Fundamental polytopes of metric trees via parallel connections of matroids

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A B S T R A C T

We tackle the problem of a combinatorial classification of finite metric spaces via their *fundamental polytopes*, as suggested by Vershik (2010).

In this paper we consider a hyperplane arrangement associated to every split pseudometric and, for tree-like metrics, we study the combinatorics of its underlying matroid.

- We give explicit formulas for the face numbers of fundamental polytopes and Lipschitz polytopes of all tree-like metrics.
- We characterize the metric trees for which the fundamental polytope is simplicial.

1. Introduction

1.1. Polytopes associated to metric spaces

The study of *fundamental polytopes* of finite metric spaces was proposed by Vershik [24] as an approach to a combinatorial classification of metric spaces, motivated by its connections to the transportation problem. Indeed, the Kantorovich–Rubinstein norm associated to the finite case of the transportation problem is an extension (uniquely determined by some conditions) of the Minkowski–Banach norm associated to the fundamental polytope (see [17, Theorem 1] for details). The polar dual of the fundamental polytope affords a more direct description: it consists of all real-valued functions with Lipschitz constant 1, and it is called *Lipschitz polytope*. As polar duality

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preserves all combinatorial data, the combinatorial classification of Lipschitz polytopes is equivalent to that of fundamental polytopes.

Very little is known to date about the combinatorics of these polytopes, aside from the aforementioned work of Vershik. For instance, their f-vectors[1] are unknown in general. Gordon and Petrov [8] obtained bounds for the number of possible different f-vectors given the size of the metric space. The same authors also examined "generic metric spaces" (see Definition 5.4), computing their f-vectors (which, in this class, only depend on the number of elements in the space). Further study of fundamental polytopes of generic metrics appeared in [12,13] especially around a connection with duals of cyclohedra and Bier spheres. The special case of fundamental polytopes of full trees (Definition 2.8) fits into the framework of symmetric edge polytopes, which are themselves in the focus of active research, see e.g. [10,19].

In this paper we compute the f-vectors of Lipschitz polytopes for all tree-like pseudometric spaces, hence also of fundamental polytopes of tree-like metric spaces. Moreover, we characterize exactly which metric trees give rise to fundamental polytopes that are simplicial. In particular, this characterization shows that our computations do not fall under the case considered in [8].

1.2. Arrangements of hyperplanes and matroids

We call "arrangement of hyperplanes" a finite set of hyperplanes (i.e., linear codimension 1 subspaces) of a real vector space and refer to Section 2.2 for some basics about these well-studied objects. Here we only point out that the enumerative combinatorics of an arrangement is governed by the associated *matroid*, an abstract combinatorial object encoding the intersection pattern of the hyperplanes (see Section 2.6).

In particular, such an arrangement subdivides the unit sphere into a polyhedral complex $K_{\mathscr{A}}$ which is "combinatorially dual" (see Remark 2.15) to the zonotopes arising as Minkowski sum of any choice of normal vectors for the hyperplanes. The enumeration of the faces of these polyhedral complexes in terms of the arrangement's matroid, due to Zaslavsky [26], has been one of the earliest successful applications of matroid theory. More recently, Cuntz and Gies [5] have given a necessary and sufficient condition for $K_{\mathscr{A}}$ to be a simplicial complex, again in terms of enumerative invariants of the associated matroid.

In this paper we notice that fundamental polytopes and Lipschitz polytopes of finite metric trees are related to the complex $K_{\mathscr{A}}$ of an arrangement that is canonically associated to the metric space, and then to exploit the fact that the arrangement's matroid has a nice decomposition as a parallel connection of simple sub-matroids.

1.3. Structure of the paper and main results

We start Section 2 by recalling the main definitions and some results about polytopes associated to metric spaces, tree-like metrics, systems of splits, arrangements of hyperplanes and matroids. In particular, we focus on an arrangement of hyperplanes $\mathscr{A}(\mathcal{S})$ that can be associated to any system of splits $\mathcal{S}$. This arrangement and the associated matroid $\mathscr{M}(\mathcal{S})$ (which were already considered in a different context [16]), provide the combinatorial underpinning of our considerations.

(1) We notice that the fundamental polytope of any tree-like finite metric space is combinatorially isomorphic to the complex $K_{\mathscr{A}(\mathcal{S})}$, where $\mathcal{S}$ is the unique system of splits in the Bandelt–Dress decomposition of the given space. In fact, the Lipschitz polytope of such spaces is the zonotope defined as the Minkowski sum of a certain choice of normal vectors for the hyperplanes of $\mathscr{A}(\mathcal{S})$: this is the content of Theorem 3.1, see also Remark 3.2.
(2) We compute the intersection poset of $\mathscr{A}(\mathcal{S})$ (and the closure operator of the associated matroid) from the combinatorics of the split system (Theorem 4.4).

[1] The f-*vector* of a polytope (or of any polyhedral complex) is the list of integers encoding the number of faces of each dimension.

(3) We prove that the matroid of $\mathscr{A}(\mathcal{S})$ decomposes as a "parallel connection" of elementary building blocks that can be read off the (unique) metric tree representing the metric at hand. This allows us to give explicit formulas for the face numbers of fundamental and Lipschitz polytopes in terms of the combinatorial structure of the tree (Theorem 4.21), building on Zaslavsky's theorems and on results of Bonin and De Mier on characteristic polynomials of parallel connections.
(4) Our formulas allow us to use Cuntz and Gies' criterion for simpliciality of arrangements in order to prove that the fundamental polytope of a tree-like metric is simplicial if and only if to every vertex of the metric tree is associated an element of the metric space (Theorem 5.3). Our characterization shows in particular that no tree-like metric is generic in the sense of Gordon and Petrov [8] (Corollary 5.5).

1.4. *Related work*

The study of metric spaces by means of associated polyhedral complexes is a classical topic, going back at least to work of Buneman [3] and Bandelt-Dress [1], and driven in part by application to the study of phylogenetic trees. After a first version of this paper circulated, we learned about further recent literature that helped us to contextualize our work. Koichi [16] recently gave a uniform description of the approaches by Buneman and Bandelt–Dress, building on Hirai's [11] polyhedral split decomposition method, where a metric is viewed as a polyhedral "height function" defined on a point configuration. In [16] we also find the defining forms of the arrangement $\mathscr{A}(\mathcal{S})$. (On the other hand, the hyperplanes associated to splits in [11, p. 350] do not coincide with ours.) Motivated by the connections to tropical convexity [6,23], Herrmann and Joswig [9] studied split complexes of general polytopes and, in the process, consider an arrangement of "split hyperplanes" associated to every split metric. In this respect we notice that, even if each of our hyperplanes can be expressed in the form [9, Equation (9)], the arrangement $\mathscr{A}(\mathcal{S})$ is not one of the arrangements considered in [9] (see Remark 2.16). Moreover, the matroid we consider is different from the matroid whose basis polytope is cut from the hypersimplex by a set of compatible split hyperplanes, which is studied by Joswig and Schröter in [15].

Lipschitz polytopes of finite metric spaces are weighted digraph polyhedra in the sense of Joswig and Loho [14], who give some general results about dimension, face structure and projections [14, §2.1, 2.2, 2.6] but mostly focus on the case of "braid cones" which does not apply to our context. We close by mentioning that the polyhedra considered, e.g., in the above-mentioned work of Hirai [11, Formula (4.1)] are different from the Lipschitz polytopes we consider here: in fact, such polyhedra are (translated) zonotopes for all split-decomposable metrics [11, Remark 4.10], while – for instance – the Lipschitz polytope of any split-decomposable metric on 4 points is only a zonotope if the associated split system is compatible.

2. The main characters

2.1. *Metric spaces and their polytopes*

Definition 2.1. Let X be a set. A *metric* on X is a symmetric function $d : X \times X \to \mathbb{R}_{\geq 0}$ with the following properties.

(1) For all $x, y \in X$, $d(x, y) = 0$ implies $x = y$.
(2) For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ ("triangle inequality").

If requirement (1) is dropped, then d is called a *pseudometric*. The pair (X, d) is then called a metric space (resp. pseudometric space).

In this paper we will focus on *finite* metric spaces, i.e., metric spaces (X, d) where $|X| < \infty$. We will tacitly assume so throughout.

Definition 2.2. Let (X, d) be a (finite) metric space with $|X| > 1$. Consider the vector space $\mathbb{R}^X$ with its standard basis $\{\mathbb{1}_k\}_{k \in X}$, i.e.,

$$(\mathbb{1}_k)_i := \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Following Vershik [24] we define the *fundamental polytope* of (X, d) as

$$P_d(X) := \operatorname{conv}\{e_{i,j} \mid i, j \in X,\ i \neq j\},$$

where

$$e_{i,j} := \frac{\mathbb{1}_i - \mathbb{1}_j}{d(i,j)}.$$

This polytope is contained (and full-dimensional) in the subspace

$$V_0(X) = \{x \in \mathbb{R}^X \mid \textstyle\sum_i x_i = 0\}.$$

Definition 2.3. Let (X, d) be a (finite) pseudometric space with $|X| > 1$.
The *Lipschitz polytope* of (X, d) is given as an intersection of halfspaces by

$$\operatorname{LIP}(X, d) := \left\{x \in \mathbb{R}^X \mid \textstyle\sum_i x_i = 0,\ x_i - x_j \leq d(i,j)\ \forall i, j \in X\right\}. \tag{1}$$

This polytope is contained (and full-dimensional) in the subspace

$$V(X, d) := \{x \in \mathbb{R}^X \mid \textstyle\sum_i x_i = 0,\ \ x_i = x_j \text{ whenever } d(i,j) = 0\}.$$

Remark 2.4 (*On Lipschitz Polytopes*)**.** For metric spaces our definition specializes to the standard definition of the Lipschitz polytope, e.g., as given in [24]. We remark that, although related, this is not the set of Lipschitz functions considered in the work of Wu, Xu and Zhu on graph indexed random walks [25].

Remark 2.5 (*On Polytopes*)**.** We point the reader to the book by Ziegler [27] for terminology and basic facts about polytopes and fans. Here let us only mention that the combinatorics of a given polytope P is encoded in its *poset of faces* $\mathscr{F}(P)$ which, here, we take to be the set of all faces of P including the empty face ordered by inclusion. A rougher, but very important enumerative invariant of a polytope is its *face numbers* $f_0^P, \ldots, f_{\dim(P)}^P$, where

$$f_i^P = |\{i - \text{dimensional faces of } P\}|.$$

It is customary to consider the empty face as a face of "dimension -1", thus to write $f_{-1}^P = 1$ and to fit these numbers into the *f-polynomial* of P, defined as

$$f^P(t) := f_{-1}^P t^{m+1} + f_0^P t^m + \cdots + f_m^P$$

where we write $m := \dim(P)$.

The problem posed by Vershik [24] is to study the face numbers and face structure of the fundamental polytope of a metric space. We will do so by focussing on the associated Lipschitz polytope, whose combinatorics is "dual" to that of the fundamental polytope in the following sense.

Remark 2.6. A look at Theorem 2.11.(vi) of [27] shows that indeed, for every *metric* space (X, d) the polytopes $P_d(X)$ and $\operatorname{LIP}(X, d)$ are *polar dual* to each other (with respect to the ambient space $V_0(X)$, cf. [27, Definition 2.10]). Polar duality induces an isomorphism of posets $\mathscr{F}(P_d(X)) \cong \mathscr{F}(\operatorname{LIP}(X, d))^{op}$ and, in particular, the equality $f_i^{P_d(X)} = f_{\dim(P_d(X))-1-i}^{\operatorname{LIP}(X,d)}$.

Example 2.7 (*Metric Spaces from Weighted Graphs*)**.** Let G be a finite, connected and simple graph. Write $V(G)$ and $E(G)$ for the set of vertices, resp. edges of G.

A *weighting* of G is any function $w : E(G) \to \mathbb{R}_{>0}$, and the pair (G, w) is called a *weighted graph*. Then, setting

$$d_w(v, v') := \min \left\{ w(e_1) + \cdots + w(e_k) \mid e_1, \ldots, e_k \text{ an edge-path joining } v \text{ with } v' \right\}$$

the pair $(V(G), d_w)$ is a metric space.

Recall that the *neighborhood* $N(v)$ of a vertex v in a (simple) graph is the set of edges incident to v. The *degree* of v is then the number $\deg(v) := |N(v)|$ of such edges. A *tree* is a graph in which every pair of vertices is connected by a unique path. A *leaf* in a tree is any vertex of degree 1.

Definition 2.8 (*Tree-Like Metrics and X-Trees*)**.** Let X be a finite set. An X*-tree* is a pair (T, ϕ), where T is a tree and $\phi : X \to V(T)$ is a map whose image contains every vertex of V that is incident to at most two edges, i.e., $\{v \in V(T) \mid \deg(v) \leq 2\} \subseteq \phi(X)$.

A (pseudo)metric d on a set X is called a *tree-like (pseudo)metric* if there exists an X-tree (T, ϕ) and a weighting w of T such that for all $x, y \in X$

$$d(x, y) = d_w(\phi(x), \phi(y)).$$

("d is induced by a weighted X-tree"). The pseudometric d is a metric if and only if ϕ is injective. When ϕ is bijective, we call (X, d) a *full tree*.

2.2. Arrangements of hyperplanes

Let V denote a finite-dimensional real vector space, say of dimension m. An arrangement of hyperplanes (or, for short, *arrangement*) in V, is a finite set $\mathscr{A}$ of hyperplanes (i.e., linear subspaces of codimension 1). Such an arrangement defines a polyhedral fan in V, and we let $\mathscr{F}(\mathscr{A})$ denote the poset of all faces of this fan, partially ordered by inclusion. We write $f_i^{\mathscr{A}}$ for the number of faces of this fan of dimension i, for all $i = 0, \ldots, m$, and we arrange these numbers into the f*-polynomial* of $\mathscr{A}$,

$$f^{\mathscr{A}}(t) := f_0^{\mathscr{A}} t^m + f_1^{\mathscr{A}} t^{m-1} + \cdots + f_m^{\mathscr{A}}.$$

The *poset of intersections* of $\mathscr{A}$ is the set

$$\mathscr{L}(\mathscr{A}) := \{\cap \mathscr{B} | \mathscr{B} \subseteq \mathscr{A}\}, \quad x \leq y \Leftrightarrow x \supseteq y$$

of all subspaces that arise as intersections of hyperplanes in $\mathscr{A}$, partially ordered by reverse inclusion. The poset $\mathscr{L}(\mathscr{A})$ is *ranked* by the function $\mathrm{rk}(x) := m - \dim(x)$. and we define the *rank of* $\mathscr{A}$ to be $r := \mathrm{rk}(\cap \mathscr{A})$. The *Möbius polynomial* of $\mathscr{A}$ is

$$M_{\mathscr{A}}(u, v) := \sum_{x, y \in \mathscr{L}(\mathscr{A})} \mu(x, y) u^{\mathrm{rk}(x)} v^{r - \mathrm{rk}(y)} \tag{2}$$

where μ denotes the Möbius function of $\mathscr{L}(\mathscr{A})$ (see e.g. [22, (3.17)]).

Theorem 2.9 (*Zaslavsky [26, Theorem A]*)**.**

$$f^{\mathscr{A}}(x) = (-1)^r M_{\mathscr{A}}(-x, -1).$$

2.3. Zonotopes

Associated to every set of nonzero real vectors $v_1, \ldots, v_k \in \mathbb{R}^m \setminus \{0\}$ there is a polytope obtained as the Minkowski (i.e., pointwise) sum

$$Z(v_1, \ldots, v_k) := \sum_{i=1}^{k} [-1, 1] v_i$$

where $[-1, 1] \subseteq \mathbb{R}$ denotes the 1-dimensional unit cube (see [27, §1.1]). Polytopes of this form are called *zonotopes*. Strongly related to $Z(v_1, \ldots, v_k)$ is the arrangement of normal hyperplanes to the

v_i, i.e., $\mathscr{A} := \{v_i^\perp \mid i = 1, \ldots, k\}$. In particular, there is an isomorphism of posets (see, e.g., [27, Corollary 7.18])

$$\mathscr{F}(\mathscr{A})^{\mathrm{op}} \cong \mathscr{F}(Z(v_1, \ldots, v_k)) \setminus \{\emptyset\}$$

which implies the following relationship among the f-polynomials.

$$f^{Z(v_1,\ldots,v_k)}(t) - t^{r+1} = t^r f^{\mathscr{A}}(\frac{1}{t}) \tag{3}$$

2.4. Split systems

We introduce a special class of pseudometric spaces, keeping the terminology that is in use in the literature (e.g., [1,3]).

Definition 2.10. Let X be a finite set. A *split* of X is a bipartition of X, i.e., a pair of nonempty and disjoint subsets $A, B \subseteq X$ (the *sides* of the split) such that $A \cup B = X$. Such a pair will be written $A|B$. Clearly, $A|B$ and $B|A$ describe the same split. In fact, every split $\sigma = A|B$ corresponds to a nontrivial equivalence relation $\sim_\sigma$ on X, whose equivalence classes are A and B. Given a split σ and any element $i \in X$ we write $[i]_\sigma$ for the equivalence class of i with respect to the equivalence relation $\sim_\sigma$. Thus, to any split σ we can associate the function

$$\delta_\sigma(i, j) = \begin{cases} 0 & i \sim_\sigma j \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

A split σ is called *trivial* if one of its sides is a singleton. We will use the shorthand $\sigma = k|k^c$ in order to denote a trivial split whose singleton side is $\{k\}$.

Two splits $A|B$ and $C|D$ are *compatible* if at least one of the sets $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$ is empty.

A *system of splits* on X is just a set of splits of X; the system is called *compatible* if its elements are pairwise compatible.

Definition 2.11. A *weighted split system* is a pair $(\mathcal{S}, \alpha)$ where $\mathcal{S}$ is a system of splits on X and $\alpha \in (\mathbb{R}_{\geq 0})^{\mathcal{S}}$ is any weighting. Any such weighted split system defines a symmetric nonnegative function $d_\alpha : X \times X \to \mathbb{R}$ via

$$d_\alpha(x, y) = \sum_{\sigma \in \mathcal{S}} \alpha_\sigma \delta_\sigma(x, y)$$

where δ_σ is as in Eq. (4). The functions of the form d_α are called *split-decomposable pseudometrics* associated to $\mathcal{S}$. In fact, the pair (X, d_α) is a pseudometric space. A *positively weighted* split system is one where $\alpha_\sigma > 0$ for all $\sigma \in \mathcal{S}$. In this case, we will write

$$V(\mathcal{S}) := V(X, d_\alpha).$$

as this subspace clearly does not depend on α.

Such metric spaces are also known as *cut (pseudo)metrics* [7].

Theorem 2.12 (*See [21, Theorems 3.1.4, 7.1.8, 7.2.6, 7.3.2]*)**.** *Let (X, d) be a pseudometric space. The following are equivalent:*

(i) d satisfies the "four point condition": for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max |\{d(x, z) + d(y, w), d(x, w) + d(z, y)\}|$$

(ii) d is a tree-like pseudo-metric on X (in the sense of Definition 2.8).

(iii) d is a split-decomposable pseudometric associated to a positively weighted system of compatible splits. Moreover, this system is unique.

Remark 2.13. Under the equivalence of (ii) with (iii), splits in the decomposition of the metric correspond bijectively to edges in the tree.

2.5. Arrangements associated to split systems

We now define an arrangement of hyperplanes associated to any split system. This set of hyperplanes appeared already in [16, p. 10], see Remark 4.7.

Definition 2.14. Let X be a finite set and consider a split $\sigma = A|B$ of X, where $|X| = n$. To σ we associate the line segment (one-dimensional polytope)

$$S_\sigma := \operatorname{conv}\left\{ \frac{|B|}{n} \cdot \mathbb{1}_A - \frac{|A|}{n} \cdot \mathbb{1}_B, \frac{|A|}{n} \cdot \mathbb{1}_B - \frac{|B|}{n} \cdot \mathbb{1}_A \right\} \subseteq V(\mathcal{S}) \subseteq \mathbb{R}^X$$

where $\mathbb{1}_A := \sum_{x\in A} \mathbb{1}_x$, as well as a hyperplane $H_\sigma := (S_\sigma)^\perp$.

Accordingly, the hyperplane arrangement and the zonotope associated to $\mathcal{S}$ are

$$\mathscr{A}(\mathcal{S}) := \{H_\sigma \mid \sigma \in \mathcal{S}\}; \qquad Z(\mathcal{S}) := \sum_{\sigma\in\mathcal{S}} S_\sigma .$$

Remark 2.15. Both the arrangement $\mathscr{A}(\mathcal{S})$ and the zonotope $Z(\mathcal{S})$ are full-rank, resp. full-dimensional, inside the natural "ambient space" $V(\mathcal{S})$.

Remark 2.16. Each of our hyperplanes H_σ has the form of an (A, B, μ)-hyperplane as described in [9, Equation 8], for $\mu = p|B|$ and $k = pn$ where p is any positive integer. However, such values of μ, k are excluded in [9].

2.6. Matroids

The abstract combinatorial objects on which our enumerative considerations rest are *matroids*. Technicalities about matroids will only be needed in few proofs, therefore we only give a partial review of the definitions and the terminology and simply refer to Oxley's textbook [20].

Let E be a finite set. A matroid $\mathscr{M}$ on E can be given by a collection of subsets of E that contains the full set E and which has the structure of a geometric lattice when partially ordered by inclusion. The elements of this collection are the *flats* of the matroid, and the poset of all flats ordered by inclusion is called $\mathscr{L}(\mathscr{M})$. Since geometric lattices are ranked posets, for every $A \subseteq E$ we can define a *rank* $\rho_{\mathscr{M}}(A)$ as the poset rank of the smallest element of $\mathscr{L}(\mathscr{M})$ that contains A.

The *characteristic polynomial* of a matroid $\mathscr{M}$ on the ground set E is

$$\chi_{\mathscr{M}}(t) := \sum_{A\subseteq E} (-1)^{|A|} t^{\rho(E)-\rho(A)}.$$

The matroid is called *simple* if the minimal flat is the empty set and every minimal nonempty flat is a singleton set. In this case, the structure of $\mathscr{L}(\mathscr{M})$ determines the matroid fully.

Example 2.17. Let k be a positive integer and E any k-element set. The set of all subsets of E is the set of flats of a matroid on E that we denote by $\mathscr{F}(k)$ and call the free matroid on k elements. The set of all subsets of E of cardinality other than $k - 1$ is also the set of flats of a matroid: we denote this by $\mathscr{C}(k)$ and call it the k-cycle matroid. (The reader familiar with matroid theory will recognize $\mathscr{C}(k)$ as the uniform matroid $U_{k,k-1}$, and notice that $\mathscr{F}(k) \simeq U_{k,k}$.) The characteristic polynomials of those matroids have the following form.

$$\chi_{\mathscr{F}(k)}(t) = (t - 1)^k, \qquad \chi_{\mathscr{C}(k)}(t) = (-1)^{k-1} \sum_{i=1}^{k-1} (1 - t)^i \tag{5}$$

Example 2.18. To every arrangement $\mathscr{A}$ of hyperplanes in the sense of Section 2.2 is associated a matroid on the ground set $\mathscr{A}$ by declaring any $\mathcal{K} \subseteq \mathscr{A}$ to be a flat if and only if there is a linear subspace X of V such that $\mathcal{K}$ is the set of all hyperplanes containing X. In particular, there is a poset isomorphism

$$\mathscr{L}(\mathscr{A}) \to \mathscr{L}(\mathscr{M}); \quad X \mapsto \{H \in \mathscr{A} \mid X \subseteq H\}$$

and, for every $\mathcal{K} \subseteq \mathscr{A}$, $\rho_{\mathscr{M}}(\mathcal{K}) = \operatorname{codim} \cap \mathcal{K}$.

From Zaslavsky's Theorem 2.9 and elementary computations we see that

$$f_i^{\mathscr{A}} = (-1)^i \sum_{\substack{\mathcal{K} \in \mathscr{L}(\mathscr{M}) \\ \rho(\mathscr{A}) - \rho(\mathcal{K}) = i}} \chi_{\mathscr{M}/\mathcal{K}}(-1) \tag{6}$$

where $\mathscr{M}/\mathcal{K}$ denotes the *contraction* of the flat $\mathcal{K}$ (see [20, Section 3.1]).

We conclude this brief overview with two matroid operations.

Definition 2.19. Let E_1, E_2 be two disjoint finite sets, and for $i = 1, 2$ let $\mathscr{M}_i$, be a matroid on the ground set E_i with lattice of flats $\mathscr{L}_i$. The *direct sum* $\mathscr{M}_1 \oplus \mathscr{M}_2$ is the matroid on $E_1 \cup E_2$ whose flats are precisely the unions of flats of $\mathscr{M}_1$ and $\mathscr{M}_2$. In particular, there is an isomorphism of posets $\mathscr{L}(\mathscr{M}_1 \oplus \mathscr{M}_2) \simeq \mathscr{L}_1 \times \mathscr{L}_2$.

The characteristic polynomial of a direct sum decomposes as a product.

$$\chi_{\mathscr{M}_1 \oplus \mathscr{M}_2}(t) = \chi_{\mathscr{M}_1}(t)\chi_{\mathscr{M}_2}(t) \tag{7}$$

Definition 2.20. Let E_1, E_2 be two finite sets such that $E_1 \cap E_2 = \{e\}$ for some e. For $i = 1, 2$ let $\mathscr{M}_i$, be a matroid on the ground set E_i with lattice of flats $\mathscr{L}_i$ and rank function ρ_i. If $\rho_1(e) = \rho_2(e)$, the *parallel connection* of $\mathscr{M}_1$ and $\mathscr{M}_2$ along e is the matroid $\mathscr{M}_1 \oplus_e \mathscr{M}_2$ on the ground set $E_1 \cup E_2$ whose flats are precisely the subsets of the form $F_1 \cup F_2$ for $(F_1, F_2) \in \mathscr{L}_1 \times \mathscr{L}_2$ and either $e \in F_1 \cap F_2$ or $e \notin F_1 \cup F_2$.

Characteristic polynomials of parallel connections behave naturally, e.g., as in the following sample result which we state for later reference.

Remark 2.21 (*[2, Theorem 5]*)**.** In the setting of Definition 2.20, if $\rho_{\mathscr{M}_1}(\{e\}) = \rho_{\mathscr{M}_2}(\{e\}) \neq 0$ and F is any flat of $\mathscr{M}_1 \oplus_e \mathscr{M}_2$, then

$$\chi_{(\mathscr{M}_1 \oplus_e \mathscr{M}_2)/F}(t) = \frac{\chi_{\mathscr{M}_1/(F \cap E_1)}(t)\chi_{\mathscr{M}_2/(F \cap E_2)}(t)}{(t-1)^{|\{e\} \setminus F|}}.$$

Again we refer to [20, Section 3.1] for the definition of the contraction of a matroid.

3. Lipschitz polytopes of compatible systems of splits

Theorem 3.1. *Let (X, d) be a tree-like pseudometric space. Then,*

$$LIP(X, d) = \sum_{\sigma \in \mathcal{S}} \alpha_\sigma S_\sigma$$

where $(\mathcal{S}, \alpha)$ is the unique weighted system of compatible splits of X such that $d = d_\alpha$ (cf. Theorem 2.12).

Remark 3.2. We thank an anonymous referee for pointing out to us that this theorem can be deduced from work of Koichi [16] and Murota's book on convex discrete analysis [18], as we explain in Proof A. For the benefit of the reader who might not be familiar with this apparatus, we also offer an elementary direct argument (Proof below).

Proof. Following, e.g., Murota [18], we identify any $C \subset \mathbb{R}^X$ via its "indicator function" $\delta_C : \mathbb{R}^X \to \{0, +\infty\}$, defined as

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & \text{otherwise.} \end{cases} \tag{8}$$

When C is convex, [18, Theorem 3.2 and (3.31)] says that the "conjugate" (or Legendre–Fenchel transform) $\delta_C^\bullet$ of δ_C satisfies

$$\delta_C^\bullet(x) = \sup_{y \in C} x^T y.$$

This expression allows for an explicit verification of the fact that for every Minkowski-sum decomposition $C = A + B$ of C into convex sets A and B and for every $\alpha > 0$ we have

$$\delta_C^\bullet(x) = \delta_A^\bullet(x) + \delta_B^\bullet(x), \quad \alpha \cdot \delta_C^\bullet(x) = \delta_{\alpha C}^\bullet(x) \tag{9}$$

Moreover, following [16], to any finite (pseudo)metric space (X, d) one can associate the finite vector configuration $K := \{\mathbb{1}_u - \mathbb{1}_v\}_{u,v \in X} \subseteq \mathbb{R}^X$ and consider the function $\overline{d} : \mathbb{R}^X \to \mathbb{R}$ defined as the homogeneous convex closure [16, §2.3] of the discrete function $K \to \mathbb{R}$, $(\mathbb{1}_u - \mathbb{1}_v) \mapsto d(u, v)$. An explicit expression for $\overline{d}$ is given in [16, (3.1)], and a direct check shows that in this case

$$\overline{d} = \delta_{\mathrm{LIP}(X,d)}^\bullet. \tag{10}$$

In particular, since for every split σ of X the function δ_σ from Eq. (8) is a pseudometric on X, we have

$$\overline{\delta_\sigma} = \delta_{\mathrm{LIP}(X,\delta_\sigma)}^\bullet. \tag{11}$$

When (X, d) is a tree-like metric space with associated split system $(\mathcal{S}, \alpha)$, [16, Proposition 3.6] shows that

$$\overline{d} = \sum_{\sigma \in \mathcal{S}} \alpha_\sigma \overline{\delta_\sigma}, \tag{12}$$

where $\overline{\delta_\sigma}$ is as in Eq. (11). Summing up, we can write

$$\delta_{\mathrm{LIP}(X,d)}^\bullet \overset{(10)}{=} \overline{d} \overset{(12)}{=} \sum_{\sigma \in \mathcal{S}} \alpha_\sigma \overline{\delta_\sigma} \overset{(11)}{=} \sum_{\sigma \in \mathcal{S}} \alpha_\sigma \delta_{\mathrm{LIP}(X,\delta_\sigma)}^\bullet \overset{(9)}{=} \delta_{(\sum_{\sigma \in \mathcal{S}} \mathrm{LIP}(X,\alpha_\sigma \delta_\sigma))}^\bullet.$$

From the equality of the Legendre–Fenchel transforms of the indicator functions one then deduces equality of the polytopes, proving the claim. □

Proof. The proof is by induction on the cardinality of $\mathcal{S}$. If $|\mathcal{S}| = 0$ there is nothing to prove.

Let then $|\mathcal{S}| > 0$ and suppose that the theorem holds for all weighted systems of compatible splits of smaller cardinality. By Theorem 2.12, to the space (X, d) is associated a weighted X-tree (T, ϕ) in the sense of Definition 2.8, and the corresponding tree metric can be expressed as a split metric with a split for every edge in the tree. The uniqueness part of Bandelt and Dress' decomposition theorem ([1, Theorem 2]) says that the associated split system must be $\mathcal{S}$. In particular, the tree T has at least one edge, and thus at least one leaf vertex (i.e., a vertex incident to exactly one edge). Choose then such a leaf vertex, say v, and let $\sigma \in \mathcal{S}$ be the split corresponding to the unique edge incident to v. Then,

$$\sigma = A \mid A^c \text{ with } A := \phi^{-1}(v).$$

Let $\mathcal{S}' := \mathcal{S} \setminus \{\sigma\}$ and let (X, d') be the pseudometric space defined by $\mathcal{S}'$ and the appropriate restriction of α. Now notice that $d = d' + \alpha_\sigma \delta_\sigma$ and that, for all $i, j \in A$, we have $d'(i, j) = 0$. The claim then follows by induction hypothesis applied to $\mathcal{S}'$ via the following identity.

$$\mathrm{LIP}(X, d' + \alpha_\sigma \delta_\sigma) = \mathrm{LIP}(X, d') + \alpha_\sigma S_\sigma.$$

The right-to-left containment is verified directly. In order to check the left-to-right containment we consider a point $x \in \mathrm{LIP}(X, d' + \alpha_\sigma \delta_\sigma)$ and prove that it is contained in the right-hand side. The definition of the Lipschitz polytope implies immediately that, for all $i, j \in X$, $x_i - x_j \leq d'(i, j) + \alpha_\sigma$.

Define

$$\alpha := max_{i \in A, j \in A^c}\{0, x_i - x_j - d'(i, j), x_j - x_i - d'(i, j)\}.$$

If $\alpha = 0$, then $x \in \mathrm{LIP}(X, d')$. Otherwise, choose i_0, j_0 such that $\alpha = x_{i_0} - x_{j_0} - d'(i_0, j_0)$. Assume w.l.o.g. $i_0 \in A$ and $j_0 \in A^c$ (otherwise switch A and A^c in the following). Since now $0 \le \alpha \le \alpha_\sigma$, it is enough to show that

$$y := x - \alpha v_\sigma \in \mathrm{LIP}(X, d')$$

where $v_\sigma := \frac{|A^c|}{n} \cdot \mathbb{1}_A - \frac{|A|}{n} \cdot \mathbb{1}_{A^c}$. This is proved by verifying, with a direct computation, that y satisfies Eq. (1). $\square$

Theorem 3.3. *Let (X, d) be a tree-like pseudometric space with associated system of compatible splits $\mathcal{S}$. Then the f-vector of the associated Lipschitz polytope is as follows.*

$$f^{\mathrm{LIP}(X,d)}(x) = (-x)^{\mathrm{rk}(\cap \mathscr{A}(\mathcal{S}))} M_{\mathscr{A}(\mathcal{S})}\left(-\frac{1}{x}, -1\right) + x^{\mathrm{rk}(\cap \mathscr{A}(\mathcal{S}))+1}$$

If additionally (X, d) is a metric space, then the f-vector of the associated fundamental polytope is

$$f^{P_d(X)}(x) = (-1)^{\mathrm{rk}(\cap \mathscr{A}(\mathcal{S}))} M_{\mathscr{A}(\mathcal{S})}(-x, -1)x + 1.$$

Proof. Theorem 3.1 implies that $\mathscr{F}(\mathrm{LIP}(X, d)) \simeq \mathscr{F}(Z(\mathcal{S}))$, and thus with Remark 2.15, Theorem 2.9 and Eq. (3) we can compute

$$f^{\mathrm{LIP}(X,d)}(x) = (-x)^{\mathrm{rk}(\cap \mathscr{A}(\mathcal{S}))} M_{\mathscr{A}(\mathcal{S})}(-\frac{1}{x}, -1) + x^{\mathrm{rk}(\cap \mathscr{A}(\mathcal{S}))+1}$$

This proves the first of the claimed equalities. The second follows by duality (Remark 2.6). $\square$

Corollary 3.4. *For any tree-like metric space (X, d)*

$$f_i^{P_d(X)} = f_{i+1}^{\mathscr{A}(\mathcal{S})} = f_{|X|-2-i}^{\mathrm{LIP}(X,d)}$$

where $\mathcal{S}$ denotes the associated system of (compatible) splits and the index i runs from -1 to $\dim(P_d(X)) = |X| - 1$.

4. Computation of face numbers

We turn to the problem of an effective computation of the f-vectors of fundamental polytopes. The main result of this section are explicit formulas for the face numbers of fundamental polytopes of tree-like metric spaces.

We will start with two easy cases and then offer a general tool allowing to compute the intersection lattice of the associated hyperplane arrangement. From there, we will study the structure of the matroid $\mathscr{M}(\mathcal{S})$ in order to set up our formulas.

Example 4.1 (*Points in* $\mathbb{R}^1$)**.** We can represent the metric space defined by any set of n points in $\mathbb{R}^1$ by just taking its metric graph in a line, considering the associated set of splits and choosing the coefficients in the split-metric accordingly. The arrangement corresponds to $(n-1)$ independent vectors in $n-1$-dimensional space, i.e., it is isomorphic to the coordinate arrangement. The corresponding matroid is the uniform matroid $\mathcal{U}_{n-1}^{n-1}$ and, in particular, $f_i^{\mathscr{A}} = 2^i\binom{n-1}{i}$.

Example 4.2 (*The Root Polytope of Type A_{n-1}*)**.** Let us consider a star graph, i.e., a tree with $n > 2$ leaves and a unique internal vertex. If we assign each edge the length $\frac{1}{2}$, we define the structure of a metric space on the set X of leaves of our star graph.

The corresponding split system consists exactly of all the trivial splits, and any two points are at distance 1. Then, by definition, the fundamental polytope of this space is the convex hull of the vectors $e_{i,j} = \mathbb{1}_i - \mathbb{1}_j$, where $i \neq j \in [n]$. This is also called the *root polytope of type A_{n-1}*, and its face numbers have been computed via algebraic-combinatorial considerations by Cellini and Marietti [4, Proposition 4.3]. Of course, one could compute these numbers by computing the Möbius function of the corresponding matroid, i.e., the uniform matroid $\mathcal{U}_n^{n-1}$.

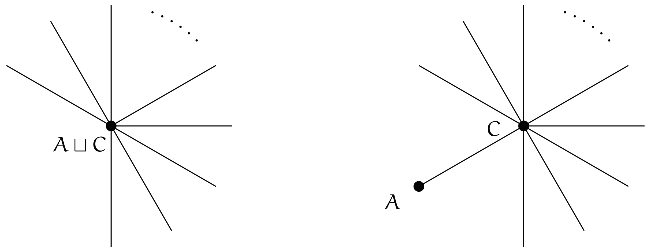

Fig. 1. The neighborhood of the vertex v' in the X-tree T' (left-hand side) and T (right-hand side).

4.1. The intersection lattice of $\mathscr{A}(\mathcal{S})$

We will start by describing the intersection poset of $\mathscr{A}(\mathcal{S})$ by means of partitions. Work in this direction can be implicitly found in [16], but for our purposes it will be convenient to give explicit statements and direct proofs (see Remark 4.7 for details).

Definition 4.3. Let (X, d) be a pseudometric space. The function d induces a partition $\pi(d)$ of the set X given as the set of equivalence classes of the equivalence relation in which i and j are equivalent if and only if $d(i, j) = 0$. If the space (X, d) arises from a positively weighted system of splits $(\mathcal{S}, \alpha)$, the partition $\pi(d)$ does not depend on α and we only write $\pi(\mathcal{S})$.

We have an order-reversing map of posets

$$\pi : 2^{\mathcal{S}} \to \Pi_X; \qquad \mathcal{S}' \mapsto \pi(\mathcal{S}')$$

where $2^{\mathcal{S}}$ denotes the poset of all subsets of $\mathcal{S}$ ordered by inclusion, and Π_X is the poset of all partitions of X ordered by refinement.

Theorem 4.4. *Let $(\mathcal{S}, \alpha)$ be an arbitrary positively weighted system of compatible splits of a finite set X and write $\pi := \pi(\mathcal{S})$. Then,*

$$\cap_{\sigma\in\mathcal{S}} H_\sigma = \langle e_{i,j} : i \text{ and } j \text{ in the same block of } \pi(\mathcal{S})\rangle,$$

where $e_{i,j} = (\mathbb{1}_i - \mathbb{1}_j)/d(i, j)$, see Definition 2.2.

Proof. The right-to-left inclusion holds by definition. We will prove the left-to-right inclusion by induction on the cardinality of the system of splits.

If $|\mathcal{S}| = 1$ the claim is evident. Let then $m > 0$, assume that the statement holds for any weighted system of up to m compatible splits and consider a weighted system of splits $(\mathcal{S}, \alpha)$ with $|\mathcal{S}| = m+1$.

By Theorem 2.12, $(\mathcal{S}, \alpha)$ can be represented by an X-tree (T, ϕ) with at least one edge, hence with at least one leaf vertex v. In particular, with $A := \phi^{-1}(v)$, we know that $\sigma := A|A^c \in \mathcal{S}$ and we can consider

$$\mathcal{S}' := \mathcal{S} \setminus \{\sigma\}, \qquad \alpha' := \alpha_{|\mathcal{S}'}, \qquad d' := d_{\alpha'}, \qquad \pi' := \pi(\mathcal{S}').$$

The X-tree (T', ϕ') associated to $(\mathcal{S}', \alpha')$ must have a vertex v' with $\phi'(A) = v'$ (otherwise there would be $i, j \in A$ with $d_\alpha(i, j) \geq d'_\alpha(i, j) > 0$).

In a neighborhood of v', the X-trees associated to $(\mathcal{S}', \alpha')$, resp. $(\mathcal{S}, \alpha)$, differ as in Fig. 1. In particular, the partitions associated to v and v' are of the form

$$\pi = \{A, C, \pi_1, \ldots, \pi_k\}; \qquad \pi' = \{A \sqcup C, \pi_1, \ldots, \pi_k\} \qquad (*)$$

where, as in the following, we think of a partition as a set of blocks and we denote by C the unique block of π that is merged in order to form π'. Moreover, given a partition π of a set let $\sim_\pi$ denote the equivalence relation on the same set whose equivalence classes are the blocks of π.

Let $v_\sigma := \frac{|A|}{n} \cdot \mathbb{1}_{A^c} - \frac{|A^c|}{n} \cdot \mathbb{1}_A$, so that $(v_\sigma)^\perp = H_\sigma$. By induction hypothesis,

$$\bigcap_{\tau \in \mathcal{S}} H_\tau = \bigcap_{\tau \in \mathcal{S}'} H_\tau \cap H_\sigma = \langle e_{i,j} : i \sim_{\pi'} j \rangle \cap (v_\sigma)^\perp. \tag{13}$$

In view of $(*)$, the subspace $\langle e_{i,j} : i \sim_{\pi'} j \rangle$ decomposes as

$$\bigoplus_{b \in \pi'} \langle e_{i,j} \mid i, j \in b \rangle = \langle e_{i,j} \mid i, j \in A \sqcup C \rangle \oplus \underbrace{\bigoplus_{b \in \pi' \setminus \{A \sqcup C\}} \langle e_{i,j} \mid i, j \in b \rangle}_{=:W}.$$

Since σ does not split any block of $\pi' \setminus \{A \sqcup C\}$, we have $W \subseteq (v_\sigma)^\perp$. Therefore, the right-hand side of Eq. (13) equals $\left(\langle e_{i,j} \mid i, j \in A \sqcup C \rangle \cap (v_\sigma)^\perp\right) \oplus W$.

On the other hand,

$$\langle e_{i,j} \mid i, j \in A \sqcup C \rangle \cap (v_\sigma)^\perp = \langle e_{i,j} \mid i, j \in A \rangle \oplus \langle e_{i,j} \mid i, j \in C \rangle$$

Thus, we can rewrite the right-hand side of Eq. (13) as

$$\langle e_{i,j} \mid i, j \in A \rangle \oplus \langle e_{i,j} \mid i, j \in C \rangle \oplus W$$

and in particular, recalling the block structure of π from $(*)$,

$$\langle e_{i,j} \mid i \sim_{\pi'} j \rangle \cap (v_\sigma)^\perp = \langle e_{i,j} \mid i \sim_\pi j \rangle$$

which, together with Eq. (13), concludes the proof. □

Recall that the posets of intersections of $\mathscr{A}(\mathcal{S})$ is the lattice of flats of the matroid $\mathscr{M}(\mathcal{S})$.

Corollary 4.5. *There is a poset isomorphism*

$$\mathscr{L}(\mathscr{A}(\mathcal{S})) \simeq \operatorname{im} \pi$$

where the right-hand side is considered as an induced sub-poset of Π_X^{op}.

More precisely, if we identify the ground set of the matroid $\mathscr{M}(\mathcal{S})$ with $\mathcal{S}$ itself, we can write the closure operator of the matroid as

$$cl(\mathcal{S}') = \max_{\subseteq}\{\mathcal{S}'' \subseteq \mathcal{S} | \pi(\mathcal{S}'') = \pi(\mathcal{S}')\}$$

with $\rho(\mathcal{S}') = |\pi(\mathcal{S}')| - 1$.

Example 4.6 (*Full Trees*)**.** If (X, d) is a "full tree" (in the sense of Definition 2.8), then it can be represented by an X-tree where each vertex is labeled by exactly one point of X. Therefore it is apparent that π is injective, and thus the poset $\mathcal{L}(\mathscr{A}(\mathcal{S}))$ is boolean. With Eq. (6) and Corollary 3.4 we immediately obtain

$$f_i^{P_d(X)} = 2^i \binom{n-1}{i},$$

generalizing, as expected, Example 4.1.

Remark 4.7. Koichi [16] considers lattices of flats $\mathscr{L}$ of matroids of linear dependencies of centrally symmetric vector configurations and characterizes the families $\mathcal{C} \subseteq \mathscr{L} \setminus (\max \mathscr{L})$ such that the subposet $\{\wedge C \mid C \subseteq \mathcal{C}\}$ of $\mathscr{L}$ (i.e., generated by meets of subfamilies of $\mathcal{C}$) is anti-isomorphic to a geometric lattice [16, Theorem 4.1]. Our case corresponds to point configurations of "type Ω" in [16], for which [16, Theorem 4.6] establishes that the set of all flats consisting of the vectors contained in one of the hyperplanes from Definition 2.14 satisfies indeed this condition. Moreover, in [16, §4.3.1] it is hinted at a description of $\mathscr{L}(\mathscr{A}(\mathcal{S}))$ in terms of partitions of X. We thought it helpful to give explicit statements and a direct proof in this paper.

4.2. A graph-theoretic description of $\mathscr{M}(\mathcal{S})$

In order to give explicit formulas for the face numbers of the fundamental polytope of a tree-like metric space (X, d) we study the structure of the matroid $\mathscr{M}(\mathcal{S})$ in terms of the associated X-tree. First, note that since the ground set of $\mathscr{M}(\mathcal{S})$ is the set of splits, via Theorem 2.12 we can naturally think of $\mathscr{M}(\mathcal{S})$ as having the set of edges of the X-tree as a ground set.

In this section let then X be a finite set, and let T denote any X-tree (Definition 2.8). For simplicity we identify a labeled vertex with its label, thus regarding X as a subset of $V(T)$. Given any $F \subseteq E(T)$, the associated *edge-induced subgraph* is $T[F] = (V(T), F)$, i.e., the graph consisting of all vertices of T but only the edges in F.

Every edge $e \in E(T)$ defines a split $\mathcal{S}_e$ of X by partitioning X into two parts according to which connected component of $T[E \setminus \{e\}]$ they are in. Let $\mathcal{S}(F) := \{\mathcal{S}_e \mid e \in F\}$ denote the system of splits associated to an edge set F.

Definition 4.8. Call a subset $F \subseteq E(T)$ of edges of an X-tree *flat* if the induced subgraph $T[E \setminus F]$ has no connected component with a labeled vertex and an unlabeled leaf.

Proposition 4.9. *Let T be an X-tree and let $\mathcal{S}$ denote the associated system of splits of X. Then, a set $F \subseteq E$ is flat if and only if $\mathcal{S}(F)$ is closed in $\mathscr{M}(S)$. Moreover, the rank of $\mathcal{S}(F)$ in $\mathcal{M}(T)$ is one less than the number of connected components of $T[E \setminus F]$ that contain at least an X-labeled vertex.*

Proof. The partition $\pi(\mathcal{S}(F))$ is the set of equivalence classes of the relation defined on X by $x \simeq_F y$ if x, y in the same connected component of $T[E \setminus F]$. Thus, $\mathcal{S}(F)$ is not closed if and only if there is a split σ in $\mathcal{S} \setminus \mathcal{S}(F)$ with $\pi(\mathcal{S}(F) \cup \{\sigma\}) = \pi(\mathcal{S}(F))$. Equivalently, there is an edge $e \notin F$ whose removal from $T[E \setminus F]$ does not increase the number of connected components containing X-vertices: this can only happen if $T[E \setminus F]$ contains a non-X-labeled leaf. □

Definition 4.10. For any given X-tree T, let $\mathscr{M}(T)$ denote the unique simple matroid on the ground set $E(T)$ where a set is closed if and only if it is flat in T.

Corollary 4.11. *For every tree-like metric space with associated split-system $\mathcal{S}$ and tree T, the matroids $\mathscr{M}(\mathcal{S})$ and $\mathscr{M}(T)$ are isomorphic.*

In the following we will study how the structure of the tree T leads to a decomposition of the matroid $\mathscr{M}(T)$. The decomposition is in terms of *parallel connections*. Recall that

Definition 4.12. Given any tree T let $\widehat{T}$ be the tree obtained by removing all leaves from T. We call $\widehat{T}$ the "core" of T.

For every leaf c of $\widehat{T}$ let $\ell(c)$ be the set of leaves of T adjacent to c. Let c' be the vertex of $\widehat{T}$ adjacent to c. Then set $T_{[c]}$ to be the $(\ell(c) \cup \{c'\})$-tree obtained from the neighborhood of c in T by, if necessary, labeling c'. Moreover, let $T^{[c]}$ denote the $((X \setminus \ell(c)) \cup \{c\})$-tree obtained by pruning $\ell(c)$ from T and, if necessary, labeling c.

Lemma 4.13. *Let T be an X-tree, let c be a leaf of $\widehat{T}$, $e(c)$ the edge connecting c to $\widehat{T}$. Then,*

$$\mathscr{M}(T) = \mathscr{M}(T^{[c]}) \oplus_{e(c)} \mathscr{M}(T_{[c]}).$$

Proof. We check the definition via the set of flats. Fix $F \subseteq E$ and let $F' := F \cap E(T^{[c]})$, $F'' := F \cap T_{[c]}$. Then, F is flat in T if and only if F' and F'' are both flat in the respective graphs (where unlabeled vertices have the same neighborhood as in T). □

In the following, given any vertex v of a tree T and any $A \subseteq E(T)$ we will write $\deg_A(v)$ for the degree of v in the graph $T[A]$. More generally, for any given set $W \subseteq V(T)$ of vertices, we write $\deg_A(W) := \sum_{v \in W} \deg_A(v)$. If no precision is necessary, we will write deg for $\deg_E$, the degree in the full graph T.

Remark 4.14. If $c \in X$, then $\mathscr{M}(T_{[c]}) \simeq \mathscr{F}(\deg(c))$. Otherwise, $\mathscr{M}(T_{[c]}) \simeq \mathscr{C}(\deg(c))$.

Theorem 4.15. *Let T be an X-tree. Fix an enumeration $c_1, \dots, c_m$ of the vertices of $\widehat{T}$ such that $\widehat{T}_i$, the graph induced on the vertices $c_1, \dots, c_i$, is connected for all $i = 1, \dots, m$. In particular, for every i let e_i denote the unique edge $\{c_j, c_{i+1}\}$ with $j \leq i \leq m$. Then,*

$$\mathscr{M}(T) = \mathscr{R}(\deg(c_1)) \oplus_{e_1} \cdots \oplus_{e_{m-1}} \mathscr{R}(\deg(c_m))$$

where $\mathscr{R}(\deg(c))$ is a matroid on the ground set $N(c)$ and equals $\mathscr{F}(\deg(c))$ if $c \in X$ and equals $\mathscr{C}(\deg(c))$ otherwise.

Proof. We apply Lemma 4.13 to the leaf c_m of $\widehat{T}$, then to the leaf c_{m-1} of $\widehat{T^{[c_m]}}$ and, recursively, to the leaf c_{m-j} of the core of $T^{[c_m]\cdots[c_{m-j-1}]}$. We obtain

$$\mathscr{M}(T) = \mathscr{M}(T^{[c_m]\cdots[c_2]}) \oplus_{e_1} \mathscr{M}(T^{[c_m]\cdots[c_3]}_{[c_2]}) \dots \oplus_{e_{m-1}} \mathscr{M}(T_{[c_m]}).$$

From this expansion the claim follows by Remark 4.14, noticing that $T^{[c_m]\cdots[c_i]}_{[c_{i-1}]} = T_{[c_{i-1}]}$ for all $i = 2, \dots, m$, and that $T^{[c_m]\cdots[c_2]} = T_{[c_1]}$. $\square$

Definition 4.16. Let T be an X-tree as above. We denote by L, resp. U, the set of labeled, resp. unlabeled vertices of the core $\widehat{T}$ (hence $L = V(\widehat{T}) \cap X$ and $V(\widehat{T}) = L \uplus U$). Moreover, given any $A \subseteq E$, let $\epsilon(A) := |A \cap E(\widehat{T})|$ be the number of edges of $\widehat{T}$ contained in A, and write $\varphi(A)$ for the number of unlabeled vertices that are isolated in $T[A^c]$ (hence $\varphi(A) = |\{c \in U \mid \deg_A(c) = \deg(c)\}|$.

Lemma 4.17. *The rank in $\mathscr{M}(T)$ of any flat $F \subseteq E$ is*

$$\deg_F(V(\widehat{T})) - \varphi(F) - \epsilon(F) \tag{14}$$

Moreover, the characteristic polynomial of the contraction $\mathscr{M}(T)/F$ can be expressed as follows (where we write $G := E \setminus F$).

$$(-1)^{\deg_G(V(\widehat{T})) - \epsilon(G)}(1-t)^{\deg_G(L) - \epsilon(G)} \prod_{\substack{c \in U \\ \deg_G(c) > 0}} \frac{(1-t)^{\deg_G(c)} - (1-t)}{t} \tag{15}$$

Proof. The rank of F is the sum of the ranks of $F \cap N(c)$ in $\mathscr{R}(c)$ to which one has to subtract the number of e_i contained in F. The rank of $F \cap N(c)$ is its cardinality ($\deg_F(c)$), except in the case where $N(c) \subseteq F$ ($\deg_F(c) = \deg_T(c)$) and c is unlabeled, where one has to subtract one. The first claim follows.

For the second claim, repeated application of Remark 2.21 yields

$$\chi_{\mathscr{M}(T)/F}(t) = \frac{\prod_{i=1}^m \chi_{\mathscr{R}(c_i)/(F \cap N(c_i))}(t)}{(t-1)^{|E(\widehat{T}) \setminus F|}}$$

For any $I \subseteq [k]$ with $i = |I|$ we have $\mathscr{F}(k)/I = \mathscr{F}(k-i)$ and $\mathscr{C}(k)/I = \mathscr{C}(k-i)$. Moreover, notice that $\chi_{\mathscr{C}(0)}(t) = 1$. With this, a direct computation proves the second claim. $\square$

Corollary 4.18. *Let T be an X-tree. Then the characteristic polynomial of $\mathscr{M}(T)$ is as follows.*

$$\chi_{\mathscr{M}(T)}(t) = (-1)^{|V(T)|-1}(1-t)^{\deg(L) - |E(\widehat{T})|} \prod_{c \in U} \frac{(1-t)^{\deg(c)} - (1-t)}{t}$$

The number of facets of the fundamental polytope P_d (i.e., vertices of the Lipschitz polytope) is thus the following.

$$2^{\deg(L) - |E(\widehat{T})|} \prod_{c \in U} (2^{\deg(c)} - 2)$$

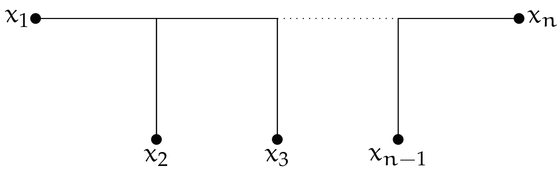

Fig. 2. The n-caterpillar graph.

Proof. The first claim is immediate from Lemma 4.17, noticing that in an X-tree there are no unlabeled vertices of degree less than 3. The second claim follows using Corollary 3.4 with $i = |X| - 1$. □

Definition 4.19. For a given X-tree T and fixed $k \in \mathbb{N}^{V(\widehat{T})}$, $i, \epsilon \in \mathbb{N}$, let $\Gamma_T(k, \epsilon, i)$ denote the number of subgraphs $T[G]$ of T with no unlabeled leaves and such that

- exactly $i + 1$ connected components of $T[G]$ contain labeled vertices
- $\deg_G(c) = k_c$ for all $c \in V(\widehat{T})$.
- G contains ϵ "core edges" ($\epsilon(G) = \epsilon$).

Remark 4.20. Notice that only $G = E(T)$ satisfies the conditions when $i = 0$; i.e., $\Gamma_T(k, \epsilon, 0) = 1$ if $k_c = \deg_T(c)$ for all c and $\epsilon = |E(\widehat{T})|$, and 0 otherwise.

Theorem 4.21. *Let (X, d) be a tree-like metric space with associated X-tree T. Then, for all $i \geq 0$,*

$$f_i^{\mathrm{LIP}(X,d)} = f_{|X|-1-i}^{P_d} = \sum_{(k,\epsilon)\in\mathbb{N}^{V(\widehat{T})+1}} \Gamma_T(k, \epsilon, i) 2^{\sum_{c\in L} k_c - \epsilon} \prod_{\substack{c\in U \\ k_c>0}} (2^{k_c} - 2)$$

Proof. This formula is a direct consequence of Corollary 3.4, via Formula (6) and the explicit expression of Lemma 4.17. □

Example 4.22. Let $n \in \mathbb{N}$, $n \geq 3$, and let us consider any tree metric (X, γ) whose underlying X-tree is an n-caterpillar graph (see Fig. 2) with every leaf labeled by exactly one of the n points of X, and no internal vertices labeled.

Then, $\widehat{T}$ is an $(n-2)$-point path and our formula immediately computes the number of vertices of the associated Lipschitz-polytope as

$$f_0^{\mathrm{LIP}(X,\gamma)} = 2 \cdot 3^{n-2}.$$

We can also compute the number of edges: notice that for $i = 1$, the only subgraphs that are counted by $\Gamma_T(k, \epsilon, i)$ are of the form $T[E \setminus \{e\}]$ for some $e \in E$.

If $e \notin E(\widehat{T})$ then $\epsilon = |E(\widehat{T})| = n - 3$ and the subgraph has exactly one unlabeled vertex of degree 2, while all others have degree 3. This results in a contribution to the number of edges in the amount of $2^{3-n} \cdot 2 \cdot 6^{n-3}$ for each of these n cases. If $e \in E(\widehat{T})$, then $\epsilon = n - 4$ and there are exactly two unlabeled vertices of degree two, while all others have degree 3. This gives a contribution of $2^{4-n} \cdot 2^2 \cdot 6^{n-4}$ in each of these $n - 3$ cases.

In total, the number of edges of the Lipschitz-polytope of the n-caterpillar graph then equals

$$f_1^{\mathrm{LIP}(X,\gamma)} = n2 \cdot 3^{n-3} + (n-3)2^2 \cdot 3^{n-4} \text{ for all } n \geq 3.$$

These formulas do in fact correctly predict some of the numbers in Table 1, which shows the f-polynomials of the fundamental polytopes of these metric spaces for the first few values of n as computed with SAGE via Corollary 4.5. It took around 10 s to compute the f-polynomial of the biggest example, the 6-caterpillar graph, on the sage cloud (run on a free server).

Table 1
The f-polynomials of the fundamental polytope of some caterpillar trees (compare Fig. 2).

Metric space	f-polynomial of $P_d(X)$
3-caterpillar	$t^3 + 6t^2 + 6t + 1$
4-caterpillar	$t^4 + 12t^3 + 28t^2 + 18t + 1$
5-caterpillar	$t^5 + 20t^4 + 80t^3 + 114t^2 + 54t + 1$
6-caterpillar	$t^6 + 30t^5 + 180t^4 + 422t^3 + 432t^2 + 162t + 1$

5. A characterization of simpliciality

We turn to characterizing the tree-like metric spaces whose fundamental polytope is simplicial. Recall that a polytope is called simplicial if each of its faces is (combinatorially equivalent to) a simplex [27, Section 2.5]. Equivalently, a polytope P is simplicial if, for every face $F \in \mathscr{F}(P)$, the lower interval $\mathscr{F}(P)_{\leq F}$ is a boolean poset. Analogously, an arrangement $\mathscr{A}$ of hyperplanes in a real vector space is called simplicial if each of the cones of the fan determined by $\mathscr{A}$ (see Section 2.2) is a cone over a simplex or, equivalently, if $\mathscr{F}(\mathscr{A})_{\leq F}$ is a boolean poset for each $F \in \mathscr{F}(\mathscr{A})$.

Consider a tree-like metric space (X, d) with associated split system $\mathscr{A}(\mathcal{S})$. We know that the posets of faces of the fundamental polytope and of the associated hyperplane arrangement are isomorphic: $\mathscr{F}(P_d(X)) \simeq \mathscr{F}(\mathscr{A}(\mathcal{S}))$. Therefore, our characterization of simpliciality for fundamental polytopes of metric trees will build upon the following characterization of simpliciality of arrangements of hyperplanes.

Theorem 5.1 (*Cuntz–Geis [5, Corollary 2.4]*)**.** *Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbb{R}^r$. Suppose that $\cap\mathscr{A}$ is a single point, so that the matroid $\mathscr{M}(\mathscr{A})$ has rank r.*

The arrangement $\mathscr{A}$ is simplicial if and only if the characteristic polynomial satisfies

$$r\,\chi_{\mathscr{M}(\mathscr{A})}(-1) + 2\sum_{H\in\mathscr{A}} \chi_{\mathscr{M}(\mathscr{A})/H}(-1) = 0. \tag{16}$$

Lemma 5.2. *Let T be an X-tree and $\mathscr{M}(T)$ the associated matroid. Let e be an edge of T and let $L(e)$, resp. $U(e)$, be the set of labeled, resp. unlabeled vertices of the edge e. Then*

$$\frac{\chi_{\mathscr{M}(T)/e}(-1)}{\chi_{\mathscr{M}(T)}(-1)} = -2^{|U(e)|-1} \prod_{v\in U(e)} \frac{2^{\deg(v)-1}-2}{2^{\deg(v)}-2}$$

Proof. The negative sign appears because the sign of the characteristic polynomial evaluated at -1 is the parity of the rank of the matroid, and contracting by a non-loop element decreases the matroid's rank by one. The formula for the absolute value follows from a case-by-case inspection according to the cardinality of $U(e)$ and $L(e)$, using Lemma 4.17 and the fact that here $\{e\}$ is a flat of $\mathscr{M}(T)$. □

Theorem 5.3. *Let (X, d) be a tree-like metric space. The fundamental polytope $P_d(X)$ is simplicial if and only if the space is a full tree. (In this case the face numbers are computed in Example 4.6.)*

Proof. With Lemma 5.2, and using the fact that $\mathscr{M}(\mathscr{A}(\mathcal{S}))$ has rank $|X|-1$, Cuntz and Geis' condition (Eq. (16)) is equivalent to

$$\left(|X| - 1 - 2\sum_{e\in E(T)} q(e)\right) = 0. \tag{17}$$

where we write $q(e)$ for the absolute value of the quantity at the right-hand side of the claim in Lemma 5.2. Now notice that

$$q(e) = \frac{1}{2} \quad \text{if } U(e) = \emptyset, \qquad q(e) > \frac{1}{2} \quad \text{otherwise},$$

since the degree of an unlabeled vertex in an X-tree is always at least 3. Thus the left-hand side of Eq. (17) is

$$\left(|X| - 1 - 2 \sum_{e \in E(T)} q(e) \right) \le |X| - 1 - |E(T)| \le 0$$

where the first inequality is an equality if and only if $U(e) = \emptyset$ for all $e \in E(T)$. This means that no unlabeled vertices exist, hence $|X| = V(T)$ and the second inequality is also an equality because every finite tree has one more vertex than it has edges. The claim follows. □

Definition 5.4 (*Gordon and Petrov [8]*)**.** A metric space (X, d) is called *generic* if the triangle inequality is always strict (i.e., $d(x, z) < d(x, y)+d(y, z)$ for all pairwise distinct $x, y, z \in X$) and the fundamental polytope $P_d(X)$ is simplicial.

Corollary 5.5. *No tree-like metric on more than* 2 *points is generic in the sense of Definition* 5.4*.*

Proof. We know by Theorem 5.3 that a tree metric for which the polytope is simplicial must be a full tree (i.e., without unlabeled vertices). But every full tree on 3 or more vertices contains two adjacent edges, and the triangle inequality applied to the tree vertices incident to those two edges is in fact an equality. □

CRediT authorship contribution statement

Emanuele Delucchi: Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Writing - original draft, Writing - review & editing. **Linard Hoessly:** Conceptualization, Investigation, Methodology, Project administration, Writing - original draft, Writing - review & editing.

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