Discrete Mathematics 344 (2021) 112169

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Reducing the domination number of graphs via edge contractions and vertex deletions

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ARTICLE INFO

Article history:
Received 16 October 2019
Received in revised form 20 September 2020
Accepted 21 September 2020
Available online xxxx

Keywords:
Blocker problem
Graph classes
Domination number
Graph modification problem

ABSTRACT

In this work, we study the following problem: given a connected graph G, can we reduce the domination number of G by at least one using k edge contractions, for some fixed integer $k > 0$? We show that for $k = 1$ (resp. $k = 2$), the problem is NP-hard (resp. coNP-hard). We further prove that for $k = 1$, the problem is W[1]-hard parameterized by domination number plus the mim-width of the input graph, and that it remains NP-hard when restricted to chordal $\{P_6, P_4 + P_2\}$-free graphs, bipartite graphs and $\{C_3, \ldots, C_\ell\}$-free graphs for any $\ell \geq 3$. We also show that for $k = 1$, the problem is coNP-hard on subcubic claw-free graphs, subcubic planar graphs and on $2P_3$-free graphs. On the positive side, we show that for any $k \geq 1$, the problem is polynomial-time solvable on $(P_5 + pK_1)$-free graphs for any $p \geq 0$ and that it can be solved in FPT-time and XP-time when parameterized by treewidth and mim-width, respectively. Finally, we start the study of the problem of reducing the domination number of a graph via vertex deletions and edge additions and, in this case, present a complexity dichotomy on H-free graphs.

1. Introduction

In a graph modification problem, we are usually interested in modifying a given graph G, via a small number of operations, into some other graph G' that has a certain desired property. This property often describes a specific graph class to which G' must belong. Such graph modification problems allow to capture a variety of classical graph-theoretic problems; for instance, if only k vertex deletions are allowed and G' must be an independent set or a clique, we obtain the Independent Set or Clique problem, respectively.

Now, instead of specifying a graph class to which G' should belong, we may ask for a specific graph parameter π to decrease. In other words, given a graph G, a set $\mathcal{O}$ of one or more graph operations and an integer $k \geq 1$, the question is whether G can be transformed into a graph G' by using at most k operations from $\mathcal{O}$ such that $\pi(G') \leq \pi(G) - d$ for some *threshold* $d \geq 0$. Such problems are called *blocker problems* as the set of vertices or edges involved can be viewed as "blocking" the parameter π. Identifying such sets may provide important information about the structure of the input graph; for instance, if $\pi = \alpha$, $k = d = 1$ and $\mathcal{O} = \{\text{vertex deletion}\}$, the problem is equivalent to testing whether the input graph contains a vertex that is in every maximum independent set (see [24]).

Blocker problems have received much attention in the recent literature (see for instance [1–4,9,13,14,21,23–25]) and have been related to other well-known graph problems such as Hadwiger Number, ClubContraction and several graph transversal problems (see for instance [13,23]). The graph parameters considered so far in the literature include the chromatic number, the independence number, the clique number, the matching number and the vertex cover number,

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https://doi.org/10.1016/j.disc.2020.112169

while the set $\mathcal{O}$ consists of a single graph operation, namely either vertex deletion, edge contraction, edge deletion or edge addition. Since these blocker problems are often NP-hard in general graphs, particular attention has been paid to their computational complexity when restricted to special graph classes.

In this paper, we study another parameter, namely the domination number γ, and we consider the following three operations: edge contraction, vertex deletion and edge addition, the first one being the main focus of our work. To the best of our knowledge, a systematic study of the computational complexity of the blocker problem for the domination number has not yet been attempted in the literature. We point out that a related problem (that of characterizing the graphs for which the contraction of *any* edge decreases the domination number) has already been considered in the literature (see for instance [6–8,26]). We initiate here a study of the blocker problem for this parameter, as it has been done for other parameters and several graph operations.

Formally, let $G = (V, E)$ be a graph. The *contraction* of an edge $uv \in E$ removes vertices u and v from G and replaces them by a new vertex that is made adjacent to precisely those vertices that were adjacent to u or v in G (without introducing self-loops nor multiple edges). We say that a graph G can be *k-contracted* into a graph G', if G can be transformed into G' by a sequence of at most k edge contractions, for an integer $k \geq 1$ (note that contracting an edge cannot increase the domination number). The first problem we consider is the following, where $k \geq 1$ is a fixed integer.

k-Edge Contraction(γ)	
Instance:	A connected graph $G = (V, E)$.
Question:	Can G be k-edge contracted into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$?

Interestingly, Huang and Xu [20] showed that three edge contractions are always enough to decrease the domination number of a graph. Hence, if $k \geq 3$, k-Edge Contraction(γ) has a simple answer: every graph with domination number at least two is a Yes-instance to this problem. For this reason, we only consider the problem for $k = 1, 2$. We show that this problem is NP-hard and W[1]-hard parameterized by γ + mim-width for $k = 1$, and coNP-hard for $k = 2$. We then consider the problem on specific classes of graphs. We show that 1-Edge Contraction(γ) is NP-hard and W[1]-hard parameterized by γ on $\{P_6, P_4 + P_2\}$-free chordal graphs. We also show that it remains NP-hard when restricted to bipartite graphs and to $\{C_3, \ldots, C_\ell\}$-free graphs, for every $\ell \geq 3$, and that it is coNP-hard when restricted to $2P_3$-free graphs, subcubic planar graphs and subcubic claw-free graphs. Followingly, we present some positive results. We show that 1-Edge Contraction(γ) can be solved in polynomial time on P_5-free graphs and that it can be solved in FPT-time and XP-time when parameterized by tree-width and mim-width, respectively. We also show that 2-Edge Contraction(γ) is polynomial-time solvable on P_6-free graphs, thus providing a graph class in which the complexities of 1-Edge Contraction(γ) and 2-Edge Contraction(γ) are not the same. Together, these results also provide a complexity dichotomy for 1-Edge Contraction(γ) on H-free graphs, when H is a connected graph.

We then turn our attention to the other two operations: vertex deletion and edge addition. As opposed to the case of edge contractions, there is no constant upper bound on the number of vertex deletions or edge additions necessary to decrease the domination number of a graph, as can be seen by a graph consisting of two stars having a common leaf. We are therefore interested in the following two problems, where for a set $S \subseteq V(G)$ (resp. $S \subseteq E(G)$), $G - S$ (resp. $G + S$) denotes the graph obtained from G by the deletion (resp. addition) of the elements of S.

Vertex Deletion(γ)	
Instance:	A connected graph G and an integer k.
Question:	Is there $S \subseteq V(G)$ such that $\lvert S\rvert \leq k$ and $\gamma(G - S) \leq \gamma(G) - 1$?

Edge Addition(γ)	
Instance:	A connected graph G and an integer k.
Question:	Is there $S \subseteq E(G)$ such that $\lvert S\rvert \leq k$ and $\gamma(G + S) \leq \gamma(G) - 1$?

When k is fixed and thus not part of the input, we denote by k-Vertex Deletion(γ) and k-Edge Addition(γ) the corresponding problems.

We first show that Vertex Deletion(γ) and Edge Addition(γ) are in fact equivalent problems, a rather surprising result as the vertex deletion and edge addition operations behave differently: while edge additions can only decrease the domination number of a graph, vertex deletions can both increase or decrease it. Due to this equivalence, we only focus on Vertex Deletion(γ). We show that even for $k = 1$, Vertex Deletion(γ) is NP-hard and W[1]-hard parameterized by γ on split graphs, which rules out the possibility of algorithms running in FPT- or even XP-time parameterized by k for this problem, unless P=NP. In view of this, we solely focus on Vertex Deletion(γ) with $k = 1$. We show that 1-Vertex Deletion(γ) remains NP-hard (resp. coNP-hard) on bipartite graphs and $\{C_3, \ldots, C_\ell\}$-free graphs, for $\ell \geq 3$ (resp. claw-free graphs). Finally, we provide a few cases in which 1-Vertex Deletion(γ) becomes polynomial-time solvable; in particular, we show that this is the case for $(P_4 + kP_1)$-free graphs. Together, these results lead to a complexity dichotomy for 1-Vertex Deletion(γ) on H-free graphs.

The paper is organized as follows: In Section 2, we present definitions and notations that are used throughout the paper, and also provide some preliminary results, including the equivalence between Vertex Deletion(γ) and Edge Addition(γ). Section 3 is devoted to the complexity study of k-Edge Contraction(γ), while Section 4 explores the complexity of the 1-Vertex Deletion(γ) problem. We conclude the paper in Section 5 with some final remarks and future research directions. Extended abstracts of distinct parts of this work appeared in the proceedings of MFCS 2019 [17] and ISAAC 2019 [16].

2. Preliminaries

Definitions and notation. Throughout the paper, we only consider finite, undirected, connected graphs that have no self-loops or multiple edges. We refer the reader to [12] for any terminology and notation not defined here and to [10] for basic definitions and terminology regarding parameterized complexity.

Let $G = (V, E)$ be a graph and let $n = |V|$. For any $u \in V$, we denote by $N_G(u)$, or simply $N(u)$ if it is clear from the context, the set of vertices that are adjacent to u i.e., the *neighbors* of u, and let $N[u] = N(u) \cup \{u\}$. Two vertices $u, v \in V$ are said to be *true twins* (resp. *false twins*), if $N[u] = N[v]$ (resp. if $N(u) = N(v)$). If $|N(v)| = 1$, we say v is a *leaf* of G and we denote by leaves(G) the set of leaves of G.

For a subset $V' \subseteq V$, we let $G[V']$ denote the subgraph of G *induced* by V', which has vertex set V' and edge set $\{uv \in E \mid u, v \in V'\}$. A subset $K \subseteq V(G)$ is a *clique* if any two vertices of K are adjacent in G. A vertex $v \in V(G)$ is *simplicial* if $N(v)$ is a clique.

A graph is a *chordal graph* if it has no induced cycle of length at least four or, equivalently, if it admits a *perfect elimination ordering*, that is, an ordering $v_1v_2 \dots v_n$ of V such that for every $1 \leq i \leq n$, v_i is simplicial in the graph $G[\{v_i, v_{i+1}, \dots, v_n\}]$.

Given two vertex disjoint graphs H and H', the graph obtained from the *disjoint union* of H and H' (denoted by $H+H'$) is the graph with vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. We say that $G = kH$ if G is the graph obtained from the disjoint union of k vertex-disjoint copies of a graph H. For a family $\{H_1, \dots, H_p\}$ of graphs, G is said to be $\{H_1, \dots, H_p\}$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$ we may write H_1-free instead of $\{H_1\}$-free. For $k \geq 1$, the path and cycle on k vertices are denoted by P_k and C_k respectively. A *tree* is a graph that is connected and acyclic. A graph is *bipartite* if every cycle contains an even number of vertices. The *complete graph* on $k \geq 1$ vertices is denoted by K_k; K_3 is also referred to as a *triangle*. The *diamond* is the graph obtained from K_4 by deleting an edge.

We denote by $d_G(u, v)$, or simply $d(u, v)$ if it is clear from the context, the length of a shortest path from u to v in G. Similarly, for any subset $V' \subseteq V$, we denote by $d_G(u, V')$, or simply $d(u, V')$ if it is clear from the context, the minimum length of a shortest path from u to some vertex in V' i.e., $d(u, V') = \min_{v \in V'} d(u, v)$; and for $V'' \subseteq V$, we denote by $d_G(V', V'')$, or simply $d(V', V'')$ if it is clear from context, the minimum length of a shortest path from a vertex of V' to some vertex of V'' i.e., $d(V', V'') = \min_{x \in V', y \in V''} d(x, y)$.

For a vertex $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. For an edge $e \in E$, we denote by $G \backslash e$ the graph obtained from G by contracting the edge e. The *k-subdivision* of an edge uv consists in replacing it by a path u-v_1-…-v_k-v, where $v_1, \dots, v_k$ are new vertices.

Given a graph H and an integer $d \geq 0$, we say that G is *at distance at most d from H* if there exists a subset $X \subseteq V$ such that $|X| \leq d$ and $G[V \setminus X]$ is isomorphic to H.

A subset $S \subseteq V$ is called an *independent set*, or is said to be *independent*, if any two vertices in S are nonadjacent. A subset $D \subseteq V$ is called a *dominating set*, if every vertex in $V \setminus D$ is adjacent to at least one vertex in D; the *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set. For any $v \in D$ and $u \in N[v]$, v is said to *dominate* u (in particular, v dominates itself); furthermore, u is a *private neighbor of v with respect to D* if u has no neighbor in $D \backslash \{v\}$. We say that D *contains an edge* (or more) if the graph $G[D]$ contains an edge (or more). A vertex $v \in V(G)$ is a *domination-critical vertex* if its deletion results in a graph with smaller domination number. The Dominating Set problem is to test whether a given graph G has a dominating set of size at most ℓ, for some given integer $\ell \geq 0$. If Π is a problem that takes as input a graph G and an integer k, we denote by (G, k) an instance of Π.

Parameterized Complexity. Let Σ be an alphabet. A *parameterized problem* is a set $\Pi \subseteq \Sigma^* \times \mathbb{N}$. A parameterized problem Π is said to be *fixed-parameter tractable*, or contained in the complexity class FPT, if there exists an algorithm that for each $(x, k) \in \Sigma^* \times \mathbb{N}$ decides whether $(x, k) \in \Pi$ in time $f(k) \cdot |x|^c$ for some computable function f and fixed integer $c \in \mathbb{N}$. A parameterized problem Π is said to be contained in the complexity class XP if there is an algorithm that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ decides whether $(x, k) \in \Pi$ in time $f(k) \cdot |x|^{g(k)}$ for some computable functions f and g. The basic way to show a parameterized problem Π is unlikely to admit an FPT algorithm when parameterized by k is to show that the problem is W[1]-hard under this parameterization. Analogously to the classical P versus NP setting, this is done by providing a parameterized reduction from a known W[1]-hard problem to Π. For more detailed definitions regarding parameterized complexity, we refer the reader to [10].

Mim-width. Let $M \subseteq E(G)$ and V_M be the set of vertices that are endpoints of M. We say M is an *induced matching* of G if $G[V_M]$ is isomorphic to pK_2 with $p = |M|$. Given a graph G and a partition of V into two subsets A, B, we denote by $G[A, B]$ the bipartite graph with vertex set V and edge set $\{uv \in E \mid u \in A, v \in B\}$. A *branch decomposition* of a graph G is a pair $(T, \mathcal{L})$ where T is a tree of maximum degree three and $\mathcal{L}$ is a bijection $\mathcal{L}\colon V(G) \rightarrow$ leaves(T).

For any subtree T' of T, we denote by $A(T')$ the set of vertices of G that $\mathcal{L}$ maps to the leaves of T', i.e. $A(T') = \mathcal{L}^{-1}(\text{leaves}(T'))$. For every $e = uv \in E(T)$, let T_u (resp. T_v) be the subtree of $T - e$ that contains u (resp. v). Let $mimval_G(uv) = \max\{|M|: M$ is an induced matching in $G[A(T_u), A(T_v)]\}$. The mim-width of a branch decomposition $(T, \mathcal{L})$ is defined as $\max_{uv\in E(T)} mimval_G(uv)$. The *mim-width of* G, denoted by $mimw(G)$, is the minimum mim-width over all branch decompositions of G.

Preliminary results. Reducing the domination number of a graph via edge contractions was first considered by Huang and Xu [20]; they prove that for a connected graph G such that $\gamma(G) \geq 2$, we have $ct_\gamma(G) \leq 3$ where $ct_\gamma(G)$ the minimum number of edge contractions required to transform G into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$. It follows that a connected graph G with $\gamma(G) \geq 2$ is always a Yes-instance of k-Edge Contraction(γ), if $k \geq 3$. The authors [20] further prove the following theorem:

Theorem 2.1 ([20])**.** *For a connected graph G, the following hold:*

(i) $ct_\gamma(G) = 1$ *if and only if there exists a minimum dominating set in G that is not an independent set.*
(ii) $ct_\gamma(G) = 2$ *if and only if every minimum dominating set in G is an independent set and there exists a dominating set D in G of size* $\gamma(G) + 1$ *such that* $G[D]$ *contains at least two edges.*

Burton and Sumner [8] initiated the study of the problem of reducing the domination number of a graph via one single vertex deletion. In their work, they prove the following result, which nicely connects domination-critical vertices to edges whose contraction decreases the domination number.

Lemma 2.2 ([8])**.** *Let uv be an edge in a graph G. Then* $\gamma(G\backslash uv) < \gamma(G)$ *if and only if either there exists a minimum dominating set D of G such that* $u, v \in D$ *or at least one of u or v is a domination-critical vertex in G.*

While Lemma 2.2 shows that if G is a Yes-instance for 1-Vertex Deletion(γ) then G is a Yes-instance for 1-Edge Contraction(γ), the converse is not true, as can be seen by a graph G formed by a clique K of size at least two with two pendant vertices attached to each vertex of K. A minimum dominating set of this graph consists of all the vertices of K; hence by Theorem 2.1(i), G is a Yes-instance for 1-Edge Contraction(γ). However, it is easy to see that there is no vertex in G whose deletion results in a graph with smaller domination number.

In the same work, Burton and Sumner [8] also introduce the notion of a *selfish* vertex in a minimum dominating set D: a vertex v is selfish in D if it has no private neighbor outside D, that is, if $D \setminus \{v\}$ is a dominating set for $G - v$. This allows to obtain the following characterization of domination-critical vertices in a graph.

Lemma 2.3 ([8])**.** *For any graph G and* $v \in V(G)$, $\gamma(G - v) < \gamma(G)$ *if and only if there is a minimum dominating set D of G in which v is selfish.*

In addition to edge contraction and vertex deletion, the present work also considers edge addition. To the best of our knowledge, decreasing the domination number of a graph using this operation has not yet been studied in the literature. As we next show, Edge Addition(γ) and Vertex Deletion(γ) turn out to be equivalent problems.

Proposition 2.4. *A graph is a* Yes*-instance to* Vertex Deletion*(γ) if and only if it is a* Yes*-instance to* Edge Addition*(γ).*

Proof. Let (G, k) be a Yes-instance for Vertex Deletion(γ) and let $S = \{v_1, \ldots, v_k\}$ be a set of vertices such that $\gamma(G - S) < \gamma(G)$. Let D' be a minimum dominating set of $G - S$ and let v be a vertex of D'. Consider the set of edges $F = \{vv_1, \ldots, vv_k\} \setminus E(G)$. Then $|F| \leq k$. Moreover, D' is a dominating set for the graph $G + F$ such that $|D'| < \gamma(G)$. This implies that $\gamma(G + F) < \gamma(G)$ and thus, (G, k) is a Yes-instance for Edge Addition(γ).

Now suppose that (G, k) is a Yes-instance for Edge Addition(γ) and let $F \subseteq E(G)$ be such that $|F| \leq k$ and $\gamma(G + F) < \gamma(G)$. Let $F' \subseteq F$ be a minimal subset of F with the property that $\gamma(G + F') < \gamma(G)$. Then, $|F'| \leq k$ and for any $F'' \subset F'$, $\gamma(G + F'') = \gamma(G)$ (recall that edge addition cannot increase the domination number of a graph). Let D' be a minimum dominating set of $\gamma(G + F')$ and let $uv \in F'$. If $D' \cap \{u, v\} = \emptyset$ or $D' \cap \{u, v\} = \{u, v\}$, then we would have a dominating set in $G + (F' \setminus \{uv\})$ of size $|D'| < \gamma(G)$, a contradiction to the minimality of F'. This implies that for any $uv \in F'$, $|D' \cap \{u, v\}| = 1$. Now, let $S = \{u \in V(G) \mid uv \in F'$ and $D' \cap \{u, v\} = \{v\}\}$. Then $|S| \leq |F'| \leq k$. Furthermore, D' is a dominating set of $G - S$ as $D' \cap S = \emptyset$. Since $|D'| < \gamma(G)$, we thus conclude that (G, k) is a Yes-instance for Vertex Deletion(γ). □

3. Complexity of k-Edge Contraction(γ)

In this section, we present hardness results and algorithms for k-Edge Contraction(γ). Recall that if $k \geq 3$, any graph with domination number at least two is a Yes-instance for k-Edge Contraction(γ). We investigate the complexity of the problem when $k = 1, 2$.

We first prove hardness results for 1-Edge Contraction(γ) in specific classes of graphs (see Section 3.1). Followingly, the hardness of 2-Edge Contraction(γ) is presented in Section 3.2, while Section 3.3 contains algorithms for k-Edge Contraction(γ), with $k = 1, 2$. Finally, the results obtained in Sections 3.1 and 3.3 lead to a complexity dichotomy for H-free graphs, when H is connected; we state and discuss this result in Section 3.4.

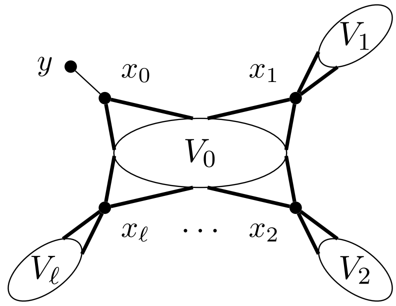


Fig. 1. The graph G' (thick lines indicate that the vertex x_i is adjacent to every vertex in V_0 and V_i, for $i = 0, \ldots, \ell$).

3.1. Hardness of 1-Edge Contraction(γ)

In this section, we present hardness results for the 1-Edge Contraction(γ) problem. We first consider the class of $\{P_6, P_4 + P_2\}$-free chordal graphs.

Theorem 3.1. *1-Edge Contraction(γ) is* NP*-hard and* W[1]*-hard parameterized by $\gamma + mimw$ on $\{P_6, P_4+P_2\}$-free chordal graphs.*

Proof. We give a reduction from Dominating Set. Given an instance (G, ℓ) for Dominating Set, we construct an instance G' for 1-Edge Contraction(γ) as follows. We denote by $\{v_1, \ldots, v_n\}$ the vertex set of G. The vertex set of the graph G' is given by $V(G') = V_0 \cup \cdots \cup V_\ell \cup \{x_0, \ldots, x_\ell, y\}$, where each V_i is a copy of the vertex set of G. We denote the vertices of V_i by $v^i_1, v^i_2, \ldots, v^i_n$. The adjacencies in G' are then defined as follows:

- $V_0 \cup \{x_0\}$ is a clique;
- $yx_0 \in E(G')$;

and for $1 \leq i \leq \ell$,

- V_i is an independent set;
- x_i is adjacent to all the vertices of $V_0 \cup V_i$;
- v^i_j is adjacent to $\{v^0_a \mid v_a \in N_G[v_j]\}$ for any $1 \leq j \leq n$ (see Fig. 1).

Claim 1. $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

Proof. It is clear that $\{x_0, x_1, \ldots, x_\ell\}$ is a dominating set of G'; thus, $\gamma(G') \leq \ell + 1$. If $\gamma(G) \leq \ell$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, it is easily seen that $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_0\}$ is a dominating set of G'. Thus, $\gamma(G') \leq \gamma(G) + 1$ and so, $\gamma(G') \leq \min\{\gamma(G) + 1, \ell + 1\}$. Now, suppose to the contrary that $\gamma(G') < \min\{\gamma(G) + 1, \ell + 1\}$ and consider a minimum dominating set D' of G'. We first make the following simple observation.

Observation 1. *For any dominating set D of G', $D \cap \{y, x_0\} \neq \emptyset$.*

Now, since $\gamma(G') < \ell + 1$, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$ (otherwise, $\{x_1, \ldots, x_\ell\} \subset D'$ and combined with Observation 1, D' would be of size at least $\ell + 1$). But then, $D'' = D' \cap V_0$ must dominate every vertex in V_i, and so $|D''| \geq \gamma(G)$. Since $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$), we then have $\gamma(G) \leq |D'| - 1$, a contradiction. Thus, $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$. ▲

We now show that (G, ℓ) is a Yes-instance for Dominating Set if and only if G' is a Yes-instance for 1-Edge Contraction(γ).

Assume first that $\gamma(G) \leq \ell$. Then $\gamma(G') = \gamma(G)+1$ by the previous claim, and if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, then $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_0\}$ is a minimum dominating set of G' which is not independent. Hence, by Theorem 2.1(i), G' is a Yes-instance for 1-Edge Contraction(γ).

Conversely, assume that G' is a Yes-instance for 1-Edge Contraction(γ) i.e., there exists a minimum dominating set D' of G' which is not independent (see Theorem 2.1(i)). Then, Observation 1 implies that there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$; indeed, if it were not the case, we would then have by Claim 1 $\gamma(G') = \ell + 1$ and thus, D' would consist of $x_1, \ldots, x_\ell$ and either y or x_0. In both cases, D' would be independent, a contradiction. It follows that $D'' = D' \cap V_0$ must dominate every vertex in V_i and thus, $|D''| \geq \gamma(G)$. But $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$) and so by Claim 1, $\gamma(G) \leq |D'| - 1 \leq (\ell + 1) - 1$ that is, (G, ℓ) is a Yes-instance for Dominating Set.

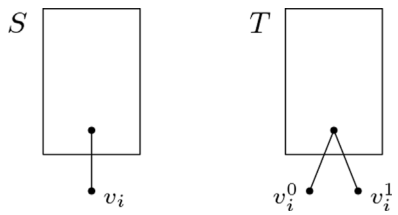


Fig. 2. Constructing a branch decomposition $(T, \mathcal{L})$ for $H = G'[V_0 \cup V_1]$ from the branch decomposition $(S, \mathcal{L}')$ of G.

We next show that G' is a P_6-free graph. Let P be an induced path of G'. First observe that since V_0 is a clique, $|V(P) \cap V_0| \leq 2$. If $|V(P) \cap V_0| = 0$, since each V_i is independent and the same holds for $\{x_0, \ldots, x_\ell\}$, we have that $|V(P)| \leq 3$. We now consider the following two cases.

Case 1. $|V(P) \cap V_0| = 2$.
Let $u, v \in V_0$ be the vertices of $V(P) \cap V_0$. Since P is an induced path, u and v appear consecutively in P, that is, $uv \in E(P)$. Furthermore, $V(P) \cap \{x_0, \ldots, x_\ell\} = \emptyset$ since u and v are adjacent to all the vertices of $\{x_0, \ldots, x_\ell\}$. If u has another neighbor $w \in V_i$ in P, for some $i > 0$, then since $N(w) \subset V_0 \cup \{x_i\}$, w can have no neighbor in P other than u, that is, w is an endpoint of the path. Symmetrically, the same holds for a neighbor of v in P different from u. Hence, we conclude that $|V(P)| \leq 4$.

Case 2. $|V(P) \cap V_0| = 1$.
Let $u \in V_0$ be the vertex of $V(P) \cap V_0$. If $V(P) \cap \{x_0, \ldots, x_\ell\} = \emptyset$, then it is easy to see that $|V(P)| \leq 3$, since any neighbor of u in the path must belong to $\cup_{1 \leq i \leq \ell} V_i$ and, by the same argument as in Case 1, such a neighbor would have to be an endpoint of the path. If $V(P) \cap \{x_0, \ldots, x_\ell\} \neq \emptyset$, let x_i be a vertex that is in P. Note that since $ux_i \in E(G')$, we necessarily have that $ux_i \in E(P)$. Now suppose that x_i has another neighbor w in P. Then $w \in V_i$ since $N(x_i) = V_0 \cup V_i$. But then, by the argument used above, we conclude that w is an endpoint of the path; and since u can have at most two neighbors in $\{x_0, \ldots, x_\ell\}$, it follows that $|V(P)| \leq 5$.

Now, to see that G' is also a $P_4 + P_2$-free graph, it suffices to note that any induced P_4 of G' contains at least one vertex of the clique V_0. Finally, we provide a perfect elimination ordering for G' which would prove that G' is chordal. Observe first that any vertex in $\{y\} \cup V_1 \cup \cdots \cup V_\ell$ is simplicial in G'. Once those vertices have been deleted from G', $x_0, x_1, \ldots, x_\ell$ become simplicial in the resulting graph; and since V_0 is a clique, we obtain a perfect elimination ordering for G'.

Since Dominating Set is NP-hard, the above proves that 1-Edge Contraction(γ) is NP-hard on $\{P_6, P_4+P_2\}$-free chordal graphs.

We will now show that 1-Edge Contraction(γ) is also W[1]-hard parameterized by domination number plus mim-width. To do so, we will use the fact recently shown by Fomin et al. [15] that Dominating Set is W[1]-hard parameterized by solution size plus mim-width. There remains to show that $\gamma(G') + mimw(G')$ is bounded by a function of $\gamma(G) + mimw(G)$. We first show the following.

Lemma 3.2. *Let $H = G'[V_0 \cup V_1]$. Then $mimw(H) \leq mimw(G)$.*

Proof. Let $(S, \mathcal{L}')$ be a branch decomposition for G with width $mimw(G)$. To show the stated result we will construct a branch decomposition for H of width $mimw(G)$ as well. We construct such a branch decomposition $(T, \mathcal{L})$ for H as follows. Let T be the tree obtained from S in the following way: for every $v_i \in V(G)$, we replace the leaf corresponding to v_i by two leaves, one corresponding to v_i^0 and the other to v_i^1 (see Fig. 2). We will now show that the width of this decomposition is at most $mimw(G)$.

Recall that for any subtree T' of T, we denote by $A(T')$ the set of vertices of H that $\mathcal{L}'$ maps to the leaves of T', i.e. $A(T') = \mathcal{L}^{-1}(\text{leaves}(T'))$. For every $e = uv \in E(T)$, we let T_u (resp. T_v) be the subtree of $T - e$ that contains u (resp. v).

First, if $e = uv \in T$ is an edge such that either u or v is a leaf of T, then a maximum induced matching of $H[A(T_u), A(T_v)]$ has size one, since $\min\{|A(T_u)|, |A(T_v)|\} = 1$. Let $e = uv$ be an edge such that u and v are not leaves of T. Then, for every $1 \leq i \leq n$, $v_i^0 \in A(T_u)$ if and only if $v_i^1 \in A(T_u)$ also. Let M be a maximum induced matching of $H[A(T_u), A(T_v)]$. We want to show that $|M| \leq mimw(G)$. Note that since $N_H[v_i^1] \subseteq N_H[v_i^0]$, then only one of $\{v_i^0, v_i^1\}$ can be covered by M. Moreover, since V_0 is a clique in H and V_1 is an independent set in H, every edge of H has an endpoint in V_0. Hence, if there exists i, j such that $v_i^0 v_j^0 \in M$, then $M = \{v_i^0 v_j^0\}$, as any other edge of M will have an endpoint adjacent to either v_i^0 or v_j^0. We may therefore assume that every edge of M has one endpoint in V_0 and the other in V_1. Let $v_i^0 v_j^1 \in M$. Suppose without loss of generality that $v_i^0 \in A(T_u)$ and $v_j^1 \in A(T_v)$. Since v_i^0 is adjacent to all vertices of V_0, we can conclude that every vertex of V_0 covered by M also belongs to $A(T_u)$. Hence, every vertex of V_1 covered by M belongs to $A(T_v)$. To conclude, consider the edge uv in S and the partition of $V(G)$ into $A(S_u)$ and $A(S_v)$. By the construction of T, it holds that $v_i \in A(S_u)$ if and only if $v_i^0, v_i^1 \in A(T_u)$. Let M' be the matching in $G[A(S_u), A(S_v)]$ defined as follows: $v_i v_j \in M'$ for every $v_i^0 v_j^1 \in M$. Since M is an induced matching in $H[A(T_u), A(T_v)]$, then M' is also an induced matching of $G[A(S_u), A(S_v)]$. Moreover, $|M| = |M'|$. Hence, $|M| \leq mimw(G)$. This concludes the proof that $(T, \mathcal{L})$ is a branch decomposition of width $mimw(G)$ for H. ▲

We will also need the following two basic observations about the mim-width of a graph. Recall that two vertices $u, v \in V$ are said to be *true twins* (resp. *false twins*), if $N[u] = N[v]$ (resp. if $N(u) = N(v)$).

Observation 2. *Let H be a graph and $u, v \in V(H)$ be two vertices that are true (resp. false) twins in H. Then $mimw(H - v) = mimw(H)$.*

Observation 3. *Let H be a graph and $v \in V(H)$. Then $mimw(H) \leq mimw(H - v) + 1$.*

Now, note that G' can be constructed from $H = G'[V_0 \cup V_1]$ from the addition of false twins $(V_2, V_3, \ldots, V_\ell)$ plus the addition of $\ell + 2$ vertices $(x_0, x_1, \ldots, x_\ell, y)$. By Observation 2, the addition of false twins does not increase the mim-width of a graph and, by Observation 3, the addition of a vertex can only increase the mim-width by one; thus, $mimw(G') \leq mimw(H) + \ell + 2$. By Lemma 3.2, we have that $mimw(G') \leq mimw(G) + \ell + 2$, which together with Claim 1 concludes the proof. □

In order to obtain complexity results for further graph classes, let us now consider subdivisions of edges.

Lemma 3.3. *Let G be a graph and let G' be the graph obtained by 3-subdividing every edge of G. Then G is a* YES*-instance for 1-*EDGE CONTRACTION*(γ) if and only if G' is a* YES*-instance for 1-*EDGE CONTRACTION*(γ).*

Proof. Let $G = (V, E)$ be a graph. In the following, given an edge $e \in E$, we denote by e_1, e_2 and e_3 the three new vertices resulting from the 3-subdivision of the edge e. We first prove the following:

Claim 2. *If H is the graph obtained from G by 3-subdividing one edge, then $\gamma(H) = \gamma(G) + 1$.*

Proof. Assume that H is obtained by 3-subdividing the edge $e = uv$ (we assume in the following that e_1 is adjacent to u and e_3 is adjacent to v in H), and consider a minimum dominating set D of G. We construct a dominating set of H as follows. If $D \cap \{u, v\} = \emptyset$, then $D \cup \{e_2\}$ is a dominating set of H. If $|D \cap \{u, v\}| = 1$, then we may assume without loss of generality that $u \in D$; but then, $D \cup \{e_3\}$ is a dominating set of H. Finally, if $\{u, v\} \subset D$, then $D \cup \{e_1\}$ is a dominating set of H. Thus, $\gamma(H) \leq \gamma(G) + 1$.

Conversely, let D' be a minimum dominating set of H. First observe that at least one vertex among e_1, e_2 and e_3 belongs to D' as e_2 must be dominated. Furthermore, we may assume, without loss of generality, that $\{e_1, e_3\} \not\subset D'$; indeed, if $\{e_1, e_3\} \subset D'$ then, by minimality of D', $v \notin D'$ for otherwise $D' \backslash \{e_3\}$ would be a dominating set of G' of size strictly smaller than D', a contradiction. But then, $(D' \backslash \{e_3\}) \cup \{v\}$ is a minimum dominating set of G' not containing both e_1 and e_3. We next prove the following:

Observation 4. *If $e_1 \in D'$ (resp. $e_3 \in D'$) then $(D' \backslash \{e_1, e_2, e_3\}) \cup \{v\}$ (resp. $(D' \backslash \{e_1, e_2, e_3\}) \cup \{u\}$) is a dominating set of G of size at most $\gamma(H) - 1$.*

Indeed, if $e_1 \in D'$ then either $v \in D'$ and $(D' \backslash \{e_1, e_2, e_3\}) \cup \{v\} = D' \backslash \{e_1, e_2, e_3\}$ is a dominating set of G of size at most $\gamma(H) - 1$. Or $v \notin D'$ but then $e_2 \in D'$ since $e_3 \notin D'$ (recall that $|D' \cap \{e_1, e_3\}| \leq 1$) must be dominated. But again, $(D' \backslash \{e_1, e_2, e_3\}) \cup \{v\}$ is a dominating set of G of size at most $\gamma(H) - 1$. By symmetry, we conclude similarly if $e_3 \in D'$. ⋄

On the other hand, if $\{e_1, e_3\} \cap D' = \emptyset$, then $e_2 \in D'$ and $D' \backslash \{e_1, e_2, e_3\}$ is a dominating set of G of size $\gamma(H) - 1$, which concludes the proof of the claim. ▲

We next prove the statement of the lemma. Let G' be the graph obtained from G by 3-subdividing every edge of G. It then follows from Claim 2 that $\gamma(G') = \gamma(G) + |E|$.

First assume that G is a YES-instance for 1-EDGE CONTRACTION(γ) i.e., there exists a minimum dominating set D of G containing an edge $e = uv$ (see Theorem 2.1(i)). Let D' be the minimum dominating set of G' constructed according to the proof of Claim 2. Then by construction, the edge ue_1 is contained in D' which implies that G' is a YES-instance for 1-EDGE CONTRACTION(γ).

Conversely, assume that G' is a YES-instance for 1-EDGE CONTRACTION(γ) that is, there exists a minimum dominating set D' of G' containing an edge f (see Theorem 2.1(i)). First note that we may assume that for any edge $e = uv \in E$, $\{e_1, e_3\} \not\subset D'$; indeed, if $\{e_1, e_3\} \subset D'$, then by minimality of D' we have that $v \notin D'$ (with v adjacent to e_3) for otherwise $D' \backslash \{e_3\}$ is a dominating set of G' of size strictly smaller than D', a contradiction (also note that by minimality of D', $e_2 \notin D'$). But then, $(D' \backslash \{e_3\}) \cup \{v\}$ is also a minimum dominating set of G' containing the edge f; indeed, since both e_2 and v are not contained in D', e_3 is not an endvertex of f. In the following, we denote by $e = uv$ the edge of G such that f is an edge of the 3-subdivision of e, with e_1 adjacent to u and e_3 adjacent to v.

Now consider the minimum dominating set D of G constructed according to the proof of Claim 2. We distinguish two cases depending on whether $f = ue_1$ or $f = e_1e_2$ (note that the cases where $f = e_3v$ or $f = e_2e_3$ are symmetric to those considered).

First assume that $f = ue_1$. Then, by Observation 4, $v \in D$ and thus, uv is an edge contained in D. Now, if $f = e_1e_2$ then again, by Observation 4, $v \in D$. But then, by minimality of D', we know that $e_3 \notin D'$ as well as $v \notin D'$, for otherwise $D' \backslash \{e_2\}$ would be a dominating set of G' of size strictly smaller than D', a contradiction. Thus, v is dominated in G' by some vertex e'_1 with $e' = vw \in E$, and it follows from Observation 4 that $w \in D$. But then, D contains the edge vw, which concludes the proof. □

By 3-subdividing every edge of a graph G sufficiently many times, we deduce the following corollary from Lemma 3.3.

Corollary 3.4. *1-Edge Contraction(γ) is* NP*-hard when restricted to bipartite graphs and to* $\{C_3, \ldots, C_\ell\}$*-free graphs, for any fixed* $\ell \geq 3$.

We next determine the complexity of 1-Edge Contraction(γ) when restricted to $2P_3$-free graphs, subcubic planar graphs and subcubic claw-free graphs. To this end, we first introduce the following two problems.

All Efficient MD	
Instance:	A connected graph $G = (V, E)$.
Question:	Is every minimum dominating set of G efficient?

All Independent MD	
Instance:	A connected graph $G = (V, E)$.
Question:	Is every minimum dominating set of G independent?

The following is a straightforward consequence of Theorem 2.1(i).

Fact 3.5. *Given a graph G, G is a* Yes*-instance for 1-*Edge Contraction*(γ) if and only if G is a* No*-instance for* All Independent MD.

We may now proceed to state and prove the final hardness results of this subsection.

Theorem 3.6. All Independent MD *is* NP*-hard when restricted to* $2P_3$*-free graphs.*

Proof. We reduce from 3-Sat: given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Independent MD as follows: For any variable $x \in X$, we introduce a copy of C_3, which we denote by G_x, with one distinguished *positive literal vertex* x and one distinguished *negative literal vertex* $\bar{x}$; in the following, we denote by u_x the third vertex in G_x. For any clause $c \in C$, we introduce a *clause vertex* c; we then add an edge between c and the (positive or negative) literal vertices whose corresponding literal occurs in c. Finally, we add an edge between any two clause vertices so that the set of clause vertices induces a clique denoted by K in the following. We denote by G_Φ the resulting graph.

Observation 5. *For any dominating set D of G_Φ and any variable $x \in X$, $|D \cap V(G_x)| \geq 1$. In particular, $\gamma(G_\Phi) \geq |X|$.*

Claim 3. *Φ is satisfiable if and only if $\gamma(G_\Phi) = |X|$.*

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. For any variable $x \in X$, if x is true, add the positive literal vertex x to D; otherwise, add the negative variable vertex $\bar{x}$ to D. Clearly, D is dominating and we conclude by Observation 5 that $\gamma(G_\Phi) = |X|$.

Conversely, assume that $\gamma(G_\Phi) = |X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 5, $|D \cap V(G_x)| = 1$ for any $x \in X$. It follows that $D \cap K = \emptyset$ and so, every clause vertex must be adjacent to some (positive or negative) literal vertex belonging to D. We thus construct a truth assignment satisfying Φ as follows: for any variable $x \in X$, if the positive literal vertex x belongs to D, set x to true; otherwise, set x to false. ▲

Claim 4. *$\gamma(G_\Phi) = |X|$ if and only if every minimum dominating set of G_Φ is independent.*

Proof. Assume that $\gamma(G_\Phi) = |X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 5, $|D \cap V(G_x)| = 1$ for any $x \in X$. It follows that $D \cap K = \emptyset$ and since $N[V(G_x)] \cap N[V(G_{x'})] \subset K$ for any two $x, x' \in X$, D is independent.

Conversely, consider a minimum dominating set D of G_Φ. Since D is independent, $|D \cap V(G_x)| \leq 1$ for any $x \in X$ and we conclude by Observation 5 that in fact, equality holds. Now suppose that there exists $c \in C$, containing variables x_1, x_2 and x_3, such that the corresponding clause vertex c belongs to D (note that since D is independent, $|D \cap K| \leq 1$). Assume without loss of generality that x_1 occurs positively in c, that is, c is adjacent to the positive literal vertex x_1. Then, $x_1 \notin D$ since D is independent and so, either $u_{x_1} \in D$ or $\bar{x_1} \in D$. In the first case, we immediately obtain that $(D \setminus \{u_{x_1}\}) \cup \{x_1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. In the second case, since $c \in D$, any vertex dominated by $\bar{x_1}$ is also dominated by c; thus, $(D \setminus \{\bar{x_1}\}) \cup \{x_1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. Consequently, $D \cap K = \emptyset$ and so, $\gamma(G_\Phi) = |D| = |X|$. ▲

Now by combining Claims 3 and 4, we obtain that Φ is satisfiable if and only if every minimum dominating set of G_Φ is independent, that is, G_Φ is a Yes-instance for All Independent MD. There remains to show that G_Φ is $2P_3$-free. To see this, it suffices to observe that any induced P_3 of G_Φ contains at least one vertex in the clique K. This concludes the proof. □

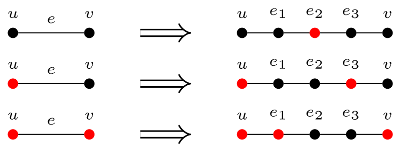


Fig. 3. Constructing a dominating set of H from the dominating set D of G (vertices in red belong to the corresponding dominating set).

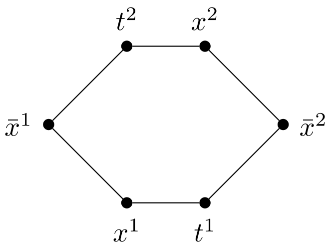


Fig. 4. The variable gadget G_x.

We now deduce the following from Theorem 3.6 and Fact 3.5.

Corollary 3.7. *1-Edge Contraction(γ) is* coNP*-hard on* $2P_3$*-free graphs.*

We next consider the class of subcubic planar graphs.

Theorem 3.8. All Independent MD *is* NP*-hard when restricted to subcubic planar graphs.*

Proof. We reduce from Planar Exactly 3-Bounded 3-Sat, where each variable appears in exactly three clauses and each clause contains at least two and at most three literals. This problem is shown to be NP-complete in [11]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Independent MD as follows. First note that we may assume that no variable occurs only positively or only negatively. Now consider the incidence graph G of Φ, that is, the bipartite graph with vertex set $X \cup C$ where $x \in X$ and $c \in C$ are adjacent if and only if c contains one of the literals x and $\bar{x}$ (note that by assumption G is planar). For any variable $x \in X$, let G_x be a copy of C_6 with two distinguished *positive literal vertices* x^1 and x^2, and two distinguished *negative literal vertices* $\bar{x}^1$ and $\bar{x}^2$ (see Fig. 4); in the following, we denote by t^1 and t^2 the two other vertices of G_x. We now replace the vertex x in G with G_x in such a way that every (positive or negative) literal vertex in G_x is adjacent to at most one vertex in C and x^i (resp. $\bar{x}^i$) for some $i = 1, 2$, is adjacent to some vertex $c \in C$ if and only if x occurs positively (resp. negatively) in c; note that this may be done so that planarity is preserved. We denote by G_Φ the resulting graph. Notice that $\Delta(G_\Phi) = 3$. Our goal is now to show that Φ is satisfiable if and only if G_Φ is a Yes-instance for All Independent MD. To this end, we first prove the following:

Observation 6. *For any dominating set D of G_Φ and any variable $x \in X$, $|D \cap V(G_x)| \geq 2$. In particular, $\gamma(G_\Phi) \geq 2|X|$.*

Indeed, as t^i must be dominated for $i = 1, 2$, it follows that $D \cap \{x^1, \bar{x}^2, t^1\} \neq \emptyset$ and $D \cap \{x^2, \bar{x}^1, t^2\} \neq \emptyset$. $\diamond$

Claim 5. *Φ is satisfiable if and only if $\gamma(G_\Phi) = 2|X|$.*

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. For any variable $x \in X$, if x is true, add x^1 and x^2 to D; otherwise, add $\bar{x}^1$ and $\bar{x}^2$ to D. Since every clause has at least one true literal, D is dominating. Thus, $\gamma(G_\Phi) \leq 2|X|$ and we conclude by Observation 6 that in fact, equality holds.

Conversely, assume that $\gamma(G_\Phi) = 2|X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 6, $|D \cap V(G_x)| = 2$ for any $x \in X$, which implies that no clause vertex belongs to D. Thus, for any $x \in X$, $D \cap V(G_x)$ is a minimum dominating set of G_x which implies that either $\{x^1, x^2\} \subset D$, $\{\bar{x}^1, \bar{x}^2\} \subset D$ or $\{t^1, t^2\} \subset D$; furthermore, every clause vertex is dominated by some (positive or negative) literal vertex. We may then construct a truth assignment satisfying Φ as follows: for any variable $x \in X$, if $\{x^1, x^2\} \subset D$, set x to true; otherwise, set x to false. $\blacktriangle$

Claim 6. *$\gamma(G_\Phi) = 2|X|$ if and only if every minimum dominating set of G_Φ is independent.*

Proof. Assume that $\gamma(G_\Phi) = 2|X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 6, $|D \cap V(G_x)| = 2$ for any $x \in X$ which implies in particular that no clause vertex belongs to D. It follows that for any $x \in X$, $D \cap V(G_x)$ is a minimum dominating set of G_x and thus, independent. Consequently, D is independent.

(a) The variable gadget G_x. (b) The clause gadget G_c.

Fig. 5. Construction of the graph G_Φ (the rectangle indicates that the corresponding set of vertices induces a clique).

Conversely, let D be a minimum dominating set of G_Φ (note that by assumption, D is independent). Suppose that there exists a clause $c \in C$ such that the corresponding clause vertex belongs to D and denote by x_1, x_2 and x_3 the variables contained in c (a similar reasoning would apply if c contained only two variables). Since D is independent, none of the (positive or negative) literal vertices adjacent to c belong to D. Now, assume without loss of generality that x_1^1 is adjacent to c. Then, we may assume the x_1^1 has a neighbor in $V(G_{x_1})$ belonging to D. Indeed, suppose that $t^1, \bar{x_1}^1 \notin D$. Then, necessarily $\bar{x_1}^2 \in D$ for otherwise t^1 would not be dominated; and since D is independent and t^2 must be dominated, $x_1^2 \notin D$ which implies that $t^2 \in D$. But then, it suffices to consider $(D \setminus \{t^2\}) \cup \{\bar{x_1}^1\}$ so that x_1^1 is dominated by some vertex in $V(G_{x_1})$. Note that if $\bar{x_1}^1$ is dominated by some clause vertex in the set $(D \setminus \{t^2\}) \cup \{\bar{x_1}^1\}$, we immediately reach a contradiction. Similarly, we may assume that the other literal vertices adjacent to c are dominated by some vertex in their gadget. But then, $(D \setminus \{c\}) \cup \{x_1^1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. Thus, no clause vertex belongs to D. Now, suppose that there exists $x \in X$ such that $|D \cap V(G_x)| \geq 3$. Then, since D is independent, there exists $i \in \{1, 2\}$ such that $t^i \in D$, say $t^1 \in D$ without loss of generality. It then follows that $x^1, \bar{x}^2 \notin D$ and since $|D \cap V(G_x)| \geq 3$, necessarily $x^2, \bar{x}^1 \in D$ and $t^2 \notin D$ (D would otherwise contain an edge). But then, $(D \setminus \{t^1\}) \cup \{x^1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. Thus, for any $x \in X$, $|D \cap V(G_x)| \leq 2$ and we conclude by Observation 6 that $\gamma(G_\Phi) = 2|X|$. ▲

Now by combining Claims 5 and 6, we obtain that Φ is satisfiable if and only if every minimum dominating set of G_Φ is independent i.e., G_Φ is aYes-instance for All Independent MD. □

We now deduce the following from Theorem 3.8 and Fact 3.5.

Theorem 3.9. *1-Edge Contraction(γ) is* coNP*-hard when restricted to subcubic planar graphs.*

Finally, we consider the class of subcubic claw-free graphs and show that when restricted to this graph class, 1-Edge Contraction(γ) is coNP-hard. To this end, we first prove that All Efficient MD is NP-hard when restricted to subcubic graphs and then use this result to show that All Independent MD is NP-hard when restricted to subcubic claw-free graphs.

Lemma 3.10. All Efficient MD *is* NP*-hard when restricted to subcubic graphs.*

Proof. We reduce from Positive Exactly 3-Bounded 1-In-3 3-Sat, where each variable appears in exactly three clauses and only positively, each clause contains three positive literals, and we want a truth assignment such that each clause contains exactly one true literal. This problem is shown to be NP-complete in [22]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Efficient MD as follows. For any variable $x \in X$, we introduce a copy of C_9, which we denote by G_x, with three distinguished *true vertices* T_x^1, T_x^2 and T_x^3, and three distinguished *false vertices* F_x^1, F_x^2 and F_x^3 (see Fig. 5(a)). For any clause $c \in C$ containing variables x_1, x_2 and x_3, we introduce the gadget G_c depicted in Fig. 5(b) which has one distinguished *clause vertex* c and three distinguished *variable vertices* x_1, x_2 and x_3 (note that G_c is not connected). For every $j \in \{1, 2, 3\}$, we then add an edge between x_j and $F_{x_j}^i$ and between c and $T_{x_j}^i$ for some $i \in \{1, 2, 3\}$ so that $F_{x_j}^i$ (resp. $T_{x_j}^i$) is adjacent to exactly one variable vertex (resp. clause vertex). We denote by G_Φ the resulting graph. Note that $\Delta(G_\Phi) = 3$.

Observation 7. *For any dominating set D of G_Φ, $|D \cap V(G_x)| \geq 3$ for any $x \in X$ and $|D \cap V(G_c)| \geq 1$ for any $c \in C$. In particular, $\gamma(G_\Phi) \geq 3|X| + |C|$.*

Indeed, for any $x \in X$, since u_x^1, u_x^2 and u_x^3 must be dominated and their neighborhoods are pairwise disjoint and contained in G_x, it follows that $|D \cap V(G_x)| \geq 3$. For any $c \in C$, since the vertices of K_c must be dominated and their neighborhoods are contained in G_c, $|D \cap V(G_c)| \geq 1$. ◇

Observation 8. *For any $x \in X$, if D is a minimum dominating set of G_x then either $D = \{u_x^1, u_x^2, u_x^3\}$, $D = \{T_x^1, T_x^2, T_x^3\}$ or $D = \{F_x^1, F_x^2, F_x^3\}$.*

Claim 7. *Φ is satisfiable if and only if $\gamma(G_\Phi) = 3|X| + |C|$.*

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. For any variable $x \in X$, if x is true, add T_x^1, T_x^2 and T_x^3 to D; otherwise, add F_x^1, F_x^2 and F_x^3 to D. For any clause $c \in C$ containing variables x_1, x_2 and x_3, exactly one variable is true, say x_1 without loss of generality; we then add $l_{\{x_1\}}$ to D. Clearly, D is dominating and we conclude by Observation 7 that $\gamma(G_\Phi) = 3|X| + |C|$.

Conversely, assume that $\gamma(G_\Phi) = 3|X| + |C|$ and consider a minimum dominating set D of G_Φ. Then by Observation 7, $|D \cap V(G_x)| = 3$ for any $x \in X$ and $|D \cap V(G_c)| = 1$ for any $c \in C$. Now, for a clause $c \in C$ containing variables x_1, x_2 and x_3, if $D \cap \{c, x_1, x_2, x_3\} \neq \emptyset$ then $D \cap V(K_c) = \emptyset$ and so, at least two vertices from K_c are not dominated; thus, $D \cap \{c, x_1, x_2, x_3\} = \emptyset$. It follows that for any $x \in X$, $D \cap V(G_x)$ is a minimum dominating set of G_x which by Observation 8 implies either $\{T_x^1, T_x^2, T_x^3\} \subset D$ or $D \cap \{T_x^1, T_x^2, T_x^3\} = \emptyset$; and we conclude similarly that either $\{F_x^1, F_x^2, F_x^3\} \subset D$ or $D \cap \{F_x^1, F_x^2, F_x^3\} = \emptyset$. Now given a clause $c \in C$ containing variables x_1, x_2 and x_3, since $D \cap \{c, x_1, x_2, x_3\} = \emptyset$, at least one true vertex adjacent to the clause vertex c must belong to D, say $T_{x_1}^i$ for some $i \in \{1, 2, 3\}$ without loss of generality. It then follows that $\{T_{x_1}^1, T_{x_1}^2, T_{x_1}^3\} \subset D$ and $D \cap \{F_{x_1}^1, F_{x_1}^2, F_{x_1}^3\} = \emptyset$ which implies that $l_{\{x_1\}} \in D$ (either x_1 or a vertex from K_c would otherwise not be dominated). But then, since x_j for $j \neq 1$, must be dominated, it follows that $\{F_{x_j}^1, F_{x_j}^2, F_{x_j}^3\} \subset D$. We thus construct a truth assignment satisfying Φ as follows: for any variable $x \in X$, if $\{T_x^1, T_x^2, T_x^3\} \subset D$, set x to true, otherwise set x to false. ▲

Claim 8. *$\gamma(G_\Phi) = 3|X| + |C|$ if and only if every minimum dominating set of G_Φ is efficient.*

Proof. Assume that $\gamma(G_\Phi) = 3|X| + |C|$ and consider a minimum dominating set D of G_Φ. Then by Observation 7, $|D \cap V(G_x)| = 3$ for any $x \in X$ and $|D \cap V(G_c)| = 1$ for any $c \in C$. As shown previously, it follows that for any clause $c \in C$ containing variables x_1, x_2 and x_3, $D \cap \{c, x_1, x_2, x_3\} = \emptyset$; and for any $x \in X$, either $\{T_x^1, T_x^2, T_x^3\} \subset D$ or $D \cap \{T_x^1, T_x^2, T_x^3\} = \emptyset$ (we conclude similarly with $\{F_x^1, F_x^2, F_x^3\}$ and $\{u_x^1, u_x^2, u_x^3\}$). Thus, for any $x \in X$, every vertex in G_x is dominated by exactly one vertex. Now given a clause $c \in C$ containing variables x_1, x_2 and x_3, since the clause vertex c does not belong to D, there exists at least one true vertex adjacent to c which belongs to D. Suppose to the contrary that c has strictly more than one neighbor in D, say $T_{x_1}^i$ and $T_{x_2}^j$ without loss of generality. Then, $\{T_{x_k}^1, T_{x_k}^2, T_{x_k}^3\} \subset D$ for $k = 1, 2$ which implies that $D \cap \{F_{x_1}^1, F_{x_1}^2, F_{x_1}^3, F_{x_2}^1, F_{x_2}^2, F_{x_2}^3\} = \emptyset$ as $|D \cap V(G_{x_k})| = 3$ for $k = 1, 2$. It follows that the variable vertices x_1 and x_2 must be dominated by some vertices in G_c; but $|D \cap V(G_c)| = 1$ and $N[x_1] \cap N[x_2] = \emptyset$ and so, either x_1 or x_2 is not dominated. Thus, c has exactly one neighbor in D, say $T_{x_1}^i$ without loss of generality. Then, necessarily $D \cap V(G_c) = \{l_{\{x_1\}}\}$ for otherwise either x_1 or some vertex in K_c would not be dominated. But then, it is clear that every vertex in G_c is dominated by exactly one vertex; thus, D is efficient.

Conversely, assume that every minimum dominating set of G_Φ is efficient and consider a minimum dominating set D of G_Φ. If for some $x \in X$, $|D \cap V(G_x)| \geq 4$, then clearly at least one vertex in G_x is dominated by two vertices in $D \cap V(G_x)$. Thus, $|D \cap V(G_x)| \leq 3$ for any $x \in X$ and we conclude by Observation 7 that in fact, equality holds. The next observation immediately follows from the fact that D is efficient.

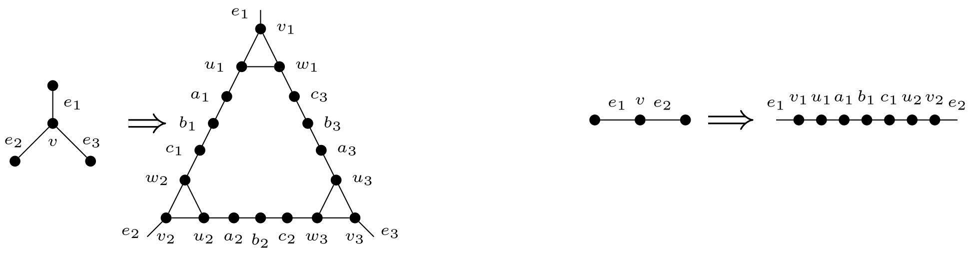
Observation 9. *For any $x \in X$, if $|D \cap V(G_x)| = 3$ then either $\{u_x^1, u_x^2, u_x^3\} \subset D$, $\{T_x^1, T_x^2, T_x^3\} \subset D$ or $\{F_x^1, F_x^2, F_x^3\} \subset D$.*

Now, consider a clause $c \in C$ containing variables x_1, x_2 and x_3 and suppose without loss of generality that $T_{x_1}^1$ is adjacent to c (note that then the variable vertex x_1 is adjacent to $F_{x_1}^1$). If the clause vertex c belongs to D then, since D is efficient, $T_{x_1}^1 \notin D$ and $u_{x_1}^1, F_{x_1}^1 \notin D$ ($T_{x_1}^1$ would otherwise be dominated by at least two vertices) which contradicts Observation 9. Thus, no clause vertex belongs to D. Similarly, suppose that there exists $i \in \{1, 2, 3\}$ such that $x_i \in D$, say $x_1 \in D$ without loss of generality. Then, since D is efficient, $F_{x_1}^1 \notin D$ and $T_{x_1}^1, u_{x_1}^2 \notin D$ ($F_{x_1}^1$ would otherwise be dominated by at least two vertices) which again contradicts Observation 9. Thus, no variable vertex belongs to D. Finally, since D is efficient, $|D \cap V(K_c)| \leq 1$ and so, $|D \cap V(G_c)| = 1$ by Observation 7. ▲

Now by combining Claims 7 and 8, we obtain that Φ is satisfiable if and only if every minimum dominating set of G_Φ is efficient, that is, G_Φ is a Yes-instance for All Efficient MD. □

Theorem 3.11. All Independent MD *is* NP*-hard when restricted to subcubic claw-free graphs.*

Proof. As in the proof of Lemma 3.10, we use a reduction from Positive Exactly 3-Bounded 1-In-3 3-Sat. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Independent MD as follows. Consider the graph $G_\Phi = (V, E)$ constructed in the proof of Lemma 3.10 and let $V_i = \{v \in V : d_{G_\Phi}(v) = i\}$ for $i = 2, 3$ (note that no vertex in G_Φ has degree one). Then, for any $v \in V_3$, we replace the vertex v by the gadget G_v depicted in Fig. 6(a); and for any $v \in V_2$, we replace the vertex v by the gadget G_v depicted in Fig. 6(b). We denote by G'_Φ the resulting graph. Note that G'_Φ is claw-free and $\Delta(G'_\Phi) = 3$ (also note



(a) $d_{G_\Phi}(v) = 3$.   **(b)** $d_{G_\Phi}(v) = 2$.

Fig. 6. The gadget G_v.

that no vertex in G'_Φ has degree one). It is shown in the proof of Lemma 3.10 that Φ is satisfiable if and only if G_Φ is a Yes-instance for All Efficient MD; we here show that G_Φ is a Yes-instance for All Efficient MD if and only if G'_Φ is a Yes-instance for All Independent MD. To this end, we first prove the following:

Claim 9. $\gamma(G'_\Phi) = \gamma(G_\Phi) + 5|V_3| + 2|V_2|$.

Proof. Let D be a minimum dominating set of G_Φ. We construct a dominating set D' of G'_Φ as follows. For any $v \in D$, if $v \in V_3$, add v_1, v_2, v_3, b_1, b_2, and b_3 to D'; otherwise, add v_1, v_2 and b_1 to D'. For any $v \in V \setminus D$, let $u \in D$ be a neighbor of v, say $e_1 = uv$ without loss of generality. Then, if $v \in V_3$, add a_1, c_3, w_2, u_3 and b_2 to D'; otherwise, add a_1 and u_2 to D'. Clearly, D' is dominating and $|D'| = \gamma(G_\Phi) + 5|V_3| + 2|V_2| \geq \gamma(G'_\Phi)$.

Observation 10. *For any dominating set D' of G'_Φ, the following hold:*

(i) For any $v \in V_2$, $|D' \cap V(G_v)| \geq 2$. Moreover, if equality holds then $D' \cap \{v_1, v_2\} = \emptyset$ and there exists $j \in \{1, 2\}$ such that $u_j \notin D'$.
(ii) For any $v \in V_3$, $|D' \cap V(G_v)| \geq 5$. Moreover, if equality holds then $D' \cap \{v_1, v_2, v_3\} = \emptyset$ and there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$.

(i) Clearly, $D' \cap \{v_1, u_1, a_1\} \neq \emptyset$ and $D' \cap \{c_1, u_2, v_2\} \neq \emptyset$ as u_1 and u_2 must be dominated. Thus, $|D' \cap V(G_v)| \geq 2$. Now, suppose that $D' \cap \{v_1, v_2\} \neq \emptyset$ say $v_1 \in D'$ without loss of generality. Then $D' \cap \{u_1, a_1, b_1\} \neq \emptyset$ as a_1 must be dominated which implies that $|D' \cap V(G_v)| \geq 3$ (recall that $D' \cap \{c_1, u_2, v_2\} \neq \emptyset$). Similarly, if both u_1 and u_2 belong to D', then $|D' \cap V(G_v)| \geq 3$ as $D' \cap \{a_1, b_1, c_1\} \neq \emptyset$ (b_1 would otherwise not be dominated).

(ii) Clearly, for any $i \in \{1, 2, 3\}$, $D' \cap \{a_i, b_i, c_i\} \neq \emptyset$ as b_i must be dominated. Now, if there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality, then $a_1, c_3 \in D'$ (one of u_1 and w_1 would otherwise not be dominated). But then, $D' \cap \{b_1, c_1, w_2\} \neq \emptyset$ as c_1 must be dominated, and $D' \cap \{a_3, b_3, u_3\} \neq \emptyset$ as a_3 must be dominated; and so, $|D' \cap V(G_v)| \geq 5$ (recall that $D' \cap \{a_2, b_2, c_2\} \neq \emptyset$). Otherwise, for any $j \in \{1, 2, 3\}$, $D' \cap \{u_j, v_j, w_j\} \neq \emptyset$ which implies that $|D' \cap V(G_v)| \geq 6$.

Now suppose that $D' \cap \{v_1, v_2, v_3\} \neq \emptyset$, say $v_1 \in D'$ without loss of generality. If there exists $j \neq 1$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 2$ without loss of generality, then $c_1, a_2 \in D'$ (one of u_2 and w_2 would otherwise not be dominated). But then, $D' \cap \{a_1, b_1, u_1\} \neq \emptyset$ as a_1 should be dominated, and $D' \cap \{b_2, c_2, w_3\} \neq \emptyset$ as c_2 must be dominated. Since $D' \cap \{a_3, b_3, c_3\} \neq \emptyset$, it then follows that $|D' \cap V(G_v)| \geq 6$. Otherwise, $D' \cap \{u_j, v_j, w_j\} \neq \emptyset$ for any $j \in \{1, 2, 3\}$ and so, $|D' \cap V(G_v)| \geq 6$ (recall that $D' \cap \{a_i, b_i, c_i\} \neq \emptyset$ for any $i \in \{1, 2, 3\}$). $\diamond$

Observation 11. *If D' is a minimum dominating set of G'_Φ, then $|D' \cap V(G_v)| \leq 3$ for any $v \in V_2$ and $|D' \cap V(G_v)| \leq 6$ for any $v \in V_3$.*

Indeed, if $v \in V_2$ then $\{v_1, b_1, v_2\}$ is a dominating set of $V(G_v)$; and if $v \in V_3$, then $\{v_1, v_2, v_3, b_1, b_2, b_3\}$ is a dominating set of $V(G_v)$. $\diamond$

Now, consider a minimum dominating set D' of G'_Φ and let $D_3 = \{v \in V_3 : |D' \cap V(G_v)| = 6\}$ and $D_2 = \{v \in V_2 : |D' \cap V(G_v)| = 3\}$. We claim that $D = D_3 \cup D_2$ is a dominating set of G_Φ. Indeed, consider a vertex $v \in V \setminus D$. We distinguish two cases depending on whether $v \in V_2$ of $v \in V_3$.

Case 1. $v \in V_2$. Then $|D' \cap V(G_v)| = 2$ by construction, which by Observation 10(i) implies that there exists $j \in \{1, 2\}$ such that $D' \cap \{v_j, u_j\} = \emptyset$, say $j = 1$ without loss of generality. Since v_1 must be dominated, v_1 must then have a neighbor x_i belonging to D', for some vertex x adjacent to v in G_Φ. But then, it follows from Observation 10 that $|D' \cap V(G_x)| > 2$ if $x \in V_2$, and $|D' \cap V(G_x)| > 5$ if $x \in V_3$ (indeed, $x_i \in D'$); thus, $x \in D$.

Case 2. $v \in V_3$. Then $|D' \cap V(G_v)| = 5$ by construction, which by Observation 10(ii) implies that there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality. Since v_1 must be dominated, v_1 must then have a neighbor x_i belonging to D', for some vertex x adjacent to v in G_Φ. But then, it follows from Observation 10 that $|D' \cap V(G_x)| > 2$ if $x \in V_2$, and $|D' \cap V(G_x)| > 5$ if $x \in V_3$ (indeed, $x_i \in D'$); thus, $x \in D$.

Hence, D is a dominating set of G_Φ. Moreover, it follows from Observations 10 and 11 that $|D'| = 6|D_3| + 5|V_3 \setminus D_3| + 3|D_2| + 2|V_2 \setminus D_2| = |D| + 5|V_3| + 2|V_2|$. Thus, $\gamma(G'_\Phi) = |D'| \geq \gamma(G_\Phi) + 5|V_3| + 2|V_2|$ and so, $\gamma(G'_\Phi) = \gamma(G_\Phi) + 5|V_3| + 2|V_2|$. Finally note that this implies that the constructed dominated set D is in fact minimum. ▲

We next show that G_Φ is a Yes-instance for All Efficient MD if and only if G'_Φ is a Yes-instance for All Independent MD. Since Φ is satisfiable if and only if G_Φ is a Yes-instance for All Efficient MD, as shown in the proof of Lemma 3.10, this would conclude the proof.

Assume first that G_Φ is a Yes-instance for All Efficient MD and suppose to the contrary that G'_Φ is a No-instance for All Independent MD that is, G'_Φ has a minimum dominating set D' which is not independent. Denote by D the minimum dominating set of G_Φ constructed from D' according to the proof of Claim 9. Let us show that D is not efficient. Consider two adjacent vertices $a, b \in D'$. If a and b belong to gadgets G_x and G_v respectively, for two adjacent vertices x and v in G_Φ, that is, a is of the form x_i and b is of the form v_j, then by Observation 10 $x, v \in D$ and so, D is not efficient. Thus, it must be that a and b both belong the same gadget G_v, for some $v \in V_2 \cup V_3$. We distinguish cases depending on whether $v \in V_2$ or $v \in V_3$.

Case 1. $v \in V_2$. Suppose that $|D' \cap V(G_v)| = 2$. Then by Observation 10(i), $D' \cap \{v_1, v_2\} = \emptyset$ and there exists $j \in \{1, 2\}$ such that $u_j \notin D'$, say $u_1 \notin D'$ without loss of generality. Then, necessarily $a_1 \in D'$ (u_1 would otherwise not be dominated) and so, $b_1 \in D'$ as $D' \cap V(G_v)$ contains an edge and $|D' \cap V(G_v)| = 2$ by assumption; but then, u_2 is not dominated. Thus, $|D' \cap V(G_v)| \geq 3$ and we conclude by Observation 11 that in fact, equality holds. Note that consequently, $v \in D$. We claim that then, $|D' \cap \{v_1, v_2\}| \leq 1$. Indeed, if both v_1 and v_2 belong to D', then $b_1 \in D'$ (since $|D' \cap V(G_v)| = 3$, D' would otherwise not be dominating) which contradicts that fact that $D' \cap V(G_v)$ contains an edge. Thus, $|D' \cap \{v_1, v_2\}| \leq 1$ and we may assume without loss of generality that $v_2 \notin D'$. Let $x_i \neq u_2$ be the other neighbor of v_2 in G'_Φ, where x is a neighbor of v in G_Φ.

Suppose first that $x \in V_2$. Then, $|D' \cap V(G_x)| = 2$ for otherwise x would belong to D and so, D would contain the edge vx. It then follows from Observation 10(i) that there exists $j \in \{1, 2\}$ such that $D' \cap \{x_j, y_j\} = \emptyset$, where y_j is the neighbor of x_j in $V(G_x)$. We claim that $j \neq i$; indeed, if $j = i$, since $v_2, x_i, y_i \notin D'$, x_i would not be dominated. But then, x_j must have a neighbor $t_k \neq y_j$, for some vertex t adjacent to x in G_Φ, which belongs to D'; it then follows from Observation 10 and the construction of D that $t \in D$ and so, x has two neighbors in D, namely v and t, a contradiction.

Second, suppose that $x \in V_3$. Then, $|D' \cap V(G_x)| = 5$ for otherwise x would belong to D and so, D would contain the edge vx. It then follows from Observation 10(ii) that there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{x_j, y_j, z_j\} = \emptyset$, where y_j and z_j are the two neighbors of x_j in $V(G_x)$. We claim that $j \neq i$; indeed, if $j = i$, since $v_2, x_i, y_i, z_i \notin D'$, x_i would not be dominated. But then, x_j must have a neighbor $t_k \neq y_j, z_j$, for some vertex t adjacent to x in G_Φ, which belongs to D'; it then follows from Observation 10 and the construction of D that $t \in D$ and so, x has two neighbors in D, namely v and t, a contradiction.

Case 2. $v \in V_3$. Suppose that $|D' \cap V(G_v)| = 5$. Then, by Observation 10(ii), $D' \cap \{v_1, v_2, v_3\} = \emptyset$ and there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality. Then, $a_1, c_3 \in D'$ (one of u_1 and w_1 would otherwise not be dominated), $D' \cap \{c_1, w_2, u_2\} \neq \emptyset$ (w_2 would otherwise not be dominated), $D' \cap \{a_3, u_3, w_3\} \neq \emptyset$ (u_3 would otherwise not be dominated) and $D' \cap \{a_2, b_2, c_2\} \neq \emptyset$ (b_2 would otherwise not be dominated); in particular, $b_1, b_3 \notin D'$ as $|D' \cap V(G_v)| = 5$ by assumption. Since $D' \cap V(G_v)$ contains an edge, it follows that either $u_2, a_2 \in D'$ or $c_2, w_3 \in D'$; but then, either c_1 or a_3 is not dominated, a contradiction. Thus, $|D' \cap V(G_v)| \geq 6$ and we conclude by Observation 11 that in fact, equality holds. Note that consequently, $v \in D$. It follows that $\{v_1, v_2, v_3\} \not\subset D'$ for otherwise $D' \cap V(G_v) = \{v_1, v_2, v_3, b_1, b_2, b_3\}$ and so, $D' \cap V(G_v)$ contains no edge. Thus, we may assume without loss of generality that $v_1 \notin D'$. Denoting by $x_i \neq u_1, w_1$ the third neighbor of v_1, where x is a neighbor of v in G_Φ, we then proceed as in the previous case to conclude that x has two neighbors in D.

Thus, D is not efficient, which contradicts the fact that G_Φ is a Yes-instance for All Efficient MD. Hence, every minimum dominating set of G'_Φ is independent i.e., G'_Φ is a Yes-instance for All Independent MD.

Conversely, assume that G'_Φ is a Yes-instance for All Independent MD and suppose to the contrary that G_Φ is a No-instance for All Efficient MD that is, G_Φ has a minimum dominating set D which is not efficient. Let us show that D either contains an edge or can be transformed into a minimum dominating set of G_Φ containing an edge. Since any minimum dominating of G'_Φ constructed according to the proof of Claim 9 from a minimum dominating set of G_Φ containing an edge, also contains an edge, this would lead to a contradiction and thus conclude the proof.

Suppose that D contains no edge. Since D is not efficient, there must then exist a vertex $v \in V \setminus D$ such that v has two neighbors in D. We distinguish cases depending on which type of vertex v is.

Case 1. *v is a variable vertex.* Suppose that $v = x_1$ in some clause gadget G_c, where $c \in C$ contains variables x_1, x_2 and x_3, and assume without loss of generality that x_1 is adjacent to $F^1_{x_1}$. By assumption, $F^1_{x_1}, l_{\{x_1\}} \in D$ which implies that $D \cap \{l_{\{x_2\}}, l_{\{x_3\}}, T^1_{x_1}, u^2_{x_1}\} = \emptyset$ (D would otherwise contain an edge). We may then assume that $F^i_{x_2}$ and $F^j_{x_3}$, where $F^i_{x_2}x_2, F^j_{x_3}x_3 \in E(G_\Phi)$, belong to D; indeed, since x_2 (resp. x_3) must be dominated, $D \cap \{F^i_{x_2}, x_2\} \neq \emptyset$ (resp. $D \cap \{F^j_{x_3}, x_3\} \neq \emptyset$) and since $l_{\{x_1\}} \in D$, $(D \setminus \{x_2\}) \cup \{F^i_{x_2}\}$ (resp. $(D \setminus \{x_3\}) \cup \{F^j_{x_3}\}$) remains dominating. We may then assume that $T^i_{x_2}, T^j_{x_3} \notin D$

for otherwise D would contain an edge. It follows that $c \in D$ (c would otherwise not be dominated); but then, it suffices to consider $(D \setminus \{c\}) \cup \{T^1_{x_1}\}$ to obtain a minimum dominating set of G_Φ containing an edge.

Case 2. *$v = u^i_x$ for some variable $x \in X$ and $i \in \{1, 2, 3\}$.* Assume without loss of generality that $i = 1$. Then $T^1_x, F^3_x \in D$ by assumption, which implies that $F^1_x, T^3_x \notin D$ (D would otherwise contain an edge). But then, $|D \cap \{u^2_x, F^2_x, T^2_x, u^3_x\}| \geq 2$ as u^2_x and u^3_x must be dominated; and so, $(D \setminus \{u^3_x, F^2_x, T^2_x, u^2_x\}) \cup \{F^2_x, T^2_x\}$ is a dominating set of G_Φ of size at most that of D which contains an edge.

Case 3. *v is a clause vertex.* Suppose that $v = c$ for some clause $c \in C$ containing variables x_1, x_2 and x_3, and assume without loss of generality that c is adjacent to $T^1_{x_i}$ for any $i \in \{1, 2, 3\}$. By assumption c has two neighbors in D, say $T^1_{x_1}$ and $T^1_{x_2}$ without loss of generality. Since D contains no edge, it follows that $F^1_{x_1}, F^1_{x_2} \notin D$; but then, $|D \cap \{x_1, x_2, l_{\{x_1\}}, l_{\{x_2\}}\}| \geq 2$ (one of x_1 and x_2 would otherwise not be dominated) and so, $(D \setminus \{x_1, x_2, l_{\{x_1\}}, l_{\{x_2\}}\}) \cup \{l_{\{x_1\}}, l_{\{x_2\}}\}$ is a dominating set of G_Φ of size at most that of D which contains an edge.

Case 4. *$v \in V(K_c)$ for some clause $c \in C$.* Denote by x_1, x_2 and x_3 the variables contained in c and assume without loss of generality that $v = l_{\{x_1\}}$. Since $l_{\{x_1\}}$ has two neighbors in D and D contains no edge, necessarily $x_1 \in D$. Now assume without loss of generality that x_1 is adjacent to $F^1_{x_1}$ (note that by construction, c is then adjacent to $T^1_{x_1}$). Then, $F^1_{x_1} \notin D$ (D would otherwise contain an edge) and $T^1_{x_1}, u^2_{x_1} \notin D$ for otherwise $(D \setminus \{x_1\}) \cup \{F^1_{x_1}\}$ would be a minimum dominating set of G_Φ containing an edge (recall that by assumption, $D \cap V(K_c) \neq \emptyset$). It follows that $T^2_{x_1} \in D$ ($u^2_{x_1}$ would otherwise not be dominated) and so, $F^2_{x_1} \notin D$ as D contains no edge. It follows that $|D \cap \{u^1_{x_1}, F^3_{x_1}, T^3_{x_1}, u^3_{x_1}\}| \geq 2$ as $u^1_{x_1}$ and $u^3_{x_1}$ must be dominated. Now if c belongs to D, then $(D \setminus \{u^1_{x_1}, F^3_{x_1}, T^3_{x_1}, u^3_{x_1}\}) \cup \{F^3_{x_1}, T^3_{x_1}\}$ is a dominating set of G_Φ of size at most that of D which contains an edge. Thus, we may assume that $c \notin D$ which implies that $u^1_{x_1} \in D$ ($T^1_{x_1}$ would otherwise not be dominated) and that there exists $j \in \{2, 3\}$ such that $T^i_{x_j} \in D$ with $cT^i_{x_j} \in E(G_\Phi)$ (c would otherwise not be dominated). Now, since $u^3_{x_1}$ must be dominated and $F^2_{x_1} \notin D$, it follows that $D \cap \{u^3_{x_1}, T^3_{x_1}\} \neq \emptyset$ and we may assume that in fact $T^3_{x_1} \in D$ (recall that $T^2_{x_1} \in D$ and so, $F^2_{x_1}$ is dominated). But then, by considering the minimum dominating set $(D \setminus \{u^1_{x_1}\}) \cup \{T^1_{x_1}\}$, we fall back into Case 3 as c is then dominated by both $T^1_{x_1}$ and $T^i_{x_j}$.

Case 5. *v is a true vertex.* Assume without loss of generality that $v = T^1_x$ for some variable $x \in X$. Suppose first that $u^1_x \in D$. Then since D contains no edge, $F^3_x \notin D$; furthermore, denoting by $t \neq u^1_x, T^3_x$ the variable vertex adjacent to F^3_x, we also have $t \notin D$ for otherwise $(D \setminus \{u^1_x\}) \cup \{F^3_x\}$ would be a minimum dominating set containing an edge (recall that T^1_x has two neighbors in D by assumption). But then, since t must be dominated, it follows that the second neighbor of t must belong to D; and so, by considering the minimum dominating set $(D \setminus \{u^1_x\}) \cup \{F^3_x\}$, we fall back into Case 1 as the variable vertex t is then dominated by two vertices. Thus, we may assume that $u^1_x \notin D$ which implies that $F^1_x, c \in D$, where c is the clause vertex adjacent to T^1_x. Now, denote by $x_1 = x$, x_2 and x_3 the variables contained in c (note that by construction, x_1 is then adjacent to $F^1_{x_1}$). Then, $x_1 \notin D$ (D would otherwise contain the edge $F^1_{x_1}x_1$) and we may assume that $l_{\{x_1\}} \notin D$ (we otherwise fall back into Case 1 as x_1 would then have two neighbors in D). It follows that $D \cap V(K_c) \neq \emptyset$ ($l_{\{x_1\}}$ would otherwise not be dominated) and since D contains no edge, in fact $|D \cap V(K_c)| = 1$, say $l_{\{x_2\}} \in D$ without loss of generality. Then, $x_2 \notin D$ as D contains no edge and we may assume that $F^j_{x_2} \notin D$, where $F^j_{x_2}$ is the false vertex adjacent to x_2, for otherwise we fall back into Case 1. In the following, we assume without loss of generality that $j = 1$, that is, x_2 is adjacent to $F^1_{x_2}$ (note that by construction, c is then adjacent to $T^1_{x_2}$). Now, since the clause vertex c belongs to D by assumption, it follows that $T^1_{x_2} \notin D$ (D would otherwise contain the edge $cT^1_{x_2}$); and as shown previously, we may assume that $u^1_{x_2} \notin D$ (indeed, $T^1_{x_2}$ would otherwise have two neighbors in D, namely c and $u^1_{x_2}$, but this case has already been dealt with). Then, since $u^1_{x_2}$ and $F^1_{x_2}$ must be dominated, necessarily $F^3_{x_2}$ and $u^2_{x_2}$ belong to D (recall that $D \cap \{x_2, F^1_{x_2}, T^1_{x_2}, u^1_{x_2}\} = \emptyset$) which implies that $T^3_{x_2}, T^2_{x_2} \notin D$ (D would otherwise contain an edge). Now since $u^3_{x_2}$ must be dominated, $D \cap \{u^3_{x_2}, F^2_{x_2}\} \neq \emptyset$ and we may assume without loss of generality that in fact, $F^2_{x_2} \in D$. But then, by considering the minimum dominating set $(D \setminus \{u^2_{x_2}\}) \cup \{F^1_{x_2}\}$, we fall back into Case 1 as x_2 is then dominated by two vertices.

Case 6. *v is a false vertex.* Assume without loss of generality that $v = F^1_{x_1}$ for some variable $x_1 \in X$ and let $c \in C$ be the clause whose corresponding clause vertex is adjacent to $T^1_{x_1}$. Denote by x_2 and x_3 the two other variables contained in c. Suppose first that $x_1 \in D$. Then, we may assume that $D \cap V(K_c) = \emptyset$ for otherwise either D contains an edge (if $l_{\{x_1\}} \in D$) or we fall back into Case 4 ($l_{\{x_1\}}$ would indeed have two neighbors in D). Since every vertex of K_c must be dominated, it then follows that $x_2, x_3 \in D$; but then, by considering the minimum dominating set $(D \setminus \{x_1\}) \cup \{l_{\{x_1\}}\}$ (recall that $F^1_{x_1}$ has two neighbors in D by assumption), we fall back into Case 4 as $l_{\{x_2\}}$ is then dominated by two vertices. Thus, we may assume that $x_1 \notin D$ which implies that $T^1_{x_1}, u^2_{x_1} \in D$ and $T^2_{x_1}, u^1_{x_1} \notin D$ as D contains no edge. Now, denote by c' the clause vertex adjacent to $T^2_{x_1}$. Then, we may assume that $c' \notin D$ for otherwise we fall back into Case 5 ($T^2_{x_1}$ would indeed have two neighbors in D); but then, there must exist a true vertex, different from $T^2_{x_1}$, adjacent to c' and belonging to D (c' would otherwise not be dominated) and by considering the minimum dominating set $(D \setminus \{u^2_{x_1}\}) \cup \{T^2_{x_1}\}$, we then fall back into Case 3 (c' would indeed be dominated by two vertices).

Consequently, G_Φ has a minimum dominating set which is not independent which implies that G'_Φ also has a minimum dominating set which is not independent, a contradiction which concludes the proof. □

The following is now a straightforward consequence of Theorem 3.11 and Fact 3.5.

Corollary 3.12. *1-Edge Contraction(γ) is* coNP*-hard on subcubic claw-free graphs.*

To conclude this section, we observe that even if an edge is given, deciding whether contracting this particular edge decreases the domination number is unlikely to be polynomial-time solvable, as shown in the following result.

Theorem 3.13. *There exists no polynomial-time algorithm deciding whether contracting a given edge decreases the domination number, unless* P = NP.

Proof. We denote by Edge Contraction(γ) the problem that takes as an input a graph $G = (V, E)$ and an edge $e \in E$, and asks whether $\gamma(G \backslash e) \leq \gamma(G) - 1$. We show that if Edge Contraction(γ) can be solved in polynomial time, then Dominating Set can also be solved in polynomial time. Since Dominating Set is a well-known NP-complete problem, the result follows:

Let (G, ℓ) be an instance for Dominating Set and let e be an edge of G. We run the polynomial time algorithm for Edge Contraction(γ) to determine if $\gamma(G \setminus e) = \gamma(G) - 1$; we then have two possible scenarios.

Case 1. *(G, e) is a* Yes*-instance for* Edge Contraction*(γ).* Since $\gamma(G \setminus e) = \gamma(G) - 1$, we know that G has a dominating set of size ℓ if and only if $G \setminus e$ has a dominating set of size $\ell - 1$. Hence, we obtain that $(G \setminus e, \ell - 1)$ is an equivalent instance for Dominating Set.

Case 2. *(G, e) is a* No*-instance for* Edge Contraction*(γ).* Since $\gamma(G \setminus e) = \gamma(G)$, we know that G has a dominating set of size ℓ if and only if $G \setminus e$ has a dominating set of size ℓ. In this case, we obtain that $(G \setminus e, \ell)$ is an equivalent instance for Dominating Set.

In both cases, the ensuing equivalent instance has one less vertex. Thus, by applying the polynomial-time algorithm for Edge Contraction(γ) at most n times, we obtain a trivial instance for Dominating Set and can therefore correctly determine its answer. □

3.2. *Hardness of 2-Edge Contraction(γ)*

In this subsection we consider the complexity of k-Edge Contraction(γ) when $k = 2$. To this end, we introduce the following problem.

Contraction Number(γ,k)	
Instance:	A connected graph $G = (V, E)$.
Question:	Is $ct_\gamma(G) = k$?

Theorem 3.14. Contraction Number*(γ, 3) is* NP*-hard.*

Proof. We reduce from 1-in-3 Positive 3-Sat, where each variable occurs only positively, each clause contains exactly three positive literals, and we want a truth assignment such that each clause contains exactly one true variable. This problem is known to be NP-complete [18]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance G_Φ of Contraction Number(γ, 3) as follows. For any variable $x \in X$, we introduce a copy of C_3, which we denote by G_x, with two distinguished *truth vertices* T_x and F_x (see Fig. 7); in the following, the third vertex of G_x is denoted by u_x. For any clause $c \in C$ containing variables x_1, x_2 and x_3, we introduce the gadget G_c depicted in Fig. 7 (where it is connected to the corresponding variable gadgets). The vertex set of the clique K_c corresponds to the set of subsets of size 1 of $\{x_1, x_2, x_3\}$ (hence the notation); for any $i \in \{1, 2, 3\}$, the vertex x_i (resp. x'_i) is connected to every vertex $v_S \in K_c$ such that $x_i \notin S$ (resp. $x_i \in S$). Finally, for $i = 1, 2, 3$, we add an edge between t_i (resp. x'_i) and the truth vertex T_{x_i} (resp. F_{x_i}). Our goal now is to show that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 3$. In the remainder of the proof, given a clause $c \in C$, we denote by x_1, x_2 and x_3 the variables occurring in c and thus assume that t_i (resp. x'_i) is adjacent to T_{x_i} (resp. F_{x_i}) for $i \in \{1, 2, 3\}$. Let us first start with some easy observations.

Observation 12. *Let D be a dominating set of G_Φ. Then for any $x \in X$, $|D \cap V(G_x)| \geq 1$ and for any $c \in C$, $|D \cap V(G_c)| \geq 4$. In particular, $|D| \geq |X| + 4|C|$.*

Clearly, for any $x \in X$, $|D \cap V(G_x)| \geq 1$ since u_x must be dominated. Also, in order to dominate vertices a_1, a_2, a_3 and $v_{\{x_1\}}$ in some gadget G_c, we need at least 4 distinct vertices, since their neighborhoods are pairwise disjoint and so, $|D \cap V(G_c)| \geq 4$, for any $c \in C$. ⋄

Observation 13. *Let D be a dominating set of G_Φ. For any clause gadget G_c and $i \in \{1, 2, 3\}$, $D \cap \{a_i, b_i, x_i\} \neq \emptyset$.*

This immediately follows from the fact that every vertex b_i needs to be dominated and its neighbors are a_i and x_i for $i \in \{1, 2, 3\}$. ⋄

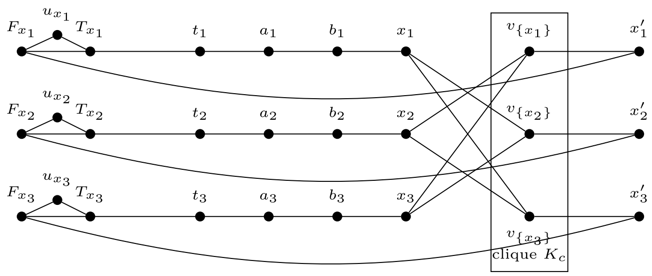


Fig. 7. The gadget G_c together with G_{x_i}, $i = 1, 2, 3$, for a clause $c \in C$ containing variables x_1, x_2 and x_3 (the rectangle indicates that the corresponding set of vertices induces a clique).

Observation 14. *Let D be a dominating set of G_Φ. For any clause gadget G_c, if $|D \cap V(G_c)| = 4$, then $D \cap \{t_i, x'_i\} = \emptyset$ and $|D \cap \{a_i, b_i, x_i\}| = 1$, for any $i \in \{1, 2, 3\}$.*

If $t_i \in D$ for some $i \in \{1, 2, 3\}$, then it follows from Observation 13 that $|D \cap \{a_j, b_j, x_j\}| = 1$ for any $j \in \{1, 2, 3\}$. This implies that at least two vertices among x_1, x_2 and x_3 belong to D for otherwise there would exist $j \in \{1, 2, 3\}$ such that $v_{\{x_j\}}$ is not dominated. In particular, there must exist $j \neq i$ such that $x_j \in D$; but then, a_j is not dominated. Similarly, if $x'_i \in D$ for some $i \in \{1, 2, 3\}$, it follows from Observation 13 that $|D \cap \{a_j, b_j, x_j\}| = 1$ for any $j \in \{1, 2, 3\}$. But then, in order to dominate the vertices of K_c, either $x_i \in D$ in which case a_i is not dominated; or $\{x_j, j \neq i\} \subset D$ and a_j with $j \neq i$, is not dominated.

Now suppose that $|D \cap \{a_i, b_i, x_i\}| \geq 2$ for some $i \in \{1, 2, 3\}$. Then by Observation 13, we conclude that $|D \cap \{a_k, b_k, x_k\}| = 1$ for $k \neq i$ and $|D \cap \{a_i, b_i, x_i\}| = 2$. This implies that $D \cap V(K_c) = \emptyset$ for otherwise we would have $|D \cap V(G_c)| \geq 5$. But then, since $x'_i \notin D$, D must contain at least two vertices among x_1, x_2 and x_3 in order to dominate the vertices of K_c; in particular, there exists $j \neq i$ such that $x_j \in D$ and so, a_j is not dominated. $\diamond$

Observation 15. *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any vertices $u, v \in D$, we have $d(u, v) \geq 3$.*

Indeed, if u, v are adjacent, we conclude by Theorem 2.1(i) that $ct_\gamma(G_\Phi) = 1$; and if u, v are at distance 2 then $D \cup \{w\}$, where w is the vertex on a shortest path from u to v, contains two edges and we conclude by Theorem 2.1(ii) that $ct_\gamma(G_\Phi) = 2$. $\diamond$

Observation 16. *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c and $i \in \{1, 2, 3\}$, $a_i \in D$ if and only if $T_{x_i} \notin D$.*

This readily follows from Observation 15. Further note that we may assume that for any $i \in \{1, 2, 3\}$, $a_i \in D$ if and only if $F_{x_i} \in D$; $T_{x_i} \notin D$ is equivalent to $\{F_{x_i}, u_{x_i}\} \cap D \neq \emptyset$ and if $T_{x_i} \notin D$, we may always replace D by $(D \backslash \{u_{x_i}\}) \cup \{F_{x_i}\}$. $\diamond$

Observation 17. *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c, $|D \cap \{a_1, a_2, a_3\}| \leq 2$.*

If it were not the case then, by Observation 15, no x_i or b_i ($i = 1, 2, 3$) would belong to D. But since x_1, x_2 and x_3 must be dominated, it follows that $D \cap V(K_c) \neq \emptyset$ and by Observation 16, we conclude that D contains two vertices at distance two (namely, $v_{\{x_i\}} \in D \cap V(K_c)$ and F_{x_i} for some $i \in \{1, 2, 3\}$), which contradicts Observation 15. $\diamond$

Observation 18. *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c, $|D \cap \{b_1, b_2, b_3\}| \leq 1$.*

Indeed, if we assume, without loss of generality, that $b_1, b_2 \in D$, then by Observation 15, $D \cap V(K_c) = \emptyset$. It then follows from Observation 15 that $x'_3 \in D$ for otherwise $V_{\{x_3\}}$ would not be dominated. But then $D \cap V(G_{x_3}) = \emptyset$ by Observation 15, which contradicts Observation 12. $\diamond$

Claim 10. *$\gamma(G_\Phi) = |X| + 4|C|$ if and only if $ct_\gamma(G_\Phi) = 3$.*

Proof. Assume that $\gamma(G_\Phi) = |X| + 4|C|$ and consider a minimum dominating set D of G_Φ. We first show that D is an independent set which would imply that $ct_\gamma(G_\Phi) > 1$ (see Theorem 2.1(i)). First note that Observation 12 implies that $|D \cap V(G_x)| = 1$ and $|D \cap V(G_c)| = 4$, for any variable $x \in X$ and any clause $c \in C$. It then follows from Observation 14 that no truth vertex is dominated by some vertex t_i or x'_i in some clause gadget G_c with $i \in \{1, 2, 3\}$; in particular, this implies that there can exist no edge in D having one endvertex in some gadget G_x ($x \in X$) and the other in some gadget G_c ($c \in C$). Hence, it is enough to show that for any $c \in C$, $D \cap V(G_c)$ is an independent set.

Now consider a clause gadget G_c. It follows from Observation 14 that if there exists $i \in \{1,2,3\}$ such that $a_i \notin D$ then $b_i \in D$ since a_i must be dominated (also note that by Observation 14, if $a_i \in D$ then $b_i \notin D$). Hence, for any $i \in \{1,2,3\}$, exactly one of a_i and b_i belongs to D. But then, by Observation 14 and since $|D \cap V(G_c)| = 4$, we immediately conclude that $D \cap V(G_c)$ is an independent set and so, D is an independent set.

Now, suppose to the contrary that $ct_\gamma(G_\Phi) = 2$ i.e., there exists a dominating set D' of G_Φ of size $\gamma(G_\Phi)+1$ containing two edges e and e' (see Theorem 2.1(ii)). First assume that there exists $x \in X$ such that $|D' \cap V(G_x)| = 2$. Then, for any $x' \neq x$, $|D' \cap V(G_{x'})| = 1$; and for any $c \in C$, $|D' \cap V(G_c)| = 4$ which by Observation 14 implies that $\{t_i, x'_i\} \cap D' = \emptyset$ for any $i \in \{1,2,3\}$. Since as shown previously, $D' \cap V(G_c)$ is then an independent set, it follows that D' contains at most one edge, a contradiction.

Thus, there must exist some $c \in C$ such that $|D' \cap V(G_c)| = 5$. We then claim that $\{a_1,a_2,a_3\} \not\subset D'$. Indeed, since $x_1, x_2, x_3, v_{\{x_1\}}, v_{\{x_2\}}$ and $v_{\{x_3\}}$ must be dominated, $D' \cap V(K_c) \neq \emptyset$ (otherwise, at least three additional vertices of G_c would be required to dominate x_1, x_2 and x_3), say $v_{\{x_1\}} \in D'$ without loss of generality. But then, $|N[x_1] \cap D'| = 1$ as x_1 must be dominated and $|D' \cap V(G_c)| = 5$ and so, D' contains at most one edge. Therefore, there must exist $i \in \{1,2,3\}$ such that $a_i \notin D'$, say $a_1 \notin D'$ without loss of generality. Then, since a_1 must be dominated, either $t_1 \in D'$ or $b_1 \in D'$.

Assume first that t_1 belongs to D' (note that $\{b_1, x_1\} \cap D' \neq \emptyset$ by Observation 13). We then claim that either e or e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$. Indeed, if it were not the case, then t_1 would be an endvertex of neither e nor e' for otherwise $T_{x_1} \in D'$ which implies that $D' \cap \{v_{\{x_1\}}, x'_1\} \neq \emptyset$ as $|D' \cap V(G_{x_1})| = 1$ and x'_1 should be dominated. But then, D' contains at most one edge as $5 = |D' \cap V(G_c)| \geq |\{t_1\}| + |D' \cap \{b_1, x_1\}| + |D' \cap \{v_{\{x_1\}}, x'_1\}| + |D' \cap \{a_j, b_j, x_j, j \neq 1\}| \geq 1+1+1+2$ and neither e nor e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$ by assumption, a contradiction. Since e and e' have at most one common endvertex, it then follows that $|D' \cap V(G_c)| \geq |\{t_1\}| + |D' \cap \{a_j, b_j, x_j, j \neq 1\}| + 3 \geq 1+2+3$, a contradiction. Thus, either e or e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$, say $j = 2$ without loss of generality. Suppose that x_2 is an endvertex of e. Then the other endvertex of e should be b_2 for otherwise it belongs to K_c and thus, a_2 would not be dominated. But then, we conclude by Observation 13 and the fact that $|D' \cap V(G_c)| = 5$, that D' contains only one edge. Thus, $e = a_2b_2$ or $e = a_2t_2$ and since $v_{\{x_1\}}$ must be dominated, necessarily $x_3 \in D'$; but then, a_3 is not dominated. Therefore, it must be that b_1 belongs to D'; and we conclude similarly that if a_2 (resp. a_3) is not in D' then b_2 (resp. b_3) belongs to D'.

Now, since $t_1, a_1 \notin D'$, it follows that $T_{x_1} \in D'$ for otherwise t_1 would not be dominated. But $|D' \cap V(G_x)| = 1$ and so, $F_{x_1} \notin D'$; thus, $D' \cap \{x'_1, v_{\{x_1\}}\} \neq \emptyset$ as x'_1 must be dominated and we may assume, without loss of generality, that in fact, $v_{\{x_1\}} \in D'$. Then, if $D' \cap \{v_{\{x_2\}}, v_{\{x_3\}}\} = \emptyset$, necessarily $F_{x_2}, F_{x_3} \in D'$; indeed, since $|D' \cap V(G_c)| = 5$, at least one among x'_2 and x'_3 does not belong to D', say x'_2 without loss of generality. But if $x'_3 \in D'$, then exactly one of a_j and b_j, for $j \neq 1$ belongs to D' (recall that if $a_j \notin D'$ then $b_j \in D'$) and therefore, D' contains at most one edge. Thus, $F_{x_2}, F_{x_3} \in D'$ which implies that $D' \cap \{t_j, a_j\} \neq \emptyset$ for $j \neq 1$ as t_j must be dominated. But by Observation 13 and the fact that $|D' \cap V(G_c)| = 5$, we have that $|D' \cap \{t_2, t_3\}| \leq 1$ and so, D' contains at most one edge. Thus, $D' \cap \{v_{\{x_2\}}, v_{\{x_3\}}\} \neq \emptyset$ and since by Observation 13 $|D' \cap V(K_c)| \leq 2$, we conclude that in fact $|D' \cap V(K_c)| = 2$. But then, exactly one among a_j and b_j belongs to D' for $j \neq 1$ and so, D' contains only one edge. Consequently, no such dominating set D' exists and thus, $ct_\gamma(G_\Phi) = 3$.

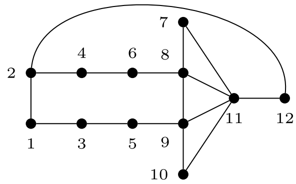
Conversely, assume that $ct_\gamma(G_\Phi) = 3$ and consider a minimum dominating set D of G_Φ. It readily follows from Observations 12 and 15 that for any variable $x \in X$, $|D \cap V(G_x)| = 1$. Now consider a clause gadget G_c. Then, by Observation 15, we obtain that $t_i \notin D$ (resp. $x'_i \notin D$) for $i \in \{1,2,3\}$, as otherwise it would be within distance at most 2 from the vertex in D belonging to the gadget G_{x_i}.

Now since for any $i \in \{1,2,3\}$, $t_i \notin D$, if $a_i \notin D$ then $b_i \in D$ as a_i must be dominated (also note that by Observation 15, if $a_i \in D$ then $b_i \notin D$). Thus, by Observations 17 and 18, we conclude that for any clause gadget G_c, $|D \cap \{a_1,a_2,a_3\}| = 2$ and $|D \cap \{b_1,b_2,b_3\}| = 1$, say $a_1, a_2, b_3 \in D$ without loss of generality. But then, $v_{\{x_3\}}$ must belong to D; indeed, since $b_3 \in D$, it follows that $T_{x_3} \in D$ for otherwise t_3 is not dominated. Observation 15 then implies that $x'_3 \notin D$ and thus, it can only be dominated by $v_{\{x_3\}}$. But then, it follows from Observation 16 that every vertex in G_c is dominated and we conclude that $|D \cap V(G_c)| = 4$ by minimality of D. Consequently, $|D| = |X| + 4|C|$ which concludes the proof of Claim 10. ▲

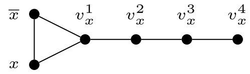
Claim 11. *$\gamma(G_\Phi) = |X| + 4|C|$ if and only if Φ is satisfiable.*

Proof. Assume first that $\gamma(G_\Phi) = |X| + 4|C|$ and consider a minimum dominating set D of G_Φ. We construct a truth assignment from D satisfying Φ as follows. For any $x \in X$, if $T_x \in D$, set x to true; otherwise, set x to false. We claim that each clause $c \in C$ has exactly one true variable. Indeed, it follows from Observation 12 that $|D \cap V(G_c)| = 4$ for any $c \in C$, and from Claim 10 that $ct_\gamma(G_\Phi) = 3$. But then, by Observation 14, for any $i \in \{1,2,3\}$, $a_i \notin D$ if and only if $b_i \in D$ (a_i would otherwise not be dominated). It then follows from Observations 17 and 18 that $|D \cap \{a_1,a_2,a_3\}| = 2$ and $|D \cap \{b_1,b_2,b_3\}| = 1$ for any $c \in C$; but by Observation 16 we conclude that $b_i \in D$ if and only if $T_{x_i} \in D$, which proves our claim.

Conversely, assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. If variable x is set to true, we add T_x to D; otherwise, we add F_x to D. For any clause $c \in C$ and $i \in \{1,2,3\}$, if $T_{x_i} \in D$, then add b_i to D; otherwise, add a_i to D. Since every clause has exactly one true variable, it follows that $|D \cap \{b_1,b_2,b_3\}| = 1$ and $|D \cap \{a_1,a_2,a_3\}| = 2$; finally add $v_{\{x_i\}}$ to D where $b_i \in D$. Now clearly $|D \cap V(G_c)| = 4$ and every vertex in G_c is dominated. Thus, $|D| = |X| + 4|C|$ and so by Observation 12, $\gamma(G_\Phi) = |X| + 4|C|$, which concludes this proof. ▲



(a) The graph H.



(b) The gadget G_x with $x \in X$.

Fig. 8. Construction of the graph G_Φ.

Now combining Claims 10 and 11, we have that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 3$ which completes the proof of Theorem 3.14. □

By observing that for any graph G, G is a Yes-instance for Contraction Number(γ, 3) if and only if G is a No-instance for 2-Edge Contraction(γ), we deduce the following corollary from Theorem 3.14.

Corollary 3.15. *2-Edge Contraction(γ) is* coNP*-hard.*

It is thus coNP-hard to decide whether $ct_\gamma(G) \leq 2$ for a graph G; and in fact, it is NP-hard to decide whether equality holds, as stated in the following.

Theorem 3.16. *Contraction Number(γ, 2) is* NP*-hard.*

Proof. We give a reduction from Exactly 3-Bounded 3-Sat, where we want to determine if a formula Φ is satisfiable, given that each variable occurs exactly three times in Φ, with both positive and negative occurrences, and each clause of Φ contains two or three literals. This problem was shown to be NP-complete by Dahlhaus et al. [11].

Given an instance Φ of Exactly 3-Bounded 3-Sat, with variable set X and clause set C, we construct an equivalent instance G_Φ of Contraction Number(γ, 2) as follows: First note that we may assume that $|X| \geq 4$ as Exactly 3-Bounded 3-Sat is otherwise polynomial-time solvable. The graph G_Φ then contains a copy of the graph H depicted in Fig. 8(a). For any variable $x \in X$, we introduce the gadget G_x which has two distinguished *literal vertices* x and $\bar{x}$, as depicted in Fig. 8(b). For any clause $c \in C$, we introduce a copy of K_2 with a distinguished *clause vertex* c and a distinguished *transmitter vertex* t_c. Finally, for each clause $c \in C$, we add an edge between the clause vertex c and the literal vertices whose corresponding literals belong to c; furthermore, we add an edge between the transmitter vertex t_c and vertices 1 and 3 of the graph H. We first prove the following:

Claim 12. $\gamma(H) = \gamma(H - \{1, 3\}) = 3$ *and* $ct_\gamma(H) = 2$.

Proof. Since $\{3, 4, 11\}$ (resp. $\{4, 5, 11\}$) is a dominating set of H (resp. $H - \{1, 3\}$), it follows that $\gamma(H) \leq 3$ and $\gamma(H - \{1, 3\}) \leq 3$. On the other hand, any dominating set of H must contain at least three vertices as $\{3, 4, 11\}$ is an independent set with $N(3) \cap N(4) = N(3) \cap N(11) = N(4) \cap N(11) = \emptyset$. Similarly, any dominating set of $H - \{1, 3\}$ must contain at least three vertices as $\{4, 5, 7\}$ is an independent set with $N(4) \cap N(5) = N(4) \cap N(11) = N(5) \cap N(11) = \emptyset$. Thus, $\gamma(H) = \gamma(H - \{1, 3\}) = 3$.

We now claim that H has a unique minimum dominating set, namely $\{3, 4, 11\}$. First observe that any minimum dominating D set of H contains vertex 11 as otherwise D would have to contain at least two vertices from $\{7, 8, 9, 10\}$ in order to dominate vertices 7 and 10, and at least two other vertices to dominate vertices 3 and 4; but then, $|D| \geq 4 > \gamma(H)$. Now if there exists a minimum dominating set D not containing vertex 4, then $\{2, 6\} \cap D \neq \emptyset$ as vertex 4 is dominated. But if $2 \in D$ then $\{6, 8\} \cap D \neq \emptyset$ as 6 must be dominated; and so, $|D| \geq 4$ as $11 \in D$ and $\{1, 3, 5\} \cap D \neq \emptyset$ (3 must be dominated). Otherwise, $6 \in S$ and similarly $\{2, 12\} \cap D \neq \emptyset$ as 2 must be dominated; and we conclude similarly that $|D| \geq 4$. Thus, every minimum dominating set contains vertex 4; we conclude similarly that every minimum dominating set contains vertex 3. It follows that $\{3, 4, 11\}$ is the only minimum dominating set of H and since it is independent, we obtain that $ct_\gamma(H) > 1$. Now, $\{1, 2, 8, 9\}$ is clearly dominating and since it contains two edges, it follows that $ct_\gamma(H) = 2$ (see Theorem 2.1(ii)). ▲

We next prove two claims which together show that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 2$.

Claim 13. $\gamma(G_\Phi) = 2|X| + 3$ *if and only if* $ct_\gamma(G_\Phi) = 2$.

Proof. Suppose that $\gamma(G_\Phi) = 2|X| + 3$ and let D be a minimum dominating set of G_Φ. Since for any $x \in X$, vertices v_x^1 and v_x^4 can only be dominated by (distinct) vertices in $V(G_x)$, it follows that $|D \cap V(G_x)| \geq 2$. Furthermore, $|D \cap V(H)| \geq 3$ as $\gamma(H) = 3$ by Claim 12 and even if vertices 1 and 3 are dominated by some transmitter vertex, we still have $\gamma(H - \{1, 3\}) = 3$ by Claim 12. Now, since $|D| = 2|X| + 3$ we have that:

- $\forall x \in X$, $|D \cap V(G_x)| = 2$;
- $|D \cap V(H)| = 3$;
- $\forall c \in C$, $D \cap V(G_c) = \emptyset$.

But then, for any $x \in X$, the set $D \cap V(G_x)$ is a minimum dominating set of G_x and therefore independent as we trivially have $ct_\gamma(G_x) = 2$. Similarly, $D \cap V(H)$ is a dominating set of H (recall that for any $c \in C$, $D \cap V(G_c) = \emptyset$) and therefore independent as $ct_\gamma(H) = 2$ by Claim 12. Thus, D is independent and since $(D \cap \bigcup_{x \in X} V(G_x)) \cup \{1, 2, 8, 9\}$ is a dominating set of G_Φ of size $\gamma(G_\Phi) + 1$ containing two edges, it follows that $ct_\gamma(G_\Phi) = 2$.

Conversely, assume that $ct_\gamma(G_\Phi) = 2$ and let D be a minimum dominating set of G_Φ (note that D is independent). Suppose that there exists $c \in C$ such that $D \cap V(G_c) \neq \emptyset$. Then, we may assume that $t_c \in D$; indeed, if $c \in D$ then no literal vertex adjacent to c is in the dominating set as D is independent. We then claim that any literal vertex adjacent to c must dominated by one of its neighbor in the gadget; if x (or $\bar{x}$) is adjacent to c and neither v_x^1 nor $\bar{x}$ belongs to D, then we necessarily have $|D \cap \{v_x^2, v_x^3, v_x^4\}| = 2$ and so, G_Φ would have a minimum dominating set which is not independent, namely $(D \backslash \{v_x^1, v_x^2, v_x^3, v_x^4\}) \cup \{v_x^2, v_x^3\}$, a contradiction. But then, $(D \backslash \{c\}) \cup \{t_c\}$ is a minimum dominating set of G_Φ. Now since $t_c \in D$, it follows that $D \cap \{1, 3\} = \emptyset$ as D is independent, which implies that $\{c', t_{c'}\} \cap D \neq \emptyset$ for any $c' \in C$. In particular, the set $D' = (D \backslash \{t_{c'}, c' \neq c\}) \cup \{c', c' \neq c\}$ is a minimum dominating set of G_Φ and thus, independent. But $|X| \geq 4$ so there must exist $x \in X$ such that both x and $\bar{x}$ are dominated in D' by some clause vertices (take any variable x not occurring in c). In particular, $\{x, \bar{x}\} \cap D' = \emptyset$ which implies that $|D' \cap \{v_x^1, v_x^2, v_x^3, v_x^4\}| = 2$; but then $(D' \backslash \{v_x^1, v_x^2, v_x^3, v_x^4\}) \cup \{v_x^2, v_x^3\}$ is a minimum dominating set of G_Φ which is not independent, a contradiction. It follows that for any $c \in C$, $D \cap V(G_c) = \emptyset$.

On the other hand, if there exists $x \in X$ such that $|D \cap V(G_x)| > 2$, it is not difficult to see that D could then be transformed into a minimum dominating set which is not independent. But since for any $x \in X$, at least two vertices are required to dominate $\{v_x^1, v_x^2, v_x^3, v_x^4\}$, we have then that $|D \cap V(G_x)| = 2$. Finally, as $D \cap V(H)$ is a minimum dominating set of H (recall that $D \cap V(G_c) = \emptyset$ and so, no vertex in $(V(G_\Phi) \backslash V(H)) \cap D$ dominates a vertex in H), $|D \cap V(H)| = \gamma(H) = 3$. Thus, $\gamma(G_\Phi) = 2|X| + 3$, which concludes the proof of the claim. ▲

Claim 14. *$\gamma(G_\Phi) = 2|X| + 3$ if and only if Φ is satisfiable.*

Proof. Assume first that $\gamma(G_\Phi) = 2|X| + 3$ and consider a minimum dominating set D of G. As shown in the proof of Claim 13, D is then independent and contains no vertex from $\bigcup_{c \in C} V(G_c)$. Therefore, any clause vertex is dominated by a literal vertex and for any $x \in X$, $|D \cap \{x, \bar{x}\}| \leq 1$. We may thus construct a truth assignment which satisfies Φ as follows:

- If $x \in D$, set variable x to true;
- if $\bar{x} \in D$, set variable x to false;
- otherwise, we may set variable x to any truth value.

Conversely, assume that Φ is satisfiable and consider a truth assignment which satisfies Φ. We construct a dominating set D of G_Φ as follows. For any $x \in X$, if x is set to true, we add x and v_x^3 to D, otherwise we add $\bar{x}$ and v_x^3 to D. We further add vertices 3, 4 and 11 of H. Then, it is not difficult to see that D is dominating (every transmitter vertex is dominated by vertex 3 and every clause vertex has an adjacent literal vertex belonging to D) and so, $\gamma(G_\Phi) \leq 2|X| + 3$. But since for any $x \in X$, $|D \cap V(G_x)| \geq 2$ and for any $c \in C$, $|D \cap V(G_C)| \geq 4$, it follows that $\gamma(G_\Phi) = 2|X| + 3$. This completes the proof of the claim. ▲

Now combining Claims 13 and 14, we have that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 2$ which concludes the proof of Theorem 3.16. □

3.3. Algorithms for *k*-Edge Contraction(γ)

We now deal with cases in which k-Edge Contraction(γ) is tractable, for $k = 1, 2$. A first simple approach to the problem, from which we obtain Proposition 3.17, is based on brute force.

Proposition 3.17. *For $k = 1, 2$, k-Edge Contraction(γ) can be solved in polynomial time for a graph class $\mathcal{C}$, if either*

(a) $\mathcal{C}$ is closed under edge contractions and Dominating Set *can be solved in polynomial time on $\mathcal{C}$; or*
(b) for every $G \in \mathcal{C}$, $\gamma(G) \leq q$, where q is some fixed constant; or
(c) $\mathcal{C}$ is the class of $(H + K_1)$-free graphs, where $|V_H| = q$ is a fixed constant and k-Edge Contraction(γ) is polynomial-time solvable on H-free graphs.

Proof. In order to prove item (a), it suffices to note that if we can compute $\gamma(G)$ and $\gamma(G \setminus e)$, for any edge e of G, in polynomial time, then we can determine whether a graph G is a Yes-instance for 1-Edge Contraction(γ) in polynomial time (we may proceed in a similar fashion for 2-Edge Contraction(γ)).

For item (b), we proceed as follows. Given a graph G of $\mathcal{C}$, we first check whether G has a dominating vertex. If it is the case, then G is a No-instance for k-Edge Contraction(γ) for both $k = 1, 2$. Otherwise, we may consider any subset $S \subset V(G)$ with $|S| \leq q$ and check whether it is a dominating set of G. Since there are at most $\mathcal{O}(n^q)$ possible such subsets, we can determine the domination number of G and check whether the conditions given in Theorem 2.1(i) or (ii) are satisfied in polynomial time.

Finally, so as to prove item (c), we provide the following algorithm that works similarly for $k = 1$ and $k = 2$. Let H and q be as stated and let G be an instance of k-Edge Contraction(γ) on $(H + K_1)$-free graphs. We first test whether G is H-free (note that this can be done in time $\mathcal{O}(n^q)$). If this is the case, we use the polynomial-time algorithm for k-Edge Contraction(γ) on H-free graphs. Otherwise, G has an induced subgraph isomorphic to H; but since G is a $(H + K_1)$-free graph, $V(H)$ must then be a dominating set of G and so, $\gamma(G) \leq q$. We then conclude by Proposition 3.17(b) that k-Edge Contraction(γ) is also polynomial-time solvable in this case. □

Proposition 3.17(b) provides an algorithm for 1-Edge Contraction(γ) parameterized by the size of a minimum dominating set of the input graph running in XP-time. Note that this result is optimal as 1-Edge Contraction(γ) is W[1]-hard with such parameterization from Theorem 3.1.

We further show that even though simple, this brute force method provides polynomial-time algorithms for a number of relevant classes of graphs, such as graphs of bounded tree-width and graphs of bounded mim-width. We first state the following result and observation.

Theorem 3.18 ([27]). *Given a graph G and a decomposition of width t,* Dominating Set *can be solved in time $\mathcal{O}^*(3^t)$ when parameterized by tree-width, and in time $\mathcal{O}^*(n^{3t})$ when parameterized by mim-width.*

Observation 19. *$mimw(G \setminus e) \leq mimw(G) + 1$.*

Indeed, note that the graph $G \setminus e$ can be obtained from G by the removal of the vertices u and v where $e = uv$, and the addition of a new vertex whose neighborhood is $N_G(u) \cup N_G(v)$. The result then follows from Observation 3 and the fact that vertex deletion does not increase the mim-width of a graph.

Proposition 3.19. *Given a decomposition of width t, k-*Edge Contraction*(γ) can be solved in time $\mathcal{O}^*(3^t)$ in graphs of tree-width at most t and in time $\mathcal{O}^*(n^{3t})$ in graphs of mim-width at most t, for $k = 1, 2$.*

Proof. We use the above-mentioned brute force approach and Theorem 3.18. That is, for $k = 1$, the algorithm first computes $\gamma(G)$ and then computes $\gamma(G \setminus e)$ for every $e \in E(G)$. For $k = 2$, the algorithm proceeds similarly for every pair of edges. We next show that the width parameters increase by a constant when contracting at most two edges. It is a well-known fact that $tw(G \setminus e) \leq tw(G)$ and so, $tw(G \setminus \{e, f\}) \leq tw(G)$. By Observation 19, $mimw(G \setminus e) \leq mimw(G) + 1$ which implies that $mimw(G \setminus \{e, f\}) \leq mimw(G) + 2$. Also note that, given a tree (resp. mim) decomposition of width t for G, we can construct in polynomial time decompositions of width t (resp. at most $t + 2$) for $G \setminus e$ and $G \setminus \{e, f\}$. This implies that $\gamma(G \setminus e)$ and $\gamma(G \setminus \{e, f\})$ can also be computed in time $\mathcal{O}^*(3^t)$ if G is a graph of tree-width at most t, and in time $\mathcal{O}^*(n^{3t})$ if G is a graph of mim-width at most t. □

Proposition 3.19 provides an algorithm for 1-Edge Contraction(γ) parameterized by mim-width running in XP-time; this result is optimal as 1-Edge Contraction(γ) is W[1]-hard parameterized by mim-width from Theorem 3.1.

Since Dominating Set is polynomial-time solvable in P_4-free graphs (see [19]), it follows from Proposition 3.17(a) that k-Edge Contraction(γ) can also be solved efficiently in this graph class. However, Dominating Set is NP-complete for P_5-free graphs (see [5]) and thus, it is natural to examine the complexity of k-Edge Contraction(γ) for this graph class. As we next show, k-Edge Contraction(γ) is in fact polynomial-time solvable on P_5-free graphs, for $k = 1, 2$.

Lemma 3.20. *Let G be a graph that is at distance at most d from a connected P_5-free graph. If $\gamma(G) \geq 2^{d+1} + d + 1$, then $ct_\gamma(G) = 1$.*

Proof. Let $G = (V, E)$ be a graph as stated above and let $X \subseteq V$ be such that $G' = G[V \setminus X]$ is a connected P_5-free graph and $|X| \leq d$. Consider the partition $(A_1, \ldots, A_\ell)$ of the vertices of $V \setminus X$ defined by their neighborhoods in X, that is, two vertices $u, v \in V \setminus X$ belong to the same set of the partition if $N_G(u) \cap X = N_G(v) \cap X$. Note that $\ell \leq 2^d$ since $|X| \leq d$. Now let D be a minimum dominating set of G of size at least $2^{d+1} + d + 1$ and suppose that D is independent. It is easy to see that there must exist $1 \leq i \leq \ell$ such that $|A_i \cap D| \geq 3$. Let $u, v \in A_i \cap D$ be such that $d_{G'}(u, v) = \max_{x,y \in A_i \cap D} d_{G'}(x, y)$. Since G' is a connected P_5-free graph, $d_{G'}(u, v) \leq 3$ and, since D is independent, $d_{G'}(u, v) \geq 2$. We thus distinguish two cases depending on this distance.

Case 1. $d_{G'}(u, v) = 3$. Let x (resp. y) be the neighbor of u (resp. v) in G' on a shortest path from u to v. Then, $N_{G'}(u) \cup N_{G'}(v) \subseteq N_{G'}(x) \cup N_{G'}(y)$; indeed, if a is a neighbor of u in G', then a is nonadjacent to v (recall that $d_{G'}(u, v) = 3$) and thus, a is adjacent to either x or y for otherwise a, u, x, y and v would induce a P_5 in G'. The same holds for any neighbor of v. Furthermore, since $|A_i \cap D| \geq 3$, the vertices of $N_G(u) \cap X$ have at least one neighbor in $D \setminus \{u, v\}$. Consequently, $(D \setminus \{u, v\}) \cup \{x, y\}$ is a minimum dominating set of G which is not independent; the result then follows from Theorem 2.1(i).

Case 2. $d_{G'}(u, v) = 2$. Since D is independent and $d_{G'}(u, v) = \max_{x,y \in D \cap A_i} d_{G'}(x, y) = 2$, it follows that every $w \in D \cap A_i \setminus \{u, v\}$ is at distance two from both u and v. Let x (resp. y) be the vertex on a shortest path from u (resp. v) to some vertex $w \in D \cap A_i \setminus \{u, v\}$ (recall that $|A_i \cap D| \geq 3$).

Suppose first that $x = y$. If every private neighbor of w with respect to D is adjacent to x then $(D\backslash\{w\}) \cup \{x\}$ is a minimum dominating set of G which is not independent (note that, again, the vertices of $N_G(w) \cap X$ are also dominated by u and v); the result then follows from Theorem 2.1(i). We conclude similarly if every private neighbor of u or v with respect to D is adjacent to x. Thus, we may assume that w (resp. u; v) has a private neighbor t (resp. r; s) with respect to D which is nonadjacent to x. Since G' is P_5-free, it then follows that r, s and t are pairwise adjacent. But then, t, r, u, x and v induce a P_5, a contradiction.

Finally, suppose that $x \neq y$ (we may also assume that $uy, vx \notin E$ as we otherwise fall back in the previous case). Then, $xy \in E$ for u, x, w, y and v would otherwise induce a P_5. Now, if a is a private neighbor of u with respect to D then a is adjacent to either x or y (a, u, x, y and v otherwise induce a P_5); we conclude similarly that any private neighbor of v with respect to D is adjacent to either x or y. If b is a vertex that is adjacent to both u and v but not w, then it is adjacent to x (and y) as v, b, u, x and w (u, b, v, y and w) would otherwise induce a P_5. But then, $(D\backslash\{u, v\}) \cup \{x, y\}$ is a minimum dominating set of G which is not independent (note that w dominates every vertex in $N_G(u) \cap X = N_G(v) \cap X$); thus, by Theorem 2.1(i), $ct_\gamma(G) = 1$ which concludes the proof. □

In particular, when $d = 0$, we obtain the following corollary.

Corollary 3.21. *If G is a P_5-free graph with $\gamma(G) \geq 3$, then $ct_\gamma(G) = 1$.*

Theorem 3.22. *k-Edge Contraction(γ) is polynomial-time solvable on graphs that are at distance at most d from a connected P_5-free graph, for any fixed $d \geq 0$ and $k = 1, 2$.*

Proof. We may proceed in a similar fashion than in the proof of Proposition 3.17(b). Specifically, if G has a dominating vertex, then G is clearly a No-instance for both $k = 1, 2$. Otherwise, we may consider every subset $S \subset V(G)$ with $|S| \leq 2^{d+1} + d + 1$ (by increasing size) and check whether it is dominating. By doing so, we can determine whether $\gamma(G) \leq 2^{d+1} + d + 1$ and, if so, check whether G has a minimum dominating set containing an edge. If it is the case, then by Theorem 2.1(i), G is a Yes-instance for k-Edge Contraction(γ) for $k = 1, 2$. If $\gamma(G) \leq 2^{d+1} + d + 1$ but G is a No-instance to 1-Edge Contraction(γ), we can determine whether G is a Yes-instance for 2-Edge Contraction(γ) by checking all sets of size $\gamma(G) + 1$ and see whether there exists one which is dominating and contains at least two edges (see Theorem 2.1(ii)). Finally, both for $k = 1$ and $k = 2$, if G has no dominating set of size at most $2^{d+1} + d + 1$, then by Lemma 3.20, G is a Yes-instance for k-Edge Contraction(γ). □

Recall that 1-Edge Contraction(γ) is NP-hard for P_6-free graphs. We show that for $k = 2$, the problem is polynomial-time solvable for this graph class. The following is a more general result.

Lemma 3.23. *Let G be a graph that is at distance at most d from a connected P_6-free graph. If $\gamma(G) \geq 2^{d+1} + d + 1$, then $ct_\gamma(G) \leq 2$.*

Proof. Let $G = (V, E)$ be a graph as stated above and let $X \subseteq V$ be such that $G' = G[V\backslash X]$ is a connected P_6-free graph and $|X| \leq d$. Consider the partition $(A_1, \ldots, A_\ell)$ of the vertices of $V\backslash X$ defined by their neighborhoods in X, that is, two vertices $u, v \in V\backslash X$ belong to the same set of the partition if $N_G(u) \cap X = N_G(v) \cap X$. Note that $\ell \leq 2^d$ since $|X| \leq d$. Now let D be a minimum dominating set of G of size at least $2^{d+1} + d + 1$ and suppose that for any $u, v \in D$, $d_G(u, v) \geq 3$, that is, no two vertices in D have a common neighbor (note that if there exist $u, v \in D$ such that $N_G(u) \cap N_G(v) \neq \emptyset$, then $D \cup \{x\}$, where $x \in N_G(u) \cap N_G(v)$, is a dominating set for G of size $\gamma(G) + 1$ containing two edges and so, $ct_\gamma(G) \leq 2$ by Theorem 2.1). Since $|D| \geq 2^{d+1} + d + 1$, there must exist $1 \leq i \leq \ell$ such that $|A_i \cap D| \geq 3$; observe first that $N_G(A_i) \cap X = \emptyset$, since $d_G(x, y) \geq 3$ for any $x, y \in A_i \cap D$, by assumption. For the same reason, $d_{G'}(N_{G'}(x), N_{G'}(y)) \geq 1$. Furthermore, $d_{G'}(N_{G'}(x), N_{G'}(y)) \leq 2$ since G' is P_6-free. Now suppose that there exists $x, y \in A_i \cap D$ such that $d_{G'}(N_{G'}(x), N_{G'}(y)) = 2$ and let $a \in N_{G'}(x)$ and $b \in N_{G'}(y)$ be such that $d_{G'}(N_{G'}(x), N_{G'}(y)) = d_{G'}(a, b)$. Let $c \in V(G')$ be the internal vertex in a shortest path from a to b in G'. Then, $D' = (D \setminus \{x, y\}) \cup \{a, b, c\}$ is a dominating set of G. Indeed, since $N_G(A_i) \cap X = \emptyset$, vertices in X remain dominated in D'; and if there exists $t \in N_{G'}(x) \cup N_{G'}(y)$ which is not dominated by a vertex in D', say $t \in N_{G'}(x)$ without loss of generality, then t, x, a, c, b, y induce a P_6, a contradiction. Thus, D' is a dominating set of size $\gamma(G) + 1$ containing two edges and so, $ct_\gamma(G) \leq 2$ by Theorem 2.1. Assume henceforth that $d_{G'}(N_{G'}(x), N_{G'}(y)) = 1$ for any $x, y \in A_i \cap D$. In the following, we let $u, v, w \in A_i \cap D$.

Suppose that for any $a \in \{u, v, w\}$, no vertex in $N_{G'}(a)$ is adjacent to both a vertex in $N_{G'}(b)$ and a vertex in $N_{G'}(c)$ for $b, c \in \{u, v, w\} \setminus \{a\}$. By assumption, there must then exist $x \in N_{G'}(a)$, $y, r \in N_{G'}(b)$ and $s \in N_{G'}(c)$, with $a, b, c \in \{u, v, w\}$, such that $xy, rs \in E(G')$. By assumption, x is nonadjacent to s, y is nonadjacent to s and r is nonadjacent to x (see Fig. 9). But then, if $yr \notin E(G')$, a, x, y, b, r, s induce a P_6; and if $yr \in E(G')$, a, x, y, r, s, c induce a P_6, a contradiction in both cases. It follows that there must exist $x \in N_{G'}(l)$, for some $l \in \{u, v, w\}$, such that x is adjacent to both a vertex $y \in N_{G'}(p)$ and a vertex $z \in N_{G'}(q)$, with $p, q \in \{u, v, w\} \setminus \{l\}$.

Suppose first that y and z are nonadjacent. Then, $D' = (D \setminus \{p, q\}) \cup \{x, y, z\}$ is a dominating set for G; indeed, if there exists $h \in N_{G'}(p)$ such that h is not dominated by a vertex in D' (the case where $h \in N_{G'}(q)$ is symmetric) then, h, p, y, x, z, q induce a P_6 (see Fig. 10(a)), a contradiction. Furthermore, since $N_G(A_i) \cap X = \emptyset$, vertices in X remain dominated in D'.

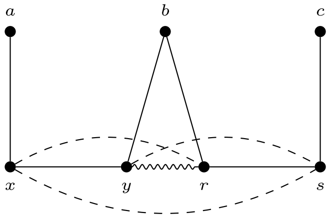
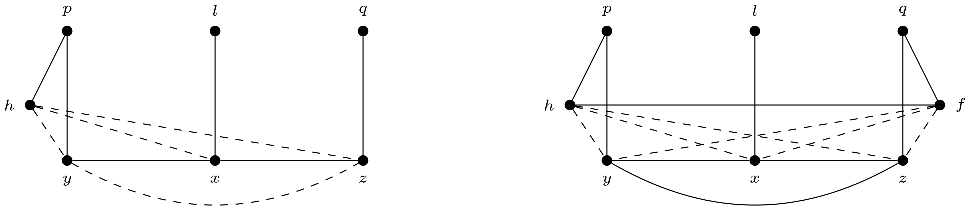


Fig. 9. No neighbor of l is adjacent to both a neighbor of p and a neighbor of q in G', for any $l, p, q \in \{u, v, w\}$ (dashed lines correspond to nonedges and the serpentine line indicates that the two vertices may or may not be adjacent).



(a) y and z are nonadjacent. (b) y and z are adjacent.

Fig. 10. There exist $x \in N_{G'}(l)$ such that x is adjacent to both a vertex $y \in N_{G'}(p)$ and a vertex $z \in N_{G'}(q)$ with $p, q, l \in \{u, v, w\}$ (dashed lines correspond to nonedges).

Thus, D' is a dominating set for G and since D' is of size $\gamma(G) + 1$ and contains two edges, we conclude by Theorem 2.1 that $ct_\gamma(G) \leq 2$.

Second, suppose that y and z are adjacent. We claim that there exists $t \in \{u, v, w\}$ such that $D' = (D \setminus \{u, v, w\}) \cup \{t, x, y, z\}$ is a dominating set for G; since D' is of size $\gamma(G)+1$ and contains two edges, this would conclude the proof (see Theorem 2.1). First recall that since $N_G(A_i) \cap X = \emptyset$, vertices in X remain dominated in D'. Now, if $D_l = (D \setminus \{p, q\}) \cup \{x, y, z\}$ is not dominating, then there exists $h \in N_{G'}(p) \cup N_{G'}(q)$ such that h is not dominated by a vertex in D_l, say $h \in N_{G'}(p)$ without loss of generality. But then, $D_p = (D \setminus \{l, q\}) \cup \{x, y, z\}$ must be dominating; indeed, if there exists $f \in N_{G'}(l) \cup N_{G'}(q)$ such that f is not dominated by a vertex in D_p, say $f \in N_{G'}(q)$ without loss of generality, then f and h must be adjacent (h, p, y, z, q, f would otherwise induce a P_6) and so, q, f, h, p, y, x induce a P_6 (see Fig. 10(b)), a contradiction which concludes the proof. □

In particular, when $d = 0$, we obtain the following corollary.

Corollary 3.24. *If G be a P_6-free graph with $\gamma(G) \geq 3$, then $ct_\gamma(G) \leq 2$.*

With a similar proof to that of Theorem 3.22, we obtain the following result from Corollary 3.24.

Theorem 3.25. *2-Edge Contraction(γ) is polynomial-time solvable on P_6-free graphs.*

3.4. *H*-free graphs

The results obtained in Sections 3.1 and 3.3 lead to a complexity dichotomy for H-free graphs when H is connected. Indeed, since 1-Edge Contraction(γ) is NP-hard when restricted to $\{C_3, \ldots, C_\ell\}$-free graphs, for any $\ell \geq 3$ (see Corollary 3.12), it follows that 1-Edge Contraction(γ) is NP-hard for H-free graphs when H contains a cycle. If H is a tree with a vertex of degree at least three, we conclude by Corollary 3.12 that 1-Edge Contraction(γ) is coNP-hard for H-free graphs. We are now left with the case in which H is a path. Theorem 3.1 shows that if H is a path of length at least 6, then 1-Edge Contraction(γ) is NP-hard for H-free graphs; and by Theorem 3.22, 1-Edge Contraction(γ) is polynomial-time solvable on H-free graphs if $H \subseteq_i P_5$. We therefore obtain the following result.

Corollary 3.26. *Let H be a connected graph. If $H \subseteq_i P_5$ then 1-Edge Contraction(γ) is polynomial-time solvable on H-free graphs, otherwise it is NP-hard or coNP-hard.*

If the graph H is not required to be connected, we know the following. As previously mentioned, 1-Edge Contraction(γ) is NP-hard (resp. coNP-hard) on H-free graphs when H contains a cycle (resp. an induced claw). Thus, there remains to consider the case where H is a linear forest. Theorem 3.1 and Corollary 3.7 show that if H contains either a P_6, a $P_4 + P_2$ or a $2P_3$ as an induced subgraph, then 1-Edge Contraction(γ) is NP-hard or coNP-hard on H-free graphs. On the other hand, by Theorem 3.22 and Proposition 3.17(c), 1-Edge Contraction(γ) is polynomial-time solvable on H-free graphs if $H \subseteq_i P_5+pK_1$. Therefore, in order to obtain a complexity dichotomy for H-free graphs, there remains to determine the complexity status of the problem restricted to H-free graphs when H is an induced subgraph of $P_3+qK_2+pK_1$ with at least one edge.

4. Complexity of Vertex Deletion(γ)

In this section, we investigate the complexity of Vertex Deletion(γ). Recall that by Proposition 2.4, Vertex Deletion(γ) is equivalent to Edge Addition(γ). Thus, the results presented in this section also hold for the Edge Addition(γ) problem. We first show that even for $k = 1$, Vertex Deletion(γ) is already a hard problem both in the classical and in the parameterized complexity setting.

Theorem 4.1. *Vertex Deletion(γ) with $k = 1$ is* NP*-hard and* W[1]*-hard parameterized by γ on split graphs.*

Proof. We give a reduction from Dominating Set. Given an instance (G, ℓ) for Dominating Set, we construct an instance $(G', 1)$ for Vertex Deletion(γ) as follows. We denote by $\{v_1, \ldots, v_n\}$ the vertex set of G. The vertex set of the graph G' is given by $V(G') = V_0 \cup \cdots \cup V_\ell \cup \{x_0, \ldots, x_\ell, y\}$, where each V_i is a copy of the vertex set of G. We denote the vertices of V_i by $v^i_1, v^i_2, \ldots, v^i_n$. The adjacencies in G' are then defined as follows:

- $V_0 \cup \{x_0, \ldots, x_\ell\}$ is a clique;
- $yx_0, x_0v^1_1 \in E(G')$;
- x_1 is adjacent to all the vertices in $V_1 \setminus \{v^1_1\}$;

and for $1 \leq i \leq \ell$,

- V_i is an independent set;
- v^i_j is adjacent to $\{v^0_a \mid v_a \in N_G[v_j]\}$ for any $1 \leq j \leq n$;
- if $i \neq 1$, x_i is adjacent to all the vertices of V_i.

Claim 15. $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

Proof. It is clear that $\{x_0, x_1, \ldots, x_\ell\}$ is a dominating set of G'; thus, $\gamma(G') \leq \ell + 1$. If $\gamma(G) \leq \ell$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, it is easily seen that $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_0\}$ is a dominating set of G'. Thus, $\gamma(G') \leq \gamma(G) + 1$ and so, $\gamma(G') \leq \min\{\gamma(G) + 1, \ell + 1\}$. Now, suppose to the contrary that $\gamma(G') < \min\{\gamma(G) + 1, \ell + 1\}$ and consider a minimum dominating set D' of G'. We first make the following simple observation.

Observation 20. *For any dominating set D of G', $D \cap \{y, x_0\} \neq \emptyset$.*

Now, since $\gamma(G') < \ell + 1$, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$ (otherwise, $\{x_1, \ldots, x_\ell\} \subset D'$ and combined with Observation 20, D' would be of size at least $\ell + 1$). If x_1 is the only vertex of $\{x_1, \ldots, x_\ell\}$ not belonging to D', then $|D' \cap \{x_1, \ldots, x_\ell\}| = \ell - 1$; and since D' must dominate the vertices of $V_1 \setminus \{v^1_1\}$, combined with Observation 20, we obtain that $|D'| \geq \ell + 1$, a contradiction. Therefore, there exists $i \geq 2$ such that $x_i \notin D'$; but then, $D'' = D' \cap V_0$ must dominate every vertex in V_i, and so $|D''| \geq \gamma(G)$. Since $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$), we then have $\gamma(G) \leq |D'| - 1$, a contradiction. Thus, $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$. ▲

We now show that (G, ℓ) is a Yes-instance for Dominating Set if and only if $(G', 1)$ is a Yes-instance for Vertex Deletion(γ).

First assume that $\gamma(G) \leq \ell$. Then, $\gamma(G') = \gamma(G)+1$ by the previous claim, and if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, then $\{v^0_{i_1}, \ldots, v^0_{i_k}, y\}$ is a minimum dominating set in which y is a selfish vertex. Thus by Lemma 2.3, $(G', 1)$ is a Yes-instance for Vertex Deletion(γ).

Conversely, assume that $(G', 1)$ is a Yes-instance for Vertex Deletion(γ) i.e., there exists a minimum dominating set D' of G' which contains a selfish vertex (see Lemma 2.3). Note that it cannot be the case that $\{x_1, \ldots, x_\ell\} \subset D'$; indeed, if it were the case, since v^1_1 and y are not dominated by $\{x_1, \ldots, x_\ell\}$, we would have by Observation 20 and Claim 15, that $x_0 \in D'$. But then, $D' = \{x_0, \ldots, x_\ell\}$ contains no selfish vertex, a contradiction. Therefore, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$. If $i \neq 1$, it follows that $D'' = D' \cap V_0$ must dominate every vertex in V_i and thus, $|D''| \geq \gamma(G)$. But $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$) and so by Claim 15, $\gamma(G) \leq |D'| - 1 \leq (\ell+1) - 1$ that is, (G, ℓ) is a Yes-instance for Dominating Set. Otherwise, x_1 is the only vertex of $\{x_1, \ldots, x_\ell\}$ not belonging to D' and so, $|D' \cap \{x_1, \ldots, x_\ell\}| = \ell - 1$. But since D' must dominate the vertices of V_1, we have $|D' \cap (V_0 \cup \{x_0\})| \geq \gamma(G)$; thus, $|D'| \geq \gamma(G) + \ell - 1$. Since $\ell \geq 2$, it then follows that $|D'| \geq \gamma(G) + 1$ and so by Claim 15, $\gamma(G) \leq \ell$ that is, (G, ℓ) is a Yes-instance for Dominating Set. □

In view of Theorem 4.1, we further investigate, in the remainder of this section, the complexity of the Vertex Deletion(γ) problem restricted to the case $k = 1$, that is, the 1-Vertex Deletion(γ) problem.

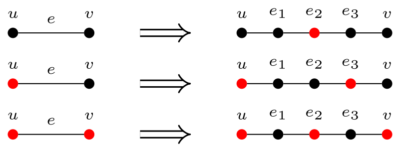


Fig. 11. Constructing a dominating set D' of G' from a dominating set D of G (vertices in red belong to the corresponding dominating set).

Proposition 4.2. *Let G be a graph and let G' be the graph obtained by 3-subdividing every edge of G. Then G is a* Yes*-instance to 1-*Edge Contraction*(γ) if and only if G' is a* Yes*-instance to 1-*Vertex Deletion*(γ).*

Proof. Let G' be the graph obtained by the 3-subdivision of every edge of G. Given an edge $e = uv$ of G, we denote by $ue_1e_2e_3v$ the path in G' resulting from the 3-subdivision of the edge uv. Suppose G is a Yes-instance to 1-Edge Contraction(γ). By Theorem 2.1(i), G has a dominating set D that is not independent. We construct from D a minimum dominating set D' for G' containing a selfish vertex, as follows. For any edge $e = uv$ of G, if $D \cap \{u, v\} = \emptyset$, then $D' \cap \{u, e_1, e_2, e_3, v\} = \{e_2\}$. If $|D \cap \{u, v\}| = 1$, then we may assume without loss of generality that $u \in D$ and we let $D' \cap \{u, e_1, e_2, e_3, v\} = \{u, e_3\}$. Finally, if $\{u, v\} \subset D$, then $D' \cap \{u, e_1, e_2, e_3, v\} = \{u, e_2, v\}$ (see Fig. 11).

It is easy to see that D' is indeed dominating; and by Claim 2, we have that $\gamma(G') = \gamma(G) + |E(G)|$, which shows that D' is also of minimum size. Furthermore, by construction, if D contains an edge $e = uv$, then the vertex e_2 in the corresponding path is selfish in D'. Thus, by Lemma 2.3, G' is a Yes-instance for 1-Vertex Deletion(γ).

Conversely, if G' is a Yes-instance for 1-Vertex Deletion(γ) then by Lemma 2.2, G' is a Yes-instance for 1-Edge Contraction(γ) and we conclude by Lemma 3.3 that G is a Yes-instance for 1-Edge Contraction(γ). □

We deduce the following from Corollary 3.4 together with Proposition 4.2.

Corollary 4.3. *1-*Vertex Deletion*(γ) is* NP*-hard on bipartite graphs and $\{C_3, \dots, C_\ell\}$-free graphs, for every fixed $\ell \geq 3$.*

We now consider the class of claw-free graphs. We first prove the following:

Lemma 4.4. *Let G be a graph and let G' be the graph obtained by 3-subdividing every edge of G not belonging to a triangle. Then G is a* Yes*-instance for 1-*Edge Contraction*(γ) if and only if G' is a* Yes*-instance for 1-*Edge Contraction*(γ).*

Proof. Let $G = (V, E)$ be a graph and let $E_2 \subseteq E$ be the set of edges not belonging to any triangle. In the following, given an edge $e \in E_2$, we denote by e_1, e_2 and e_3 the three new vertices resulting from the 3-subdivision of the edge e. Note that by Claim 2, we have that $\gamma(G') = \gamma(G) + |E_2|$.

Assume first that G is a Yes-instance for 1-Edge Contraction(γ) and let D be a minimum dominating set of G containing an edge $f = xy$ (see Theorem 2.1(i)). Let D' be the minimum dominating set of G' constructed as follows. We first add to D' every vertex in D. Then for any edge $e = uv \in E_2$, we proceed as described in the proof of Claim 2 (see Fig. 3). Now, either $f \in E_2$ in which case D' contains the edge xf_1, or $f \in E \setminus E_2$ in which case D' contains the edge f. In both cases, we conclude by Theorem 2.1(i) that G' is a Yes-instance for 1-Edge Contraction(γ).

Conversely, assume that G' is a Yes-instance for 1-Edge Contraction(γ), that is, there exists a minimum dominating set D' of G' containing an edge f (see Theorem 2.1(i)). First note that we may assume that for any edge $e = uv \in E_2$, $\{e_1, e_3\} \not\subset D'$; indeed, if $\{e_1, e_3\} \subset D'$ then, by minimality of D', we have that $v \notin D'$ (with v adjacent to e_3) for otherwise $D' \setminus \{e_3\}$ is a dominating set of G' of size strictly smaller than that of D', a contradiction (also note that by minimality of D', $e_2 \notin D'$). But then, $(D' \setminus \{e_3\}) \cup \{v\}$ is also a minimum dominating set of G' also containing the edge f; indeed, since both e_2 and v are not contained in D', e_3 is not an endvertex of f. Now let D be the minimum dominating set of G constructed as follows. We first add to D every vertex of $D' \cap V$. Now for any edge $uv \in E_2$, if $e_1 \in D'$ (with e_1 adjacent to u), we add v to D; and if $e_3 \in D'$, we add u to D. Note that by Observation 4, D has size $|D'| - |E_2|$, that is, D is a minimum dominating set of G. We now claim that D contains an edge. Indeed, if $f \in E \setminus E_2$, then D contains f. Otherwise, we distinguish cases depending on whether $f = ue_1$ or $f = e_1e_2$ for some edge $e = uv \in E_2$ (note that the cases where $f = e_3v$ or $f = e_2e_3$ are symmetric to those considered).

Suppose first that $f = ue_1$. Then by construction, $u, v \in D$ and thus, D contains the edge uv. Now if $f = e_1e_2$ then again, $v \in D$ by construction. But then, by minimality of D', both e_3 and v do not belong to D' for otherwise $D' \setminus \{e_2\}$ would be a dominating set of G' of size strictly smaller than that of D', a contradiction. It follows that v is dominated in G' by some vertex $x \in D'$ different from e_3 and v. Then, either $x \in V$ in which case $x \in D$ by construction; or $x = e'_1$ for some edge $e' = vw$ and so, $w \in D$ by construction. In both cases, D contains an edge, namely vx and vw respectively, which concludes the proof. □

Theorem 4.5. *1-*Vertex Deletion*(γ) is* **coNP***-hard on subcubic claw-free graphs.*

Proof. We reduce from 1-Edge Contraction(γ) restricted to subcubic claw-free graphs which is coNP-hard by Corollary 3.12. Consider an instance $G = (V, E)$ of this problem and let G' be the graph obtained by 3-subdividing every edge of G not belonging to a triangle. It is easy to see that G' is a subcubic claw-free graph. We next show that G' is a Yes-instance for 1-Vertex Deletion(γ) if and only if G is a Yes-instance for 1-Edge Contraction(γ).

Assume first that G' is a Yes-instance for 1-Vertex Deletion(γ). Then by Lemma 2.2, G' is a Yes-instance for 1-Edge Contraction(γ) and we conclude by Lemma 4.4 that G is a Yes-instance for 1-Edge Contraction(γ).

Conversely, assume that G is a Yes-instance for 1-Edge Contraction(γ) and let D be a minimum dominating set of G containing an edge $f = xy$ (see Theorem 2.1(i)). In the following, $E_2 \subset E$ denotes the set of edges in G not belonging to any triangle. Now let D' be the minimum dominating set of G' constructed as follows. We first add to D' every vertex in D. Then for any edge $e = uv \in E_2$, we proceed as described in the proof of Proposition 4.2 (see Fig. 11). Observe that $|D'| = |D| + |E_2|$ and so by Claim 2, D' is a minimum dominating set of G'. Now if $xy \in E_2$, then $f_2 \in D'$ is a selfish vertex and so, we conclude by Lemma 2.3 that G' is a Yes-instance for 1-Vertex Deletion(γ). Now suppose that $f \in E \setminus E_2$. We claim that either x or y is incident in G to an edge not contained in any triangle. Indeed, if every edge incident to x and y belongs to a triangle, then since G is subcubic, either x and y have degree 2 and thus belong to only one triangle, or x and y are the two vertices of degree 3 in a diamond. But then, $D \setminus \{x\}$ is a dominating set of G of size strictly smaller than that of D, a contradiction. Thus, one edge e incident to either x or y is not contained in a triangle, say e is incident to x without loss of generality. Then by construction, e_1 does not belong to D' (with e_1 adjacent to x) and $e_3 \in D'$ and so, $(D' \setminus \{x\}) \cup \{e_1\}$ is a minimum dominating set of G' containing a selfish vertex, namely e_1, which concludes the proof. $\square$

We now deal with cases in which 1-Vertex Deletion(γ) is tractable. The following statement is similar to Proposition 3.17; it is also based on brute force and relies on the fact that G is a Yes-instance for 1-Vertex Deletion(γ) if and only if G has a dominating set that contains a selfish vertex (see Lemma 2.3). Since the proof of Proposition 4.6 is similar to that of Proposition 3.17, it is omitted here.

Proposition 4.6. *1-Vertex Deletion(γ) can be solved in polynomial time for a graph class $\mathcal{C}$, if either*

(a) $\mathcal{C}$ is closed under vertex deletions and Dominating Set *can be solved in polynomial time on $\mathcal{C}$; or*
(b) for every $G \in \mathcal{C}$, $\gamma(G) \le q$, where q is some fixed constant; or
(c) $\mathcal{C}$ is the class of $\{H + K_1\}$-free graphs, where $|V(H)| = q$ is a fixed constant and 1-Vertex Deletion(γ) is polynomial-time solvable on H-free graphs.

Note that if G is a connected P_4-free graph, then G has a dominating set of size at most 2. Thus, from Propositions 4.6(b) and 4.6(c), we obtain the following corollary.

Corollary 4.7. *If $H \subseteq_i (P_4 + kP_1)$, then 1-Vertex Deletion(γ) is polynomial time solvable on H-free graphs.*

We finally note that the results in this section lead to a complexity dichotomy for 1-Vertex Deletion(γ) restricted to H-free graphs. Indeed, it follows from Theorem 4.5 (resp. Corollary 4.3) that 1-Vertex Deletion(γ) is coNP-hard (resp. NP-hard) restricted to claw-free graphs (resp. $\{C_3, \ldots, C_\ell\}$-free graphs). Thus, if H contains a cycle (resp. an induced claw), 1-Vertex Deletion(γ) is NP-hard (resp. coNP-hard) on H-free graphs. Since any split graph is $2K_2$-free, Theorem 4.1 shows that 1-Vertex Deletion(γ) is NP-hard on H-free graphs if H contains an induced $2K_2$; and by Corollary 4.7, 1-Vertex Deletion(γ)is polynomial time solvable on H-free graphs if $H \subseteq_i (P_4 + kP_1)$. We thus obtain the following dichotomy.

Corollary 4.8. *1-Vertex Deletion(γ) is polynomial time solvable on H-free graphs if and only if $H \subseteq_i (P_4 + kP_1)$.*

5. Conclusion

In this paper, we investigate the complexity of the k-Edge Contraction(γ) problem, for $k = 1, 2$. In particular, we establish a complexity dichotomy for 1-Edge Contraction(γ) on H-free graphs when H is connected. If we do not require H to be connected, there only remains to settle the complexity status of 1-Edge Contraction(γ) restricted to H-free graphs when H is an induced subgraph of $P_3 + qK_2 + pK_1$ with at least one edge.

Furthermore, we study the Vertex Deletion(γ) and Edge Addition(γ) problems and show that surprisingly, they are equivalent. As opposed to the case of edge contractions, there is no constant upper bound on the number of vertex deletions or edge contractions necessary to decrease the domination number of a graph. We show that even for $k = 1$, Vertex Deletion(γ) is NP-hard and W[1]-hard parameterized by γ, thus ruling out the possibility of algorithms running in FPT- or even XP-time parameterized by k for this problem, unless P=NP. For this reason, we focus on the 1-Vertex Deletion(γ) problem and obtain a complexity dichotomy for this problem restricted to H-free graphs. It would however be interesting to obtain such a dichotomy for higher values of k.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

P. T. Lima received funding from the Research Council of Norway via the project CLASSIS, grant number 249994. This work was developed when P. T. Lima was visiting the University of Fribourg funded by a Scholarship of the University of Fribourg.

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