Lipschitz property requires we can reach the same conclusion from the weaker assumption that f has p -weak upper gradient 1. The next result translates this condition to having an abundance of nearly geodesic curves in the space.

Theorem 1.7 *Let X be an infinitesimally doubling metric measure space and $p \geq 1$. Then X has the p -Sobolev-to-Lipschitz property if and only if it is p -thick geodesic.*

These properties are also related to *volume rigidity*, which asks whether any volume preserving 1-Lipschitz map of a given class is an isometry. Recall that a Borel map $f : (X, \mu) \rightarrow (Y, \nu)$ between metric measure spaces is volume preserving if $f_*\mu = \nu$. When X and Y are n -rectifiable, we equip them with the Hausdorff measure \mathcal{H}^n , unless otherwise explicitly stated. Volume preserving 1-Lipschitz maps between rectifiable spaces are essentially length preserving, see Proposition 4.1, but need not be homeomorphisms. For volume preserving 1-Lipschitz homeomorphisms with quasiconformal inverse, the Sobolev-to-Lipschitz property guarantees isometry.

Proposition 1.8 *Let $f : X \rightarrow Y$ be a volume preserving 1-Lipschitz homeomorphism between n -rectifiable metric spaces. If f^{-1} is quasiconformal and Y has the n -Sobolev-to-Lipschitz property, then f is an isometry.*

Note that the inverse of a quasiconformal map is not quasiconformal in general; cf. [44, Remark 4.2]. This is guaranteed under the assumption that the spaces are Ahlfors n -regular and X supports an n -Poincaré inequality. The assumption that Y has the Sobolev-to-Lipschitz property is essential for the validity of Proposition 1.8; cf. Example 4.4. Thus, Proposition 1.8 connects volume rigidity, the Sobolev-to-Lipschitz property and thick geodesicity under these fairly restrictive assumptions, and will be used in the proof of Theorem 1.2.

1.4 Application: essential minimal surfaces and essential pull-back metrics

For the definitions and terminology in the forthcoming discussion, we refer the reader to Sect. 2.

A smooth map $u : M \rightarrow X$ between Riemannian manifolds gives rise to a pull-back metric on M . In the case that M and X are merely metric spaces and u a continuous map, various nonequivalent definitions of pull-back metric are discussed in [37,38]. The metric

$$d_u(p, q) := \inf_{\gamma} \ell(u \circ \gamma), \quad p, q \in M,$$

where γ varies over all continuous curves in M joining p to q , was used in [33,38,42] to study solutions $u : \mathbb{D} \rightarrow X$ of the Plateau problem in X , also referred to as *minimal discs*. If

X satisfies a quadratic isoperimetric inequality and u bounds a Jordan curve of finite length, then d_u gives rise to a factorization

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{u} & X \\ & \searrow P_u \quad \nearrow \bar{u} & \\ & Z_u & \end{array}$$

where:

- (1) Z_u is a geodesic disc which satisfies the same quadratic isoperimetric inequality as X .
- (2) P_u is a minimal disc bounding ∂Z_u and a uniform limit of homeomorphisms.
- (3) \bar{u} is 1-Lipschitz and $\ell(P_u \circ \gamma) = \ell(u \circ \gamma)$ for every curve γ in \mathbb{D} .

The fact that Z_u satisfies the same quadratic isoperimetric inequality as X reflects the fact that the intrinsic curvature of a classical minimal surface is bounded above by that of the ambient manifold; cf. [34].

Our definition of essential metrics suggests the following variant of pull-back metrics in the minimal disc setting. Namely for a family of curves Γ in \mathbb{D} let $\text{ess}\ell_u(\Gamma)$ be the essential infimum of $\ell(u \circ \gamma)$ with respect to 2-modulus; cf. Definition 5.1. Then for $p, q \in \mathbb{D}$ we define the essential pull-back metric by

$$\widehat{d}_u(p, q) := \lim_{\delta \rightarrow 0} \text{ess}\ell_u \Gamma(B_\delta(p), B_\delta(q)).$$

This construction gives rise to the following factorization result.

Theorem 1.9 *Let X be a proper metric space satisfying a (C, l_0) -quadratic isoperimetric inequality. Let $u : \mathbb{D} \rightarrow X$ be a minimal disc bounding a Jordan curve Γ which satisfies a chord-arc condition. Then there exists a geodesic disc \widehat{Z}_u and a factorization*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{u} & X \\ & \searrow \widehat{P}_u \quad \nearrow \widehat{u} & \\ & \widehat{Z}_u & \end{array}$$

such that

- (1) \widehat{Z}_u has the Sobolev-to-Lipschitz property and satisfies a (C, l_0) -quadratic isoperimetric inequality,
- (2) \widehat{P}_u a minimal disc bounding $\partial \widehat{Z}_u$ and a uniform limit of homeomorphisms, and
- (3) $\widehat{u} : \widehat{Z}_u \rightarrow X$ is 1-Lipschitz and for 2-almost every curve γ in \mathbb{D} one has $\ell(\widehat{P}_u \circ \gamma) = \ell(u \circ \gamma)$.

Furthermore the factorization is maximal in the following sense: If \widetilde{Z} is a metric disc and $u = \widetilde{u} \circ \widetilde{P}$, where $\widetilde{P} : \mathbb{D} \rightarrow \widetilde{Z}$ and $\widetilde{u} : \widetilde{Z} \rightarrow X$ satisfy (2) and (3), then there exists a surjective 1-Lipschitz map $f : \widehat{Z}_u \rightarrow \widetilde{Z}$ such that $\widetilde{P} = f \circ \widehat{P}_u$.

A Jordan curve is said to satisfy a chord-arc condition if it is bi-Lipschitz equivalent to S^1 . The fact that the map \widehat{P}_u is a minimal disc implies additional regularity. Indeed, \widehat{P}_u is infinitesimally quasiconformal, Hölder continuous and its minimal weak upper gradient has global higher integrability; cf. [31–33, 35].

Compared to the construction in [33], we trade off some regularity of \bar{u} in exchange for the Sobolev-to-Lipschitz property on \widehat{Z}_u . Indeed, \widehat{u} only preserves the length of almost every curve, while \widehat{Z}_u is nicer in analytic and geometric terms. Note however that some geometric

properties of \widehat{Z}_u , established for Z_u in [33,42], remain open for \widehat{Z}_u . In particular we do not know whether $\mathcal{H}^1(\partial\widehat{Z}_u) < \infty$.

Note that, in contrast to Theorem 1.2, no characterization in terms of the Sobolev-to-Lipschitz property in Theorem 1.9 is possible. In fact it may happen that Z_u already has the Sobolev-to-Lipschitz property but still \widehat{Z}_u and Z_u are not isometric, see Example 6.4.

More generally, when M supports a Q -Poincaré inequality where $Q \geq 1$ satisfies (2.2), maps $u \in N_{loc}^{1,p}(M; X)$, with *a priori higher exponent* $p > Q$, give rise to a p -essential pull-back metric on M and the resulting metric measure space has the Q -Sobolev-to-Lipschitz property; cf. Definition 1.6 and Theorem 5.16. Higher integrability of minimal weak upper gradients for quasiconformal maps is known to hold when M and X both have Q -bounded geometry, and also in the situation of [31, Theorem 1.4], but not in general; cf. [20,21,26]. Together with the Poincaré inequality and Morrey's embedding, higher integrability of minimal weak upper gradients implies finiteness of the essential pull-back metric. The assumption in Theorem 1.9 that Γ is not only of finite length but satisfies a chord-arc condition is needed to guarantee this additional regularity.

1.5 Organization

The paper is organized as follows. After the preliminaries in Sect. 2, we treat the equivalence of thick geodesicity and the Sobolev-to-Lipschitz property in Sect. 3, where we prove Theorem 1.7. Volume rigidity is discussed in Sect. 4, which includes the proof of Proposition 1.8.

The construction of essential metrics is presented in Sect. 5. We develop some general tools in Sect. 5.1, and prove Theorem 1.4 and Proposition 1.5 in Sect. 5.2. In Sect. 5.3 we apply our construction to Sobolev maps from a Poincaré space, and derive some basic properties of the resulting metric measure space; cf. Theorem 5.15.

Finally, in Sect. 6, we apply the results from Sects. 5.2 and 5.3 to obtain Theorems 1.2 and 1.9. First in Sect. 6.1 we prove Theorem 1.2. Then in Sect. 6.2 we prove a weaker but more general result for discs, cf. Theorem 6.3, and finally in Sect. 6.3 we discuss Theorem 1.9.

2 Preliminaries

We refer to the monographs [2,22] for the material discussed below. Given a metric space (X, d) , balls in X are denoted $B(x, r)$ and, if $B = B(x, r) \subset X$ is an open ball and $\sigma > 0$, σB denotes the ball with the same center as B and radius σr . The Hausdorff measure on X is denoted \mathcal{H}^n or, if we want to stress the space or metric, by \mathcal{H}_X^n or \mathcal{H}_d^n . The normalizing constant is chosen so that $\mathcal{H}_{\mathbb{R}^n}^n$ agrees with the Lebesgue measure.

By a measure μ on a metric space X we mean an outer measure which is Borel regular. A triple (X, d, μ) where (X, d) is a proper metric space and μ is a measure on X which is non-trivial on balls, is called a *metric measure space*. Note that all metric measure spaces in our convention are complete, separable and every closed bounded subset is compact. Furthermore all balls have positive finite measure and in particular the measures are Radon, cf. [22, Corollary 3.3.47]. We often abbreviate $X = (X, d, \mu)$ when the metric and measure are clear from the context.

Given a curve $\gamma : [a, b] \rightarrow X$, we denote by $\ell(\gamma)$ its length. When γ is absolutely continuous we write $|\gamma'|$ for the metric speed (at $t \in [a, b]$) of γ . Moreover, when γ is rectifiable (i.e. $\ell(\gamma) < \infty$) we denote by $\tilde{\gamma} : [a, b] \rightarrow X$ the *constant speed parametrization*

of γ . This is the (Lipschitz) curve satisfying

$$\gamma(a + (b - a)\ell(\gamma|_{[a,t]})/\ell(\gamma)) = \bar{\gamma}(t), \quad t \in [a, b].$$

The *arc length parametrization* γ_s of γ is the reparametrization of $\bar{\gamma}$ to the interval $[0, \ell(\gamma)]$. With the exception of Proposition 5.7, we assume curves are defined on the interval $[0, 1]$.

Given a metric measure space X , a Banach space V , and $p \geq 1$, we denote by $L^p(X; V)$ and $L^p_{\text{loc}}(X; V)$ the a.e.-equivalence classes of p -integrable and locally p -integrable μ -measurable maps $u: X \rightarrow V$. We also denote $L^p(X) := L^p(X; \mathbb{R})$ and $L^p_{\text{loc}}(X) := L^p_{\text{loc}}(X; \mathbb{R})$ and abuse notation by writing $f \in L^p(X; V)$ or $f \in L^p_{\text{loc}}(X; V)$ for maps f (instead of equivalence classes).

Properties of measures

A measure μ on X is called *doubling* if there exists $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad (2.1)$$

for every $x \in X$ and $0 < r < \text{diam } X$. The least such constant is denoted C_μ and called the doubling constant of μ . Doubling measures satisfy a relative volume lower bound

$$C \left(\frac{\text{diam } B}{\text{diam } B'} \right)^Q \leq \frac{\mu(B)}{\mu(B')} \text{ for all balls } B \subset B' \subset X, \quad (2.2)$$

for some constants $C > 0$ and $Q \leq \log_2 C_\mu$ depending only on C_μ . The opposite inequality with the same exponent Q need not hold. If there are constants $C, Q > 0$ such that

$$\frac{1}{C}r^Q \leq \mu(B(x, r)) \leq Cr^Q, \quad 0 < r < \text{diam } X,$$

we say that μ is *Ahlfors Q -regular*.

If μ is doubling and

$$\mathcal{M}_r f(x) = \sup_{0 < s < r} \int_{B(x,s)} |f| d\mu$$

denotes the (restricted) centered maximal function of f at $x \in X$ the sublinear operator \mathcal{M}_r satisfies the usual boundedness estimates

$$\|\mathcal{M}_r f\|_{L^p(X)} \leq C\|f\|_{L^p(X)} \quad (p > 1), \quad \mu(\{\mathcal{M}_r f > \lambda\}) \leq C \frac{\|f\|_{L^1(X)}}{\lambda}, \quad \lambda > 0, \quad (2.3)$$

for some constant C depending only on C_μ and p . Consequently almost every point of a μ -measurable set $E \subset X$ is a Lebesgue density point.

If the measure μ satisfies the *infinitesimal doubling condition*

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$$

for μ -almost every $x \in X$, the claim about density points still remains true.

Proposition 2.1 [22, Theorem 3.4.3] *If (X, d, μ) is infinitesimally doubling metric measure space and $f \in L^1_{\text{loc}}(X)$ then*

$$f(x) = \lim_{r \rightarrow 0} \int_{B(x,r)} f d\mu$$

for μ -almost every $x \in X$. In particular, μ -almost every point of a Borel set $E \subset X$ is a Lebesgue density point.

Sobolev spaces

Let (X, d, μ) be a metric measure space, Y a metric space, and $p \geq 1$. If $u : X \rightarrow Y$ is a map and $g : X \rightarrow [0, \infty]$ is Borel, we say that g is an *upper gradient* of u if

$$d_Y(u(\gamma(b)), u(\gamma(a))) \leq \int_{\gamma} g \quad (2.4)$$

for every curve $\gamma : [a, b] \rightarrow X$. Recall that the *line integral* of g over γ is defined as

$$\int_{\gamma} g := \int_0^{\ell(\gamma)} g(\gamma_s(t)) dt$$

if γ is rectifiable and ∞ otherwise. Suppose $Y = V$ is a separable Banach space. A μ -measurable map $u \in L^p(X; V)$ is called *p-Newtonian* if it has an upper gradient $g \in L^p(X)$. The Newtonian seminorm is

$$\|u\|_{1,p} = \left(\|u\|_{L^p(X;V)}^p + \inf_g \|g\|_{L^p(X)}^p \right)^{1/p}, \quad (2.5)$$

where the infimum is taken over all upper gradients g of u . The Newtonian space $N^{1,p}(X; V)$ is the vector space of equivalence classes of p -Newtonian maps, where two maps $u, v : X \rightarrow V$ are equivalent if $\|u - v\|_{1,p} = 0$. The quantity (2.5) defines a norm on $N^{1,p}(X; V)$ and $(N^{1,p}(X; V), \|\cdot\|_{1,p})$ is a Banach space.

We say that a μ -measurable map $u : X \rightarrow V$ is locally p -Newtonian if every point $x \in X$ has a neighbourhood U such that

$$u|_U \in N^{1,p}(U; V),$$

and denote the vector space of locally p -Newtonian maps by $N_{loc}^{1,p}(X; V)$.

Minimal upper gradients

Let Γ be a family of curves in X . We define the p -modulus of Γ as

$$\text{Mod}_p(\Gamma) := \inf \left\{ \int_X \rho^p d\mu : \rho : X \rightarrow [0, \infty] \text{ Borel, } \int_{\gamma} \rho \geq 1 \text{ for every } \gamma \in \Gamma \right\}.$$

We say that a Borel function $g : X \rightarrow [0, \infty]$ is a p -weak upper gradient of u if there is a path family Γ_0 with $\text{Mod}_p(\Gamma_0) = 0$ so that (2.4) holds for all curves $\gamma \notin \Gamma_0$. The infimum in (2.5) need not be attained by upper gradients of u but there is a minimal p -weak upper gradient g_u of u so that

$$\|g_u\|_{L^p(X)} = \inf_g \|g\|_{L^p(X)}.$$

The minimal p -weak upper gradient is unique up to sets of measure zero.

If Y is a complete separable metric space then there is an isometric embedding $\iota : Y \rightarrow V$ into a separable Banach space. We may define

$$N_{loc}^{1,p}(X; Y) = \{u \in N_{loc}^{1,p}(X; V) : \iota(u(x)) \in \iota(Y) \text{ for } \mu\text{-almost every } x \in X\}$$

We remark that maps $u \in N_{loc}^{1,p}(X; Y)$ have minimal p -weak upper gradients that are unique up to equality almost everywhere and do not depend on the embedding.

Poincaré inequalities

A metric measure space $X = (X, d, \mu)$ supports a p -Poincaré inequality if there are constants $C, \sigma \geq 1$ so that

$$\int_B |u - u_B| d\mu \leq C \operatorname{diam} B \left(\int_{\sigma B} g^p d\mu \right)^{1/p} \quad (2.6)$$

whenever $u \in L_{loc}^1(X)$, g is an upper gradient of u , and B is a ball in X . Doubling metric measure spaces supporting a Poincaré inequality enjoy a rich theory.

Theorem 2.2 (Morrey embedding) *Let (X, d, μ) be a complete doubling metric measure space, where the measure satisfies (2.2) for $Q \geq 1$, and suppose X supports a Q -Poincaré inequality. If $p > Q$, there is a constant $C \geq 1$ depending only on the data of X so that for any $u \in N_{loc}^{1,p}(X; Y)$ and any ball $B \subset X$ we have*

$$d_Y(u(x), u(y)) \leq C(\operatorname{diam} B)^{Q/p} d(x, y)^{1-Q/p} \left(\int_{\sigma B} g_u^p d\mu \right)^{1/p}, \quad x, y \in B.$$

Here the *data* of X refers to p , the doubling constant of the measure and the constants in the Poincaré inequality.

Rectifiability and metric differentials

A separable metric space X is said to be n -rectifiable if there exist Borel sets $E_i \subset \mathbb{R}^n$ and Lipschitz maps $g_i : E_i \rightarrow X$ for each $i \in \mathbb{N}$, satisfying

$$\mathcal{H}^n \left(X \setminus \bigcup_{i \in \mathbb{N}} g_i(E_i) \right) = 0.$$

Unless otherwise stated we always equip n -rectifiable spaces with the Hausdorff n -measure \mathcal{H}^n .

Metric differentials play an important role in the study of rectifiable metric spaces. Let $E \subset \mathbb{R}^n$ be measurable and $f : E \rightarrow X$ be a Lipschitz map into a metric space X . By a fundamental result of Kirchheim [25], f admits a *metric differential* $\operatorname{md}_x f$ at almost every point $x \in E$. The metric differential is a seminorm on \mathbb{R}^n and such that

$$|\operatorname{md}_x f(y - z) - d(f(y), f(z))| = o(d(x, y) + d(x, z)) \text{ whenever } y, z \in E.$$

If $\Omega \subset \mathbb{R}^n$ is a domain and $p \geq 1$, a Sobolev map $u \in N_{loc}^{1,p}(\Omega; X)$ admits an *approximate metric differential* $\operatorname{apmd}_x u$ for almost every $x \in \Omega$; cf. [24] and [31]. If $\Omega \subset M$ is a domain in a Riemannian n -manifold, the approximate metric differential of $u \in N^{1,p}(\Omega; X)$ may be defined, for almost every $x \in \Omega$, as a seminorm on $T_x M$ by

$$\operatorname{apmd}_x u := \operatorname{apmd}_{\psi(x)}(u \circ \psi^{-1}) \circ d\psi(x) \quad (2.7)$$

for any chart ψ around x ; see the discussion after [14, Definition 2.1] and the references therein for the details.

We formulate the next results for maps defined on domains of manifolds; that is, we assume that $\Omega \subset M$ for a Riemannian n -manifold M . Note that, although in the references the statements are proved in the Euclidean setting, the definition (2.7) easily implies the corresponding statements for general Riemannian manifolds. For the following area formula for Sobolev maps with Lusin's property (N), recall that a Borel map $f : (X, \mu) \rightarrow (Y, \nu)$ is said to have Lusin's property (N) if $\nu(f(E)) = 0$ whenever $\mu(E) = 0$.

Theorem 2.3 [24, Theorems 2.4 and 3.2] *Let $u \in N_{loc}^{1,p}(\Omega; X)$ for some $p > n$. Then u has Lusin's property (N), and*

$$\int_{\Omega} \varphi(x) J(\text{apmd}_x u) dx = \int_X \left(\sum_{x \in u^{-1}(y)} \varphi(x) \right) d\mathcal{H}^n(y) \quad (2.8)$$

for any Borel function $\varphi : \Omega \rightarrow [0, \infty]$.

Here the *Jacobian* of a seminorm s on \mathbb{R}^n is defined by

$$J(s) = \alpha(n) \left(\int_{S^{n-1}} s(v)^{-n} d\mathcal{H}^{n-1}(v) \right)^{-1} \quad (2.9)$$

Following [31] we also define the *maximal stretch* of a seminorm s by

$$I_+(s) = \max\{s(v) : |v| = 1\}.$$

Note that the inequality $J(s) \leq I_+^n(s)$ always holds, and $I_+(\text{apmd}_x u) = g_u$ almost everywhere for $u \in N_{loc}^{1,n}(\Omega; X)$, see [31, Section 4]. We say that s is *K-quasiconformal* if $I_+^n(s) \leq K J(s)$. A Sobolev map $u \in N_{loc}^{1,n}(\Omega; X)$ is called *infinitesimally K-quasiconformal*, if $\text{apmd}_x u$ is *K-quasiconformal* for almost every $x \in \Omega$.

The (*Reshetnyak*) *energy* and (*parametrized Hausdorff*) *area* of $u \in N_{loc}^{1,n}(\Omega; X)$ are defined as

$$\begin{aligned} E_+^n(u) &:= \int_{\Omega} I_+^n(\text{apmd}_x u) dx \\ \text{Area}(u) &:= \int_{\Omega} J(\text{apmd}_x u) dx. \end{aligned}$$

Quasiconformality

We refer to [17–20] for various definitions of quasiconformality and their relationship in metric spaces, and only mention what is sometimes known as *geometric quasiconformality*. A homeomorphism $u : (X, \mu) \rightarrow (Y, \nu)$ between metric measure spaces is said to be *K-quasiconformal* with “index” $Q \geq 1$, if

$$\text{Mod}_Q \Gamma \leq K \text{Mod}_Q u(\Gamma) \quad (2.10)$$

for any curve family Γ in X . Without further assumptions on the geometry of the spaces, the modulus condition (2.10) is fundamentally one-sided; cf. the discussion in the introduction. We refer to [44] for an equivalent characterization in terms of *analytic quasiconformality* and note that there is a corresponding result for *monotone maps*. Recall that a map is monotone if the preimage of every point is a connected set.

Proposition 2.4 [35, Proposition 3.4] *Let M be $\overline{\mathbb{D}}$ or S^2 . If $u \in N^{1,2}(M; X)$ is a continuous, surjective, monotone and infinitesimally K -quasiconformal map into a complete metric space X , then*

$$\text{Mod}_2 \Gamma \leq K \text{Mod}_2 u(\Gamma)$$

for any curve family Γ in $\overline{\mathbb{D}}$.

We remark that in [35] the result is proven for $M = \overline{\mathbb{D}}$. The same proof however carries over to S^2 with minor modifications.

Quadratic isoperimetric inequality and minimal discs

Given a proper metric space X and a closed Jordan curve Γ in X , we denote

$$\Lambda(\Gamma, X) := \{u \in N^{1,2}(\overline{\mathbb{D}}; X) : u \text{ spans } \Gamma\}.$$

Here, a function $u \in N^{1,2}(\overline{\mathbb{D}}; X)$ is said to span Γ if $u|_{S^1}$ agrees a.e. with a monotone parametrization of Γ . We call $u \in \Lambda(\Gamma, X)$ a *minimal disc* spanning Γ if u has least area among all maps in $\Lambda(\Gamma, X)$ and is furthermore of minimal Reshetnyak energy E_+^2 among all area minimizers in $\Lambda(\Gamma, X)$. In [31] it has been shown that, as soon as $\Lambda(\Gamma, X) \neq \emptyset$, there exists a minimal disc u spanning Γ and every such u is infinitesimally $\sqrt{2}$ -quasiconformal.

Definition 2.5 A metric space X is said to satisfy a (C, l_0) -quadratic isoperimetric inequality if, for any Lipschitz curve $\gamma : S^1 \rightarrow X$ with $\ell(\gamma) < l_0$, there exists $u \in N^{1,2}(\overline{\mathbb{D}}; X)$ with $u|_{S^1} = \gamma$ and

$$\text{Area}(u) \leq C \cdot \ell(\gamma)^2.$$

If X satisfies a quadratic isoperimetric inequality then minimal discs $u : \overline{\mathbb{D}} \rightarrow X$ lie in $N_{loc}^{1,p}(\overline{\mathbb{D}}, X)$ for some $p > 2$ and, if furthermore Γ satisfies a chord-arc condition, then $u \in N^{1,p}(\overline{\mathbb{D}}, X)$.

If a proper geodesic metric space Z is homeomorphic to a 2-manifold, then it satisfies a (C, l_0) -quadratic isoperimetric inequality if and only if every Jordan curve Γ in Z such that $\ell(\Gamma) < l_0$ bounds a Jordan domain $U \subset Z$ which satisfies

$$\mathcal{H}^2(U) \leq C \cdot \ell(\Gamma)^2. \quad (2.11)$$

This follows from the proof of [35, Theorem 1.4] together with the observation leading to [6, Corollary 1.5]. We will use this fact in Sect. 6 to establish the quadratic isoperimetric inequality for \widehat{Z}_u in Theorem 1.9.

3 A geometric characterization of the Sobolev-to-Lipschitz property

In this section we prove Theorem 1.7. We assume throughout this section that X is a metric measure space and $p \geq 1$. We will need the following consequence of the Sobolev-to-Lipschitz property for measurable maps with 1 as p -weak upper gradients.

Lemma 3.1 *Suppose X has the p -Sobolev-to-Lipschitz property, and let V be a separable Banach space. Then any measurable function $f : X \rightarrow V$ with 1 as p -weak upper gradient has a 1-Lipschitz representative.*

Proof We prove the claim first when $V = \mathbb{R}$. Fix $x_0 \in X$ and consider the function

$$w_k := (k - \text{dist}(\cdot, B(x_0, k)))_+$$

for each $k \in \mathbb{N}$. Note that w_k is 1-Lipschitz and $w_k \in N^{1,p}(X)$. The functions

$$f_k := \min\{w_k, f\}$$

have 1 as p -weak upper gradient (see [22, Proposition 7.1.8]) and $f_k \in N^{1,p}(X)$. By the Sobolev-to-Lipschitz property there is a set $N \subset X$ with $\mu(N) = 0$ and 1-Lipschitz functions \tilde{f}_k such that $f_k(x) = \tilde{f}_k(x)$ if $x \notin N$, for each $k \in \mathbb{N}$. Since $f_k \rightarrow f$ pointwise everywhere we have, for $x, y \notin N$, that

$$|f(x) - f(y)| = \lim_{k \rightarrow \infty} |\tilde{f}_k(x) - \tilde{f}_k(y)| \leq d(x, y).$$

Thus f has a 1-Lipschitz representative.

Next, let V be a separable Banach space and let $\{x_n\} \subset V$ be a countable dense set. Given f as in the claim, the functions $f_n := \text{dist}(x_n, f)$ have 1 as p -weak upper gradient, and thus there is a null set $E \subset X$ and 1-Lipschitz functions \tilde{f}_n so that $f_n(x) = \tilde{f}_n(x)$ whenever $n \in \mathbb{N}$ and $x \notin E$. For $x, y \notin E$, we have

$$\|f(x) - f(y)\|_V = \sup_n |f_n(x) - f_n(y)| = \sup_n |\tilde{f}_n(x) - \tilde{f}_n(y)| \leq d(x, y).$$

We again conclude that f has a 1-Lipschitz representative. \square

Proposition 3.2 *Let X be infinitesimally doubling and p -thick quasiconvex with constant C . Then every $u \in N^{1,p}(X)$ satisfying $g_u \leq 1$, has a C -Lipschitz representative.*

Proof Let Γ_0 a path family of zero p -modulus such that

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_{\gamma} 1 = \ell(\gamma)$$

whenever $\gamma \notin \Gamma_0$. By Lusin's theorem there is a decreasing sequence of open sets $U_m \subset X$ such that, for each $m \in \mathbb{N}$, $\mu(U_m) < 2^{-m}$ and $u|_{X \setminus U_m}$ is continuous. Fix $m \in \mathbb{N}$, and let x and y be distinct density points of $X \setminus U_m$. Then for every $\delta > 0$ we have that

$$\text{Mod}_p(\Gamma(B(x, \delta) \setminus U_m, B(y, \delta) \setminus U_m; C)) > 0.$$

For $\gamma^\delta \in \Gamma(B(x, \delta) \setminus U_m, B(y, \delta) \setminus U_m; C) \setminus \Gamma_0$ one has

$$|u(\gamma^\delta(1)) - u(\gamma^\delta(0))| \leq \ell(\gamma) \leq C \cdot d(\gamma^\delta(0), \gamma^\delta(1)) \leq Cd(x, y) + 2C\delta.$$

Letting $\delta \rightarrow 0$ and using the continuity of $u|_{X \setminus U_m}$ we obtain

$$|u(x) - u(y)| \leq Cd(x, y). \quad (3.1)$$

Letting $m \rightarrow \infty$ one sees that (3.1) holds for μ -almost every $x, y \in X$ and hence u has a C -Lipschitz representative. \square

Proof of Theorem 1.7 One implication is directly implied by the preceding proposition. Indeed, assume X is p -thick geodesic and let $f \in N^{1,p}(X)$ have 1 as a p -weak upper gradient. By Proposition 3.2, f has a C -Lipschitz representative for every $C > 1$. So the continuous representative of f is 1-Lipschitz.

Now assume X has the p -Sobolev-to-Lipschitz property but is not p -thick geodesic. Let $E, F \subset X$ measurable and $C > 1$ be such that $\mu(E)\mu(F) > 0$ and $\text{Mod}_p(\Gamma(E, F; C)) = 0$.

By looking at density points of E and F respectively we may assume without loss of generality that

$$0 < \text{dist}(E, F) =: D \quad \text{and} \quad \text{diam}(E), \text{diam}(F) \leq \frac{(C-1)D}{4C}.$$

There exists a non-negative Borel function $g \in L^p(X)$ for which $\int_{\gamma} g = \infty$ for every $\gamma \in \Gamma(E, F; C)$. Denote

$$\Gamma_0 := \left\{ \gamma : \int_{\gamma} g = \infty \right\},$$

whence $\Gamma(E, F; C) \subset \Gamma_0$, $\text{Mod}_p(\Gamma_0) = 0$ and for $\gamma_1, \gamma_2 \notin \Gamma_0$ one has $\gamma_1 \gamma_2 \notin \Gamma_0$ whenever the concatenation $\gamma_1 \gamma_2$ is defined. Define the function $v : X \rightarrow [0, \infty]$ by

$$v(x) = \lim_{n \rightarrow \infty} \inf \left\{ \int_{\gamma} (1 + g/n) : \gamma \in \Gamma(E, x) \right\}.$$

The function v is measurable by [22, Theorem 9.3.1] and satisfies $v|_E \equiv 0$. We also have

$$|v(y) - v(x)| \geq C \cdot D \geq C \cdot \left(d(x, y) - \frac{(C-1)D}{2C} \right) \geq \frac{C+1}{2} d(x, y) \quad (3.2)$$

whenever $y \in F$ and $x \in E$.

For any $\gamma \notin \Gamma_0$ by closedness under composition one has $v(\gamma(0)) = \infty$ iff $v(\gamma(1)) = \infty$. Furthermore if $v(\gamma(0)) \neq \infty$, then

$$|v(\gamma(1)) - v(\gamma(0))| \leq \ell(\gamma) = \int_{\gamma} 1 \quad (3.3)$$

So $g \equiv 1$ is a p -weak upper gradient of v .

If we had that $v \in N^{1,p}(X)$, the Sobolev-to-Lipschitz property and (3.2) would lead to a contradiction. A simple cut off argument remains to complete the proof.

Define $w : X \rightarrow [0, \infty]$ by

$$w(x) := ((C+2)D - d(E, x))^+.$$

Then w has 1 as a upper gradient, $w \in N^{1,p}(X)$ is compactly supported and for $x \in E$, $y \in F$ one has

$$w(y) = (C+2)D - d(E, y) \geq CD \geq \frac{C+1}{2} d(x, y). \quad (3.4)$$

Set $u := \min\{v, w\}$. Then $u \in N^{1,p}(X)$ and $g_u \leq 1$. By the Sobolev-to-Lipschitz property u has a 1-Lipschitz representative. But for every $x \in E$, $y \in F$

$$|u(x) - u(y)| = u(y) \geq \frac{C+1}{2} d(x, y).$$

As $\mu(E)\mu(F) > 0$ these two observations lead to a contradiction. \square

Example 3.3 For $p \in [1, \infty)$ let $Z_p := \{(x, y) \in \mathbb{R}^2 : |y| \leq |x|^p\}$ be endowed with the intrinsic length metric and Lebesgue measure. Then Z is q -thick geodesic if and only if $q > p + 1$. This follows by Example 1 in [12]. Since Z_p is doubling, it also has the q -Sobolev-to-Lipschitz property if and only if $q > p + 1$.

4 Volume rigidity

We prove Proposition 1.8 using the fact that volume preserving 1-Lipschitz maps between rectifiable spaces are essentially length preserving. We say that a map $f : X \rightarrow Y$ between n -rectifiable spaces is volume preserving, if $f_*\mathcal{H}_X^n = \mathcal{H}_Y^n$.

Proposition 4.1 *Let $f : X \rightarrow Y$ volume preserving 1-Lipschitz map between n -rectifiable metric spaces. There exists a Borel set $N \subset X$ with $\mathcal{H}^n(N) = 0$ so that for every absolutely continuous curve γ in X with $|\gamma^{-1}N| = 0$ we have*

$$\ell(\gamma) = \ell(f \circ \gamma).$$

Proof Let $E_i \subset \mathbb{R}^n$ and $g_i : E_i \rightarrow X$ be Borel sets and Lipschitz maps, respectively, satisfying $\mathcal{H}^n(E) = 0$, where

$$E := X \setminus \bigcup_i g_i(E_i).$$

We may assume that the sets $g_i(E_i)$ are pairwise disjoint. For each i , fix Lipschitz extensions $\bar{g}_i : \mathbb{R}^n \rightarrow l^\infty$ and $\bar{f}_i : \mathbb{R}^n \rightarrow l^\infty$ of g_i and $f \circ g_i$, respectively (here we embed X and Y isometrically into l^∞).

Lemma 4.2 *Let $i \in \mathbb{N}$. For almost every $x \in E_i$, we have*

$$\text{md}_x \bar{g}_i = \text{md}_x \bar{f}_i. \quad (4.1)$$

Proof of Lemma 4.2 Since f is 1-Lipschitz we have, at almost every point $x \in E_i$ where both $\text{md}_x \bar{g}_i$ and $\text{md}_x \bar{f}_i$ exist, the inequality $\text{md}_x \bar{f}_i \leq \text{md}_x \bar{g}_i$, which in particular implies $J(\text{md}_x \bar{f}_i) \leq J(\text{md}_x \bar{g}_i)$ for \mathcal{L}^n -almost every $x \in E_i$. By the area formula [25, Corollary 8] and the fact that f is volume preserving, it follows that

$$\begin{aligned} \int_{E_i} J(\text{md}_x \bar{g}_i) d\mathcal{L}^n(x) &= \mathcal{H}^n(g_i(E_i)) = \mathcal{H}^n(f(g_i(E_i))) \\ &= \int_{E_i} J(\text{md}_x \bar{f}_i) d\mathcal{L}^n(x); \end{aligned}$$

cf. Theorem 2.3. Consequently $J(\text{md}_x \bar{g}_i) = J(\text{md}_x \bar{f}_i)$ for \mathcal{L}^n -almost every $x \in E_i$. Thus $\text{md}_x \bar{g}_i = \text{md}_x \bar{f}_i$ for \mathcal{L}^n -almost every $x \in E_i$. \square

For each $i \in \mathbb{N}$, let U_i denote the set of those density points x of E_i for which $\text{md}_x \bar{g}_i = \text{md}_x \bar{f}_i$. Lemma 4.2 implies that $\mathcal{L}^n(E_i \setminus U_i) = 0$. Set

$$N := E \cup \bigcup_i g_i(E_i \setminus U_i),$$

whence $\mathcal{H}^n(N) = 0$. Let $\gamma : [0, 1] \rightarrow X$ be a Lipschitz path. For each $i \in \mathbb{N}$, denote $K_i = \gamma^{-1}(g_i(U_i)) \subset [0, 1]$, and suppose $\gamma_i : [0, 1] \rightarrow \mathbb{R}^n$ is a Lipschitz extension of the “curve fragment” $g_i^{-1} \circ \gamma : K_i \rightarrow \mathbb{R}^n$.

Let $t \in K_i$ be a density point of K_i , for which $|\dot{\gamma}_t|$, $|(f \circ \gamma)'_t|$ and $\gamma'_t(t)$ exist, and $\gamma(t) \notin N$. Then $\gamma_i(t) \in U_i$ and we have

$$\text{md}_{\gamma_i(t)} \bar{g}_i(\gamma'_i(t)) = \lim_{h \rightarrow 0} \frac{\|\bar{g}_i(\gamma_i(t) + h\gamma'_i(t)) - \bar{g}_i(\gamma_i(t))\|_{l^\infty}}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{\|\bar{g}_i(\gamma_i(t+h)) - \bar{g}_i(\gamma_i(t))\|_{l^\infty} + o(h)}{h} \\
&= \lim_{\substack{h \rightarrow 0 \\ t+h \in K_i}} \frac{\|\bar{g}_i(\gamma_i(t+h)) - \bar{g}_i(\gamma_i(t))\|_{l^\infty}}{h} \\
&= \lim_{\substack{h \rightarrow 0 \\ t+h \in K_i}} \frac{d(\gamma(t+h), \gamma(t))}{h} = |\dot{\gamma}_t|.
\end{aligned}$$

The same argument with \bar{f}_i in place of \bar{g}_i yields

$$\text{md}_{\gamma_i(t)} \bar{f}_i(\gamma'_i(t)) = |(f \circ \gamma)'_t|.$$

Since $\gamma_i(t) \in U_i$, we have

$$|\dot{\gamma}_t| = \text{md}_{\gamma_i(t)} \bar{g}_i(\gamma'_i(t)) = \text{md}_{\gamma_i(t)} \bar{f}_i(\gamma'_i(t)) = |(f \circ \gamma)'_t|.$$

Suppose now that $\gamma : [0, 1] \rightarrow X$ is a absolutely continuous path with $|\gamma^{-1}(N)| = 0$. We may assume that γ is constant speed parametrized. Since $|\gamma^{-1}(N)| = 0$ it follows that the union of the sets K_i over $i \in \mathbb{N}$ has full measure in $[0, 1]$. Since, for each $i \in \mathbb{N}$, a.e. K_i satisfies the conditions listed above, we may compute

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}_t| dt = \sum_i \int_{K_i} |\dot{\gamma}_t| dt = \sum_i \int_{K_i} |(f \circ \gamma)'_t| dt = \ell(f \circ \gamma).$$

□

The proof yields the following corollary.

Corollary 4.3 *Let $f : X \rightarrow Y$ be a surjective 1-Lipschitz map between rectifiable spaces with $\mathcal{H}^n(X) = \mathcal{H}^n(Y) < \infty$. Then f is volume preserving and, in particular, $\ell(f \circ \gamma) = \ell(\gamma)$ for ∞ -almost every γ in X .*

We close this section with the proof of Proposition 1.8, which connects volume rigidity and the Sobolev-to-Lipschitz property.

Proof of Proposition 1.8 Since f^{-1} is quasiconformal, $f^{-1} \circ \gamma$ is rectifiable for n -almost every curve γ in Y . Proposition 4.1 implies that

$$g_{f^{-1}} \leq 1.$$

Since Y has the n -Sobolev-to-Lipschitz property, it follows that f^{-1} has a 1-Lipschitz representative, cf. Lemma 3.1. By continuity of f^{-1} it coincides with this representative. □

The Sobolev-to-Lipschitz property is crucial for the conclusion of Proposition 1.8, as the next example shows.

Example 4.4 Let $Y = (\mathbb{D}, d_w)$, where

$$w(x) = 1 - (1 - |x|)\chi_{[0,1] \times \{0\}}, \quad x \in \mathbb{D},$$

and the metric d_w is given by,

$$d_w(x, y) = \inf_{\gamma \in \Gamma(x, y)} \int_{\gamma} w, \quad x, y \in \mathbb{D}.$$

The map $\text{id} : \mathbb{D} \rightarrow Y$ is a volume preserving 1-Lipschitz homeomorphism and the inverse is quasiconformal, but not Lipschitz continuous.

Indeed, since $[0, 1] \times \{0\}$ has zero measure, we have $J(\text{apmd id}) = 1$ almost everywhere, whence the area formula implies that $\mathcal{H}^2(\overline{\mathbb{D}}) = \mathcal{H}_{d_w}^2(\overline{\mathbb{D}})$.

5 Construction of essential pull-back metrics

Let X be a metric measure space and let $\overline{\Gamma}(X)$ denote the set of Lipschitz curves $[0, 1] \rightarrow X$. Recall that p -almost every curve in X admits a Lipschitz reparametrization. In this section we construct essential pull-back distances by Sobolev maps. The key notion here is the essential infimum of a functional over a path family.

Definition 5.1 Let $p \geq 1$, $F : \overline{\Gamma}(X) \rightarrow [-\infty, \infty]$ be a function and $\Gamma \subset \overline{\Gamma}(X)$ a path family. Define the p -essential infimum of F over Γ by

$$\operatorname{ess\,inf}_{\Gamma} F = \operatorname{ess\,inf}_{\gamma \in \Gamma} F(\gamma) := \sup_{\operatorname{Mod}_p(\Gamma_0)=0} \inf\{F(\gamma) : \gamma \in \Gamma \setminus \Gamma_0\}$$

with the usual convention $\inf \emptyset = \infty$.

It is clear that the supremum may be taken over curve families Γ_ρ for non-negative Borel functions $\rho \in L^p(X)$, where

$$\Gamma_\rho := \left\{ \gamma : \int_\gamma \rho = \infty \right\}.$$

Remark 5.2 We have the following alternative expression for $\operatorname{ess\,inf}_p$:

$$\begin{aligned} \operatorname{ess\,inf}_{\Gamma} F &= \max\{\lambda > 0 : \operatorname{Mod}_p(\Gamma \cap \Gamma_F(\lambda)) = 0\} \\ &= \inf\{\lambda > 0 : \operatorname{Mod}_p(\Gamma \cap \Gamma_F(\lambda)) > 0\}. \end{aligned}$$

Here

$$\Gamma_F(\lambda) := \{\gamma \in \overline{\Gamma}(X) : F(\gamma) < \lambda\}.$$

Let Y be a complete metric space, $p \geq 1$, and $u \in N_{loc}^{1,p}(X; Y)$. Given a path family Γ in X we set

$$\operatorname{ess\,el}_{u,p}(\Gamma) := \operatorname{ess\,inf}_{\gamma \in \Gamma} \ell(u \circ \gamma).$$

Definition 5.3 Let $u \in N_{loc}^{1,p}(X; Y)$. Define the *essential pull-back distance* $d'_{u,p} : X \times X \rightarrow [0, \infty]$ by

$$d'_{u,p}(x, y) := \lim_{\delta \rightarrow 0} \operatorname{ess\,el}_{u,p}(\Gamma(\bar{B}(x, \delta), \bar{B}(y, \delta))), \quad x, y \in X.$$

In general, the essential pull-back distance may assume both values 0 and ∞ for distinct points, and it need not satisfy the triangle inequality. We denote by $d_{u,p}$ the maximal pseudometric not greater than $d'_{u,p}$, given by

$$d_{u,p}(x, y) = \inf \left\{ \sum_{i=1}^n d'_{u,p}(x_i, x_{i-1}) : x_0, \dots, x_n \in X, x_0 = x, x_n = y \right\}, \quad x, y \in X.$$

The maximal pseudometric below $d'_{u,p}$ may also fail to be a finite valued. We give two situations that guarantee finiteness and nice properties of the arising metric space. Firstly, in Sect. 5.2 we consider the simple case $u = \text{id} : X \rightarrow X$ and use it to prove Theorem 1.4. Secondly, in Proposition 5.12 we prove that, when X supports a Poincaré inequality and $u \in N^{1,p}_{\text{loc}}(X; Y)$ for large enough p , the distance in Definition 5.3 is a finite valued pseudometric.

For both we need the notion of regular curves, whose properties we study next.

5.1 Regular curves

Throughout this subsection (X, d, μ) is a metric measure space. We define a metric on $\overline{\Gamma}(X)$ by setting

$$d_{\infty}(\alpha, \beta) := \sup_{0 \leq t \leq 1} d(\alpha(t), \beta(t))$$

for any two Lipschitz curves α, β . By a simple application of the Arzela-Ascoli theorem it follows that $(\overline{\Gamma}(X), d_{\infty})$ is separable.

Definition 5.4 A curve $\gamma \in \overline{\Gamma}(X)$ is called (u, p) -regular if $u \circ \gamma$ is absolutely continuous and

$$\text{ess}\ell_{u,p} B(\gamma, \delta) \leq \ell(u \circ \gamma)$$

for every $\delta > 0$.

Proposition 5.5 p -almost every Lipschitz curve is (u, p) -regular.

For the proof we will denote $\Gamma_{u,p}(\lambda) = \{\gamma : \ell(u \circ \gamma) < \lambda\}$.

Proof of Proposition 5.5 Denote by Γ_0 the set of curves in $\overline{\Gamma}(X)$ which are not (u, p) -regular. We may write $\Gamma_0 = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{\gamma : u \circ \gamma \text{ not absolutely continuous}\}, \quad \Gamma_2 = \Gamma_0 \setminus \Gamma_1.$$

By the fact that $u \in N^{1,p}_{\text{loc}}(X; Y)$ we have $\text{Mod}_p(\Gamma_1) = 0$. It remains to show that $\text{Mod}_p(\Gamma_2) = 0$. For any $\delta > 0$ and $\gamma \in \overline{\Gamma}(X)$, set

$$\begin{aligned} \varepsilon(\delta, \gamma) &:= \text{ess}\ell_{u,p} B(\gamma, \delta) - \ell(u \circ \gamma) \\ \delta(\gamma) &:= \sup\{\delta > 0 : \varepsilon(\delta, \gamma) > 0\}. \end{aligned}$$

Note that $\delta(\gamma) > 0$ if and only if $\gamma \in \Gamma_2$. Using Remark 5.2 we make the following observation which holds for $\delta, \varepsilon > 0$:

$$\text{Mod}_p(B(\gamma, \delta) \cap \Gamma_{u,p}(\ell(u \circ \gamma) + \varepsilon)) = 0 \text{ implies } \delta \leq \delta(\gamma) \text{ and } \varepsilon \leq \varepsilon(\delta, \gamma). \quad (5.1)$$

Moreover, for any $\gamma \in \Gamma_2$ and $\delta > 0$, we have

$$\text{Mod}_p(B(\gamma, \delta) \cap \Gamma_{u,p}(\ell(u \circ \gamma) + \varepsilon(\delta, \gamma))) = 0. \quad (5.2)$$

Let $\{\gamma_i\}_{i \in \mathbb{N}} \subset \Gamma_2$ be a countable dense set. For each $i, k \in \mathbb{N}$ and rational $r > 0$ let $\gamma_{i,k,r} \in B(\gamma_i, r) \cap \Gamma_2$ satisfy

$$\ell(u \circ \gamma_{i,k,r}) < \inf\{\ell(u \circ \beta) : \beta \in B(\gamma_i, r) \cap \Gamma_2\} + 1/k.$$

By (5.2) it suffices to prove that

$$\Gamma_2 \subset \bigcup_{i,k \in \mathbb{N}} \bigcup_{r, \delta \in \mathbb{Q}_+} B(\gamma_{i,k,r}, \delta) \cap \Gamma_{u,p}(\ell(u \circ \gamma_{i,k,r}) + \varepsilon(\delta, \gamma_{i,k,r})).$$

For any $\gamma \in \Gamma_2$ let $i \in \mathbb{N}$ be such that $d_\infty(\gamma, \gamma_i) < \delta(\gamma)/8$. Choose rational numbers $r, \delta \in \mathbb{Q}_+$ such that $d_\infty(\gamma_i, \gamma) < r < \delta(\gamma)/8$ and $\delta(\gamma)/4 < \delta < \delta(\gamma)/2$, and a natural number $k \in \mathbb{N}$ so that $1/k < \varepsilon(2\delta, \gamma)$.

We will show that

$$\gamma \in B(\gamma_{i,k,r}, \delta) \cap \Gamma_{u,p}(\ell(u \circ \gamma_{i,k,r}) + \varepsilon(\delta, \gamma_{i,k,r})). \quad (5.3)$$

Indeed, since $\gamma_{i,k,r} \in B(\gamma_i, r) \cap \Gamma_2$, the triangle inequality yields

$$d_\infty(\gamma, \gamma_{i,k,r}) \leq d_\infty(\gamma, \gamma_i) + d_\infty(\gamma_i, \gamma_{i,k,r}) < 2r < \delta(\gamma)/4 < \delta.$$

In particular

$$\gamma \in B(\gamma_{i,k,r}, \delta) \quad (5.4)$$

and also

$$B(\gamma_{i,k,r}, \delta) \subset B(\gamma, 2\delta) \quad (5.5)$$

To bound the length of $u \circ \gamma$, observe that

$$\begin{aligned} \ell(u \circ \gamma_{i,k,r}) &< \inf\{\ell(u \circ \beta) : \beta \in B(\gamma_i, r) \cap \Gamma_2\} + 1/k < \ell(u \circ \gamma) + 1/k \\ &< \ell(u \circ \gamma) + \varepsilon(2\delta, \gamma) \end{aligned}$$

Setting $\varepsilon := \ell(u \circ \gamma) + 1/k - \ell(u \circ \gamma_{i,k,r}) > 0$ it follows that

$$\Gamma_{u,p}(\ell(u \circ \gamma_{i,k,r}) + \varepsilon) \subset \Gamma_{u,p}(\ell(u \circ \gamma) + \varepsilon(2\delta, \gamma)) \quad (5.6)$$

Combining (5.5) and (5.6) with (5.2) we obtain

$$\text{Mod}_p(B(\gamma_{i,k,r}, \delta) \cap \Gamma_{u,p}(\ell(u \circ \gamma_{i,k,r}) + \varepsilon)) = 0,$$

which, by (5.1) yields $\varepsilon \leq \varepsilon(\delta, \gamma_{i,k,r})$. Thus

$$\ell(u \circ \gamma) < \ell(u \circ \gamma) + 1/k = \ell(u \circ \gamma_{i,k,r}) + \varepsilon \leq \ell(u \circ \gamma_{i,k,r}) + \varepsilon(\delta, \gamma_{i,k,r})$$

which, together with (5.4), implies (5.3). This completes the proof. \square

For the next two results we assume that $u \in N_{loc}^{1,p}(X; Y)$ is continuous. We record the following straightforward consequence of the definition of $d'_{u,p}$ and the continuity of u as a lemma.

Lemma 5.6 *For each $x, y \in X$ we have*

$$d(u(x), u(y)) \leq d'_{u,p}(x, y)$$

For the next proposition, we denote by $\ell_{u,p}(\gamma)$ the length of a curve γ with respect to the pseudometric $d_{u,p}$.

Proposition 5.7 *If $\gamma : [a, b] \rightarrow X$ is (u, p) -regular and $a \leq t \leq s \leq b$ then $\gamma|_{[t,s]}$ is (u, p) -regular and*

$$\ell_{u,p}(\gamma) = \ell(u \circ \gamma).$$

Proof Denote $\eta := \gamma|_{[s,t]}$. Let

$$\Gamma_n := \left\{ c|_{[s,t]} : c \in B\left(\gamma, \frac{1}{n}\right) \cap \Gamma_{u,p}\left(\ell(u \circ \gamma) + \frac{1}{n}\right) \right\}.$$

Since γ is (u, p) -regular we have

$$\text{Mod}_p(\Gamma_n) \geq \text{Mod}_p\left(B\left(\gamma, \frac{1}{n}\right) \cap \Gamma_{u,p}\left(\ell(u \circ \gamma) + \frac{1}{n}\right)\right) > 0.$$

We claim that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that

$$\Gamma_n \subset B\left(\eta, \frac{1}{n}\right) \cap \Gamma_{u,p}(\ell(u \circ \eta) + \varepsilon) \quad (5.7)$$

for all $n \geq n_0$. Indeed, otherwise there exists $\varepsilon_0 > 0$ and a sequence

$$\gamma_{n_k} \in B\left(\gamma, \frac{1}{n_k}\right) \cap \Gamma_{u,p}\left(\ell(u \circ \gamma) + \frac{1}{n_k}\right)$$

so that

$$\ell(u \circ \gamma_{n_k}|_{[s,t]}) \geq \ell(u \circ \eta) + \varepsilon_0.$$

Thus

$$\begin{aligned} \ell(u \circ \gamma) + \frac{1}{n_k} &\geq \ell(u \circ \gamma_{n_k}) = \ell(u \circ \gamma_{n_k}|_{[s,t]}) + \ell(u \circ \gamma_{n_k}|_{[s,t]^c}) \\ &\geq \ell(u \circ \eta) + \varepsilon_0 + \ell(u \circ \gamma_{n_k}|_{[s,t]^c}) \end{aligned}$$

yielding

$$\ell(u \circ \gamma|_{[s,t]^c}) + \frac{1}{n_k} \geq \ell(u \circ \gamma_{n_k}|_{[s,t]^c}) + \varepsilon_0. \quad (5.8)$$

By taking $\liminf_{k \rightarrow \infty}$ in (5.8) we obtain

$$\ell(u \circ \gamma|_{[s,t]^c}) \geq \ell(u \circ \gamma|_{[s,t]^c}) + \varepsilon_0,$$

which is a contradiction.

Thus (5.7) holds true. If $\varepsilon, \delta > 0$, let $n \in \mathbb{N}$ be such that (5.7) holds and $\delta > 1/n$. We have

$$0 < \text{Mod}_p \Gamma_n \leq \text{Mod}_p(B(\eta, \delta) \cap \Gamma_{u,p}(\ell(u \circ \eta) + \varepsilon)),$$

implying $\text{ess}\ell_{u,p} B(\eta, \delta) \leq \ell(u \circ \eta) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, η is (u, p) -regular. To prove the equality in the claim note that, since γ is (u, p) -regular we have

$$d_{u,p}(\gamma(t), \gamma(s)) \leq d'_{u,p}(\gamma(t), \gamma(s)) \leq \lim_{\delta \rightarrow 0} \text{ess}\ell_{u,p} B(\gamma|_{[s,t]}, \delta) \leq \ell(u \circ \gamma|_{[s,t]})$$

for any $s \leq t$. It follows that

$$\ell_{u,p}(\gamma) \leq \ell(u \circ \gamma).$$

On the other hand Lemma 5.6 implies that

$$d(u(\gamma(t)), u(\gamma(s))) \leq d_{u,p}(\gamma(t), \gamma(s))$$

whenever $s \leq t$, from which the opposite inequality readily follows. \square

5.2 The Sobolev-to-Lipschitz property in thick quasiconvex spaces

In this subsection let $p \in [1, \infty]$ and $X = (X, d, \mu)$ be p -thick quasiconvex with constant $C \geq 1$. Consider the map $u = \text{id} \in N_{loc}^{1,p}(X; X)$. We denote by d_p the pseudometric $d_{u,p}$ associated to u .

Lemma 5.8 d_p is a metric on X , and satisfies $d \leq d_p \leq Cd$.

Proof Let $x, y \in X$ be distinct and $\delta > 0$. Since

$$\text{Mod}_p \Gamma(B(x, \delta), B(y, \delta); C) > 0$$

it follows that

$$\text{ess} \ell_{\text{id},p} \Gamma(B(x, \delta), B(y, \delta)) \leq C(d(x, y) + 2\delta),$$

implying $d'_{\text{id},p}(x, y) \leq Cd(x, y)$. Thus $d_p \leq Cd$ and in particular d_p is finite-valued. Lemma 5.6 implies that $d(x, y) \leq d_p(x, y)$ for every $x, y \in X$. These estimates together prove the claim. \square

We are now ready to prove Theorem 1.4. For the proof, we denote by X_p the space (X, d_p, μ) and by $B_p(x, r)$ balls in X with respect to the metric d_p ; ℓ_p and $\text{Mod}_{X_p,p} \Gamma$ refer to the length of curves and p -modulus taken with respect to X_p .

Proof of Theorem 1.4 By Lemma 5.8 it follows that X_p is an infinitesimally doubling metric measure space and $\text{Mod}_{X_p,p} \Gamma = 0$ if and only if $\text{Mod}_p \Gamma = 0$. In particular if Γ_* denotes the set of curves γ in X such that $\ell(\gamma) \neq \ell_p(\gamma)$ then by Propositions 5.5 and 5.7

$$\text{Mod}_{X_p,p} \Gamma_* = \text{Mod}_p \Gamma_* = 0. \quad (5.9)$$

From (5.9) and the definition of modulus it follows that $\text{Mod}_{X_p,p} \Gamma = \text{Mod}_p \Gamma$ for every family of curves Γ in X .

By (5.9) and Proposition 3.2 every $f \in N^{1,p}(X_p)$ with $g_f \leq 1$ has a C -Lipschitz representative. If $x, y \in X$ are distinct and $\delta, \varepsilon > 0$ note that the curve family

$$\Gamma_1 = \{\gamma \in \Gamma(B(x, \delta), B(y, \delta)) : \ell(\gamma) \leq \text{ess} \ell_{\text{id},p} \Gamma(B(x, \delta), B(y, \delta)) + \varepsilon\}$$

satisfies

$$\text{Mod}_p \Gamma_1 > 0$$

by Remark 5.2 and the fact that $\varepsilon > 0$. Note also that

$$\ell(\gamma) \leq \text{ess} \ell_{\text{id},p} \Gamma(B(x, \delta), B(y, \delta)) + \varepsilon \leq d'_{\text{id},p}(\gamma(1), \gamma(0)) + \varepsilon$$

for $\gamma \in \Gamma_1$. Let $f \in N^{1,p}(X_p)$ satisfy $g_f \leq 1$ almost everywhere. Let \tilde{f} be the Lipschitz representative of f , and Γ_0 a curve family with $\text{Mod}_p \Gamma_0 = 0$ and

$$|f(\gamma(1)) - f(\gamma(0))| \leq \ell_p(\gamma) = \ell(\gamma)$$

whenever $\gamma \notin \Gamma_0$. We have

$$\text{Mod}_p(\Gamma_1 \setminus \Gamma_0) > 0,$$

so that there exists $\gamma_\delta \in \Gamma_1 \setminus \Gamma_0$. We obtain

$$|\tilde{f}(\gamma_\delta(1)) - \tilde{f}(\gamma_\delta(0))| \leq \ell_p(\gamma_\delta) \leq d'_{\text{id},p}(\gamma_\delta(1), \gamma_\delta(0)) + \varepsilon.$$

Letting $\delta \rightarrow 0$ yields $|\tilde{f}(x) - \tilde{f}(y)| \leq d'_{\text{id},p}(x, y) + \varepsilon$. Since $x, y \in X$ and $\varepsilon > 0$ are arbitrary it follows that \tilde{f} is 1-Lipschitz with respect to d_p . In particular X_p has the p -Sobolev-to-Lipschitz property and hence Theorem 1.7 implies it is p -thick geodesic. We prove the minimality in Proposition 5.9. \square

The metric d_p is the *minimal* metric above d which has the p -Sobolev-to-Lipschitz property. Proposition 5.9 provides a more general statement, from which the minimality discussed in the introduction immediately follows.

Proposition 5.9 *Let Y be a p -thick geodesic metric measure space and $f : Y \rightarrow X$ be a volume preserving 1-Lipschitz map. Then the map*

$$f_p := f : Y \rightarrow (X, d_p)$$

is volume preserving and 1-Lipschitz.

Proof It suffices to show that

$$d'_{\text{id},p}(f(x), f(y)) \leq d(x, y), \quad x, y \in Y.$$

For any $A > 1$, the curve family

$$\Gamma := \{\gamma \in \Gamma(B(x, \delta), B(y, \delta)) : \ell(\gamma) \leq Ad(\gamma(1), \gamma(0))\}$$

has positive p -modulus in Y . Since f is volume preserving and 1-Lipschitz we have

$$0 < \text{Mod}_{Y,p} \Gamma \leq \text{Mod}_{X,p} f\Gamma.$$

If $\gamma \in \Gamma$ then $f \circ \gamma \in \Gamma(B(f(x), \delta), B(f(y), \delta))$ and, moreover

$$\ell(f \circ \gamma) \leq \ell(\gamma) \leq Ad(\gamma(1), \gamma(0)) \leq Ad(x, y) + 2A\delta.$$

It follows that $\text{ess} \ell_{\text{id},p} \Gamma(B(f(x), \delta), B(f(y), \delta)) \leq Ad(x, y) + 2A\delta$, and thus

$$d'_{\text{id},p}(f(x), f(y)) \leq Ad(x, y) + 2A\delta.$$

Since $A > 1$ and $\delta > 0$ are arbitrary, the claim follows. \square

We have the following immediate corollary.

Corollary 5.10 *Assume X is infinitesimally doubling and p -thick geodesic. Then $d = d_p$.*

Before considering essential pull-back metrics by non-trivial maps, we prove Proposition 1.5. The proof is based on the fact that spaces with Poincaré inequality are thick quasiconvex, and on the independence of the minimal weak upper gradient on the exponent.

Proposition 5.11 ([12]) *Let $p \in [1, \infty]$ and X be a doubling metric measure space satisfying a p -Poincaré inequality. There is a constant $C \geq 1$ depending only on the data of X so that X is p -thick quasiconvex with constant C .*

The inverse implication in Proposition 5.11 only holds if $p = \infty$, see [11, 12].

Proof of Proposition 1.5 Assume X is a doubling metric measure space supporting a p -Poincaré inequality, and $q \geq p \geq 1$. Since $\text{Mod}_q \Gamma = 0$ implies $\text{Mod}_p \Gamma = 0$, see [2, Proposition 2.45], we have

$$d \leq d_q \leq d_p \leq Cd$$

for some constant C depending only on p and the data of X .

Fix $x_0 \in X$ and consider the function

$$f : X \rightarrow \mathbb{R}, \quad x \mapsto d_p(x_0, x).$$

Then f is Lipschitz and, by Propositions 5.5 and 5.7, it has 1 as a p -weak upper gradient. Since X is doubling and supports a p -Poincaré inequality, [2, Corollary A.8] implies that the minimal q -weak and p -weak upper gradients of f agree almost everywhere, and thus 1 is a q -weak upper gradient of f , i.e.

$$|f(\gamma(1)) - f(\gamma(0))| \leq \ell(\gamma) \leq \ell_q(\gamma)$$

for q -almost every curve. The space X_q has the q -Sobolev-to-Lipschitz property, see Theorem 1.7, and the 1-Lipschitz representative of f agrees with f everywhere, since f is continuous. By this and the definition of d_q we obtain

$$d_p(x_0, x) \leq d_q(x_0, x), \quad x \in X.$$

Since $x_0 \in X$ is arbitrary the equality $d_p = d_q$ follows.

For the remaining equality, note that $d \leq d_\infty \leq d_{ess} \leq Cd$. Indeed, X supports an ∞ -Poincaré inequality, whence [11, Theorem 3.1] implies the rightmost estimate with a constant C depending only on the data of the ∞ -Poincaré inequality. As above, the function

$$g : X \rightarrow \mathbb{R}, \quad x \mapsto d_{ess}(x_0, x)$$

satisfies

$$|g(\gamma(1)) - g(\gamma(0))| \leq \ell(\gamma) \leq \ell_\infty(\gamma)$$

for ∞ -almost every γ , from which the inequality $d_{ess} \leq d_\infty$ follows. \square

Note that in the proof of Proposition 5.11 we use that in p -Poincaré spaces the q -weak upper gradient does not depend on $q \geq p$. In general such an equality is not true, see [9], and we do not know whether we can weaken the assumptions in Proposition 1.5 from p -Poincaré inequality to p -thick quasiconvexity.

5.3 Essential pull-back metrics by Sobolev maps

Throughout this subsection (X, d, μ) will be a doubling metric measure space satisfying (2.2) with $Q \geq 1$ and supporting a weak $(1, Q)$ -Poincaré inequality, and $Y = (Y, d)$ a proper metric space.

We will use the following observation without further mention. If $p > Q$, and $u : X \rightarrow Y$ has a Q -weak upper gradient in $L^p_{loc}(X)$, then u has a representative $\bar{u} \in N^{1,p}_{loc}(X; Y)$ and the minimal p -weak upper gradient of \bar{u} coincides with the minimal Q -weak upper gradient of u almost everywhere. See [2, Chapter 2.9 and Appendix A] and [22, Chapter 13.5] for more details.

The next proposition states that higher regularity of a map is enough to guarantee that the essential pull-back distance in Definition 5.3 is a finite-valued pseudometric.

Proposition 5.12 *Let $p > Q$, and suppose $u \in N^{1,p}_{loc}(X; Y)$. The pull-back distance $d_u := d'_{u,Q}$ in Definition 5.3 is a pseudometric satisfying*

$$d_u(x, y) \leq C \operatorname{diam} B^{Q/p} d(x, y)^{1-Q/p} \left(\int_{\sigma B} g_u^p d\mu \right)^{1/p} \quad (5.10)$$

whenever $B \subset X$ is a ball and $x, y \in B$, where the constant C depends only on p and the data of X .

For the proof of Proposition 5.12 we define the following auxiliary functions. Let $x \in X$, $\delta > 0$ and Γ_0 a curve family with $\text{Mod}_Q \Gamma_0 = 0$. Set $f := f_{x,\delta,\Gamma_0} : X \rightarrow \mathbb{R}$ by

$$f(y) = \inf \{ \ell(u \circ \gamma) : \gamma \in \Gamma(\bar{B}(x, \delta), y) \setminus \Gamma_0 \}, \quad y \in X.$$

When $\rho \in L^Q(X)$ is a non-negative Borel function, whence $\text{Mod}_Q(\Gamma_\rho) = 0$, we denote $f_{x,\delta,\rho} := f_{x,\delta,\Gamma_\rho}$.

Lemma 5.13 *Let x, δ and ρ be as above. The function $f = f_{x,\delta,\rho}$ is finite μ -almost everywhere and has a representative in $N_{loc}^{1,p}(X)$ with p -weak upper gradient g_u . The continuous representative \bar{f} of f satisfies*

$$|\bar{f}(y) - \bar{f}(z)| \leq C(\text{diam } B)^{Q/p} d(y, z)^{1-Q/p} \left(\int_{\sigma B} g_u^p d\mu \right)^{1/p}. \quad (5.11)$$

Proof Let $g \in L_{loc}^p(X)$ be a genuine upper gradient of u and let $\varepsilon > 0$ be arbitrary. We fix a large ball $B \subset X$ containing $\bar{B}(x, \delta)$ and note that there exists $x_0 \in \bar{B}(x, \delta)$ for which $\mathcal{M}_B(g + \rho)^Q(x_0) < \infty$, since $(g + \rho)|_B \in L^Q(B)$; cf. (2.3). Arguing as in [41, Lemma 4.6] we have that

$$f(y) \leq \inf \left\{ \int_\gamma (g + \rho) : \gamma \in \Gamma_{x_0 y} \setminus \Gamma_\rho \right\} < \infty$$

for almost every $y \in B$.

Let $\gamma \notin \Gamma_\rho$ be a curve such that $f(\gamma(1)), f(\gamma(0)) < \infty$ and $\int_\gamma g < \infty$. We may assume that $|f(\gamma(1)) - f(\gamma(0))| = f(\gamma(1)) - f(\gamma(0)) \geq 0$. If β is an element of $\bar{\Gamma}(B(x, \delta), \gamma(0)) \setminus \Gamma_\rho$ such that $\ell(u \circ \beta) < f(\gamma(0)) + \varepsilon$ then the concatenation $\gamma\beta$ satisfies $\gamma\beta \in \Gamma(\bar{B}(x, \delta), \gamma(1)) \setminus \Gamma_\rho$. Thus

$$|f(\gamma(1)) - f(\gamma(0))| \leq \ell(u \circ \gamma\beta) - \ell(u \circ \beta) + \varepsilon = \ell(u \circ \gamma) + \varepsilon \leq \int_\gamma g + \varepsilon.$$

It follows that $g \in L_{loc}^p(X)$ is a Q -weak upper gradient for f . By [2, Corollary 1.70] we have that $f(y) < \infty$ for Q -quasievery $y \in B$, and $f \in N_{loc}^{1,Q}(X)$; cf. [22, Theorem 9.3.4].

Moreover, since $g \in L_{loc}^p(X)$, f has a continuous representative $\bar{f} \in N_{loc}^{1,p}(X)$ which satisfies (5.11), cf. [22, Theorem 9.2.14]. \square

Proof of Proposition 5.12 To prove the triangle inequality, let $x, y, z \in X$ be distinct. Take $\delta > 0$ small, $\rho \in L^Q(X)$ non-negative, and let $E \subset X$ be a set of Q -capacity zero such that f_{x,δ,Γ_ρ} and f_{y,δ,Γ_ρ} agree with their continuous representatives outside E . Remember that $\alpha\beta \notin \Gamma_\rho$ whenever $\alpha, \beta \notin \Gamma_\rho$. We have that

$$\begin{aligned} d'_{u,Q}(x, z) + d'_{u,Q}(z, y) &\geq \inf_{w \in \bar{B}(z, \delta)} f_{x,\delta,\Gamma_\rho \cup \Gamma_E}(w) + \inf_{v \in \bar{B}(z, \delta)} f_{y,\delta,\Gamma_\rho \cup \Gamma_E}(v) \\ &\geq \inf_{w, v \in \bar{B}(z, \delta) \setminus E} [f_{x,\delta,\rho}(w) + f_{y,\delta,\rho}(v)]. \end{aligned}$$

Together with the estimate (5.11) this yields

$$d'_{u,Q}(x, z) + d'_{u,Q}(z, y) \geq \inf_{w \in \bar{B}(z, \delta) \setminus E} [f_{x,\delta,\rho}(w) + f_{y,\delta,\rho}(w)] - C\delta^{1-Q/p}$$

$$\geq \inf\{\ell(u \circ \gamma) : \gamma \in \Gamma(\bar{B}(x, \delta), \bar{B}(y, \delta)) \setminus \Gamma_\rho\} - C\delta^{1-Q/p},$$

where C depends on u , x and y as well as the data. Since $\delta > 0$ and ρ are arbitrary it follows that

$$d'_{u,Q}(x, z) + d'_{u,Q}(z, y) \geq d'_{u,Q}(x, y).$$

Moreover, for any $y' \in \bar{B}(y, \delta) \setminus E$,

$$\begin{aligned} |f_{x,\delta,\rho}(y')| &\leq C \operatorname{diam} B^{p/Q} d(x, y')^{1-Q/p} \left(\int_{\sigma B} g_u^p d\mu \right)^{1/p} \\ &\leq C \operatorname{diam} B^{p/Q} (d(x, y) + \delta)^{1-Q/p} \left(\int_{\sigma B} g_u^p d\mu \right)^{1/p} \end{aligned}$$

Taking supremum over ρ , and letting δ tend to zero, we obtain (5.10). \square

Remark 5.14 A slight variation of the proof of Proposition 5.12 shows that also in the setting of Theorem 1.4 the essential pull-back distance defines a metric. In particular it satisfies the triangle inequality and one does not have to pass to the maximal semimetric below. In this case in the argument instead of the Morrey embedding one applies Proposition 3.2.

Fix a continuous map $u \in N_{loc}^{1,p}(X; Y)$, where $p > Q$, and denote by Y_u the set of equivalence classes $[x]$ of points $x \in X$, where x and y are set to be equivalent if $d'_{u,Q}(x, y) = 0$. The pseudometric $d'_{u,Q}$ defines a metric d_u on Y_u by

$$d_u([x], [y]) := d'_{u,Q}(x, y), \quad x, y \in X.$$

The natural projection map

$$\widehat{P}_u : X \rightarrow Y_u, \quad x \mapsto [x]$$

is continuous by (5.10). The map u factors as $u = \widehat{u} \circ \widehat{P}_u$, where

$$\widehat{u} : Y_u \rightarrow Y, \quad [x] \mapsto u(x)$$

is well-defined and 1-Lipschitz; cf. Lemma 5.6.

Lemma 5.15 *Under the given assumptions we obtain the following properties.*

- (1) $\widehat{P}_u \in N_{loc}^{1,p}(X; Y_u)$ and $g_{\widehat{P}_u} = g_u$ μ -almost everywhere.
- (2) If u is proper then Y_u is proper and the projection $\widehat{P}_u : X \rightarrow Y_u$ is proper and monotone.

Recall here that a map is called proper if the preimage of every compact set is compact or equivalently if the preimage of every singleton is compact.

Proof (1) Suppose γ is a (u, Q) -regular curve such that g_u is an upper gradient of u along γ . Then by Proposition 5.7

$$d_u(\widehat{P}_u \circ \gamma(1), \widehat{P}_u \circ \gamma(0)) \leq \ell(u \circ \gamma) \leq \int_\gamma g_u.$$

By Proposition 5.5 this implies that $\widehat{P}_u \in N_{loc}^{1,Q}(X; Y_u)$ and $g_{\widehat{P}_u} \leq g_u$. The opposite inequality follows because \widehat{u} is 1-Lipschitz.

(2) The factorization implies that $K \subset \widehat{P}_u(u^{-1}(\widehat{u}(K)))$, for $K \subset Y_u$ and hence Y_u is proper. Similarly $\widehat{P}_u^{-1}(K) \subset u^{-1}(\widehat{u}(K))$ and hence \widehat{P}_u is proper. The proof of monotonicity is an adaptation of the proof of [33, Lemma 6.3].

Assume $\widehat{P}_u^{-1}(y)$ is not connected for some $y \in Y_u$. Then there are compact sets K_1, K_2 for which $\text{dist}(K_1, K_2) > 0$ and $\widehat{P}_u^{-1}(y) = K_1 \cup K_2$. Let S denote the closed and non-empty set of points X whose distance to K_1 and K_2 agree. Let a be the minimum of $x \mapsto \text{dist}_u(y, \widehat{P}_u(x))$ on S . Note here that $\widehat{P}_u(S)$ is closed as Y_u and \widehat{P}_u are proper and hence the infimum is attained and positive. Let $k_i \in K_i$ for $i = 1, 2$. Since $d'_{u,Q}(k_1, k_2) = 0$, for every $\varepsilon > 0$ and small enough $\delta > 0$, Proposition 5.5 implies the existence of a (u, Q) -regular curve $\gamma \in \Gamma(B(k_1, \delta), B(k_2, \delta))$ with $\ell(u \circ \gamma) < \varepsilon$. If δ is chosen small enough the curve must intersect S at some point $s := \gamma(t)$. Since γ is (u, Q) -regular, $\gamma|_{[0,t]}$ is (u, Q) -regular and it follows that

$$a \leq d'_{u,Q}(s, k_1) \leq d'_{u,Q}(s, \gamma(0)) + C\delta^{1-\frac{Q}{p}} \leq \ell(u \circ \gamma) + C\delta^{1-\frac{Q}{p}} \leq \varepsilon + C\delta^{1-\frac{Q}{p}};$$

cf. Proposition 5.7 and (5.10). Choosing $\varepsilon > 0$ and $\delta > 0$ small enough this yields a contradiction. Thus $\widehat{P}_u^{-1}(y)$ is connected for every $y \in Y_u$. \square

Assume additionally that Y is endowed with a measure ν such that $Y = (Y, d, \nu)$ is a metric measure space. We equip (Y_u, d_u) with the measure $\nu_u := \widehat{u}^* \nu$, which is characterized by the property

$$\nu_u(E) = \int_Y \#(\widehat{u}^{-1}(y) \cap E) d\nu(y), \quad E \subset Y_u \text{ Borel}; \quad (5.12)$$

cf. [13, Theorem 2.10.10]. Note that in general ν_u is not a σ -finite measure. For the next theorem, we say that a Borel map $u : X \rightarrow Y$ has Jacobian Ju , if there exists a Borel function $Ju : X \rightarrow [0, \infty]$ for which

$$\int_E Ju d\mu = \int_Y \#(u^{-1}(y) \cap E) d\nu(y) \quad (5.13)$$

holds for every Borel set $E \subset X$. The Jacobian Ju , if it exists, is unique up to sets of μ -measure zero.

Theorem 5.16 *Assume u is proper, nonconstant and has a locally integrable Jacobian Ju which satisfies*

$$g_u^Q \leq K Ju \quad (5.14)$$

μ -almost everywhere for some $K \geq 1$. Then the following properties hold.

- (1) Y_u is a metric measure space and has the Q -Sobolev-to-Lipschitz property.
- (2) Ju is the locally integrable Jacobian of \widehat{P}_u and $\# \widehat{P}_u^{-1}(y) = 1$ for ν_u -almost every $y \in Y_u$.

Proof Note that

$$\int_{\widehat{P}_u^{-1}E} Ju d\mu = \int_Y \#(u^{-1}(y) \cap \widehat{P}_u^{-1}E) d\nu(y) \geq \int_Y \#(\widehat{u}^{-1}(y) \cap E) d\nu(y) = \nu_u(E)$$

for every Borel set $E \subset Y_u$. Thus ν_u is a locally finite measure, and the estimate above implies that \widehat{P}_u has a locally integrable Jacobian $J\widehat{P}_u \leq Ju$. For any Borel set $E \subset X$ we have

$$\begin{aligned}
\int_E J \widehat{P}_u d\mu &= \int_{Y_u} \#(\widehat{P}_u^{-1}(y) \cap E) dv_u(y) = \int_Y \left(\sum_{y \in \widehat{u}^{-1}(z)} \#(\widehat{P}_u^{-1}(y) \cap E) \right) dv(z) \\
&= \int_Y \#(u^{-1}(z) \cap E) dv(z) = \int_E J u d\mu,
\end{aligned}$$

which yields $J \widehat{P}_u = Ju$ almost everywhere. Since \widehat{P}_u is monotone and satisfies (5.13) we have that $\#P_u^{-1}(y) = 1$ for v_u -almost every $y \in Y_u$.

By Lemma 5.15 to see that Y_u is a metric measure space it remains to show that balls in Y_u have positive measure. The idea of proof is borrowed from [33, Lemma 6.11]. Assume $B = B(z, r)$ is a ball in Y_u such that $v_u(B) = 0$. Then we would have $J \widehat{P}_u = J u = 0$ almost everywhere on $U := \widehat{P}_u^{-1}(B)$. By (5.14) and Lemma 5.15 $g_{\widehat{P}_u} = g_u = 0$ almost everywhere on U . Thus \widehat{P}_u is locally constant on U . But U is connected and hence \widehat{P}_u is constant.

To see that U is connected let $x \in U$ satisfy $\widehat{P}_u(x) = z$. Then, for $y \in U$ and $\delta > 0$ arbitrary, there is a (u, Q) -regular curve $\gamma \in \Gamma(B(x, \delta), B(y, \delta))$ such that $\ell(u \circ \gamma) < r$. For δ small enough by Proposition 5.7 the image of γ is contained in U . As this holds for all small enough δ , the points x and y must lie in the same connected component of the open set U . Since y was arbitrary, U must be connected.

We have deduced that $B = \widehat{P}_u(U)$ consists only of the single point z . So either Y_u is disconnected or consists of a single point. The former is impossible because X is connected and the later by the assumption that u is nonconstant.

Before showing the Sobolev-to-Lipschitz property we first claim

$$\text{Mod}_Q \Gamma \leq K \text{Mod}_Q \widehat{P}_u(\Gamma) \quad (5.15)$$

for every curve family Γ in X . Indeed, suppose $\rho \in L^Q(Y_u)$ is admissible for $\widehat{P}_u(\Gamma)$, and set $\rho_1 = \rho \circ \widehat{P}_u$. Let Γ_0 a curve family with $\text{Mod}_Q \Gamma_0 = 0$ for which γ is (\widehat{P}_u, Q) - and (u, Q) -regular, and g_u is an upper gradient of \widehat{P}_u and u along γ , whenever $\gamma \notin \Gamma_0$. For any $\gamma \in \Gamma \setminus \Gamma_0$ we have

$$1 \leq \int_{\widehat{P}_u \circ \gamma} \rho = \int_0^1 \rho_1(\gamma(t)) |(\widehat{P}_u \circ \gamma)'_t| dt \leq \int_0^1 \rho_1(\gamma(t)) g_u(\gamma(t)) |\gamma'_t| dt,$$

i.e. $\rho_1 g_u$ is admissible for $\Gamma \setminus \Gamma_0$. We obtain

$$\begin{aligned}
\text{Mod}_Q \Gamma &= \text{Mod}_Q(\Gamma \setminus \Gamma_0) \leq \int_X \rho_1^Q g_u^Q d\mu \leq K \int_X \rho^Q \circ \widehat{P}_u J u d\mu \\
&= K \int_{Y_u} \rho^Q dv_u.
\end{aligned}$$

The last equality follows, since Lemma 5.15 implies $\widehat{P}_{u*}(Ju \mu) = v_u$. Taking infimum over admissible ρ yields (5.15).

We prove that Y_u has the Q -Sobolev-to-Lipschitz property. Suppose $f \in N^{1,Q}(Y_u)$ satisfies $g_f \leq 1$. There is a curve family Γ_0 in Y_u with $\text{Mod}_Q(\Gamma_0) = 0$ such that

$$|f(\gamma(1)) - f(\gamma(0))| \leq \ell_u(\gamma)$$

whenever $\gamma \notin \Gamma_0$. By (5.15) we have

$$\text{Mod}_Q \widehat{P}_u^{-1} \Gamma_0 \leq K \text{Mod}_Q \Gamma_0 = 0.$$

Here $\widehat{P}_u^{-1}\Gamma$ denotes the family of curves γ in X such that $\widehat{P}_u \circ \gamma \in \Gamma_0$. Together with Propositions 5.5 and 5.7 this implies that, for Mod_Q -almost every curve γ in X , we have

$$|f \circ \widehat{P}_u(\gamma(1)) - f \circ \widehat{P}_u(\gamma(0))| \leq \ell_u(\gamma) = \ell(u \circ \gamma) \leq \int_{\gamma} g_u.$$

Since $g_u \in L^p_{\text{loc}}(X)$ for $p > Q$, it follows that $f \circ \widehat{P}_u$ has a continuous representative \bar{f} . Let $x, y \in X$ be distinct, $\varepsilon > 0$ arbitrary, and $\delta > 0$ such that

$$|\bar{f}(x) - \bar{f}(z)| + |\bar{f}(y) - \bar{f}(w)| < \varepsilon$$

whenever $z \in \bar{B}(x, \delta)$ and $w \in \bar{B}(y, \delta)$. Denote by Γ_1 the curve family with $\text{Mod}_{X,Q} \Gamma_1 = 0$ so that

$$|\bar{f}(\gamma(1)) - \bar{f}(\gamma(0))| \leq \ell(u \circ \gamma) \quad \text{whenever } \gamma \notin \Gamma_1.$$

Then

$$|\bar{f}(x) - \bar{f}(y)| \leq \varepsilon + \inf_{\gamma \in \Gamma(\bar{B}(x, \delta), \bar{B}(y, \delta)) \setminus \Gamma_1} \ell(u \circ \gamma) \leq \varepsilon + d'_{u,Q}(x, y).$$

Since $x, y \in X$ and $\varepsilon > 0$ are arbitrary it follows that $\bar{f}(x) = \bar{f}(y)$ whenever $d'_{u,Q}(x, y) = 0$ and that the map $[x] \mapsto \bar{f}(x)$ is 1-Lipschitz with respect to the metric d_u , and is a ν_u -representative of f . \square

6 Essential metrics on surfaces

In this section we apply the constructions from Sect. 5 to prove Theorems 1.2 and 1.9.

6.1 Ahlfors regular spheres

In what follows, Z is an Ahlfors 2-regular metric sphere which is linearly locally connected. We consider Z as endowed with the Hausdorff 2-measure \mathcal{H}^2_Z . Denote by $\Lambda(Z)$ the set of cell-like maps $u \in N^{1,2}(S^2, Z)$. Recall that a continuous map is called cell-like if the preimage of every point is cell-like. The definition of general cell-like sets may be found in [33] but for our purposes it suffices to know that a compact subset K of S^2 is cell-like if and only if K and $S^2 \setminus K$ are connected. Let us recall the following result from [35].

Theorem 6.1 ([35]) *The set $\Lambda(Z)$ is nonempty and contains an element $u \in \Lambda(Z)$ of least Reshetnyak energy $E^2_+(u)$. Such u is a quasimetric homeomorphism and unique up to a conformal diffeomorphism of S^2 .*

In the following we fix such energy minimizing quasimetric homeomorphism $u \in N^{1,2}(S^2, Z)$. Then Z supports a 2-Poincaré inequality, and u and u^{-1} are quasiconformal; cf. [21, Corollary 8.15 and Theorem 9.8]. Moreover, by [26, Theorem 2.9 and Corollary 4.20], we have that $u \in N^{1,p}(S^2, Z)$ for some $p > 2$.

We denote by $\widehat{Z} = (Z, d_2)$ the metric space as constructed in Sect. 5.2 and by $\widehat{Z}_u = (S^2, d_{u,2})$ the metric space as constructed in Sect. 5.3. It turns out that in fact the two constructions coincide.

Lemma 6.2 *The map $\widehat{Z}_u \rightarrow \widehat{Z}$ induced by u defines an isometry.*

Proof Since u is a homeomorphism, the claim follows immediately from the construction and the fact that

$$\text{Mod}_2(\Gamma) > 0 \Leftrightarrow \text{Mod}_2(u \circ \Gamma) > 0$$

for every family of curves Γ in S^2 . \square

In the following we identify \widehat{Z} and \widehat{Z}_u under this isometry. Thus we have a canonical factorization $u = \widehat{u} \circ \widehat{P}_u$ where $\widehat{P}_u : S^2 \rightarrow \widehat{Z}$ is a homeomorphism and $\widehat{u} : \widehat{Z} \rightarrow Z$ a 1-Lipschitz homeomorphism.

Note that Z as supports a 2-Poincaré in particular it is 2-thick quasiconvex. By Theorem 1.4 the space \widehat{Z} is bi-Lipschitz equivalent to Z and hence itself an Ahlfors 2-regular metric sphere which is linearly locally connected.

Another crucial observation is the following. If Y is another complete metric space and $v \in N^{1,2}(S^2, Y)$, then

$$\text{apmd } u = \text{apmd } v$$

holds almost everywhere on S^2 if and only if

$$\ell(u \circ \gamma) = \ell(v \circ \gamma)$$

holds for 2-almost every curve γ in S^2 . In particular $Z_u = Y_v$ if either of the two conditions holds, see [33, Lemma 3.1 and Corollary 3.2].

Proof of Theorem 1.2 By Theorem 5.15 we have that $\widehat{P}_u \in N^{1,p}(S^2, \widehat{Z})$. By Proposition 5.5 and Proposition 5.7, 2-almost every curve γ in S^2 satisfies $\ell(u \circ \gamma) = \ell(\widehat{P}_u \circ \gamma)$. In particular

$$\text{apmd } u = \text{apmd } \widehat{P}_u \quad (6.1)$$

holds almost everywhere on S^2 . Assume $v \in \Lambda(\widehat{Z})$ is such that $E_+^2(v) < E_+^2(\widehat{P}_u)$. Then $\widehat{u} \circ v \in \Lambda(Z)$ and

$$E_+^2(\widehat{u} \circ v) \leq E_+^2(v) < E_+^2(\widehat{P}_u) = E_+^2(u). \quad (6.2)$$

This contradicts the assumption that u is of minimal energy and hence \widehat{P}_u is energy minimizing in $\Lambda(\widehat{Z})$. By (6.1) we obtain that \widehat{Z} is analytically equivalent to Z .

By Theorem 1.4 we know that \widehat{Z} is thick geodesic **with respect to** \mathcal{H}_Z^2 . We want thick geodecity however to hold with respect to \mathcal{H}_Z^2 . In particular to see property (2) it suffices to show that

$$\mathcal{H}_Z^2 = \mathcal{H}_{\widehat{Z}}^2. \quad (6.3)$$

Equality (6.3) follows however immediately from (6.1), Theorem 2.3 and the fact that \widehat{P}_u and \widehat{u} are homeomorphisms which satisfy higher integrability.

Since \widehat{Z}_u is Ahlfors 2-regular, Theorem 1.7 implies that \widehat{Z} has the 2-Sobolev-to-Lipschitz property. (We could derive this also directly from Theorem 5.16 by showing $v_u = \mathcal{H}_Z^2$ where $v = \mathcal{H}_Z^2$.)

Let Y be analytically equivalent to \widehat{Z} and hence also to Z . Let $v \in \Lambda(Y)$ be the energy minimizing homeomorphism. Then $Y_v = \widehat{Z}_u = \widehat{Z}$ and hence the desired surjective 1-Lipschitz homeomorphism $\widehat{Z} \rightarrow Y$ is given by $f = \widehat{v}$. This proves (3).

Next we prove (4), i.e. volume rigidity. Let Y be an Ahlfors 2-regular metric sphere which is linearly locally connected. Suppose $f : Y \rightarrow \widehat{Z}$ is 1-Lipschitz, cell-like and volume

preserving with respect to \mathcal{H}_Y^2 and \mathcal{H}_Z^2 . Let $v \in \Lambda(Y)$ be an energy minimizer. Then v is a quasisymmetric homeomorphism, and

$$\mathcal{H}^2(Y) = \mathcal{H}^2(\widehat{Z}) \leq \int_{S^2} J(\text{apmd}_p f \circ v) \leq \int_{S^2} J(\text{apmd}_p v) = \mathcal{H}^2(Y).$$

Thus $f \circ v$ is an area minimizer in $\Lambda(\widehat{Z})$ and $\text{apmd } f \circ v = \text{apmd } v$ almost everywhere. Since v is infinitesimally quasiconformal, see [35, Theorem 6.6], the same holds for $f \circ v$. By the proof of [35, Theorem 6.3] this shows that $f \circ v$ is quasisymmetric. It follows that f is quasisymmetric and thus f^{-1} is quasiconformal. Proposition 1.8 implies that f is an isometry.

It remains to check that \widehat{Z} is characterized uniquely by any of the listed properties. For the forthcoming discussion, we fix a metric sphere Y which is analytically and bi-Lipschitz equivalent to Z . We moreover fix an energy minimal homeomorphism $v : S^2 \rightarrow Y$ such that $\text{apmd } u = \text{apmd } v$ almost everywhere.

If Y satisfies (1) (resp. (2)), then by Theorem 1.7 it satisfies (2) (resp. (1)) and, by Theorem 1.4 (see Corollary 5.10), we have that $\widehat{Y} = Y$. But under the usual identifications $\widehat{Y} = \widehat{Y}_v = \widehat{Z}_u = \widehat{Z}$ and hence Y is isometric to \widehat{Z} .

If Y satisfies (3), then there are surjective 1-Lipschitz maps $f : Y \rightarrow \widehat{Z}$ and $g : \widehat{Z} \rightarrow Y$. Since Y and \widehat{Z} are compact, the composition $f \circ g$ is an isometry, see [4, Theorem 1.6.15]. By the fact that f is 1-Lipschitz it follows that g is an isometry.

Assume Y satisfies (4). The canonical map $\widehat{Y} \rightarrow Y$ is 1-Lipschitz and volume preserving. In particular by volume rigidity \widehat{Y} is isometric to Y . However the former is isometric to \widehat{Z} . This concludes the proof. \square

6.2 Metric discs

In this subsection let Z be a geodesic metric disc such that (Z, \mathcal{H}_Z^2) is a metric measure space. First we discuss the following weaker variant of Theorem 1.2 in the disc setting.

Theorem 6.3 *Assume $u \in \Lambda(\partial Z, Z)$ is a minimizer of E_+^2 which is moreover monotone and lies in $N^{1,p}(\overline{\mathbb{D}}, Z)$ for some $p > 2$. Then there is a geodesic metric disc \widehat{Z}_u and a factorization $u = \widehat{u} \circ \widehat{P}_u$ such that:*

- (1) \widehat{Z}_u satisfies the Sobolev-to-Lipschitz property.
- (2) $\widehat{P}_u \in \Lambda(\partial \widehat{Z}_u, \widehat{Z}_u)$ is a minimizer of E_+^2 , contained in $N^{1,p}(\overline{\mathbb{D}}, \widehat{Z}_u)$, and a uniform limit of homeomorphisms.
- (3) For 2-almost every curve γ in $\overline{\mathbb{D}}$ one has $\ell(\widehat{P}_u \circ \gamma) = \ell(u \circ \gamma)$.
- (4) If Y is a disc, and $v \in \Lambda(\partial Y, Y)$ satisfies $\text{apmd } u = \text{apmd } v$ almost everywhere, then there is a surjective 1-Lipschitz map $j : \widehat{Z} \rightarrow Y$ such that $v = j \circ \widehat{P}_u$.
- (5) If Z satisfies a (C, l_0) -quadratic isoperimetric inequality then \widehat{Z}_u satisfies a (C, l_0) -quadratic isoperimetric inequality.

Note that the existence of such u is guaranteed if Z satisfies a quadratic isoperimetric inequality and ∂Z satisfies a chord-arc condition, see [31, 35].

Proof Let $\widehat{Z}_u = (\overline{\mathbb{D}}, d_u)$ be the space constructed as in Sect. 5.3 and $u = \widehat{u} \circ \widehat{P}_u$ be the associated factorization as discussed therein. Then as in the previous subsection we see that $\text{apmd } u = \text{apmd } \widehat{P}_u$ holds almost everywhere and hence in particular $\ell(u \circ \gamma) = \ell(\widehat{P}_u \circ \gamma)$ for 2-almost every curve γ in $\overline{\mathbb{D}}$. As in the previous subsection this implies that u and \widehat{P}_u are infinitesimally quasiconformal.

Next we show that \widehat{Z}_u is a metric disc, and that $\widehat{P}_u \in \Lambda(\partial\widehat{Z}_u, \widehat{Z}_u)$ is a uniform limit of homeomorphisms. By [33, Corollary 7.12], to see this it suffices to prove that \widehat{P}_u is cell-like and its restriction $S^1 \rightarrow \widehat{P}_u(S^1)$ is monotone. Since by Lemma 5.15 \widehat{P}_u is monotone, the proof that \widehat{P}_u is cell-like follows by the same argument as in the proof of [33, Theorem 8.1]. Thus it suffices to show that $\widehat{P}_u|_{S^1} : S^1 \rightarrow \widehat{P}_u(S^1)$ is monotone, i.e. that $\widehat{P}_u^{-1}(y) \cap S^1$ is connected for every $y \in \widehat{Z}_u$.

We identify \mathbb{D} with the lower hemisphere of S^2 . Since \widehat{P}_u is cell-like, the set $K := \widehat{P}_u^{-1}(y)$ is a cell-like subset of S^2 and in particular K and $S^2 \setminus K$ are connected. Assume $K \cap S^1$ is not connected. Then there exist $z, w \in S^1$ such that neither of the arcs connecting z and w is contained in K . However $u(z) = u(w) \in \partial Z$. Then since $u \in \Lambda(\partial Z, Z)$, one of the arcs $A \subset S^1$ connecting z and w gets mapped constantly to $u(w)$. Set $M := K \cup A$. Clearly M is connected. We claim that $S^2 \setminus M$ is also connected. Assume it was not, and let O be a connected component of $S^2 \setminus M$ contained in \mathbb{D} . Then as in the proof of [33, Theorem 8.1], the restriction of u to ∂O is identically constant $u(w)$, and hence $u|_O$ is identically constant $u(w)$. By the definition of \widehat{P}_u it follows that \widehat{P}_u is constant on O . As $\partial O \cap K \neq \emptyset$ we obtain $O \subset K$, which is a contradiction.

Thus M is a cell-like subset of S^2 . By Moore's quotient theorem, see e.g. [33, Theorem 7.11], S^2/M is homeomorphic to S^2 . Since S^2/M is obtained from $S^2/K \cong S^2$ by quotienting out a closed curve that only self-intersects at one point, the arising space would have a topological cutpoint which is not the case for S^2 . This contradiction shows that the restriction of \widehat{P}_u to S^1 must be monotone and hence \widehat{Z}_u is a metric disc, and $\widehat{P}_u \in \Lambda(\partial\widehat{Z}_u, \widehat{Z}_u)$ a uniform limit of homeomorphisms.

If \widehat{P}_u is not an energy minimizer, then there exists $v \in \Lambda(\partial\widehat{Z}_u, \widehat{Z}_u)$ such that $E_+^2(v) < E_+^2(\widehat{P}_u)$. This implies that $\widehat{u} \circ v \in \Lambda(\partial Z, Z)$, and that

$$E_+^2(\widehat{u} \circ v) \leq E_+^2(v) < E_+^2(\widehat{P}_u) = E_+^2(u). \quad (6.4)$$

This contradiction shows that \widehat{P}_u is an energy minimizer.

Set $v := \mathcal{H}_Z^2$. By Theorem 5.16 we know that \widehat{Z}_u has the Sobolev-to-Lipschitz property with respect to v_u . Recall that u is infinitesimally quasiconformal and that, by Theorem 2.3 and monotonicity of u , the equality $J(\text{apmd } u) = Ju$ holds almost everywhere. We want however the Sobolev-to-Lipschitz property to hold with respect to $\mathcal{H}_{\widehat{Z}_u}^2$. To this end we show that $v_u = \mathcal{H}_{\widehat{Z}_u}^2$.

Since u and \widehat{P}_u are monotone and surjective, it follows that the map $\widehat{u} : \widehat{Z}_u \rightarrow Z$ is monotone. This and (5.12) imply that $\#\widehat{u}^{-1}(z) = 1$ for \mathcal{H}_Z^2 -almost every $z \in Z$. Using the equality of the approximate metric differentials, we have

$$\begin{aligned} v_u(A) &= \int_Z \#(\widehat{u}^{-1}(z) \cap A) d\mathcal{H}_Z^2(z) = \mathcal{H}_Z^2(\widehat{u}(A)) \\ &= \int_{u^{-1}(\widehat{u}(A))} J(\text{apmd}_x \widehat{P}_u) dx = \int \#(\widehat{P}_u^{-1}(y) \cap u^{-1}(\widehat{u}(A))) d\mathcal{H}_{\widehat{Z}_u}^2(y) \\ &\geq \mathcal{H}_{\widehat{Z}_u}^2(A) \end{aligned}$$

for any Borel set $A \subset \widehat{Z}_u$. On the other hand

$$v_u(A) = \mathcal{H}_Z^2(\widehat{u}(A)) \leq \mathcal{H}_{\widehat{Z}_u}^2(A). \quad (6.5)$$

These two inequalities imply that $v_u = \mathcal{H}_{\widehat{Z}_u}^2$. In particular we have shown that \widehat{Z}_u has the Sobolev-to-Lipschitz property with respect to $\mathcal{H}_{\widehat{Z}_u}^2$.

The proof of the maximality statement (4) is identical to the proof of maximality in Theorem 1.2.

Finally assume that Z satisfies a (C, l_0) -quadratic isoperimetric inequality. Note first that, by [33, Proposition 5.1] and Theorem 2.3, for every Jordan domain $V \subset \mathbb{D}$ with $\ell(u|_{\partial V}) < l_0$, one has

$$\text{Area}(\widehat{P}_u|_V) = \text{Area}(u|_V) \leq C \cdot \ell(u|_{\partial V})^2 \leq C \cdot \ell(\widehat{P}_u|_{\partial V})^2. \quad (6.6)$$

The remaining proof is a variation of the proof of [33, Theorem 8.2]. We say that points $p, q \in \overline{\mathbb{D}}$ are sufficiently connected if for every $\epsilon > 0$ and there exists a curve η joining p to q such that

$$\ell(\widehat{P}_u \circ \eta) < \widehat{d}_u(p, q) + \epsilon. \quad (6.7)$$

If we could show that all points $p, q \in \overline{\mathbb{D}}$ are sufficiently connected then the proof of [33, Theorem 8.2] would go through without changes. Unfortunately we do not know whether this is true.

Let A be the set of those $p \in \overline{\mathbb{D}}$ such that p either lies in the interior of $\overline{\mathbb{D}}$ or p lies in an open interval $I \subset S^1$ satisfying $\ell(\widehat{P}_u \circ I) < \infty$. We claim that if $p, q \in A$ then p and q are sufficiently connected.

So let $p, q \in A$. The proof of [33, Corollary 5.4] shows that for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y \in B_\delta(p)$ and $z \in B_\delta(q)$, there are curves γ_p, γ_q connecting y to p and z to q , respectively, such that

$$\ell(\widehat{P}_u \circ \gamma_p) < \epsilon, \quad \ell(\widehat{P}_u \circ \gamma_q) < \epsilon. \quad (6.8)$$

By construction of the metric \widehat{d}_u , there exists a curve γ connecting $B_\delta(p)$ to $B_\delta(q)$ such that

$$\ell(\widehat{P}_u \circ \gamma) < \widehat{d}_u(p, q) + \epsilon.$$

The concatenation of γ with correspondingly chosen γ_p and γ_q yields a curve η connecting p to q which satisfies

$$\ell(\widehat{P}_u \circ \eta) < \widehat{d}_u(p, q) + 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary the claim follows.

Now let $U \subset \widehat{Z}_u$ a Jordan domain satisfying $\ell(\partial U) < l_0$. Then the set of points in ∂U which have a preimage in A is dense in ∂U . In particular we may choose the points t_i and x_i in the proof of [33, Theorem 8.2] such that all the points x_i lie in A and are hence pairwise sufficiently connected. Under this modification the argument in the proof of [33, Theorem 8.2] shows that

$$\mathcal{H}^2(U) \leq C \cdot \ell(\partial U)^2 \quad (6.9)$$

which completes the proof. \square

6.3 Essential minimal surfaces

Proof of Theorem 1.9 Let X be a proper metric space which satisfies a (C, l_0) -quadratic isoperimetric inequality, and $u \in \Lambda(\Gamma, X)$ a minimal disc spanning a Jordan curve Γ which satisfies a chord-arc condition. By [30, Theorem 3.1] one has that $u \in N^{1,p}(\overline{\mathbb{D}}; X)$ for some $p > 2$. Let \widehat{Z}_u be the metric space discussed in Sect. 5.3 and $u = \widehat{u} \circ \widehat{P}_u$ be the corresponding factorization. Let Z_u be the metric disc discussed in Sect. 1.4 and $u = \bar{u} \circ P_u$ the corresponding factorization. Then, by [33] and the proof of [5, Theorem 2.7], the pair (Z_u, P_u) satisfies

the assumptions of Theorem 6.3. On the other hand, since $\ell(P_u \circ \gamma) = \ell(u \circ \gamma)$ for every curve γ in \mathbb{D} , it follows that $\widehat{Z}_u = \widehat{Z}_{\widehat{P}_u}$ and $\widehat{P}_u = \widehat{P}_{P_u}$. Theorem 6.3 now implies the claim. \square

The following example from [35, Example 5.9], demonstrates that a unique characterization statement in Theorem 1.9 in terms of the Sobolev-to-Lipschitz property does not hold.

Example 6.4 Let Z be the metric space obtained from the standard Euclidean disc by collapsing a segment I in its interior to a point. Then Z is a geodesic metric disc satisfying a quadratic isoperimetric inequality with constant $\frac{1}{2\pi}$, compare the proof of [5, Theorem 3.2]. The canonical quotient map $u : \mathbb{D} \rightarrow Z$ is an energy minimizer in $\Lambda(\partial Z, Z)$ and satisfies $\text{apmd } u = \text{apmd id}_{\mathbb{D}}$ almost everywhere. In particular \widehat{Z}_u is isometric to \mathbb{D} . However Z_u is isometric to Z , see [5, Theorem 1.2]. It is also straightforward to see that Z_u has the Sobolev-to-Lipschitz property.

To see that Z is thick geodesic let $E \subset Z$ be a measurable subset of positive measure and $C > 1$. Furthermore let $p \in \mathbb{D}$ be such that $u(p) \neq u(I)$ is a density point of E and $q \in I$ the point which is closest to p . Then for $\delta > 0$ sufficiently small

$$0 < \text{Mod}_2 \Gamma(u^{-1}(E) \cap B(p, \delta), I \cap B(q, \delta); 1) \leq \text{Mod}_2 \Gamma(E, u(I); C), \quad (6.10)$$

see [12, Remark 3.4] for the first inequality. Now having Eq. (6.10) it is not hard to deduce that Z is thick geodesic. The example also shows that being thick quasiconvex with constant 1 is a strictly stronger condition than being thick geodesic.

The metric disc Z is not Ahlfors regular, since $\mathcal{H}^2(B_Z(p, r))$ grows linearly in r , and thus Example 6.4 does not contradict Theorem 1.2. Note that, for Z in the example, the construction in [33] yields the original space Z , while the space \widehat{Z} constructed in the proof of Theorem 6.3 coincides with \mathbb{D} . This need not always be the case when collapsing a cell-like subset in the interior of \mathbb{D} ; the Euclidean disc with a small ball (in the interior) collapsed is a metric disc satisfying the assumptions of Theorem 6.3, where both constructions yield the original space; cf. [33, Example 11.3].

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