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Some properties of edge intersection graphs of single-bend paths on a grid

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ABSTRACT

In this paper we consider graphs G whose vertices can be represented as single-bend paths (i.e., paths with at most one turn) on a rectangular grid, such that two vertices are adjacent in G if and only if the corresponding paths share at least one edge of the grid. These graphs, called B_1-EPG graphs, were first introduced in Golumbic et al. (2009) [13]. Here we show that the neighborhood of every vertex in a B_1-EPG graph induces a weakly chordal graph. From this we conclude that the family $\mathcal{F}$ of B_1-EPG graphs satisfies the Erdős–Hajnal property with $\epsilon(\mathcal{F}) = \frac{1}{3}$, i.e., that every B_1-EPG graph on n vertices contains either a clique or a stable set of size at least $n^{\frac{1}{3}}$. Finally we give a characterization of B_1-EPG graphs among some subclasses of chordal graphs, namely chordal bull-free graphs, chordal claw-free graphs, chordal diamond-free graphs, and special cases of split graphs.

1. Introduction

Edge intersection graphs of paths on a grid (or for short EPG graphs) are graphs whose vertices can be represented as paths on a rectangular grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. These graphs were first introduced in [13] and have also been studied by several authors (see [2,6,14]). The motivation for studying these graphs comes from circuit layout problems (see for instance [3]). In [13] the authors show that every graph G is an EPG graph. They also introduce some subclasses of EPG graphs, namely B_k-EPG graphs, for $k \geq 0$. For these graphs, the paths on the grid that represent the vertices of G are allowed to have at most k bends, i.e., at most k turns. They show that every tree is a B_1-EPG graph and they also give some examples of graphs which are not B_1-EPG. Furthermore, the representation of cliques and cycles in B_1-EPG graphs is considered. In [2] the authors study questions related to the size of the grid that is needed in order to represent every n vertex graph as an edge intersection of paths on a grid. Furthermore, they show that for any k, only a small fraction of all labeled graphs on n vertices are B_k-EPG. Some results of [2] were also proved in [6]. In addition the authors in [6] consider different classes of graphs and show in particular that every planar graph is a B_5-EPG graph. In [14], the authors prove that recognizing B_1-EPG graphs is $\mathcal{NP}$-complete.

In this paper we focus on B_1-EPG graphs. The paper is organized as follows. In Section 2 we give definitions and notation as well as some useful results from [13]. In Section 3 we present some properties of the neighborhood of a vertex in a B_1-EPG graph. In Section 4 we prove that B_1-EPG graphs satisfy the Erdős–Hajnal property with $\epsilon(\mathcal{F}) = \frac{1}{3}$, i.e., that every B_1-EPG graph on n vertices contains either a clique or a stable set of size at least $n^{\frac{1}{3}}$. Section 5 focuses on some subclasses of chordal

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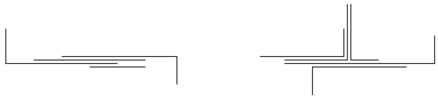

Fig. 1. Left: an edge clique. Right: a claw clique.

graphs and gives a characterization of B_1-EPG graphs of these subclasses. Finally we conclude with some open questions in Section 6. For graph theoretical terms that are not defined here, we refer the reader to [4].

2. Preliminaries

All graphs in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph. For a vertex $v \in V(G)$, we let $N_G(v)$ denote the set of vertices in G that are adjacent to v, i.e., the neighbors of v. $N_G(v)$ is called the *neighborhood* of vertex v. We will write $N_G[v] = N_G(v) \cup \{v\}$, and call $N_G[v]$ the *closed neighborhood* of vertex v. Whenever it is clear from the context what G is, we will drop the subscripts and write $N(v) = N_G(v)$ and $N[v] = N_G[v]$. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise nonadjacent vertices. For $X \subseteq V(G)$, we denote by $G|X$ the subgraph induced by X. For graphs G and H, G is said to be *H-free* if G has no induced subgraph isomorphic to H.

For disjoint sets $A, B \subseteq V(G)$, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and that A is *anticomplete* to B if every vertex in A is nonadjacent to every vertex in B. A *hole* is an induced cycle on at least four vertices. An *antihole* is the complement of a hole. The *length* of a hole (antihole) is the number of vertices inducing the hole (antihole). A hole (antihole) is *odd* if it has odd length. A graph G is *chordal* if G does not contain any hole. A graph G is *weakly chordal* if it contains no hole of length at least 5 and no antihole of length at least 5. For a graph G, let $\omega(G)$ denote the size of a largest clique in G, and let $\chi(G)$ denote the chromatic number of G. A graph G is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G. It has been shown in [7] that a graph is perfect if and only if it contains no odd hole and no odd antihole. Three vertices u, v, w of a graph G form an *asteroidal triple* (AT) of G if for every pair of them there exists a path connecting the two vertices and such that the path avoids the neighborhood of the remaining vertex. We denote by P_k an induced path on k vertices, and by H_k a hole on k vertices. The complement of a graph G is denoted by $\overline{G}$.

Let $\mathcal{G}$ be a rectangular grid of size $(2m+1) \times (2m+1)$. The horizontal grid lines will be referred to as *rows* and denoted by $y_{-m}, y_{-m+1}, \ldots, y_0, \ldots, y_{m-1}, y_m$, and the vertical grid lines will be referred to as *columns* and denoted by $x_{-m}, x_{-m+1}, \ldots, x_0, \ldots, x_{m-1}, x_m$. Let $\mathcal{P}$ be a collection of nontrivial simple paths on $\mathcal{G}$. We define the edge intersection graph EPG($\mathcal{P}$) of $\mathcal{P}$ to have vertices which correspond to the members of $\mathcal{P}$, such that two vertices are adjacent in EPG($\mathcal{P}$) if and only if the corresponding paths in $\mathcal{P}$ share at least one edge in $\mathcal{G}$. An undirected graph G is called an *edge intersection graph of paths on a grid (EPG)* if $G = \mathrm{EPG}(\mathcal{P})$ for some $\mathcal{P}$ and $\mathcal{G}$, and $\langle \mathcal{P}, \mathcal{G} \rangle$ is an *EPG representation of* G. For any vertex $v \in V(G)$, we denote by P_v the corresponding path in the EPG representation of G. In this paper we will always assume that the size of the grid $\mathcal{G}$, in particular m, is sufficiently large such that the B_1-EPG graphs that we are interested in admit an EPG representation on $\mathcal{G}$.

A turn of a path at a grid point is called a *bend* and the grid point is called a *bend point*. An EPG representation is a B_k-EPG representation if each path has at most k bends. A graph that has a B_k-EPG representation is called B_k-EPG. In this paper we only consider B_1-EPG graphs. We define a *$\ulcorner$-path* P as a bended path with some bend point (x_i, y_j) such that P uses column x_i between rows y_k and y_j, for some $k < j$, and P uses row y_j between columns x_i and x_l, for some $l > i$. *$\urcorner$-paths*, *$\llcorner$-paths*, and *$\lrcorner$-paths* are defined in a similar way. We say that a path P on the grid *contains* a grid point (x_i, y_j) if $(x_i, y_j) \subseteq P$ and (x_i, y_j) is not an endpoint of P. A horizontal segment (resp. vertical segment) on a row y (resp. a column x) between columns x_i and x_j, $i < j$ (resp. between rows y_p and y_q, $p < q$) is denoted by $[x_i, x_j] \times \{y\}$ (resp. $[y_p, y_q] \times \{x\}$).

Let K be a clique of a B_1-EPG graph G. If there exists an edge ϵ in the grid $\mathcal{G}$ such that $\epsilon \subseteq P_v$ for all $v \in K$, we say that K is represented as an *edge clique*. Another way to represent K in a B_1-EPG representation of G is as follows. Every path P_v, $v \in K$, has one of the following properties: (i) P_v is a $\lrcorner$-path (resp. $\urcorner$-path) with bend point (x, y); (ii) P_v is a $\llcorner$-path (resp. $\ulcorner$-path) with bend point (x, y); (iii) P_v uses row y and contains the grid point (x, y). If there exists a path P_v, $v \in K$, of each of these three types, we say that K is represented as a *claw clique* with horizontal basis y and center (x, y). See Fig. 1 for an example of an edge clique and one of a claw clique. We define in a similar way claw cliques with vertical basis.

In [13] the authors showed the following result.

Theorem 1. *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation on a grid $\mathcal{G}$ of a graph G. Every clique in G corresponds to either an edge clique or a claw clique in $\langle \mathcal{P}, \mathcal{G} \rangle$.*

Consider H_4 with edge set $\{ab, bc, cd, ad\}$ in a B_1-EPG graph G, and consider a B_1-EPG representation of G. If P_a is a $\lrcorner$-path with bend point (x, y), P_b is a $\llcorner$-path with bend point (x, y), P_c is a $\ulcorner$-path with bend point (x, y), and P_d is a $\urcorner$-path with bend point (x, y), then we say that H_4 is represented as a *true pie with center* (x, y). Another possible representation of H_4 is as follows. P_a is a $\lrcorner$-path (resp. $\llcorner$-path) with bend point (x, y), P_b uses column x and contains (x, y), P_c is a $\ulcorner$-path (resp. $\urcorner$-path) with bend point (x, y), and P_d uses row y and contains (x, y). Such a representation of H_4 is called a *false pie with center* (x, y). Finally a third possible way to represent H_4 is to use what the authors in [13] call a *frame*. A frame is a rectangle in $\mathcal{G}$ such that each corner is the bend for a different member of P_a, P_b, P_c, P_d, the subpaths $P_a \cap P_b$, $P_b \cap P_c$, $P_c \cap P_d$, $P_d \cap P_a$ are

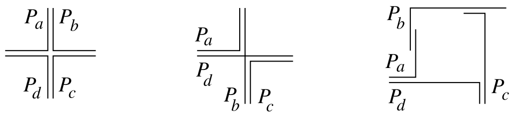

Fig. 2. H_4 represented as a true pie, a false pie, and a frame.

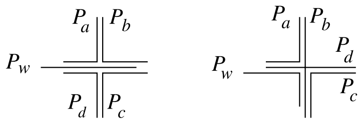

Fig. 3. Two possible B_1-EPG representations of $H_4 \cup \{w\}$. Left: H_4 corresponds to a true pie. Right: H_4 corresponds to a false pie.

nonempty, and the subpaths $P_b \cap P_d$, $P_a \cap P_c$ are empty. See Fig. 2 for an example of a true pie, a false pie, and a frame. In [13] the authors showed the following result.

Theorem 2. *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation on a grid $\mathcal{G}$ of a graph G. Every hole of length 4 in G corresponds to a true pie or a false pie or a frame in $\langle \mathcal{P}, \mathcal{G} \rangle$.*

3. Some properties of the neighborhood

In this section, we focus on the neighborhood of a vertex in a B_1-EPG graph. In particular we will give some properties of the subgraph induced by the neighborhood of any vertex in a B_1-EPG graph. First we start with an easy observation.

Lemma 3. *Consider H_4 with edge set $\{ab, bc, cd, ad\}$ and let w be adjacent to a, b, c and d. Then in any B_1-EPG representation on a grid $\mathcal{G}$ of $H_4 \cup \{w\}$, H_4 corresponds either to a true pie or to a false pie.*

Proof. From Theorem 2 it follows that in any B_1-EPG representation, H_4 corresponds to a true pie, or a false pie, or a frame. Suppose that H_4 corresponds to a frame. Thus every path P_i, $i \in V(H)$, is bended and two paths cannot have a same bend point. So we may assume, without loss of generality, that P_a, P_b, P_c, P_d have bend points, respectively in (x_1, y_1), (x_2, y_1), (x_2, y_2), (x_1, y_2). Now in order to intersect both paths P_a and P_c, the path P_w must be bended either in (x_1, y_2) or in (x_2, y_1). In the first case P_w will clearly not intersect P_b and in the second case P_w will clearly not intersect P_d. Thus H_4 cannot be represented as a frame.

If H_4 corresponds to a true pie with center (x, y), then P_w intersects the paths of the pie either on column x or on row y (see Fig. 3). If H_4 corresponds to a false pie with center (x, y), we may assume, without loss of generality, that P_a is represented as a $\lrcorner$-path, P_b uses column x and contains (x, y), P_c is represented as a $\ulcorner$-path, and P_d uses row y and contains (x, y). Then P_w is either a $\llcorner$-path with bend point (x, y) or a $\urcorner$-path with bend point (x, y) (see Fig. 3). □

Let us now focus on some configurations which can never occur in the neighborhood of a vertex in a B_1-EPG graph.

Lemma 4. *Let G be a B_1-EPG graph. Then $G|N(v)$ does not contain $\overline{H}_6$, $\overline{P}_6$ as induced subgraphs, for all $v \in V(G)$.*

Proof. The proof is by contradiction. We will prove that $G|N(v)$ does not contain $\overline{H}_6$. The proof for $\overline{P}_6$ is similar. Consider $\overline{H}_6$ with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, bc, cd, ad, ae, de, ef, bf, cf\}$. Suppose a vertex $v \in V(G)$ is complete to $V(H_6)$. It follows from Lemma 3 that the hole H_4 with vertex set a, b, c, d is represented either as a true pie or as a false pie. First suppose that H_4 is represented as a true pie with center (x_i, y_j). Without loss of generality, we may assume that P_a is represented as a $\lrcorner$-path, P_b as a $\llcorner$-path, P_c as a $\ulcorner$-path, and P_d as a $\urcorner$-path. Since e is adjacent to a and d, but not to b and c, P_e must use row y_j only at the left of column x_i and cannot use column x_i. Similarly since f is adjacent to b and c, but not to a and d, P_f must use row y_j only at the right of column x_i and cannot use column x_i. But now clearly P_e and P_f cannot intersect, a contradiction (see Fig. 4). So suppose that H_4 is represented as a false pie with center (x_i, y_j). Without loss of generality, we may assume that P_a is represented as a $\lrcorner$-path, P_b uses column x_i and contains (x_i, y_j), P_c is represented as a $\ulcorner$-path, and P_d uses row y_j and contains (x_i, y_j). Using the same arguments as before we obtain that P_e must use row y_j only at the left of x_i and cannot use column x_i, and P_f must use column x_i only below row y_j and cannot use row y_j. Thus P_e and P_f cannot intersect, a contradiction (see Fig. 4). □

Since every antihole of length at least 7 contains $\overline{P}_6$ as an induced subgraph, the following result follows immediately from Lemma 4.

Corollary 5. *Let G be a B_1-EPG graph. Then $G|N(v)$ contains no $\overline{H}_k$, $k \geq 6$, for all $v \in V(G)$.*

We prove the following lemma.

Lemma 6. *Let G be a B_1-EPG graph. Then $G|N(v)$ contains no H_k, $k \geq 5$, for all $v \in V(G)$.*

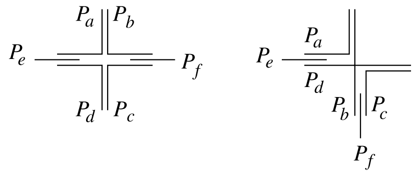

Fig. 4. Left: H_4 represented as a true pie; P_e and P_f cannot intersect. Right: H_4 represented as a false pie; P_e and P_f cannot intersect.

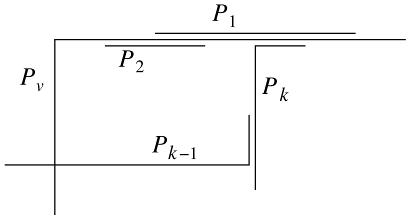

Fig. 5. P_{k-1} cannot intersect P_k and P_v.

Proof. The proof is by contradiction. Consider H_k, $k \geq 5$, with vertex set $\{a_1, a_2, \ldots, a_k\}$ and edge set $\{a_1a_2, a_2a_3, \ldots, a_{k-1}a_k, a_ka_1\}$. Suppose that a vertex $v \in V(G)$ is complete to $V(H_k)$. Without loss of generality, we may assume that P_v is represented as a $\ulcorner$-path with bend point (x_i, y_j) and endpoints $(x_{i'}, y_j)$, $(x_i, y_{j'})$. Furthermore, since $k \geq 5$, we may assume that there are at least three paths representing vertices of H_k which intersect P_v on row y_j. Thus we may assume that P_{a_1} is the path having the rightmost intersection with P_v on row y_j among the paths P_{a_i}, $i \in \{1, \ldots, k\}$. We may also assume that the intersection of P_{a_1} with P_v lies between columns x_l and $x_{l'}$, $i < l < l' \leq i'$. Since P_{a_2} and P_{a_k} do not intersect, we may assume that the intersection of P_{a_k} with P_{a_1} is at the right of the intersection of P_{a_2} with P_{a_1}. This implies that P_{a_k} intersects P_v only on row y_j. Since P_{a_1} is the path having the rightmost intersection with P_v on row y_j, and since $P_{a_{k-1}}$ does not intersect either P_{a_1} or P_{a_2}, it follows that $P_{a_{k-1}}$ must intersect P_{a_k} on some column x_p, $l \leq p \leq l'$, and $P_{a_{k-1}}$ must intersect P_v on column x_i. But this is clearly impossible since $P_{a_{k-1}}$ has a single bend (see Fig. 5). □

We can now state our main result of this section.

Theorem 7. *Let G be a B_1-EPG graph. Then $G|N(v)$ is weakly chordal, for all $v \in V(G)$.*

Proof. It follows from Lemma 6 that $G|N(v)$ contains no H_k, $k \geq 5$, and does not contain $\overline{H}_5$ (since H_5 is isomorphic to its complement), for all $v \in V(G)$. From Corollary 5 it follows that $G|N(v)$ contains no $\overline{H}_k$, $k \geq 6$, for all $v \in V(G)$. This proves the theorem. □

Since weakly chordal graphs are perfect graphs, we obtain the following result.

Lemma 8. *Let G be a B_1-EPG graph. Then $G|N(v)$ is perfect, for all $v \in V(G)$.*

Finally let us show the following result which we will use later in the paper.

Theorem 9. *Let G be a B_1-EPG graph. Then $G|N(v)$ is AT-free, for all $v \in V(G)$.*

Proof. The proof is by contradiction. Suppose that $G|N(v)$ contains an asteroidal triple a, b, c for some vertex $v \in V(G)$. Suppose, without loss of generality, that P_v is represented as a $\ulcorner$-path with bend point (x_i, y_j) and endpoints $(x_i, y_{j'})$, $(x_{i'}, y_j)$. We may assume that P_a and P_b intersect P_v on row y_j, that P_a is at the right of P_b, and that P_c intersects P_v on row y_j at the left of P_b or on column x_i. Let (x_l, y_j) be the rightmost intersection point of P_b with P_v, $i < l < i'$. In $G|N(v)$ there exists a path $\pi = \{av_1, v_1v_2, \ldots, v_{q-1}v_q, v_qc\}$ from a to c avoiding the neighborhood of b. Clearly there exists at least one vertex v_i, $i \in \{1, \ldots, q\}$, such that the corresponding path P_{v_i} contains a grid point lying at the left of column x_l. Let v_p be the first vertex in the order $\{v_1, v_2, \ldots, v_q\}$ whose corresponding path contains a grid point lying at the left of column x_l. P_{v_p} cannot contain a grid point on row y_j at the left of column x_l, otherwise it intersects P_b. Thus P_{v_p} uses some row $y_{j''}$, $j'' \neq j$ (see Fig. 6). Since P_{v_p} must intersect P_v, it necessarily uses column x_i. Now consider $P_{v_{p-1}}$. By definition of v_p, $P_{v_{p-1}}$ lies at the right of column x_l and must intersect P_{v_p} on row $y_{j''}$. But then clearly $P_{v_{p-1}}$ cannot intersect P_v, since they must necessarily intersect on row y_j. □

As a consequence we get the following result.

Corollary 10. *Let G be a B_1-EPG graph not containing H_4. Then for every vertex $v \in V(G)$, $G|N(v)$ is an interval graph.*

Proof. Let $v \in V(G)$. Since G does not contain H_4 and because $G|N(v)$ cannot contain H_k, $k \geq 5$ (see Lemma 6), $G|N(v)$ is a chordal graph. From Theorem 9, we know that $G|N(v)$ is AT-free. By a result from [15], a graph is an interval graph if and only if it is chordal and AT-free. Therefore $G|N(v)$ is an interval graph. □

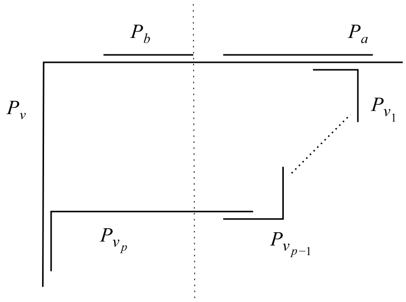

Fig. 6. Illustrating the proof of Theorem 9.

4. The Erdős–Hajnal property for B_1-EPG graphs

In [10], Erdős and Hajnal made the following conjecture.

Conjecture 11. *For every graph H, there exists $\delta(H) > 0$ such that if G is a graph and no induced subgraph of G is isomorphic to H, then G contains either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$.*

Related to this, we consider the following definition (see [12]).

Definition 12. A family $\mathcal{F}$ of finite graphs has the *Erdős–Hajnal property* if there exists a constant $\epsilon(\mathcal{F}) > 0$ such that every graph in $\mathcal{F}$ on n vertices contains either a clique or a stable set of size at least $n^{\epsilon(\mathcal{F})}$.

Notice that the family $\mathcal{F}$ of perfect graphs satisfies the Erdős–Hajnal property with $\epsilon(\mathcal{F}) = \frac{1}{2}$. Indeed for any perfect graph G on n vertices we have $n \leq \omega(G)\alpha(G)$, where $\omega(G)$ is the size of a largest clique in G and $\alpha(G)$ is the size of a largest stable set in G. It follows that either $\omega(G)$ or $\alpha(G)$ has value at least $n^{\frac{1}{2}}$.

It follows from [1,12] that the family $\mathcal{F}$ of B_1-EPG graphs satisfies the Erdős–Hajnal property but without giving any fixed value of $\epsilon(\mathcal{F})$. Here we will prove that the family $\mathcal{F}$ of B_1-EPG graphs satisfy the property with $\epsilon(\mathcal{F}) = \frac{1}{3}$.

Theorem 13. *Let $G = (V, E)$ be a B_1-EPG graph with $|V| = n$. Then G contains either a clique or a stable set of size at least $n^{\frac{1}{3}}$.*

Proof. First suppose that there exists a vertex $v \in V(G)$ with degree at least $n^{\frac{2}{3}}$. Thus it follows from Lemma 8 that $G|N(v)$ is a perfect graph of size at least $n^{\frac{2}{3}}$. From the remark above we deduce that G contains either a clique or a stable set of size $(n^{\frac{2}{3}})^{\frac{1}{2}} = n^{\frac{1}{3}}$.

Now suppose that G does not contain any vertex with degree at least $n^{\frac{2}{3}}$. Thus G has maximum degree at most $n^{\frac{2}{3}} - 1$. In [5] it was proven that every maximal stable set in a graph on n vertices and of maximum degree h has size at least $\lceil \frac{n}{h+1} \rceil$. It follows that in our case G contains a stable set of size at least $\left\lceil \frac{n}{(n^{\frac{2}{3}}-1)+1} \right\rceil \geq n^{\frac{1}{3}}$. $\square$

5. Subclasses of chordal graphs

In this section we focus on subclasses of chordal graphs. We consider several classes of chordal graphs: chordal bull-free graphs, chordal claw-free graphs, chordal diamond-free graphs (see Fig. 7; formal definitions will be given in the corresponding sections), and finally special cases of split graphs. We give necessary and sufficient conditions for such graphs to be B_1-EPG graphs or we prove that all graphs of the class considered are B_1-EPG.

We start with two easy observations.

Lemma 14. *Let G be a B_1-EPG graph and let K be a maximal clique in G.*

(1) *If K is represented as an edge clique, then there exists a segment on the grid which is used by all paths corresponding to the vertices of K and no other path uses that segment.*
(2) *If K is represented as a claw clique with a horizontal basis and with center $C = (x_i, y_j)$ and if G is C_4-free, then there exists a B_1-EPG representation of G in which all the paths corresponding to the vertices of K use row y_j between column x_{i-1} and column x_{i+1} and no other path uses that segment.*

Proof. (1) By definition of an edge clique there exists an edge on the grid such that all paths corresponding to vertices of K use this edge. If there is a path P_v with $v \notin K$ using this same edge, then K is not maximal, a contradiction. This proves (1).

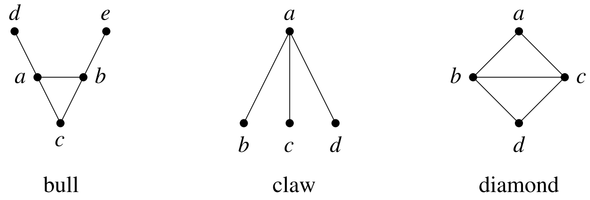

Fig. 7. Bull, claw, and diamond graphs.

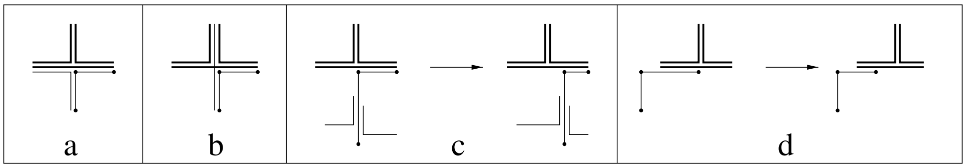

Fig. 8. In this figure, P_v is represented by a path with small dots as endpoints and the bend point. (a) P_v intersects a $\urcorner$-path with bend point C. (b) The vertical segment of P_v intersects a path using the grid segment $\{x_1\} \times [y_0, y_1]$. (c) The vertical segment of P_v intersects a path not containing C. (d) C is an endpoint of P_v.

(2) Denote by $\mathcal{K}$ the set of paths corresponding to vertices of K. Assume, without loss of generality, that the basis of the claw clique is $[x_0, x_2] \times \{y_0\}$ and that $C = (x_1, y_0)$. If no path $P_v \notin \mathcal{K}$ uses row y_0 between columns x_0 and x_2, then we are done. So we may assume now that there exists a path $P_v \notin \mathcal{K}$ such that P_v uses the grid segment $[x_1, x_2] \times \{y_0\}$. Clearly P_v cannot use either the segment $[x_0, x_1] \times \{y_0\}$ or the segment $\{x_1\} \times [y_0, y_1]$, otherwise $v \in K$, a contradiction. Thus we may distinguish two cases: (i) C is an endpoint of P_v; (ii) P_v is a $\ulcorner$-path with bend point C.

First we transform the grid $\mathcal{G}$ into a new grid $\mathcal{G}'$ by inserting between every two columns x_i, x_{i+1} a new column $x_{i'}$. Thus in the new grid $\mathcal{G}'$ all paths P have doubled in length as regards their horizontal part.

If C is an endpoint of P_v, then we may shorten P on row y_0 without changing the adjacencies, i.e., we may delete the horizontal part of P_v using the segment $[x_1, x_2] \times \{y_0\}$ such that $(x_{1'}, y_0)$ becomes a new endpoint of P_v (see Fig. 8(d) for an example).

So we may assume now that P_v is a $\ulcorner$-path with bend point C. Then we distinguish three cases: (i) the vertical segment of P_v intersects a $\urcorner$-path with bend point C (see Fig. 8(a)); (ii) the vertical segment of P_v intersects a path using the grid segment $\{x_1\} \times [y_0, y_1]$ (see Fig. 8(b)); (iii) the vertical segment of P_v intersects a path not containing C (see Fig. 8(c)). Clearly in the first two cases we get a C_4 represented as a true pie in case (i) and represented as a false pie in case (ii), a contradiction. In case (iii) we may transform the representation so that all the paths using x_1 and not contained in $\mathcal{K}$ will use instead column $x_{1'}$. To do so, we have to extend or shorten these paths in order to keep the same adjacencies (see Fig. 8(c)).

Note that similar transformations are applied if P_v is a path using the grid segment $[x_0, x_1] \times \{y_0\}$ in the original grid $\mathcal{G}$. Thus we obtain a representation in which only paths corresponding to vertices in K use the grid segment $[x_{0'}, x_{1'}] \times \{y_0\}$. □

Notice that Lemma 14(2) can also be stated with the claw clique having a vertical basis.

5.1. Chordal bull-free graphs

A *bull* is a graph with vertex set $\{a, b, c, d, e\}$ and with edge set $\{ab, ac, ad, bc, be\}$. A graph is bull-free if it does not contain any induced subgraph isomorphic to a bull.

A *homogeneous set* in a graph G is a subset $H \subseteq V(G)$ with $2 \leq |H| < |V(G)|$, such that every vertex in $G \setminus H$ is either complete to H or anticomplete to H. For such an H, let $N(H)$ denote the set of vertices in $G \setminus H$ which are complete to H and $\overline{N}(H)$ the set of vertices in $G \setminus H$ which are anticomplete to H.

In [16], the authors show that every connected bull-free graph G satisfies one of the following five properties: (i) G contains H_5, (ii) G is triangle-free, (iii) $\overline{G}$ is triangle-free, (iv) G has a homogeneous set, (v) G or $\overline{G}$ contains an induced subgraph isomorphic to $F_0 = \{ab, bc, cd, ad, ae, be, cf\}$. This result will be used in order to prove the main result of this subsection.

We will first start with the following observation.

Lemma 15. *Let G be a* {bull, C_4}*-free B_1-EPG graph. Then G admits a B_1-EPG representation in which every clique is represented as an edge clique.*

Proof. Suppose the result is false, i.e., for every B_1-EPG representation of G there exists at least one clique in G which is represented as a claw clique. Consider one such representation and let K denote a clique in G which is represented as a claw clique. Without loss of generality, we may assume that K is maximal, and that the claw clique has horizontal basis $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. From Lemma 14(2) it follows that we may assume that no path P_v, $v \notin K$, uses the grid segment $[x_0, x_2] \times \{y_0\}$. Let us denote by $\mathcal{K}$ the set of paths corresponding to the vertices of K. For every $\lrcorner$-path $P_v \in \mathcal{K}$

(resp. $\llcorner$-path $P_{v'} \in \mathcal{K}$), we do the following: if P_v (resp. $P_{v'}$) does not intersect any path $P_w \notin \mathcal{K}$ on column x_1, then we delete its vertical segment and add the grid segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_2] \times \{y_0\}$). If after these transformations either there exist no more $\lrcorner$-paths in $\mathcal{K}$ or there exist no more $\llcorner$-paths in $\mathcal{K}$, then we are done since we have obtained an edge clique. So we may assume that there exists at least one $\lrcorner$-path $P_v \in \mathcal{K}$ and at least one $\llcorner$-path $P_{v'} \in \mathcal{K}$. Since both paths $P_v, P_{v'}$ intersect a path on column x_1, they necessarily intersect a common path, say $P_t \notin \mathcal{K}$. Now if all $\lrcorner$-paths (resp. $\llcorner$-paths) do not intersect any path on row y_0, then we may transform them into $\llcorner$-paths (resp. $\lrcorner$-paths) with rightmost endpoint (x_2, y_0) (resp. with leftmost endpoint (x_0, y_0)) and thus obtain an edge clique. So we may assume that there exists at least one $\lrcorner$-path intersecting a path $P_{t'} \notin \mathcal{K}$ on row y_0 and there exists at least one $\llcorner$-path intersecting a path $P_{t''} \notin \mathcal{K}$ on row y_0. Without loss of generality, we may assume that P_v intersects $P_{t'}$ and that $P_{v'}$ intersects $P_{t''}$. Clearly $P_{t'}$ and $P_{t''}$ cannot intersect since $t', t'' \notin K$ and all paths are single-bended. By the same arguments, $P_{t'}, P_{t''}$ cannot intersect P_t. But now v, v', t, t', t'' induce a bull, a contradiction. □

We are now in a position to prove our main result of this subsection.

Theorem 16. *Let G be a chordal bull-free graph. Then G is a B_1-EPG graph if and only if it does not contain any induced subgraph isomorphic to $T_2 = \{ab, bc, cd, de, cf, fg\}$ in the neighborhood of a vertex.*

Proof. From Theorem 9 it follows that if G is a B_1-EPG graph then T_2 is a forbidden induced subgraph for $G|N(v), \forall v \in V(G)$, since a, e and g form an asteroidal triple. For the "if" direction, assume that the result is false and consider a minimal counterexample G'. Clearly G' must be connected. If G' is AT-free, then G' is an interval graph (see [15]) and thus it is B_1-EPG, a contradiction. So suppose that G' contains an asteroidal triple. Since G' is chordal and bull-free, it must contain T_2 as an induced subgraph. Indeed, T_2 is the only graph among the forbidden induced subgraphs that characterize AT-free graphs which does not contain either a hole or a bull (see [8]). From the aforementioned result [16], it follows that G' must contain a homogeneous set H. Indeed, G' does not contain H_5 since it is chordal; G' is not triangle-free since otherwise it is a tree and thus B_1-EPG (see [13]); $\bar{G'}$ is not triangle-free since G' must contain T_2 as an induced subgraph (b, d, f induce a triangle in $\overline{G'}$); neither G' nor $\overline{G'}$ contains F_0 as an induced subgraph since both F_0 and $\overline{F}_0$ contain H_4. Therefore G' must satisfy the remaining property, i.e., G' must contain a homogeneous set H. Since $|H| < |V(G)|$ and G' is connected, we must have $|N(H)| \geq 1$. Next notice that H does not contain T_2 as an induced subgraph, since otherwise T_2 is in the neighborhood of every vertex of $N(H)$. Thus $G'|H$ is an interval graph since it is chordal and AT-free. Furthermore either $G'|H$ or $G'|N(H)$ is a clique. Indeed, if neither of them is a clique, i.e., if there exist two vertices $v_1, v_2 \in H$ and two vertices $w_1, w_2 \in N(H)$ such that $v_1v_2, w_1w_2 \notin E(G')$, then these vertices induce a C_4, a contradiction. First suppose that $G'|H$ is a clique. Delete $v_1 \in H$. $G' \setminus \{v_1\}$ admits a B_1-EPG representation by minimality of G'. Now add P_{v_1} to that representation such that P_{v_1} coincides with P_{v_2} for some $v_2 \in H$ (v_2 exists since $|H| \geq 2$). This clearly gives us a feasible B_1-EPG representation of G', a contradiction. So suppose $G'|N(H)$ is a clique. Consider a B_1-EPG representation of $G'' = G'|(V \setminus H) \cup \{v'\}$ where $v' \in H$ (v' exists since $|H| \geq 2$). It follows from Lemma 15 that we may assume that the clique K induced by $N(H) \cup \{v'\}$ is represented as an edge clique. Since K is maximal in G'', it follows from Lemma 14 that we may assume that only paths representing vertices of K use row y_0 between columns x_i and x_j, $i < j$. Now delete $P_{v'}$ and add a B_1-EPG representation of $G'|H$ on row y_0 between columns x_i and x_j. This is possible because $G'|H$ is an interval graph and we may choose x_i, x_j such that a B_1-EPG representation of $G'|H$ fits between these columns on row y_0. Thus we get a B_1-EPG representation of G', a contradiction. □

5.2. Chordal claw-free graphs

A *claw* is a graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, ac, ad\}$. A graph is claw-free if it does not contain any induced subgraph isomorphic to the claw. A claw with vertices a, b, c, d and edges ab, ac, ad will be denoted by $(a; b, c, d)$. A *simplicial vertex* in a graph is a vertex whose neighbors induce a clique. From [9], we know that every chordal graph has a simplicial vertex.

Theorem 17. *Every chordal claw-free graph G is a B_1-EPG graph.*

Proof. Suppose that the statement is wrong. Then let G be a minimal counterexample, i.e., let G be a chordal claw-free graph such that G has no B_1-EPG representation, but $G \setminus \{v\}$ has a B_1-EPG representation for every vertex $v \in V(G)$. We will consider several cases and in all of them we will deduce that G actually *has* a B_1-EPG representation; thus we get a contradiction.

According to the result mentioned above from [9], G has a simplicial vertex. Let v be such a vertex. Let $G' = G \setminus \{v\}$ and let $\langle \mathcal{P}', \mathcal{G} \rangle$ be a B_1-EPG representation of G'. Let K be the clique induced by $N(v)$ and denote by $\mathcal{K}$ the set of paths in $\langle \mathcal{P}', \mathcal{G} \rangle$ corresponding to the vertices in K. If K is a maximal clique in G', then it follows from Lemma 14 that there exists a segment on the grid used by all the paths in $\mathcal{K}$ and no other path uses this segment. Thus we may add a path P_v on this segment and hence we obtain a B_1-EPG representation of G, a contradiction.

Therefore we may assume that K is not maximal. Thus there exists at least one vertex $w \in V(G')$ which is complete to K. Let W be the set of all such vertices w and denote by $\mathcal{W}$ the set of paths in $\langle \mathcal{P}', \mathcal{G} \rangle$ corresponding to the vertices in W. We make the following easy observations:

Observation I. If $w_1, w_2 \in W$, then $w_1w_2 \in E(G)$. Indeed, if $w_1w_2 \notin E(G)$, then $(u; v, w_1, w_2)$ is a claw for any $u \in K$, a contradiction.

Observation II. If $ut \in E(G)$, for $u \in K$ and $t \notin W \cup K \cup \{v\}$, then $wt \in E(G)$ for all $w \in W$. Indeed, if $wt \notin E(G)$, then $(u; v, t, w)$ is a claw, a contradiction.

Observation III. If $ut_1, ut_2 \in E(G)$ for $u \in K$ and $t_1, t_2 \notin W \cup K \cup \{v\}$, then $t_1t_2 \in E(G)$. Indeed, if $t_1t_2 \notin E(G)$, then $(u; v, t_1, t_2)$ is a claw, a contradiction.

It follows from Observation I that $K \cup W$ is a clique—the unique maximal clique including K in G'. We will distinguish two cases according to whether $K \cup W$ is represented in $\langle \mathcal{P}', \mathcal{G} \rangle$ by an edge clique or by a claw clique.

Case 1: $K \cup W$ is represented in $\langle \mathcal{P}', \mathcal{G} \rangle$ by an edge clique.

It follows from Lemma 14 that we may assume, without loss of generality, that the segment $[x_0, x_2] \times \{y_0\}$ is used by all paths of $\mathcal{K} \cup \mathcal{W}$ and no other path uses that segment. From this and from the fact that all paths have at most one bend we obtain the following observation.

Observation IV. If a path P_t intersects (on at least one edge) a path from $\mathcal{K} \cup \mathcal{W}$ and $t \notin K \cup W$, then the intersection lies either at the left of x_0 or at the right of x_2.

This observation allows us to define the following subsets of $V(G')$.

T_ℓ is the set of vertices $t \notin K \cup W$ such that P_t intersects (on at least one edge) a path of $\mathcal{K}$ at the left of x_0;

T_r is the set of vertices $t \notin K \cup W$ such that P_t intersects (on at least one edge) a path of $\mathcal{K}$ at the right of x_2;

K_ℓ is the set of vertices $u \in K$ with at least one neighbor in T_ℓ;

K_r is the set of vertices $u \in K$ with at least one neighbor in T_r;

T_1 is the set of vertices $t \notin K \cup W \cup T_\ell \cup T_r$ such that P_t intersects (on at least one edge) a path of $\mathcal{W}$ at the left of x_0;

T_2 is the set of vertices $t \notin K \cup W \cup T_\ell \cup T_r$ such that P_t intersects (on at least one edge) a path of $\mathcal{W}$ at the right of x_2.

In what follows we will prove several claims.

Claim 1. *W is complete to $T_\ell \cup T_r$.*

This immediately follows from Observation II.

Claim 2. *$T_\ell \cup T_1$ is anticomplete to $T_r \cup T_2$.*

This follows from the fact that no path P_t with $t \in T_\ell \cup T_r \cup T_1 \cup T_2$ uses row y_0 between columns x_0 and x_1 and from the fact that all paths have at most one bend.

Claim 3. *$K_\ell \cap K_r = \emptyset$.*

Suppose that $u \in K_\ell \cap K_r$. Let $t \in T_\ell$ and $t' \in T_r$ be two neighbors of u. It follows from Claim 2 that t and t' are nonadjacent. But now $(u; v, t, t')$ is a claw, a contradiction. This proves Claim 3.

Claim 4. *$T_\ell \cup T_1, T_r \cup T_2 \neq \emptyset$.*

By symmetry it is enough to prove the claim for $T_\ell \cup T_1$. Suppose that $T_\ell \cup T_1 = \emptyset$. Then we may assume that all paths in $\mathcal{K} \cup \mathcal{W}$ have their leftmost endpoint in (x_0, y_0). Furthermore we may assume, without loss of generality, that no path is using the grid segment $[x_{-1}, x_0] \times \{y_0\}$. Thus we may extend the paths of $\mathcal{K}$ to the grid point (x_{-1}, y_0) and then add P_v as a path using only segment $[x_{-1}, x_0] \times \{y_0\}$. Hence we obtain a B_1-EPG representation of G, a contradiction. This proves Claim 4.

Claim 5. *$K_\ell, K_r \neq \emptyset$.*

If $K_\ell, K_r = \emptyset$, then it follows from Observation IV that no vertex in K has a neighbor outside of $K \cup W \cup \{v\}$. Thus we may assume that all paths in $\mathcal{K}$ only use the segment $[x_0, x_2] \times \{y_0\}$. Without loss of generality, we may assume that no path uses the segment $\{x_0\} \times [y_0, y_1]$. Now we may add a vertical part $\{x_0\} \times [y_0, y_1]$ to all the paths in $\mathcal{K}$ and add P_v as a path using only this segment, $\{x_0\} \times [y_0, y_1]$. Hence we obtain a B_1-EPG representation of G, a contradiction.

By symmetry we may assume that $K_\ell \neq \emptyset$. Let $u \in K_\ell$ and let $t \in T_\ell$ be a neighbor of u. Assume that $K_r = \emptyset$. This clearly implies that $T_r = \emptyset$. It follows from Claim 4 that $T_2 \neq \emptyset$. Let $t_2 \in T_2$ and let $w \in W$ be a neighbor of t_2. It follows from Observation II that $wt \in E(G)$ and it follows from Claim 2 that $tt_2 \notin E(G)$. Furthermore it follows from Claim 3 that u is nonadjacent to T_2. Since $t \notin K \cup W$, we conclude (by Observation II) that there exists a vertex $u^* \in K$ which is nonadjacent to t. It follows that $u^*t_2 \in E(G)$, since otherwise $(w; t, u^*, t_2)$ is a claw, a contradiction. But this implies that P_{u^*} intersects P_{t_2} necessarily at the right of x_2, and hence $u^* \in K_r$, a contradiction. This proves Claim 5.

It follows from Claim 5 that $T_\ell, T_r \neq \emptyset$.

Claim 6. *$T_1, T_2 = \emptyset$.*

By symmetry it is enough to show that $T_1 = \emptyset$. Suppose that there exists a vertex $t_1 \in T_1$ and let $w \in W$ such that $wt_1 \in E(G)$. Consider $u \in K_\ell$. Since $t_1 \notin T_\ell$, it follows that $ut_1 \notin E(G)$. But now $(w; u, t_1, t')$ is a claw for any vertex $t' \in T_r$ (recall that $wt' \in E(G)$ by Observation II and that t_1, t' are nonadjacent due to Claim 2), a contradiction. Thus $T_1 = \emptyset$. This proves Claim 6.

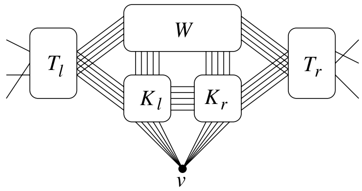

Fig. 9. The different sets of vertices and their relation.

Claim 7. *K_ℓ is complete to T_ℓ and K_r is complete to T_r.*

By symmetry it is enough to show that K_ℓ is complete to T_ℓ. Suppose $u \in K_\ell$ is nonadjacent to some vertex $t \in T_\ell$. Let $w \in W$ such that $wt' \in E(G)$ for some $t' \in T_r \cup T_2$ (Claim 4 implies that such a vertex t' exists). It follows from Claim 1 that $wt \in E(G)$ and it follows from Claim 2 that t and t' are nonadjacent. Furthermore it follows from Claim 3 that u is nonadjacent to t'. But now $(w; u', t, t')$ is a claw, a contradiction. This proves Claim 7.

Claim 8. *T_ℓ and T_r are cliques.*

This immediately follows from Claim 7 and Observation III.

Claim 9. *$K_\ell \cup K_r = K$.*

Let $u \in K \setminus (K_\ell \cup K_r)$ and let $t \in T_\ell, t' \in T_r, w \in W$. Then t is nonadjacent to t' (see Claim 2) and w is adjacent to both t and t' (see Claim 1). Thus $(w; u, t, t')$ is a claw, a contradiction. This proves Claim 8.

Fig. 9 illustrates the different sets of vertices and their relation.

From the above it follows that for each $u' \in K_\ell$, we have $N_{G'}[u] = K \cup W \cup T_\ell$, and that for each $u \in K_r$, we have $N_{G'}[u] = K \cup W \cup T_r$. Therefore all the vertices in K_ℓ have exactly the same neighbors and all the vertices in K_r have exactly the same neighbors.

It follows that we may assume that $|K_\ell| = |K_r| = 1$. Indeed if for instance $u_1, u_2 \in K_\ell$, then we obtain a B_1-EPG representation of G by adding P_{u_2} coinciding with P_{u_1} to a B_1-EPG representation of $G \setminus \{u_2\}$. Thus we assume from now on that $K_\ell = \{u'\}, K_r = \{u''\}$.

It also follows that for each $w \in W, N_{G'}[w] = K \cup W \cup T_\ell \cup T_r$. Thus all the vertices in W have exactly the same neighbors. Similarly to the assumption we did above, we may assume that $|W| = 1$ and $W = \{w\}$.

Notice that at least one of the paths $P_{u'}, P_{u''}$ is bended. For suppose not. We may assume that $P_{u'}$ has its rightmost endpoint in (x_2, y_0) and $P_{u''}$ has its leftmost endpoint in (x_0, y_0). Without loss of generality, we may assume that no path uses the segment $\{x_1\} \times [y_0, y_1]$. Now we may bend $P_{u'}$ and $P_{u''}$ at (x_1, y_0) such that they will intersect along the segment $\{x_1\} \times [y_0, y_1]$ by deleting the part $[x_1, x_2] \times \{y_0\}$ from $P_{u'}$ and the part $[x_0, x_1] \times y_0$ from $P_{u''}$. Adding P_v on exactly this segment gives us a B_1-EPG representation of G, a contradiction. So, without loss of generality, we may assume that $P_{u'}$ is a $\llcorner$-path with bend point (x_{-1}, y_0). It follows that there exists $t \in T_\ell$ such that $u't \in E(G)$ and the intersection of $P_{u'}$ and P_t is on column x_{-1}, since otherwise $P_{u'}$ does not need to be bended.

Clearly at the left of x_0, P_w and $P_{u'}$ intersect exactly the same paths (namely $P_{t'}$ for all $t' \in T_\ell$). Thus we may assume that P_w coincides with $P_{u'}$ at the left of x_0. In particular, they are bended at (x_{-1}, y_0). Since at the right of x_2, P_w and $P_{u''}$ intersect precisely the same paths, namely $P_{t''}$ for all $t'' \in T_r$, we may assume that $P_{u''}$ is not bended and coincides with P_w.

Recall that T_ℓ is a clique (see Claim 8). Thus it is represented either by an edge clique or by a claw clique. Recall that at least one path $P_t, t \in T_\ell$, intersects $P_{u'}$ on column x_{-1}. We distinguish several cases which are also shown in Fig. 10. The bold paths correspond to $P_w, P_{u'}, P_{u''}$ and $P_{t''}$ for some $t'' \in T_r$. Thin paths represent paths P_t for $t \in T_\ell$. Dotted vertical lines are $x = x_0$ and $x = x_2$. For each case that may occur, the figure presents a transformation which yields a B_1-EPG representation of G.

a. T_ℓ is represented by an edge clique and the common segment of all the paths $P_t, t \in T_\ell$, is on column x_{-1} (see Fig. 10(a)). Notice that we may assume, without loss of generality, that no path $P_t, t \in T_\ell$, uses the grid segment $[x_{-2}, x_{-1}] \times \{y_0\}$. If a path $P_t, t \in T_\ell$, is a $\ulcorner$-path or a $\llcorner$-path with bend point (x_{-1}, y_0), then its horizontal segment may be deleted. Under these assumptions, we may transform $P_{u'}$ into a $\lrcorner$-path with bend point (x_{-1}, y_0) and leftmost endpoint (x_{-2}, y_0). Next we extend $P_{u''}$ such that its left endpoint becomes (x_{-2}, y_0). Now it is possible to add P_v on the grid segment $[x_{-2}, x_{-1}] \times \{y_0\}$ and thus we obtain a B_1-EPG representation of G, a contradiction.
b. T_ℓ is represented by a claw clique with its center on column x_{-1} but not (x_{-1}, y_0) (see Fig. 10(b)). Notice that the claw clique can only have a vertical basis. The same transformation as in case a. will lead to a contradiction.
c. T_ℓ is represented by a claw clique with a vertical basis and center (x_{-1}, y_0) (see Fig. 10(c)). Then the horizontal segments of all the paths $P_t, t \in T_\ell$, which use row y_0 may be deleted and their vertical segments slightly extended to keep all the intersections, which gives us an edge clique for T_ℓ. Then we are in case a. again and thus we obtain a contradiction.
d. T_ℓ is represented by a claw clique with a horizontal basis and center (x_{-1}, y_0) (see Fig. 10(d)). We delete the vertical segment of $P_{u'}$ and extend the horizontal segment slightly to the left. Notice that we may assume that no path except $P_t, t \in T_\ell$, uses the grid segment $[x_{-2}, x_{-1}] \times \{y_0\}$. Now $P_{u'}$ and $P_{u''}$ are both not bended. Then we conclude by using the transformation mentioned above.

This completes the proof for the case where $K \cup W$ is represented as an edge clique.

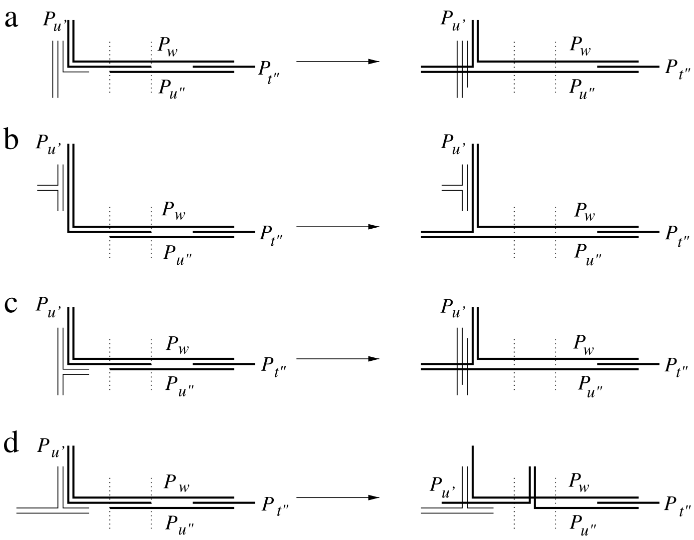

Fig. 10. Several cases that may occur and their transformation.

Case 2: $K \cup W$ is represented by a claw clique.

From Lemma 14 it follows that we may assume that the basis of the claw clique is on row y_0, the center is (x_1, y_0), and the segments $[x_0, x_2] \times y_0$ and $x_1 \times [y_0, y_1]$ are only used by paths corresponding to vertices in $K \cup W$.

First suppose there is no path $P_t \notin \mathcal{K} \cup \mathcal{W}$ which intersects a path $P_u \in \mathcal{K}$ on column x_1. Then we proceed as follows. If no $P_w \in \mathcal{W}$ uses column x_1, we can delete vertical segments on column x_1 of all the paths in $\mathcal{K}$ and extend their horizontal segments slightly (either to (x_2, y_0) or to (x_0, y_0)) to preserve the intersections. Thus $K \cup W$ is represented as an edge clique and we are done. Notice that the same transformation can be applied in the case where all the paths $P_w \in \mathcal{W}$ which are bended at (x_1, y_0) are $\lrcorner$-paths and in the case where all of them are $\llcorner$-paths.

The remaining case is when there are paths $P_w, P_{w'} \in \mathcal{W}$ such that P_w is a $\urcorner$-path and $P_{w'}$ is a $\llcorner$-path. Then we may assume that all paths $P_u \in \mathcal{K}$ are not bended. Indeed, first notice that no P_u can intersect some $P_t \notin \mathcal{K} \cup \mathcal{W}$ at the left of x_0 or at the right of x_2, otherwise $P_{w'}$ (resp. P_w) cannot intersect P_t (resp. $P_{t'}$), thus contradicting Observation II. Furthermore if P_u is a $\lrcorner$-path, then we may delete its vertical part and add the horizontal segment $[x_1, x_2] \times \{y_0\}$, and if P_u is a $\llcorner$-path, then we may delete its vertical part and add the horizontal segment $[x_0, x_1] \times \{y_0\}$. Thus all the vertices $u \in K$ have the same neighbors, namely W. So we may assume that $|K| = 1$, say $K = \{u\}$, and P_u is not bended. Without loss of generality, we may assume that no path uses the segment on $\{x_2\} \times [y_{-1}, y_0]$. But now we can add to P_u the vertical segment on $\{x_2\} \times [y_{-1}, y_0]$ and add P_v on this segment to get a B_1-EPG representation of G, a contradiction.

Thus we may assume now that there exists $P_t \notin \mathcal{K} \cup \mathcal{W}$, which intersects a path $P_u \in \mathcal{K}$ on column x_1. Notice that there cannot exist $P_{t'}, P_{t''} \notin \mathcal{K} \cup \mathcal{W}$ such that $P_{t'}$ intersects some path $P_{u'} \in \mathcal{K}$ at the left of x_1 and $P_{t''}$ intersects some path $P_{u''} \in \mathcal{K}$ at the right of x_1. Indeed, every path $P_w \in \mathcal{W}$ must intersect all these paths $P_t, P_{t'}, P_{t''}$ which is clearly not possible since the paths are single-bended and $t, t', t'' \notin K \cup W$. Thus we may assume that there is no path $P_{t''} \notin \cap K \cup \mathcal{W}$ intersecting some path $P_{u''} \in \mathcal{K}$ at the right of x_1.

First assume that there exists a path $P_{t'} \notin \mathcal{K} \cup \mathcal{W}$ intersecting some path $P_u \in \mathcal{K}$ at the left of x_1. Notice that P_t and $P_{t'}$ cannot intersect because $t, t' \notin K \cup W$ and because all paths are single-bended. Furthermore no path $P_u \in \mathcal{K}$ intersects both P_t and $P_{t'}$, otherwise $(u; t, t', v)$ is a claw, a contradiction. Also all paths $P_w \in \mathcal{W}$ must be $\lrcorner$-paths in order to intersect both P_t and $P_{t'}$. Now distinguish two cases: (i) if $P_{u'} \in \mathcal{K}$ is a $\lrcorner$-path and intersects P_t, then we may assume that its left-most endpoint is (x_0, y_0); now transform $P_{u'}$ into a $\llcorner$-path with rightmost endpoint (x_2, y_0); (ii) if $P_{u'} \in \mathcal{K}$ is a $\lrcorner$-path and intersects $P_{t'}$, then delete its vertical part and add the horizontal segment $[x_1, x_2] \times \{y_0\}$; but now all the paths $P \in \mathcal{K}$ use the horizontal segment $[x_1, x_2] \times \{y_0\}$ and no other path uses it; thus we may add P_v on that segment to obtain a B_1-EPG representation of G, a contradiction.

Finally consider the case where there exists a path $P_t \notin \mathcal{K} \cup \mathcal{W}$ such that P_t intersects a path $P_u \in \mathcal{K}$ on column x_1, but there exist no path $P_{t'} \notin \mathcal{K} \cup \mathcal{W}$ intersecting a path $P_{u'} \in \mathcal{K}$ at the left of x_1 or at the right of x_1. Clearly all paths $P_w \in \mathcal{W}$ are bended at (x_1, y_0) in order to intersect P_t. If all the paths $P_w \in \mathcal{W}$ are $\lrcorner$-paths (resp. $\llcorner$-paths), we may proceed as in the previous case. Thus we may assume that there exists $P_w \in \mathcal{W}$ which is a $\lrcorner$-path and there exists $P_{w'} \in \mathcal{W}$ which is a $\llcorner$-path. We may also assume that for every path $P_u \in \mathcal{K}$, either it is a $\llcorner$-path and intersects P_t, or it is non-bended. This follows from our hypotheses and from the fact that if $P_u \in \mathcal{K}$ is a $\lrcorner$-path and intersects P_t, we may transform it into a $\llcorner$-path with rightmost endpoint (x_2, y_0). But now we may delete the horizontal parts from all the bended paths $P_u \in \mathcal{K}$ (recall that no path $P_u \in \mathcal{K}$ intersects a path $P_{t'} \notin \mathcal{K} \cup \mathcal{W}$ at the left of x_1 or the right of x_1) and we may transform all non-bended P_u paths (for which we may assume that their endpoints are (x_0, y_0) and (x_2, y_0)) into non-bended paths using the grid segment $\{x_1\} \times [y_0, y_1]$. Thus $K \cup W$ is represented as an edge clique and we are done.

This proves the case where $K \cup W$ is a claw clique. □

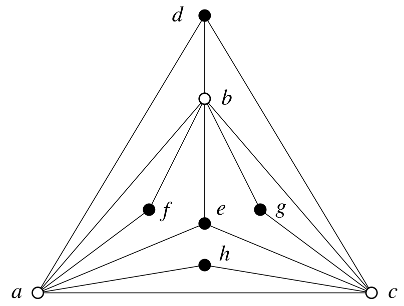

Fig. 11. The split graph G_0.

5.3. Chordal diamond-free graphs

A *diamond* is a graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab = ac, bc, bd, cd\}$. A graph is diamond-free if it does not contain any induced subgraph isomorphic to the diamond.

We will first start with an easy observation which will be helpful in the proof of the main result of this subsection.

Lemma 18. *Let G be a C_4-free B_1-EPG graph and let $v \in V(G)$ be a simplicial vertex. Then there exists a B_1-EPG representation of G in which P_v is a non-bended path.*

Proof. Consider a B_1-EPG representation of $G \setminus \{v\}$. If the clique $N(v)$ is represented as an edge clique, then it follows from Lemma 14(1) that we may assume that there exists an edge e in the grid only used by paths corresponding to vertices of $N(v)$. Now P_v may be added such that it coincides with exactly this edge e. If the clique $N(v)$ is represented as a claw clique, then it follows from Lemma 14(2) that we may assume that only paths corresponding to vertices of $N(v)$ use some row y_j between columns x_{i-1} and x_{i+1}, for some i and j. Now P_v may be added such that it uses row y_j between columns x_{i-1} and x_{i+1}. $\square$

Theorem 19. *Every chordal diamond-free graph G is a B_1-EPG graph.*

Proof. Suppose the result is false and let G' be a minimal counterexample. Let $v \in V(G')$ be a simplicial vertex. Denote by K the clique induced by $N(v)$. As in the proof of Theorem 17, we are done if K is maximal in $G' \setminus \{v\}$. Thus we may assume that K is not maximal, i.e., there exists at least one vertex $w \in V(G') \setminus (K \cup \{v\})$ which is complete to K. This implies that $|N(v)| = 1$. Indeed, if $u_1, u_2 \in N(v)$, we get a diamond induced by $\{v, w, u_1, u_2\}$, a contradiction. So let $N(v) = \{u\}$ and consider a B_1-EPG representation of $G'' = G' \setminus \{v\}$. Clearly the graph $G_u = G''|N(u)$ must be connected, otherwise there exists a grid point (x_i, y_j) such that P_u is the only path using that grid point. But then we may assume that there is an edge either on column x_i or on row y_j which is only used by P_u (we may have to shift some paths to obtain this configuration). Thus we may add P_v on that particular edge and get a B_1-EPG representation of G', a contradiction. So we may assume that G_u is connected. Furthermore, G_u must be a clique. Indeed, if $z, z' \in V(G_u)$ and $zz' \notin E(G_u)$, then there must be an induced path $\{zz_1, z_1z_2, \ldots, z_kz'\}$ in G_u from z to z' since G_u is connected. But now $\{u, z, z_1, z_2\}$ induce a diamond, a contradiction. Thus u is a simplicial vertex in G''. It follows from Lemma 18 that we may assume that P_u is represented as a non-bended path in the B_1-EPG representation of G''. Suppose P_u uses some row y_k between columns x_i and x_j, $i < j$. Next we insert an additional column $x_{i'}$ between x_i and x_{i+1}. Clearly no path uses column $x_{i'}$. Now we delete the horizontal part of P_u between columns x_i and $x_{i'}$ and we add a vertical part on column $x_{i'}$ between rows y_k and y_{k-1} (P_u becomes a $\ulcorner$-path with bend point $(x_{i'}, y_k)$). Hence we may represent P_v on that vertical part. Thus we obtain a B_1-EPG representation of G', which is a contradiction. $\square$

5.4. Split graphs

A *split graph* is a graph G such that $V(G)$ can be partitioned into a clique K and a stable set S. In [11] the following characterization of split graphs is given. A graph G is a split graph if and only if G is $\{H_4, H_5, \overline{H}_4\}$-free. It is easy to see that all split graphs with either $|K| \leq 2$ or $|S| \leq 2$ are B_1-EPG. Here we will consider split graphs for which either $|K| = 3$ or $|S| = 3$ and we will give a characterization of those that are B_1-EPG. We may assume, without loss of generality, that the split graphs G that we consider are connected.

In the following figures, the vertices of the clique K will be represented as white nodes (in some figures we represent a clique by a shaded area instead of drawing the edges of the clique) and the vertices of the independent set S will be represented as black nodes. First we consider the split graph G_0 represented in Fig. 11. We obtain the following result.

Lemma 20. *The graph G_0 is not B_1-EPG.*

Proof. Suppose for a contradiction that G_0 is B_1-EPG and consider a B_1-EPG representation. First assume that the clique K is represented as a claw clique with center (x_i, y_j). It follows that P_d and P_e must both contain the grid point (x_i, y_j) and intersect all three paths P_a, P_b and P_c. But then P_d and P_e necessarily intersect, which is a contradiction.

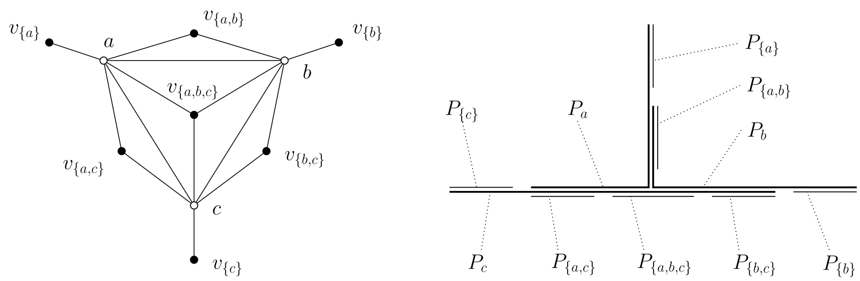

Fig. 12. Split graph G with $|K| = 3$ (a path corresponding to v_J is denoted by P_J, rather than by P_{v_J}).

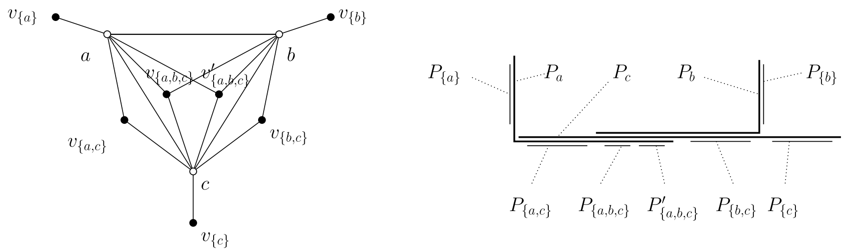

Fig. 13. Split graph G with $|K| = 3$; case where $|N_{\{a,b,c\}}| \geq 2$ and $N_{\{a,b\}} = \emptyset$.

Thus we may assume now that K is represented as an edge clique. Let $[x_i, x_{i+1}] \times \{y_j\}$ be an edge on the grid contained in all three paths P_a, P_b, P_c. Notice that $G_0|\{a, b, c, f, g, h\}$ is isomorphic to the 3-sun S_3 which is not an interval graph since f, g, h form an asteroidal triple (see [15]). Thus at least one path of P_a, P_b, P_c must be bended and intersect a path of P_f, P_g, P_h on a column x_k. Without loss of generality, we may assume that P_f uses column x_k and intersects P_a on that column, with $k < i$.

Let us now distinguish two cases. First assume that P_b also intersects P_f on column x_k. Then necessarily P_g must intersect P_b and P_c on row y_j at the right of x_{i+1}. But now clearly P_h cannot intersect P_a and P_c without intersecting any other path. Thus we may assume now that P_b intersects P_f on row y_j. Hence $G_0|\{a, b, f\}$ is represented as a claw clique with center (x_k, y_j). It follows that both P_g and P_h must lie at the right of x_{i+1} and in fact P_h must intersect P_a and P_c on row y_j. But now P_g cannot intersect P_b and P_c without intersecting any other path. Thus we conclude that G_0 is not B_1-EPG. □

We are now ready to give a characterization of all split graphs with $|K| = 3$ that are B_1-EPG.

Theorem 21. *Let G be a split graph with $V(G) = K \cup S$, $K \cap S = \emptyset$, where K is a clique, S is a stable set and $|K| = 3$. Then G is B_1-EPG if and only if G is G_0-free.*

Proof. It follows from Lemma 20 that G_0 is not B_1-EPG. Let G be a split graph with $|K| = 3$ and not containing any induced subgraph isomorphic to G_0. Let a, b, c be the vertices of K. For each $J \subseteq \{a, b, c\}$, denote by N_J the set of vertices of S that are adjacent only to the members of J. We may assume that $N_\emptyset = \emptyset$. Indeed it is always possible to add isolated paths to a B_1-EPG representation. Assume first that for each nonempty $J \subseteq \{a, b, c\}$, we have $|N_J| = 1$; define $N_J = \{v_J\}$. Then we obtain the graph represented in Fig. 12 and a feasible B_1-EPG representation of it.

Consider now the general case, i.e., $|N_J| \geq 0$ for all nonempty sets $J \subseteq \{a, b, c\}$. We will distinguish two subcases.

If $|N_{\{a,b,c\}}| \leq 1$, we modify the previous construction as follows. For every $\emptyset \neq J \subseteq \{a, b, c\}$ such that $N_J = \emptyset$, we just delete P_J; for every $\emptyset \neq J \subset \{a, b, c\}$ such that $|N_J| > 1$, we split the path P_J into $|N_J|$ non-overlapping segments.

If $|N_{\{a,b,c\}}| \geq 2$, we proceed as follows. Since G is G_0-free, at least one of the sets $N_{\{a,b\}}, N_{\{b,c\}}, N_{\{a,c\}}$ is empty. Without loss of generality, we may assume that $N_{\{a,b\}} = \emptyset$. Then we clearly obtain a feasible B_1-EPG representation of G as shown in Fig. 13. Notice that, for the sake of simplicity, we took $|N_{\{a,b,c\}}| = 2$ and $|N_J| = 1$ for each $J \subseteq \{a, b, c\}, J \neq \{a, b, c\}, \{a, b\}$; we can easily transform this representation if we have $|N_J| > 1$ for some sets $J \subseteq \{a, b, c\}, J \neq \{a, b, c\}, \{a, b\}$, by splitting the corresponding paths P_J into $|N_J|$ non-overlapping segments (as mentioned already above). □

Let us now consider split graphs with $|S| = 3$. Consider the three split graphs G_1, G_2, G_3 shown in Fig. 14. Notice that in all these graphs the vertices of S form an asteroidal triple contained in the neighborhood of the vertex denoted by v. It follows from Theorem 9 that G_1, G_2 and G_3 are not B_1-EPG. In particular, these graphs show that in Theorem 21 $|K| = 3$ cannot simply be replaced by $|K| = 4$.

We obtain the following result.

Theorem 22. *Let G be a split graph with $V(G) = K \cup S$, $K \cap S = \emptyset$, where K is a clique, S is a stable set and $|S| = 3$. Then G is B_1-EPG if and only if G is $\{G_1, G_2, G_3\}$-free.*

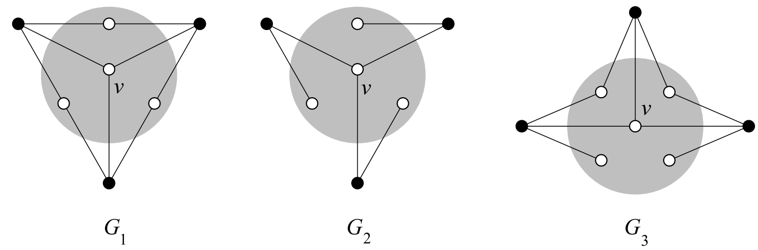

Fig. 14. The graphs G_1, G_2, G_3.

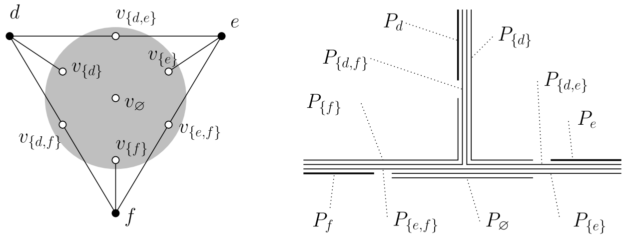

Fig. 15. Split graph G with $|S| = 3$, $N_{\{d,e,f\}} = \emptyset$.

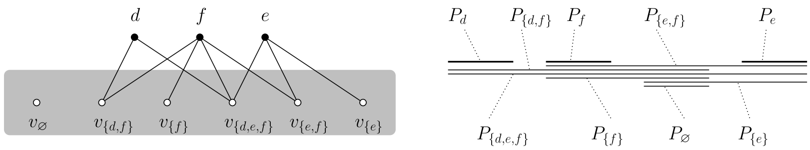

Fig. 16. Split graph G with $|S| = 3$, $N_{\{d,e,f\}} \neq \emptyset$, $N_{\{d,e\}} = N_{\{d\}} = \emptyset$.

(2a)

(2b)

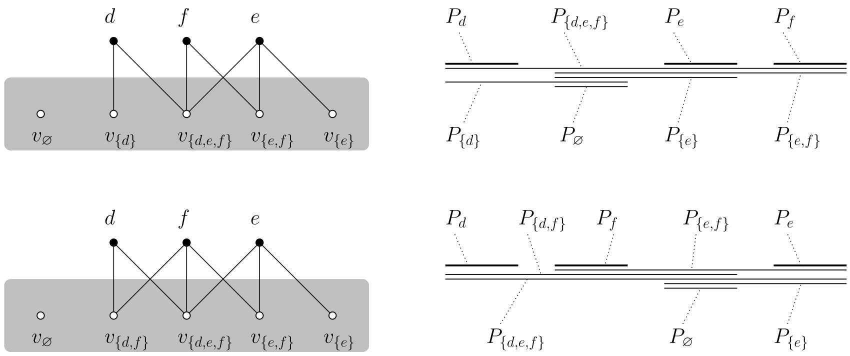

Fig. 17. Split graph G with $|S| = 3$, $N_{\{d,e,f\}} \neq \emptyset$, $N_{\{d,e\}} = N_{\{f\}} = \emptyset$.

Proof. Let d, e, f be the vertices of S. For each $J \subseteq \{d, e, f\}$, denote by N_J the set of vertices of K that are adjacent only to the members of J. Assume first that $N_{\{d,e,f\}} = \emptyset$. We may assume that for each $J \subsetneq \{d, e, f\}$ we have $|N_J| = 1$. Let $N_J = \{v_J\}$ (otherwise we shall duplicate or delete the corresponding paths in the following construction). Then we obtain the graph represented in Fig. 15 and a B_1-EPG representation of it.

Assume now that $N_{\{d,e,f\}} \neq \emptyset$. At least one among the sets $N_{\{d,e\}}, N_{\{d,f\}}, N_{\{e,f\}}$ is empty since otherwise G contains an induced subgraph isomorphic to G_1. Moreover, at least one among the sets $N_{\{d\}}, N_{\{e\}}, N_{\{f\}}$ is empty since otherwise G contains an induced subgraph isomorphic to G_2. Thus there are, up to relabeling, two cases to distinguish: (1) $N_{\{d,e\}} = N_{\{d\}} = \emptyset$; (2) $N_{\{d,e\}} = N_{\{f\}} = \emptyset$.

In the first case, we may assume that for all $J \subseteq \{d, e, f\}, J \neq \{d, e\}, \{d\}$, we have $|N_J| = 1, N_J = \{v_J\}$. Then G has a B_1-EPG representation as shown in Fig. 16.

In the second case, it is impossible that all the sets $N_{\{d,e,f\}}, N_{\{d,f\}}, N_{\{e,f\}}, N_{\{d\}}, N_{\{e\}}$ are nonempty since otherwise G contains an induced subgraph isomorphic to G_3. Then we have essentially two cases: (2a) $N_{\{d,f\}} = \emptyset$ and (2b) $N_{\{d\}} = \emptyset$ (recall that we are dealing with the case $N_{\{d,e,f\}} \neq \emptyset$). In both cases we assume that for all the sets N_J (except those assumed to be empty) we have $|N_J| = 1, N_J = \{v_J\}$ and obtain a B_1-EPG representation of G as shown in Fig. 17.

Notice that in the case where $N_{\{d,e,f\}} \neq \emptyset$, we could (instead of giving a B_1-EPG representation) note that G is a chordal AT-free graph and therefore it is an interval graph, and hence B_1-EPG. □

6. Conclusion

In this paper we considered edge intersection graphs of single-bend paths on a grid (B_1-EPG graphs). We showed that in B_1-EPG graphs the subgraph induced by the neighborhood of any vertex is weakly chordal, and thus perfect. This allowed us to prove that these graphs satisfy the Erdős–Hajnal property, i.e., they contain either a large clique or a large stable set. Then we considered some subclasses of chordal graphs and characterized the B_1-EPG graphs of these subclasses. Some of these results are constructive and explain how to obtain a B_1-EPG representation if one exists.

There remain a lot of open questions concerning B_1-EPG graphs and more generally B_k-EPG graphs for $k \geq 1$. For instance it would be interesting if one could characterize B_1-EPG graphs of special classes of graphs other than those presented here. Chordal graphs would be of major interest. Furthermore, a question arising from our results is that of whether $\frac{1}{3}$ is best possible for the Erdős–Hajnal property for B_1-EPG graphs.

Finally let us mention that it is still unknown whether, for instance, the vertex coloring problem or the maximum clique problem is polynomially solvable in B_1-EPG graphs.

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