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Blockers and transversals in some subclasses of bipartite graphs: When caterpillars are dancing on a grid

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ABSTRACT

Given an undirected graph $G = (V, E)$ with matching number $\nu(G)$, a d-blocker is a subset of edges B such that $\nu((V, E \setminus B)) \leq \nu(G) - d$ and a d-transversal T is a subset of edges such that every maximum matching M has $|M \cap T| \geq d$. While the associated decision problem is NP-complete in bipartite graphs we show how to construct efficiently minimum d-transversals and minimum d-blockers in the special cases where G is a grid graph or a tree.

1. Introduction

Given a collection $\mathcal{C}$ of subsets of a ground set E we define a transversal as a subset of E which meets every member of $\mathcal{C}$. Transversals have an interest for themselves but also for their numerous applications (for instance surveying when $\mathcal{C}$ is a collection of paths from the origin to destination in a graph). Such sets have been extensively studied for various collections $\mathcal{C}$ (see for instance [3] and the chapter 22 in [7]).

A family of problems which follows a similar spirit to transversal problems are the class of edge deletion problems [1,4,6,8]. In [10] a generalization of transversals called (d-transversals) and a closely related edge deletion problem (d-blockers) were introduced for matchings; complexity results have been derived and some polynomially solvable cases have been presented.

For general bipartite graphs finding minimum d-blockers or d-transversals is NP-hard [10]. Our purpose in this paper is to show how minimum d-transversals and minimum d-blockers can be constructed in some specific subclasses of bipartite graphs: the grid graphs and the trees. For trees, the algorithms to be presented will essentially be based on dynamic programming. For grid graphs the technique will be different: the structural simplicity of such graphs will allow us to construct directly d-blockers and d-transversals. Most of the effort will then be spent to show that no smaller subset of

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edges can be a d-blocker or d-transversal. Depending on the parity of parameters m and n describing the size of the grid graphs, various proof techniques (bounding procedures) will be necessary.

In Section 2, we will recall the basic definitions of d-blocker and d-transversal as well as some results of [10]; then we will introduce specific notations for grid graphs. In Section 3 we will state and prove the formulas giving the sizes of minimum d-transversals in grid graphs. Section 4 will study minimum d-blockers in grid graphs. Section 5 will be dedicated to the case of trees (minimum d-transversals and minimum d-blockers) and conclusions will follow in Section 6.

2. Definitions and previous results

All graph theoretical terms not defined here can be found in [2]. Throughout this paper we are concerned with undirected simple loopless graphs $G = (V, E)$. A **matching** M is a set of pairwise non-adjacent edges. A matching M is called **maximum** if its cardinality $|M|$ is maximum. The largest cardinality of a matching in G, its **matching number**, will be denoted by $\nu(G)$. Let $\boldsymbol{P_0(G)} = \{[v, w] \in E| \; \forall$ maximum matching M, $[v, w] \notin M\}$ and $\boldsymbol{P_1(G)} = \{[v, w] \in E| \; \forall$ maximum matching M, $[v, w] \in M\}$. A vertex $v \in V$ is called **saturated by a matching** M if there exists an edge $[v, w] \in M$. A vertex $v \in V$ is called **strongly saturated** if for all maximum matchings M, v is saturated by M. We denote by $\boldsymbol{S(G)}$ the set of strongly saturated vertices of a graph G. We will be interested in subsets of edges which will intersect maximum matchings in G or whose removal will reduce the matching number by a given number.

We shall say that a subset $T \subseteq E$ is a $\boldsymbol{d}$**-transversal** of G if for every maximum matching M we have $|M \cap T| \geq d$. Thus a d-transversal is a subset of edges which intersect each maximum matching in at least d edges.

A subset $B \subseteq E$ will be called a $\boldsymbol{d}$**-blocker** of G if $\nu(G') \leq \nu(G) - d$ where G' is the partial graph $G' = (V, E \setminus B)$. So B is a subset of edges whose removal reduces the cardinality of a maximum matching by at least d.

In case where $d = 1$, a d-transversal or a d-blocker is called a **transversal** or a **blocker**, respectively. We remark that in this case our definition of a transversal coincides with the definition of a transversal in the hypergraph of maximum matchings of G.

We denote by $\boldsymbol{\beta_d(G)}$ the minimum cardinality of a d-blocker in G and by $\boldsymbol{\tau_d(G)}$ the minimum cardinality of a d-transversal in G ($\beta(G)$ and $\tau(G)$ in case of a blocker or a transversal). A d-blocker (resp. d-transversal) will be *minimum* if it is of minimum size.

Let v be a vertex in graph G. The **bundle** of v, denoted by $\omega(v)$, is the set of edges which are incident to v. So $|\omega(v)| = d(v)$ is the degree of v. As we will see, bundles play an important role in finding d-transversals and d-blockers.

A **grid graph** (or shortly a **grid**) $G_{m,n} = (V, E)$ is constructed on vertices x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$; its edge set consists of horizontal edges $h_{ij} = [x_{ij}, x_{i,j+1}]$, $1 \leq j \leq n - 1$ in each row i, $1 \leq i \leq m$, and of vertical edges $v_{ij} = [x_{ij}, x_{i+1,j}]$, $1 \leq i \leq m - 1$, in each column j, $1 \leq j \leq n$.

Notice that $G_{m,n}$ is a bipartite graph; let $\mathcal{B}$, $\mathcal{W}$ be the associated partition of its vertex set. When mn is even, $|\mathcal{B}| = |\mathcal{W}| = \frac{mn}{2}$ and the maximum matchings are perfect (all vertices are saturated), i.e. $\nu(G_{m,n}) = \frac{mn}{2}$. When mn is odd, assuming that the four corners (vertices of degree 2) are in $\mathcal{B}$, we have $|\mathcal{B}| = \frac{mn+1}{2}$ and $|\mathcal{W}| = \frac{mn-1}{2}$, so $|\mathcal{B}| = |\mathcal{W}| + 1$; every maximum matching will saturate all vertices but one, i.e. $\nu(G_{m,n}) = \lfloor \frac{mn}{2} \rfloor$. Moreover for every vertex v in $\mathcal{B}h$, there is a maximum matching saturating all vertices except v.

We give some properties and results concerning d-transversals and d-blockers (see [10] for their proofs).

Property 2.1. *In any graph G and for any $d \geq 1$, a d-blocker B is a d-transversal.*

Property 2.2. *In any graph $G = (V, E)$ a set T is a transversal if and only if it is a blocker.*

Property 2.3. *For any independent set $\{v_1, v_2, \ldots, v_d\} \subseteq S(G)$ the set $T = \cup_{i=1}^{d} \omega(v_i)$ is a d-transversal.*

For the special case of $G_{1,n}$, a grid graph with a unique row, we have the following.

Property 2.4. *Let $G_{1,n}$ be a chain on n vertices $v_1, v_2, \ldots, v_n$ (i.e., $E = \{[v_i, v_{i+1}]|i = 1, \ldots, n-1\}$) and $d \geq 1$ an integer. Then*

- *$\beta_d(G) = 2d - 1$ and $\tau_d(G) = d$ if n is even,*
- *$\beta_d(G) = \tau_d(G) = 2d$ if n is odd.*

One can observe from the previous property that for the case where n is even and $d > 1$ we have $\tau_d(G) < \beta_d(G)$: so a d-transversal is not necessarily a d-blocker (i.e. the converse of Property 2.1 is not necessarily true).

In case where G is bipartite:

Theorem 2.1. *For every fixed $d \in \{1, 2, \ldots, \nu(G)\}$ finding a minimum d-blocker or a minimum d-transversal is $\mathcal{NP}$-hard even if G is bipartite.*

3. Minimum d-transversal in grid graphs

We show here how to construct a minimum d-transversal in a grid graph $G_{m,n}$. In the case where mn is even, the d-transversals constructed will generally consist of d bundles whose centers form a stable set. In some cases other constructions will be needed.

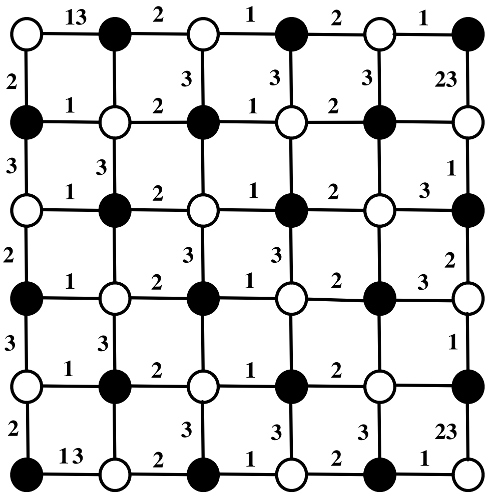

Fig. 1. Example for $m = n = 6$.

First, we establish the two following lemmas.

Lemma 3.1. *For* $m = 2$, $\tau_1(G_{2,n}) = 2$ *and* $\tau_d(G_{2,n}) = 3d - 2$, $2 \le d \le n = \nu(G_{2,n})$.

Proof. We clearly have $\tau_1(G_{2,n}) = 2$ by taking the bundle $\omega(x_{11})$. Then for $2 \le d \le n$, a d-transversal is obtained by taking a set of $d-1$ bundles $\omega(x_{11}), \omega(x_{22}), \omega(x_{13}), \omega(x_{24}), \ldots$ together with $\omega(x_{1n})$ if n is odd or $\omega(x_{2n})$ if n is even. This set T is indeed a d-transversal and it satisfies $|T| = 3d - 2$. So $\tau_d(G_{2,n}) \le 3d - 2$. But we can construct matchings M_1, M_2, M_3 such that $M_1 \cap M_2 = \emptyset$ and $|M_3 \cap (M_1 \cup M_2)| = 2$; so any d-transversal T should satisfy $|T \cap M_1|, |T \cap M_2| \ge d$ and $|T \cap (M_3 \setminus (M_1 \cup M_2))| \ge d-2$, i.e., $|T| \ge 3d-2$. Here we take $M_1 \cup M_2 = \{h_{11}, \ldots, h_{1,n-1}, v_{1n}, h_{2,n-1}, h_{2,n-2}, \ldots, h_{21}, v_{11}\}$ and $M_3 = \{v_{11}, v_{12}, \ldots, v_{1n}\}$. □

Lemma 3.2. *For* $m \ge 4$ *even and* $n \ge 3$, $\tau_d(G_{m,n}) = 2d$, $1 \le d \le 4$.

Proof. For $m \ge 4$ even, $n \ge 3$ and $1 \le d \le 4$, a d-transversal T is obtained by taking d bundles of corners (vertices $x_{11}, x_{1n}, x_{m1}, x_{mn}$); its size $|T|$ is $2d$. Every d-transversal T must have $|T| \ge 2d$ since $G_{m,n}$ contains two maximum matchings M_1, M_2 with $M_1 \cap M_2 = \emptyset$. M_1 consists of the odd horizontal edges $h_{i1}, h_{i3}, h_{i5}, \ldots$ of each row i (together with the odd vertical edges $v_{1n}, v_{3n}, \ldots$ if n is odd); M_2 consists of the odd vertical edges $v_{1j}, v_{3j}, \ldots$ of each column j if n is even or of the even horizontal edges $h_{i2}, h_{i4}, \ldots$ of each row i together with the odd vertical edges $v_{11}, v_{31}, \ldots, v_{m-1,1}$ if n is odd. □

We will now distinguish between three cases: m, n even, m, n odd and m even, n odd.

3.1. m and n even

Lemma 3.3. *For* $m, n \ge 4$ *with* $m + n \ge 10$ *and* $5 \le d \le m + n - 4$, $\tau_d(G_{m,n}) = 3d - 4$.

Proof. We construct a set T by taking the four bundles of the corners $\omega(x_{11}), \omega(x_{1n}), \omega(x_{m1}), \omega(x_{mn})$ together with $d-4$ bundles of vertices of degree 3 forming all together a stable set in $G_{m,n}$. This is possible since $d - 4 \le m + n - 8$: there are indeed $2m + 2n - 8$ vertices of degree 3 in $G_{m,n}$. We can take $m - 4$ such independent vertices in the first and the last columns and $n - 4$ in the first and the last rows. Such a set T is a d-transversal with $|T| = 3d - 4$.

We construct M_1, M_2, M_3 with $M_1 \cap M_2 = \emptyset$ and $|M_3 \cap (M_1 \cup M_2)| = 4$ as follows (see Fig. 1):

$M_1 \cup M_2$ contains all horizontal edges except $h_{2,n-1}, h_{3,n-1}, \ldots, h_{m-1,n-1}$ and we also introduce the odd vertical edges $v_{11}, v_{31}, \ldots, v_{m-1,1}$, the even vertical edges $v_{1n}, v_{2n}, \ldots, v_{m-1,n}$ into $M_1 \cup M_2$. M_3 consists of the horizontal edges $h_{11}, h_{m1}, h_{3,n-1}, \ldots, h_{m-2,n-1}$ together with the vertical edges $v_{1,n-1}, v_{m-1,n-1}, v_{21}, v_{22}, v_{41}, v_{42}, \ldots, v_{m-2,1}, v_{m-2,2}, v_{1n}, v_{m-1,n}$ and $v_{i3}, v_{i4}, \ldots, v_{i,n-2}$ $(i = 1, 3, \ldots, m-1)$. We have $M_3 \cap (M_1 \cup M_2) = \{h_{11}, v_{1n}, h_{m1}, v_{m-1,n}\}$. □

Lemma 3.4. *For* $m, n \ge 4$ *and* $m + n - 3 \le d \le \frac{mn}{2}$, $\tau_d(G_{m,n}) = 4d - m - n$.

Proof. Consider now that $m + n - 3 \le d \le \frac{mn}{2} - 2$ with m, n even and at least 4. We construct a set T consisting of the bundles of the four corners $x_{11}, x_{1n}, x_{m1}, x_{mn}$ together with $m + n - 8$ bundles of vertices of degree 3. In addition we take $d - (m + n - 4)$ bundles of vertices of degree 4 forming a stable set with the vertices already chosen. This gives a set T with $|T| = 4d - (m + n)$.

For $d = \frac{mn}{2}$ we take all edges, i.e., $|T| = |E| = 2mn - (m+n) = 4d - (m+n)$ and finally for $d = \frac{mn}{2} - 1$, we take the bundles of all vertices on the same side of the bipartition except one vertex of degree 4. This gives $|T| = 2mn - (m+n-4) = 4d - (m+n)$.

Now we can construct matchings M_1, M_2, M_3, M_4 satisfying $M_1 \cap M_2 = M_3 \cap M_4 = M_1 \cap M_4 = M_2 \cap M_3 = \emptyset$, $M_1 \cap M_3 = m$ and $M_2 \cap M_4 = n$ as follows: $M_1 = \{v_{i1}, v_{i2}, \ldots, v_{in}; i = 1, 3, 5, \ldots, m-1\}$, $M_2 = \{h_{1j}, h_{2j}, \ldots, h_{mj}; j = 1, 3, 5, \ldots, n-1\}$, $M_3 = \{v_{i1}, v_{in}\ (i = 1, 3, 5, \ldots, m-1)\} \cup \{h_{i2}, h_{i4}, \ldots, h_{i,n-1}\ (i = 1, \ldots, m)\}$, $M_4 = \{h_{1j}, h_{mj}\ (j = 1, 3, 5, \ldots, n-1)\} \cup \{v_{2j}, v_{4j}, \ldots, v_{m-2,j}\ (j = 1, \ldots, n)\}$. Then clearly $M_1 \cap M_3 = \{v_{i1}, v_{in}\ (i = 1, 3, 5, \ldots, m-1)\}$ and $M_2 \cap M_4 = \{h_{1j}, h_{mj}\ (j = 1, 3, 5, \ldots, n-1)\}$. So for every d-transversal T we must have $|T| \geq 4d - (m+n)$ and the T constructed above is optimal. $\square$

From Lemmas 3.1–3.4 we obtain the following.

Theorem 3.5. *Let $G_{m,n}$ be a grid with m and n even. The minimum cardinality of a d-transversal is*

1. *for $m = 2$ or $n = 2$*
   (a) $\tau_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq 2$
   (b) $\tau_d(G_{m,n}) = 3d - 2$ *for* $3 \leq d \leq \nu(G_{m,n})$
2. *for $m, n \geq 4$*
   (a) $\tau_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq 4$
   (b) $\tau_d(G_{m,n}) = 3d - 4$ *for* $5 \leq d \leq m + n - 4$ *and* $(m, n) \neq (4, 4)$
   (c) $\tau_d(G_{m,n}) = 4d - m - n$ *for* $m + n - 3 \leq d \leq \nu(G_{m,n})$.

3.2. m and n odd

We are now in the case where m and n are odd. The bipartition $(\mathcal{B}, \mathcal{W})$ of $G_{m,n}$ will be used and we recall that the 4 corners (vertices of degree 2) are in $\mathcal{B}$. So we have $|\mathcal{B}| = |\mathcal{W}| + 1 = \nu(G_{m,n}) + 1$. We first prove a general result when m and n are both odd.

Lemma 3.6. *For $m, n \geq 3$ every d-transversal T we have $|T| \geq \frac{8d}{3}$.*

Proof. We consider four matchings M_1, M_2, M_3, M_4 as shown in the right part of Fig. 2, i.e., $M_1 = \{v_{ij},\ i = 1, 3, 5, \ldots, m-2;\ j = 1, \ldots, n;\ h_{mj},\ j = 1, 3, \ldots, n-2\}$, $M_2 = \{h_{1j},\ j = 2, 4, \ldots, n-1;\ v_{ij},\ j = 1, \ldots, n \text{ and } i = 2, 4, \ldots, m-1\}$, $M_3 = \{h_{ij},\ i = 1, 2, \ldots, m \text{ and } j = 1, 3, \ldots, n-2;\ v_{in},\ i = 2, 4, \ldots, m-1\}$ and $M_4 = \{v_{i1},\ i = 1, 3, \ldots, m-2;\ h_{ij},\ i = 1, \ldots, m \text{ and } j = 2, 4, \ldots, n-1\}$. Then we construct four other maximum matchings. For $i = 1, \ldots 4$, the matching M_{4+i} is obtained from M_i by taking the symmetric of M_i with respect to the horizontal axis $y = \frac{m+1}{2}$. Notice that for each edge e, we have $|\{M_k \mid e \in M_k, k \in \{1, \ldots, 8\}\}| \leq 3$. So for every d-transversal T we have $3|T| \geq \sum_{e\in T} \sum_{k=1}^{8} |\{e\} \cap M_k| \geq 8d$. Hence, $|T| \geq \frac{8d}{3}$. $\square$

Using Lemma 3.6, we get the following two results.

Lemma 3.7. *For $m, n \geq 3$ and $d = 1$, $\tau_d(G_{m,n}) = 3$.*

Proof. We take the bundle of a vertex of degree 3 in $\mathcal{W}$ and we get a 1-transversal T with $|T| = 3$. $\square$

Lemma 3.8. *For $m, n \geq 3$ and $2 \leq d \leq 3$, $\tau_d(G_{m,n}) = 2d + 2$.*

Proof. We take $d + 1$ bundles of corners and we get a d-transversal T with $|T| = 2d + 2$. $\square$

Lemma 3.9. *For $m = n = 3$ and $d = 4$, $\tau_d(G_{m,n}) = 12$.*

Proof. Since $\nu(G_{3,3}) = 4$, a 4-transversal consists of all the twelve edges of $G_{3,3}$. $\square$

Lemma 3.10. *For $m, n \geq 3$ with $m + n \geq 8$ and $4 \leq d \leq m + n - 3$, $\tau_d(G_{m,n}) = 3d - 1$.*

Proof. We take $d + 1$ bundles of vertices in $\mathcal{B}$: the bundles of the four corners and $d - 3$ bundles of vertices of degree 3. This gives a d-transversal T with $|T| = 3d - 1$. It is minimum since we can construct 3 matchings M_1, M_2, M_3 with $|M_1 \cap M_2| = |M_1 \cap M_3| = 0$ and $|M_2 \cap M_3| = 1$ by taking $M_1 = \{h_{1j},\ j = 1, 3, 5, \ldots, n-2;\ h_{ij},\ j = 2, 4, 6, \ldots, n-1 \text{ and } i = 2, 3, \ldots, m;\ v_{i1},\ i = 2, 4, \ldots, m-1\}$, $M_2 = \{h_{1j},\ j = 2, 4, \ldots, n-1;\ h_{i1},\ i = 2, 3, \ldots, m;\ v_{ij},\ j = 3, 4, \ldots, n \text{ and } i = 2, 4, \ldots, m-1\}$ and $M_3 = \{v_{ij},\ j = 1, \ldots, n \text{ and } i = 1, 3, 5, \ldots, m-2;\ h_{mj},\ j = 1, 3, 5, \ldots, n-2\}$ (see left of Fig. 2). Here we have $M_2 \cap M_3 = \{h_{m1}\}$. Since there are $m+n-6$ vertices of degree 3 in B, the construction is valid for $d \leq m+n-3$. $\square$

Lemma 3.11. *For $m, n \geq 3$ and $m + n - 2 \leq d \leq \frac{mn-1}{2}$, $\tau_d(G_{m,n}) = 4d - m - n + 2$.*

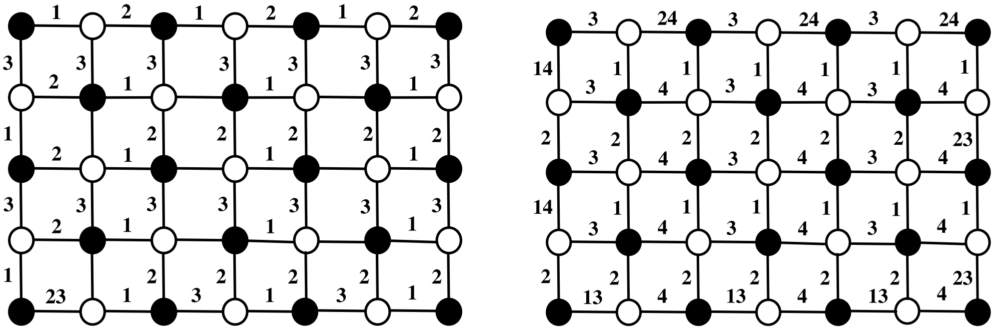

Fig. 2. Matchings for $m = 5$ and $n = 7$.

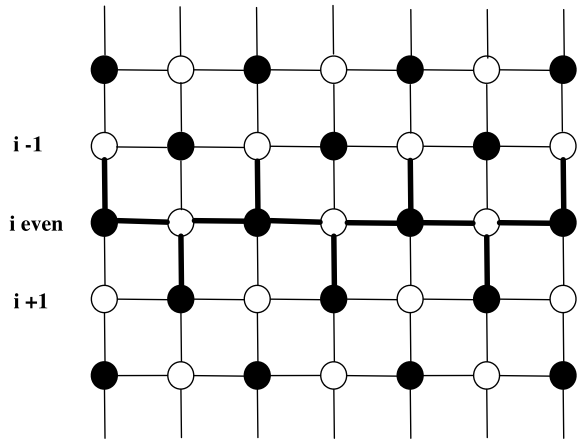

Fig. 3. A dancat D_i.

Proof. We construct a d-transversal T by taking $d+1$ bundles of vertices in $\mathcal{B}$: all the $m+n-2$ bundles of vertices of degree ≤ 3 together with $d-(m+n-3)$ bundles of vertices of degree 4. This will give a d-transversal T with $|T| = 4d-(m+n-2)$. It is minimum since we can construct four matchings M_1, M_2, M_3, M_4 satisfying $M_1 \cap M_2 = M_3 \cap M_4 = \emptyset$, $|M_1 \cap M_4| = |M_2 \cap M_3| = \frac{m-1}{2}$ and $|M_1 \cap M_3| = |M_2 \cap M_4| = \frac{n-1}{2}$. We define (see right of Fig. 2) $M_1 = \{v_{ij},\ i = 1, 3, 5, \ldots, m-2;\ j = 1, \ldots, n;\ h_{mj},\ j = 1, 3, \ldots, n-2\}$, $M_2 = \{h_{1j},\ j = 2, 4, \ldots, n-1;\ v_{ij},\ j = 1, \ldots, n \text{ and } i = 2, 4, \ldots, m-1\}$, $M_3 = \{h_{ij},\ i = 1, 2, \ldots, m \text{ and } j = 1, 3, \ldots, n-2;\ v_{in},\ i = 2, 4, \ldots, m-1\}$ and $M_4 = \{v_{i1},\ i = 1, 3, \ldots, m-2;\ h_{ij},\ i = 1, \ldots, m \text{ and } j = 2, 4, \ldots, n-1\}$. $\square$

From Lemmas 3.7–3.11 and Property 2.4 we get the following.

Theorem 3.12. *Let $G_{m,n}$ be a grid with m and n odd. The minimum cardinality of a d-transversal is*

1. *for $m = 1$ or $n = 1$*
   (a) $\tau_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq \nu(G_{m,n})$
2. *for $m, n \geq 3$*
   (a) $\tau_1(G_{m,n}) = 3$
   (b) $\tau_d(G_{m,n}) = 2d+2$ *for* $2 \leq d \leq 3$
   (c) $\tau_4(G_{3,3}) = 12$
   (d) $\tau_d(G_{m,n}) = 3d-1$ *for* $4 \leq d \leq m+n-3$ *and* $(m,n) \neq (3,3)$
   (e) $\tau_d(G_{m,n}) = 4d-m-n+2$ *for* $m+n-2 \leq d \leq \nu(G_{m,n})$.

3.3. *m even, n odd*

We define a family of partial subgraphs of $G_{m,n}$ which play an important role in the following.

Definition 3.13. Let $G_{m,n}$ be a grid graph and let $2 \leq i \leq m-2$ be an even integer. Then $D_i = \{v_{i-1,j} \mid j = 1, 3, \ldots, n\} \cup \{v_{i+1,k} \mid k = 2, 4, \ldots, n-1\} \cup \{h_{i,l} \mid l = 1, 2, \ldots, n-1\}$ is called a **dancat** (short name for a dancing caterpillar [5]).

An example of a dancat is shown in Fig. 3. Notice that a dancat D_i contains exactly $2n-1$ edges.

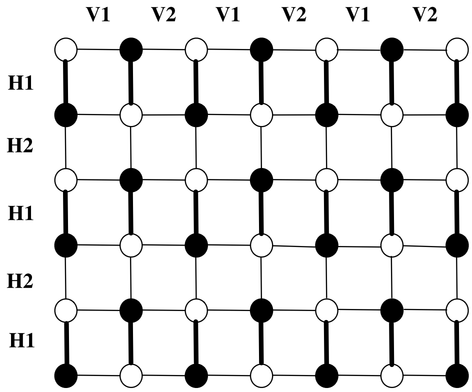

Fig. 4. Matching M_{H_1}.

Lemma 3.14. *For every maximum matching M in $G_{m,n}$, m even and n odd, and every dancat D_i, $|M \cap D_i| = \frac{n+1}{2}$.*

Proof. Consider a 3-partition of the vertices of $G_{m,n}$ $V = V_A \cup V_B \cup V_D$, where V_D is the set of vertices incident to at least one edge in the dancat D_i, V_A is the set of vertices not belonging to V_D and lying in rows $1, 2, \ldots, i-1$, and V_B is the set of vertices not belonging to V_D and lying in rows $i+1, i+2, \ldots, m$. In V_A there are $\frac{in-2n}{2}$ white vertices and $\frac{in-n-1}{2}$ black vertices (see Fig. 3). Since the white vertices of V_A can only be matched with black vertices of V_A, there remain $\frac{(in-n-1)-(in-2n)}{2} = \frac{n-1}{2}$ black vertices of V_A to be matched with $\frac{n-1}{2}$ white vertices of V_D, leaving $\frac{n+1}{2}$ white vertices of V_D unmatched with the vertices of V_A. Similarly, there are $\frac{n+1}{2}$ black vertices of V_D unmatched with the vertices of V_B. Now to get a perfect matching, these $\frac{n+1}{2}$ white unmatched vertices of V_D are matched to the $\frac{n+1}{2}$ black unmatched vertices of V_D. □

Let us now introduce a 4-partition of the edge set of $G_{m,n}$ which will play an important role in the proofs of the Lemmas which will follow. We define $H_1 = \{v_{ij} |\, j = 1, \ldots, n;\ i = 1, 3, \ldots, m-1\}$, $H_2 = \{v_{ij} |\, j = 1, \ldots, n;\ i = 2, 4, \ldots, m-2\}$, $V_1 = \{h_{ij} |\, i = 1, \ldots, m;\ j = 1, 3, \ldots, n-2\}$, $V_2 = \{h_{ij} |\, i = 1, \ldots, m;\ j = 2, 4, \ldots, n-1\}$. For each one of these sets, we define the following maximum matchings: $M_{H_1} = H_1$, $M^j_{H_2} = (H_2 \cup \{v_{ij} |\, i = 1, 3, \ldots, m-1\} \cup \{h_{ik} |\, i = 1, n;\ k = 1, 3, \ldots, j-2, j+1, j+3, \ldots, n-1\}) \setminus \{v_{ij} |\, i = 2, 4, \ldots, m-2\}$ with j odd, $M_{V_1} = V_1 \cup \{v_{in} |\, i = 1, 3, \ldots, m-1\}$ and $M_{V_2} = V_2 \cup \{v_{i1} |\, i = 1, 3, \ldots, m-1\}$. Examples of these matchings are given in Figs. 4 and 5.

Lemma 3.15. *For $m \geq 4$, $n = 3$ and $5 \leq d \leq \frac{m}{2} + 2$, $\tau_d(G_{m,3}) = 2d$.*

Proof. We show the following : let M be a perfect matching of $G_{m,3}$; for each integer i, $1 \leq i \leq m-1$, i odd, we have $|M \cap \{v_{i1}, v_{i3}\}| \geq 1$. By contradiction, we suppose that there exists an odd integer i, such that $M \cap \{v_{i1}, v_{i3}\} = \emptyset$. If $v_{i2} \notin M$ the subgrid induced by the vertex set $Z = \{x_{kj} | 1 \leq k \leq i, 1 \leq j \leq 3\}$ must contain a perfect matching, which is impossible since $|Z|$ is odd. If $v_{i2} \in M$, the subgrid induced by the vertex set $Z = \{x_{kj} | 1 \leq k \leq i, 1 \leq j \leq 3\} \setminus \{x_{i2}\}$ must contain a perfect matching, which is impossible since $|Z \cap \mathcal{B}| = |Z \cap \mathcal{W}| - 2$.

For $d \leq \frac{m}{2}$, let $T = \{v_{2i-1,1}, v_{2i-1,3} | 1 \leq i \leq d\}$. From above, it follows that for any perfect matching M we have $|T \cap M| \geq d$. For $d = \frac{m}{2} + 1$, let $T = \{v_{2i-1,1}, v_{2i-1,3} | 1 \leq i \leq \frac{m}{2}\} \cup \{h_{11}, h_{12}\}$. For any perfect matching M, if $v_{11} \notin M$ (resp. $v_{13} \notin M$) then $h_{11} \in M$ (resp. $h_{12} \in M$); thus $|M \cap \{v_{11}, v_{13}, h_{11}, h_{12}\}| \geq 2$, and we obtain $|T \cap M| \geq \frac{m}{2} + 1$. For $d = \frac{m}{2} + 2$, let $T = \{v_{2i-1,1}, v_{2i-1,3} | 1 \leq i \leq \frac{m}{2}\} \cup \{h_{11}, h_{12}, h_{m1}, h_{m2}\}$. For any perfect matching M we have $|M \cap \{v_{m-1,1}, v_{m-1,3}, h_{m1}, h_{m2}\}| \geq 2$, and we obtain $|T \cap M| \geq \frac{m}{2} + 2$. So in any case T is a d-transversal with $|T| = 2d$. Since M_{V_1} and M_{V_2} are two disjoint perfect matchings, any d-transversal has at least $2d$ edges. Thus in each of these three cases T is minimum. □

Lemma 3.16. *For $m \geq 4$, $n = 3$ and $\frac{m}{2} + 3 \leq d \leq m$, $\tau_d(G_{m,3}) = 3d - \frac{m}{2} - 2$.*

Proof. We show that any d-transversal T is such that $|T| \geq 3d - \frac{m}{2} - 2$. Let us consider the three perfect matchings M_{V_1}, M_{V_2} and $M^1_{H_2}$: M_{V_1} and M_{V_2} are disjoint and $|(M_{V_1} \cup M_{V_2}) \cap M^1_{H_2}| = \frac{m}{2} + 2$, so T has at least $3d - (\frac{m}{2} + 2)$ edges.

Let us now show how to construct such a minimum d-transversal T. We take $X = d - \frac{m}{2} - 2$ dancats $D_4, D_6, \ldots, D_{2X+2}$, then we add the edges $\{v_{11}, v_{13}\} \cup \{v_{2i+1,1}, v_{2i+1,3} | X + 1 \leq i \leq \frac{m}{2} - 1\}$ and the four edges $h_{11}, h_{12}, h_{m1}, h_{m2}$. Note that for $d \leq m$ we have $X \leq \frac{m}{2} - 2$ and the construction is valid. This gives us a set T such that $|M \cap T| = 2X + \frac{m}{2} - X + 2 = d$ for every maximum matching M (using arguments of Lemma 3.15). Thus T is a d-transversal. Concerning the cardinality of T, we have $|T| = 5X + 2(\frac{m}{2} - X) + 4 = 3d - \frac{m}{2} - 2$. □

Lemma 3.17. *For $m \geq 4$, $n \geq 5$ and $5 \leq d \leq m + n - 3$, $\tau_d(G_{m,n}) = 3d - 4$.*

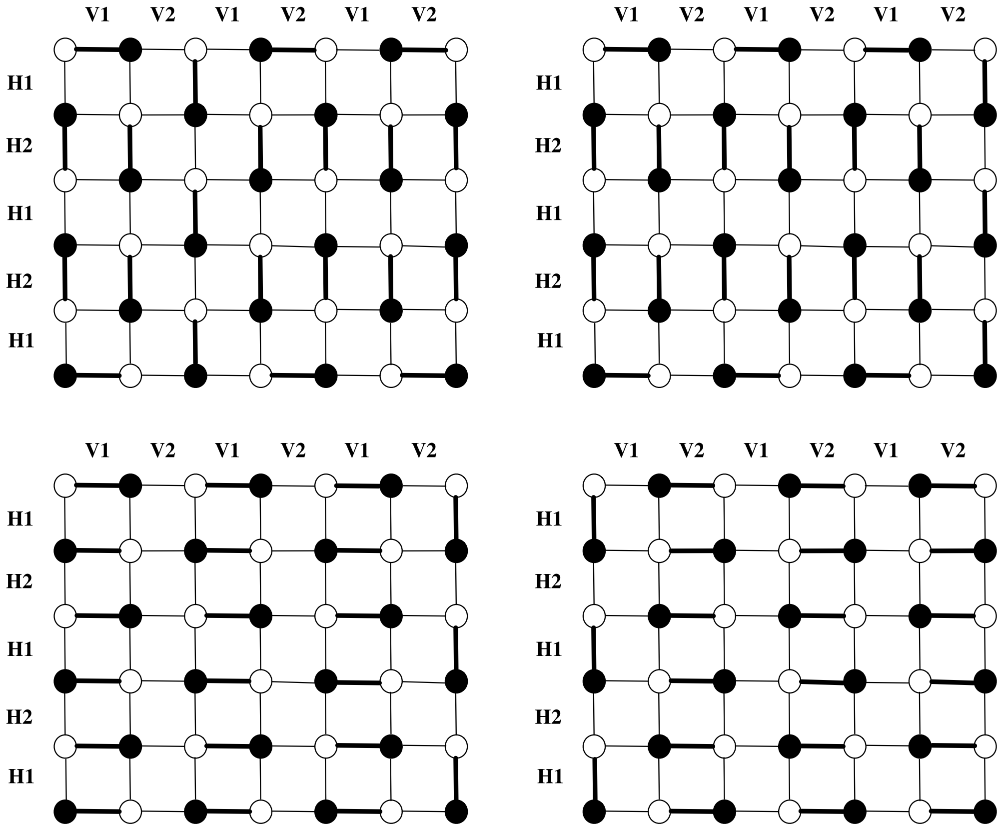

Fig. 5. $M^3_{H_2}$ (top left), $M^n_{H_2}$ (top right), M_{V_1} (bottom left), M_{V_2} (bottom right).

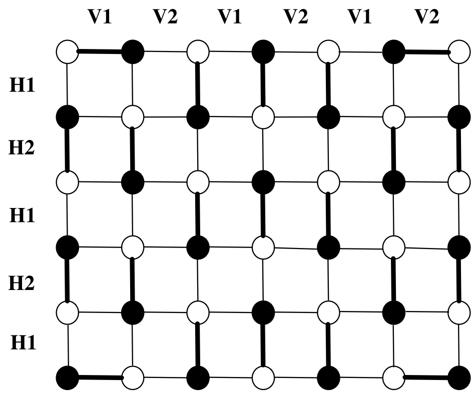

Fig. 6. Matching $M^V_{H_1}$.

Proof. We construct a set T by taking the bundles of the four corners together with those of $d-4 \leq m+n-7$ vertices of degree 3 forming a stable set with the corners. One can always choose such vertices as can be seen easily. This gives a set T with $|T| = 3d-4$ which is a d-transversal. Let $M^V_{H_1}$ be the matching shown in Fig. 6. Since $M_{V_1} \cap M_{V_2} = \emptyset$ and $|(M_{V_1} \cup M_{V_2}) \cap M^V_{H_1}| = 4$, any d-transversal T' satisfies $|T'| \geq 3d-4$. Thus T is a minimum d-transversal. □

Lemma 3.18. *For $m \geq 4$, $n \geq 5$ and $m+n-2 \leq d \leq \frac{mn}{4} + \frac{m}{4} + \frac{n-5}{2}$, $\tau_d(G_{m,n}) = 4d - m - n - 1 - \lfloor \frac{d-(m+n-3)}{\frac{n-3}{2}} \rfloor$.*

Proof. Let $m \geq 4$, $n \geq 5$ and $m+n-2 \leq d \leq \frac{mn}{4} + \frac{m}{4} + \frac{n-5}{2}$.

Let T be a d-transversal. We must clearly have $|H_1 \cap T| \geq d$ and $|V_i \cap T| \geq d - \frac{m}{2}$, $i = 1, 2$, since the matching M_{V_1} (resp. M_{V_2}) contains all edges of V_1 (resp. V_2) as well as $\frac{m}{2}$ edges of H_1. Now we can transform these inequalities into equalities by adding three nonnegative integers $x, y_1, y_2 \geq 0$. Thus we obtain: $|H_1 \cap T| = d+x$, $|V_1 \cap T| = d - \frac{m}{2} + y_1$, $|V_2 \cap T| = d - \frac{m}{2} + y_2$ and $|H_2 \cap T| = |T| - 3d + m - z$, where $z = x + y_1 + y_2$.

Consider a matching $M^j_{H_2}, j \in \{3, 5, \dots, n-2\}$: T contains at least $d - |H_2 \cap T| - (n-1) = d - (|T| - 3d + m - z) - (n-1) = 4d - m - n + 1 + z - |T|$ edges of H_1 in column j, for $j = 3, 5, \dots, n-2$, at least $\frac{m}{2} - y_2$ edges of H_1 in column 1, and at least $\frac{m}{2} - y_1$ edges of H_1 in column n. Combining all the columns, we get:

$$|H_1 \cap T| = d + x \geq m - (y_1 + y_2) + \frac{n-3}{2}(4d - m - n + 1 + z - |T|)$$

thus,

$$\frac{n-3}{2}(|T| - (4d - m - n + 1 + z)) \geq m - z - d = -(d - (m + n - 3)) - (n-3) - z$$

hence,

$$|T| - (4d - m - n + 1 + z) \geq -\frac{(d - (m + n - 3))}{\frac{n-3}{2}} - 2 - \frac{2}{n-3} z$$

finally,

$$|T| \geq 4d - m - n - 1 - \frac{(d - (m + n - 3))}{\frac{n-3}{2}} + \left(1 - \frac{2}{n-3}\right) z.$$

Since $z, 1 - \frac{2}{n-3} \geq 0$, $|T| \geq 4d - m - n - 1 - \lfloor \frac{(d-(m+n-3))}{\frac{n-3}{2}} \rfloor$.

Let us now show how to construct such a minimum d-transversal T. First we take the four bundles $\omega(x_{11})$, $\omega(x_{1n})$, $\omega(x_{m1})$, $\omega(x_{mn})$ of the corners together with the $n-3$ bundles $\omega(x_{13}), \omega(x_{15}), \dots, \omega(x_{1,n-2})$ and $\omega(x_{m3}), \omega(x_{m5}), \dots, \omega(x_{m,n-2})$. Then we add the $X = \lfloor \frac{d-(m+n-3)}{\frac{n-3}{2}} \rfloor$ dancats $D_4, D_6, \dots, D_{2X+2}$ as well as the $m - 4 - 2X$ bundles $\omega(x_{2X+3,1}), \omega(x_{2X+5,1}), \dots, \omega(x_{m-3,1})$ and $\omega(x_{2X+3,n}), \omega(x_{2X+5,n}), \dots, \omega(x_{m-3,n})$. Note that for $d = \frac{mn}{4} + \frac{m}{4} + \frac{n-5}{2}$ and $n \geq 5$ we have $\lfloor \frac{d-(m+n-3)}{\frac{n-3}{2}} \rfloor = \lfloor \frac{\frac{m}{2} \times \frac{n-3}{2} - \frac{n-1}{2}}{\frac{n-3}{2}} \rfloor = \frac{m}{2} - 2$, thus $X \leq \frac{m}{2} - 2$ and the construction is valid. Finally we complete with $d - (n + m - 3 + \frac{n-3}{2} X)$ bundles $\omega(x_{ij})$ such that $|\omega(x_{ij})| = 4$, x_{ij} is not incident to a dancat, and $i + j$ is even. This gives us a set T such that $|M \cap T| = 4 + n - 3 + X\frac{n+1}{2} + m - 4 - 2X + d - (n + m - 3 + \frac{n-3}{2} X) = d$ for every maximum matching M. Thus T is a d-transversal. Concerning the cardinality of T, we have the bundles containing $8 + 3(n-3) + 3(m - 4 - 2X) + 4(d - (n + m - 3 + \frac{n-3}{2} X)) = 4d - (n+m) - 2nX - 1$ edges and the dancats containing $(2n-1)X$ edges. This gives us $|T| = 4d - (n+m) - 1 - X = 4d - (m+n) - 1 - \lfloor \frac{d-(m+n-3)}{\frac{n-3}{2}} \rfloor$. $\square$

Lemma 3.19. *For $m \geq 4$, $n \geq 5$ and $\frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2} \leq d \leq \frac{mn}{2}$, $\tau_d(G_{m,n}) = 4d - (m+n) - \lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \rfloor$. For $m \geq 4$, $n = 3$ and $m + 1 \leq d \leq \frac{3m}{2}$, $\tau_d(G_{m,3}) = 5d - \frac{5m}{2} - 3$.*

Proof. We will show that no d-transversal T can have $4d - (m+n) - \lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \rfloor - 1$ edges. Equivalently, we show that for any $Y \subset E$ with $|Y| = 4(\frac{mn}{2} - d) + \lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \rfloor + 1$ there is a maximum matching M with $|M \cap Y| \geq (\frac{mn}{2} - d) + 1$. Since $|Y| = 4(\frac{mn}{2} - d) + \lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \rfloor + 1$, there is at least one of V_1, V_2, H_1, H_2 which contains $(\frac{mn}{2} - d) + 1$ edges of Y. If it is H_1, then we are done since $H_1 = M_{H_1}$ is a maximum matching. If it is V_1 (resp. V_2), we are also done since $V_1 \subset M_{V_1}$ (resp. $V_2 \subset M_{V_2}$). Hence we may assume that H_2 contains $(\frac{mn}{2} - d) + \lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \rfloor + 1$ edges of Y. Now $M^j_{H_2}$ contains all edges of H_2 except those of the column j, j odd. Furthermore there exists an odd column k such that the set of all columns $l \neq k$ contains at least $(\frac{mn}{2} - d) + 1$ edges of $H_2 \cap Y$: if y^j is the number of edges of $H_2 \cap Y$ in column j, we would have $\Sigma_{j \neq k}\, y^j \leq \frac{mn}{2} - d$, k *odd*. By summing

$$\sum_{k=1, k\ odd}^{n} \sum_{j \neq k} y^j \leq \sum_{k=1, k\ odd}^{n} \left(\frac{mn}{2} - d\right)$$

$$\frac{n-1}{2} \sum_j y^j \leq \left(\frac{mn}{2} - d\right) \frac{n+1}{2}$$

$$|H_2 \cap Y| \leq \left(\frac{mn}{2} - d\right) \frac{n+1}{n-1} = \left(\frac{mn}{2} - d\right) + \left(\frac{mn}{2} - d\right) \frac{2}{n-1} < \left(\frac{mn}{2} - d\right) + \left\lfloor \frac{\frac{mn}{2} - d}{\frac{n-1}{2}} \right\rfloor + 1$$

which is a contradiction.

Hence there is a maximum matching M containing $\frac{mn}{2} - d + 1$ edges of $H_2 \cap Y$ and therefore no transversal T can have $|T| < 4d - (m+n) - \lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor$.

If $n \geq 5$, then for $d = \frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2}$, we have $\lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor = \lfloor \frac{\frac{m}{2} \times \frac{n-1}{2} - \frac{n-3}{2}}{\frac{n-1}{2}} \rfloor = \frac{m}{2} - 1$. So $\lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor \leq \frac{m}{2} - 1$ for $d \geq \frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2}$. Thus the above computation is valid for $\frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2} \leq d \leq \frac{mn}{2}$.

Let us now show how to construct such a minimum d-transversal T. In the case where $\frac{mn}{4} + \frac{m}{4} + \frac{n-1}{2} \leq d \leq \frac{mn}{2}$, we do the following. We take the two bundles $\omega(x_{m1})$ and $\omega(x_{mn})$ together with the $n-2$ bundles $\omega(x_{12}), \omega(x_{14}), \ldots, \omega(x_{1,n-1})$ and $\omega(x_{m3}), \omega(x_{m5}), \ldots, \omega(x_{m,n-2})$. Then we add the $X = \lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor$ dancats $D_2, D_4, \ldots, D_{2X}$ (recall that $X \leq \frac{m}{2} - 1$) as well as the $m-2-2X$ bundles $\omega(x_{2X+2,1}), \omega(x_{2X+4,1}), \ldots, \omega(x_{m-2,1})$ and $\omega(x_{2X+2,n}), \omega(x_{2X+4,n}), \ldots, \omega(x_{m-2,n})$. Finally we complete with $d - (n + m - 2 + \frac{n-3}{2}X)$ bundles $\omega(x_{ij})$ such that $|\omega(x_{ij})| = 4$, x_{ij} is not incident to a dancat, and $i + j$ is odd. Here we note that $\frac{mn}{4} + \frac{m}{4} + \frac{n-1}{2} \leq d$ implies $d - (n + m - 2 + \frac{n-3}{2}X) \geq 0$. This gives us a set T such that $|M \cap T| = n + X\frac{n+1}{2} + m - 2 - 2X + d - (n + m - 2 + \frac{n-3}{2}X) = d$ for every maximum matching M. Thus T is a d-transversal. Concerning the cardinality of T, we have the bundles containing $3n-2+3(m-2-2X)+4d-4(n+m-2+\frac{n-3}{2}X) = 4d-(n+m)-2nX$ edges and the dancats containing $(2n-1)X$ edges. This gives us $|T| = 4d - (n+m) - X = 4d - (m+n) - \lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor$.

Now for $d = \frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2}$, we proceed as follows. We take the $n+1$ bundles $\omega(x_{11}), \omega(x_{13}), \ldots, \omega(x_{1,n})$ and $\omega(x_{m1}), \omega(x_{m3}), \ldots, \omega(x_{m,n})$, the $X - 1 = \frac{m}{2} - 2$ dancats $D_4, D_6, \ldots, D_{m-2}$ and the $\frac{n-3}{2}$ bundles $\omega(x_{22}), \omega(x_{24}), \ldots, \omega(x_{2,n-3})$. This gives a d-transversal with $|T| = 3(n+1) - 4 + (\frac{m}{2} - 2)(2n-1) + 4\frac{n-3}{2} = 4d - (m+n) - \lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \rfloor$.

If $n = 3$ then for $m + 1 \leq d \leq \frac{3m}{2}$ we use the first construction; this is possible since $X \leq \frac{m}{2} - 1$. $\square$

Having covered all possible cases we are now able to state the following.

Theorem 3.20. *Let $G_{m,n}$ be a grid with m even and n odd. The minimum cardinality of a d-transversal is*

1. *for $n = 1$*
   (a) $\tau_d(G_{m,1}) = d$ *for* $1 \leq d \leq \frac{m}{2} = \nu(G_{m,1})$
2. *for $m = 2, n \geq 3$*
   (a) $\tau_d(G_{2,n}) = 2d$ *for* $1 \leq d \leq 2$
   (b) $\tau_d(G_{2,n}) = 3d - 2$ *for* $3 \leq d \leq n = \nu(G_{2,n})$
3. *for $m \geq 4, n = 3$*
   (a) $\tau_d(G_{m,3}) = 2d$ *for* $1 \leq d \leq \frac{m}{2} + 2$
   (b) $\tau_d(G_{m,3}) = 3d - \frac{m}{2} - 2$ *for* $\frac{m}{2} + 3 \leq d \leq m$
   (c) $\tau_d(G_{m,3}) = 5d - \frac{5m}{2} - 3$ *for* $m + 1 \leq d \leq \frac{3m}{2} = \nu(G_{m,3})$
4. *for $m \geq 4, n \geq 5$*
   (a) $\tau_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq 4$
   (b) $\tau_d(G_{m,n}) = 3d - 4$ *for* $5 \leq d \leq m + n - 3$
   (c) $\tau_d(G_{m,n}) = 4d - m - n - 1 - \lfloor \frac{d-(m+n-3)}{\frac{n-3}{2}} \rfloor$ *for* $m + n - 2 \leq d \leq \frac{mn}{4} + \frac{m}{4} + \frac{n-5}{2}$
   (d) $\tau_d(G_{m,n}) = 4d - (m+n) - \left\lfloor \frac{\frac{mn}{2}-d}{\frac{n-1}{2}} \right\rfloor$ *for* $\frac{mn}{4} + \frac{m}{4} + \frac{n-3}{2} \leq d \leq \nu(G_{m,n})$.

Proof. 1(a) is Property 2.4. 2(a), 2(b) follow from Lemma 3.1. 3(a) follows from Lemmas 3.2 and 3.15. 3(b) is Lemma 3.16. 3(c) is from Lemma 3.19. 4(a) is Lemma 3.2. 4(b), 4(c), 4(d) are Lemmas 3.17–3.19, respectively. $\square$

4. Minimum d-blocker in grid graphs

We show here how to construct a minimum d-blocker in a grid graph $G_{m,n}$. We will use both our results on d-transversals proved in Section 3 and the inequality $\beta_d(G_{m,n}) \geq \tau_d(G_{m,n})$ which is an immediate corollary of Property 2.1.

First, we study the case where mn is even. W.l.o.g. we assume that m is even. Recall that in this case $\nu(G_{m,n}) = \frac{mn}{2}$.

For $m > 2$, the results will be obtained by constructing four pairwise disjoint matchings (see Fig. 7).

Both M_1 and M_2 are perfect matchings: M_1 consists of the horizontal edges $h_{i1}, h_{i3}, \ldots, h_{i,2\lfloor \frac{n}{2} \rfloor - 1}$ of each row i, together with the vertical edges $v_{1n}, v_{3n}, \ldots, v_{m-1,n}$ if n is odd; M_2 consists of either the vertical edges $v_{1j}, v_{3j}, \ldots, v_{m-1,j}$ of each column j if n is even, or of the horizontal edges $h_{i2}, h_{i4}, \ldots, h_{i,n-1}$ of each row i together with the vertical edges $v_{11}, v_{31}, \ldots, v_{m-1,1}$ if n is odd. M_3 and M_4 are not perfect matchings: if n is even, M_3 consists of the horizontal edges $h_{i2}, h_{i4}, \ldots, h_{i,n-2}$ of each row i, together with the vertical edges $v_{21}, v_{41}, \ldots, v_{m-2,1}$ and $v_{2n}, v_{4n}, \ldots, v_{m-2,n}$; if n is odd, M_3 consists of the vertical edges $v_{1j}, v_{3j}, \ldots, v_{m-1,j}, j = 2, \ldots, n-1$, together with the vertical edges $v_{21}, v_{41}, \ldots, v_{m-2,1}$ and $v_{2n}, v_{4n}, \ldots, v_{m-2,n}$. In both cases, $|M_3| = (mn/2) - 2$. Finally, M_4 consists of the vertical edges $v_{2j}, v_{4j}, \ldots, v_{m-2,j}$, $j = 2, \ldots, n-1$; we have $|M_4| = (mn/2) - m - n + 2$. Note that $M_1 \cup M_2 \cup M_3 \cup M_4 = E(G_{m,n})$.

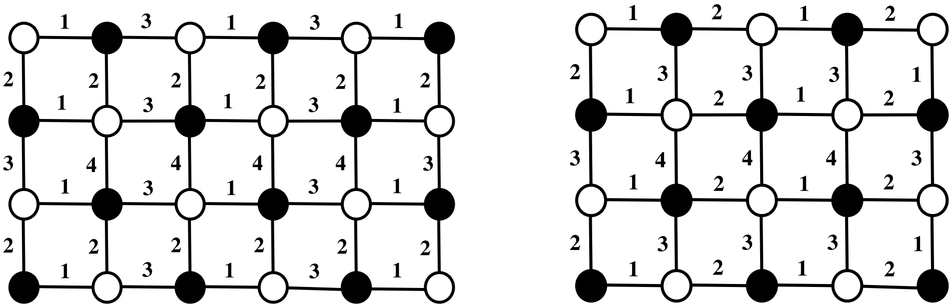

Fig. 7. Matchings for n even on the left and for n odd on the right.

Lemma 4.1. *For $d \in \{1, 2\}$, if $m \geq 2$ is even then $\beta_d(G_{m,n}) = 2d$.*

Proof. From Theorems 3.5 and 3.20 we have $\beta_d(G_{m,n}) \geq \tau_d(G_{m,n}) = 2d$. By taking the bundles of d black corners, we get a d-blocker with cardinality $2d$. □

Lemma 4.2. *For $3 \leq d \leq m + n - 2$, if $m \geq 2$ is even then $\beta_d(G_{m,n}) = 3d - 2$.*

Proof. When $m = 2$, from Theorems 3.5 and 3.20 we have $\beta_d(G_{m,n}) \geq \tau_d(G_{m,n}) = 3d - 2$. For $m > 2$, since M_1, M_2 and M_3 are pairwise disjoint a d-blocker must have at least d edges in M_1 and in M_2 and at least $d - 2$ edges in M_3, thus $\beta_d(G_{m,n}) \geq 3d - 2$. For both cases we form a d-blocker of $3d - 2$ edges by taking the bundles of two black corners together with $d - 2$ bundles of black vertices of degree 3. Note that this is possible since $d \leq m + n - 2$ and there are $m + n - 4$ disjoint bundles of degree 3. □

Lemma 4.3. *For $d > m + n - 2$, if $m > 2$ is even then $\beta_d(G_{m,n}) = 4d - m - n$.*

Proof. As previously, by considering M_1, M_2 and M_3, a d-blocker must have at least $3d - 2$ edges. In addition, since M_4 is disjoint from these three matchings, a d-blocker must have also $d - m - n + 2$ edges in M_4 and thus at least $4d - m - n$ edges. We form a d-blocker of $4d - m - n$ edges by taking the bundles of two black corners together with the $m + n - 4$ bundles of black vertices of degree 3 and $d - m - n + 2$ bundles of black vertices of degree 4. □

From Lemmas 4.1–4.3 and Property 2.4 we have:

Theorem 4.4. *Let $G_{m,n}$ be a grid with mn even. The minimum cardinality of a d-blocker is*

1. *for $m = 1$ or $n = 1$*
   (a) $\beta_d(G_{m,n}) = 2d - 1$ *for* $1 \leq d \leq \nu(G_{m,n})$
2. *for $m \geq 2$ and $n \geq 2$*
   (a) $\beta_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq 2$
   (b) $\beta_d(G_{m,n}) = 3d - 2$ *for* $3 \leq d \leq m + n - 2$
   (c) $\beta_d(G_{m,n}) = 4d - m - n$ *for* $m + n - 1 \leq d \leq \nu(G_{m,n})$.

Now, let $G_{m,n}$ be a grid graph with both m and n odd. We have he following:

Theorem 4.5. *Let $G_{m,n}$ be a grid with mn odd. The minimum cardinality of a d-blocker is*

1. *for $m = 1$ or $n = 1$*
   (a) $\beta_d(G_{m,n}) = 2d$ *for* $1 \leq d \leq \nu(G_{m,n})$
2. *for $m \geq 3$ and $n \geq 3$*
   (a) $\beta_d(G_{m,n}) = 3$ *for* $d = 1$
   (b) $\beta_d(G_{m,n}) = 2d + 2$ *for* $2 \leq d \leq 3$
   (c) $\beta_d(G_{m,n}) = 3d - 1$ *for* $4 \leq d \leq m + n - 3$ *and* $m + n \neq 6$
   (d) $\beta_d(G_{m,n}) = 4d - m - n + 2$ *for* $m + n - 2 \leq d \leq \nu(G_{m,n})$.

Proof. The case $m = 1$ or $n = 1$ is stated by Property 2.4. Now, we consider $m \geq 3$ and $n \geq 3$. By assumption the four corners are black vertices so $|\mathcal{B}| = |\mathcal{W}| + 1$. First, note that $d + 1$ bundles of black vertices form a d-blocker. Second, $\beta_d(G_{m,n}) \geq \tau_d(G_{m,n})$. Third, from Section 3.2 we know that a minimal d-transversal is formed by the $d + 1$ bundles of black vertices with the $d + 1$ smallest degrees. Thus a minimal d-blocker is a minimal d-transversal and its cardinality is given by Theorem 3.12. □

5. Minimum *d*-blocker and *d*-transversal in trees

In this section we present a dynamic programming approach that allows us to find the cardinality of a minimum d-transversal and of a minimum d-blocker in polynomial time on trees. We describe the algorithm in details for transversals. It can easily be adapted for the case of blockers. We begin by giving a property which we shall use later.

Property 5.1. *Let $G = (V, E)$ be a graph and V_1, V_2 a partition of the vertices V such that there is no edge between V_1 and V_2. We denote by $G_1 = (V_1, E_1)$, resp. $G_2 = (V_2, E_2)$, the subgraph of G induced by the vertices of V_1, resp. of V_2. For $d \in \{1, \ldots, \nu(G)\}$ we have the following results.*

(i) $\beta_d(G) = \min\{\beta_i(G_1) + \beta_{d-i}(G_2) \mid i \in \{\max\{0, d - \nu(G_2)\}, \ldots, \min\{d, \nu(G_1)\}\}\}$
(ii) $\tau_d(G) = \min\{\tau_i(G_1) + \tau_{d-i}(G_2) \mid i \in \{\max\{0, d - \nu(G_2)\}, \ldots, \min\{d, \nu(G_1)\}\}\}$.

Proof. (i) Let $B \subseteq E$, $B_1 = B \cap E_1$ and $B_2 = B \cap E_2$. The following equality implies the result.

$$\nu((V, E \setminus B)) = \nu((V_1, E_1 \setminus B_1)) + \nu((V_2, E_2 \setminus B_2)).$$

(ii) We denote by $\mathcal{M}$, $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the sets of all maximum matchings in G, G_1 (resp. G_2). The result finally follows from the following observation.

$$\mathcal{M} = \{M_1 \cup M_2 \mid M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}. \quad \Box$$

Let us now introduce some notations and terminology used throughout this section.

Let r be an arbitrary vertex in V. We orient the edges of E away from r in order to get an arborescence $T = (V, A)$ of root r. Let $D(v) = \{w \in V \mid (v, w) \in A\} = \{v_1, \ldots, v_{|D(v)|}\}$. To simplify the terminology we say that an (undirected) edge $[v, w]$ is contained in a directed graph G if either (v, w) or (w, v) is contained in G. For $v \in V$ and $a, b \in \{1, \ldots, |D(v)|\}$ with $a \le b$ we denote by $T^v_{a,b}$ the subarborescence of T over the vertices $\{v\} \cup \bigcup_{i \in \{a,\ldots,b\}} \{u \in V \mid \exists \text{ path from } v_i \text{ to } u \text{ in } T\}$. Furthermore for $v \in V$ we denote by T^v the subarborescence of T with root v, i.e., $T^v = T^v_{1,|D(v)|}$. We denote by $E^v_{a,b}$ the set of edges contained in $T^v_{a,b}$.

We can assume that the task is to find the cardinality of a minimum d-transversal in a tree G with $P_0(G) = P_1(G) = \emptyset$ because of the following observations. If $P_0(G) \neq \emptyset$, we can remove the edges of $P_0(G)$ from G since these edges are not contained in any maximum matching and therefore nor in any minimum d-transversal. The remaining graph is thus a forest. We can easily determine the cardinality of a minimum d-blocker for a forest if we have already determined the cardinality of all minimum transversals for its components by using Property 5.1. We therefore assume that the tree $G = (V, E)$ satisfies $P_0(G) = \emptyset$. Additionally it is easy to see that the only remaining case with $P_1(G) \neq \emptyset$ corresponds to a graph G consisting of only one edge which is a trivial case.

For any $v \in V$ and $a, b \in \{1, \ldots, |D(v)|\}$ with $a \le b$ we define a partition of the set of all maximum matchings of G into two sets $\mathcal{M}^{v,+}_{a,b}$ and $\mathcal{M}^{v,-}_{a,b}$, where $\mathcal{M}^{v,+}_{a,b}$ is the set of all maximum matchings in G saturating v with an edge of $E^v_{a,b}$ and $\mathcal{M}^{v,-}_{a,b}$ is the set of all maximum matchings in G not saturating v with an edge of $E^v_{a,b}$.

Finally let us define the following two notions where $U \subseteq E$:

$$m^+(T^v_{a,b}, U) = \max_{M \in \mathcal{M}^{v,+}_{a,b}} (|(M \cap E^v_{a,b}) - U|)$$

$$m^-(T^v_{a,b}, U) = \max_{M \in \mathcal{M}^{v,-}_{a,b}} (|(M \cap E^v_{a,b}) - U|).$$

In other words, $m^+(T^v_{a,b}, U)$ (resp. $m^-(T^v_{a,b}, U)$) represents the maximum number of edges of $T^v_{a,b} \setminus U$ contained in a single matching of $\mathcal{M}^{v,+}_{a,b}$ (resp. $\mathcal{M}^{v,-}_{a,b}$). We use the convention that $m^-(T, U) = 0$ if $r \in S(G)$ since in this case $\mathcal{M}^{r,-}_{1,|D(r)|} = \emptyset$. Note that no other set $\mathcal{M}^{v,+}_{a,b}$ or $\mathcal{M}^{v,-}_{a,b}$ is empty since $P_0(G) = \emptyset$.

The cardinality of a minimum d-transversal is smaller than or equal to k if and only if

$$\min_{\substack{U \subseteq E \\ |U| = k}} \max\{m^+(T, U), m^-(T, U)\} \le \nu(G) - d.$$

We say that a pair $(x^+, x^-) \in \mathbb{Z}^2$ dominates a pair $(y^+, y^-) \in \mathbb{Z}^2$ if $x^+ \le y^+$, $x^- \le y^-$ and at least one of the two inequalities is strict. Let $X \subset \mathbb{Z}^2$. We say that $x \in X$ is efficient in X if there is no element $y \in X$ that dominates x. Furthermore the efficient subset $\mathcal{E}(X)$ of X are the elements in X that are efficient in X.

We define $\widetilde{Q}^v_{a,b}(k) := \{(m^+(T^v_{a,b}, U), m^-(T^v_{a,b}, U)) \mid U \subseteq E, |U| = k\}$ and let $Q^v_{a,b}(k) = \mathcal{E}(\widetilde{Q}^v_{a,b}(k))$. In a similar way to the notation used for subarborescences of T we define $Q^v(k) = Q^v_{1,|D(v)|}(k)$. Furthermore we use the notation $Q^v_{a,b} = (Q^v_{a,b}(0), \ldots, Q^v_{a,b}(|E^v_{a,b}|))$. The algorithm we propose begins by determining $Q^v_{p,p}$ for subarborescences $T^v_{p,p}$ containing only one arc, i.e., when v_p is a leaf of T. Then the elements $Q^v_{1,p}$ corresponding to larger arborescences $T^v_{1,p}$ will be calculated on the base of sets Q corresponding to smaller arborescences by the following two types of operations.

Operation 1 (Adding an Arc). Determine $Q^v_{p,p}$ on the base of Q^{v_p}.

Operation 2 (Merging Two Subarborescences). Determine $Q^v_{1,p+1}$ on the base of $Q^v_{1,p}$ and $Q^v_{p+1,p+1}$.

If we can perform the above two operations, Q^r can easily be obtained by calculating first the values of Q for arcs $(v, w) \in A$, where w is a leaf and then combining them with the two operations. We will now give details on how we realize Operations 1 and 2.

The following results describe basic relationships that will be used to describe a simple way of executing Operation 1 and Operation 2. We start by giving a proposition which will be used in the discussion of Operation 1.

Proposition 5.1. *Let $(v, v_p) \in A$.*
If $v_p \notin S(G)$ we have
(i) $\{M \cap E^{v_p} \mid M \in \mathcal{M}^{v,+}_{p,p}\} = \{M \cap E^{v_p} \mid M \in \mathcal{M}^{v_p,-}\}$
(ii) $\{M \cap E^{v_p} \mid M \in \mathcal{M}^{v,-}_{p,p}\} = \{M \cap E^{v_p} \mid M \in \mathcal{M}\}$.
If $v_p \in S(G)$ we have
(iii) $\mathcal{M}^{v,+}_{p,p} = \mathcal{M}^{v_p,-}$
(iv) $\mathcal{M}^{v,-}_{p,p} = \mathcal{M}^{v_p,+}$.

Proof. (i) The inclusion $\subseteq$ follows from $\mathcal{M}^{v,+}_{p,p} \subseteq \mathcal{M}^{v_p,-}$. Let $M_1 \in \mathcal{M}^{v_p,-}$. We will show that there exists $M_2 \in \mathcal{M}^{v,+}_{p,p}$ with $M_1 \cap E^{v_p} = M_2 \cap E^{v_p}$. If $M_1 \in \mathcal{M}^{v,+}_{p,p}$ we are done by choosing $M_2 = M_1$. Otherwise we obtain M_2 by adding the edge $[v, v_p]$ to M_1 and removing the edge in M_1 being adjacent to $[v, v_p]$.

(ii) The inclusion $\subseteq$ follows trivially from $\mathcal{M}^{v,-}_{p,p} \subseteq \mathcal{M}$. Let M be a maximum matching in G that does not saturate v_p (such a matching exists since $v_p \notin S(G)$) and let $M_1 \in \mathcal{M}$. We will show that there exists $M_2 \in \mathcal{M}^{v,-}_{p,p}$ with $M_2 \cap E^{v_p} = M_1 \cap E^{v_p}$. Let $M_2 = (M \setminus E^{v_p}) \cup (M_1 \cap E^{v_p})$. M_2 is indeed a matching not containing $[v, v_p]$ and satisfying $M_2 \cap E^{v_p} = M_1 \cap E^{v_p}$ as desired. Furthermore the maximality of M_1 and M imply that M_2 must be a maximum matching.

(iii)/(iv) These equations follow from the observation that if every maximum matching saturates v_p, then the maximum matchings saturating v_p by the edge $[v, v_p]$ correspond exactly to the maximum matchings which do not saturate v_p by one of the edges in E^{v_p} and vice versa. □

The following two lemmas are consequences of Proposition 5.1.

Lemma 5.2. *Let $(v, v_p) \in A$, $U \subseteq E$ and $U' = U \setminus \{[v, v_p]\}$. Suppose that $v_p \notin S(G)$.*
1. *If $[v, v_p] \in U$ we have*
   (i) $m^+(T^v_{p,p}, U) = m^-(T^{v_p}, U')$
   (ii) $m^-(T^v_{p,p}, U) = \max(m^+(T^{v_p}, U'), m^-(T^{v_p}, U'))$.
2. *If $[v, v_p] \notin U$ we have*
   (iii) $m^+(T^v_{p,p}, U) = 1 + m^-(T^{v_p}, U)$
   (iv) $m^-(T^v_{p,p}, U) = \max(m^+(T^{v_p}, U), m^-(T^{v_p}, U))$.

Lemma 5.3. *Let $(v, v_p) \in A$, $U \subseteq E$ and $U' = U \setminus \{[v, v_p]\}$. Suppose that $v_p \in S(G)$.*
1. *If $[v, v_p] \in U$ we have*
   (i) $m^+(T^v_{p,p}, U) = m^-(T^{v_p}, U')$
   (ii) $m^-(T^v_{p,p}, U) = m^+(T^{v_p}, U')$.
2. *If $[v, v_p] \notin U$ we have*
   (iii) $m^+(T^v_{p,p}, U) = 1 + m^-(T^{v_p}, U)$
   (iv) $m^-(T^v_{p,p}, U) = m^+(T^{v_p}, U)$.

The next proposition will be used in the discussion of Operation 2.

Proposition 5.4. *Let $v \in V$ and $p \in \{1, \ldots, |D(v)| - 1\}$. We have*
(i)
$$\{M \cap E^v_{1,p+1} \mid M \in \mathcal{M}^{v,+}_{1,p+1}\} = \{(M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1}) \mid M_1 \in \mathcal{M}^{v,+}_{1,p}, M_2 \in \mathcal{M}^{v,-}_{p+1,p+1}\} \cup \{(M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1}) \mid M_1 \in \mathcal{M}^{v,-}_{1,p}, M_2 \in \mathcal{M}^{v,+}_{p+1,p+1}\}.$$
(ii) *If $\mathcal{M}^{v,-}_{1,p+1} \neq \emptyset$ we have*
$$\{M \cap E^v_{1,p+1} \mid M \in \mathcal{M}^{v,-}_{1,p+1}\} = \{(M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1}) \mid M_1 \in \mathcal{M}^{v,-}_{1,p}, M_2 \in \mathcal{M}^{v,-}_{p+1,p+1}\}.$$

Proof. (i) The inclusion $\subseteq$ follows from $\mathcal{M}^{v,+}_{1,p+1} = (\mathcal{M}^{v,+}_{1,p} \cap \mathcal{M}^{v,-}_{p+1,p+1}) \cup (\mathcal{M}^{v,-}_{1,p} \cap \mathcal{M}^{v,+}_{p+1,p+1})$. Let $M_1 \in \mathcal{M}^{v,+}_{1,p}$ and $M_2 \in \mathcal{M}^{v,-}_{p+1,p+1}$. We will show that there exists $M \in \mathcal{M}^{v,+}_{1,p+1}$ satisfying $M \cap E^v_{1,p+1} = (M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1})$ (the proof for the case $M_1 \in \mathcal{M}^{v,-}_{1,p}$, $M_2 \in \mathcal{M}^{v,+}_{p+1,p+1}$ is analogous). Let $M = (M_1 \setminus E^v_{p+1,p+1}) \cup (M_2 \cap E^v_{p+1,p+1})$. M is indeed a matching saturating v by an edge of $E^v_{1,p+1}$ and satisfying $M \cap E^v_{1,p+1} = (M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1})$. Furthermore the maximality of M_1 and M_2 implies that M is a maximum matching.

(ii) The inclusion $\subseteq$ follows from $\mathcal{M}^{v,-}_{1,p+1} = \mathcal{M}^{v,-}_{1,p} \cap \mathcal{M}^{v,-}_{p+1,p+1}$. Let $M_1 \in \mathcal{M}^{v,-}_{1,p}$ and $M_2 \in \mathcal{M}^{v,-}_{p+1,p+1}$. We will show that there exists $M \in \mathcal{M}^{v,-}_{1,p+1}$ satisfying $M \cap E^v_{1,p+1} = (M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1})$. Let $M_3 \in \mathcal{M}^{v,-}_{1,p+1}$ and define $M = (M_3 \setminus E^v_{1,p+1}) \cup (M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1})$. M is indeed a matching not saturating v by one of the edges in $E^v_{1,p+1}$ and satisfying $M \cap E^v_{1,p+1} = (M_1 \cap E^v_{1,p}) \cup (M_2 \cap E^v_{p+1,p+1})$. Furthermore the maximality of M_1, M_2 and M_3 implies that M is a maximum matching. □

The following lemma is a consequence of Proposition 5.4.

Lemma 5.5. *Let $v \in V$, $p \in \{1, \dots, |D(v)| - 1\}$ and $U \subseteq E$. Furthermore we define $U_1 = U \cap E^v_{1,p}$ and $U_2 = U \cap E^v_{p+1,p+1}$. We have:*

1. $m^+(T^v_{1,p+1}, U) = \max\{m^+(T^v_{1,p}, U_1) + m^-(T^v_{p+1,p+1}, U_2), m^-(T^v_{1,p}, U_1) + m^+(T^v_{p+1,p+1}, U_2)\}$
2. $m^-(T^v_{1,p+1}, U) = m^-(T^v_{1,p}, U_1) + m^-(T^v_{p+1,p+1}, U_2)$ *if $\mathcal{M}^{v,-}_{1,p+1} \neq \emptyset$ (this corresponds to the case $v \neq r$ or $p \neq |D(v)| - 1$ or $r \notin S(G)$)*
3. $m^-(T^v_{1,p+1}, U) = 0$ *if $\mathcal{M}^{v,-}_{1,p+1} = \emptyset$ (this corresponds to the case $v = r$, $p = |D(G)| - 1$ and $r \in S(G)$).*

The following proposition shows how Operation 1 can be performed in polynomial time.

Proposition 5.6. *Let $(v, v_p) \in A$ and $k \in \{1, \dots, |E^v_{p,p}|\}$.*

(i) *If $v_p \in S(G)$ we have*

$$Q^v_{p,p}(k) = \mathcal{E}\left(\{(q^-, q^+) \mid (q^+, q^-) \in Q^{v_p}(k-1)\} \cup \{(q^- + 1, q^+) \mid (q^+, q^-) \in Q^{v_p}(k)\}\right).$$

(ii) *If $v_p \notin S(G)$ we have*

$$Q^v_{p,p}(k) = \mathcal{E}\left(\{(q^-, \max\{q^-, q^+\}) \mid (q^+, q^-) \in Q^{v_p}(k-1)\} \cup \{(q^- + 1, \max\{q^-, q^+\}) \mid (q^+, q^-) \in Q^{v_p}(k)\}\right).$$

Proof. (i) As a consequence of Lemma 5.3 we have

$$\widetilde{Q}^v_{p,p}(k) = \{(q^-, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k-1)\} \cup \{(q^- + 1, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k)\}$$

which implies

$$Q^v_{p,p}(k) = \mathcal{E}(\{(q^-, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k-1)\} \cup \{(q^- + 1, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k)\}).$$

The result is finally obtained by observing that

$$\begin{aligned}&\mathcal{E}(\{(q^-, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k-1)\} \cup \{(q^- + 1, q^+) \mid (q^+, q^-) \in \widetilde{Q}^{v_p}(k)\})\\ &\quad = \mathcal{E}(\{(q^-, q^+) \mid (q^+, q^-) \in Q^{v_p}(k-1)\} \cup \{(q^- + 1, q^+) \mid (q^+, q^-) \in Q^{v_p}(k)\}).\end{aligned}$$

(ii) This part can be proved in the same way as point (i) by using Lemma 5.2 instead of Lemma 5.3. □

The following proposition shows how Operation 2 can be performed in polynomial time.

Proposition 5.7. *Let $v \in V$, $p \in \{1, \dots, |D(v)| - 1\}$ and $k \in \{0, \dots, |E^v_{1,p+1}|\}$.*

(i) *If $\mathcal{M}^{v,-}_{1,p+1} \neq \emptyset$ (this corresponds to the case $v \neq r$ or $p \neq |D(r)| - 1$ or $r \notin S(G)$) we have*

$$Q^v_{1,p+1}(k) = \mathcal{E}\left(\bigcup_{i \in \{0,\dots,k\}} \{(\max\{q_1^+ + q_2^-, q_1^- + q_2^+\}, q_1^- + q_2^-) \mid (q_1^+, q_1^-) \in Q^v_{1,p}(i), (q_2^+, q_2^-) \in Q^v_{p+1,p+1}(k-i)\}\right).$$

(ii) *If $\mathcal{M}^{v,-}_{1,p+1} = \emptyset$ (this corresponds to the case $v = r$, $p = |D(r)| - 1$ and $r \in S(G)$) we have*

$$Q^v_{1,p+1}(k) = \mathcal{E}\left(\bigcup_{i \in \{0,\dots,k\}} \{(\max\{q_1^+ + q_2^-, q_1^- + q_2^+\}, 0) \mid (q_1^+, q_1^-) \in Q^v_{1,p}(i), (q_2^+, q_2^-) \in Q^v_{p+1,p+1}(k-i)\}\right).$$

Proof. Proposition 5.7 can be proven in a similar way as Proposition 5.6 by using Lemma 5.5. □

Notice that if we are not only interested in sizes of minimum d-transversals but also in the transversals themselves, for every set $Q^v_{a,b}(k)$, we can keep track of a set of edges to remove that corresponds to $Q^v_{a,b}(k)$.

The presented algorithm can easily be adapted to the case of blockers. Wherever we considered a set of maximum matchings we now consider the set of all matchings. In particular the set $\mathcal{M}^{v,+}_{a,b}$ (respectively $\mathcal{M}^{v,-}_{a,b}$) will be replaced by the set of all matchings in G such that v is saturated (respectively not saturated) with an edge of $E^v_{a,b}$. This even simplifies the algorithm, since at each place where we had to distinguish for some vertex v whether $v \in S(G)$ or not, we now always consider the case $v \notin S(G)$, because there is no longer a vertex that has to be saturated by all matchings under consideration. The algorithm runs in exactly the same way as for the case of transversals with slightly simplified rules for performing Operation 1 and 2. Again Propositions 5.6 and 5.7 can be used to perform Operation 1 and 2 where Operation 1 is always performed by applying (ii) of Proposition 5.6 and Operation 2 is performed using always (i) of Proposition 5.7. The correctness of the algorithm for blockers is easily proven by carrying over the arguments used for the case of transversals. See [9] for more information of how the presented algorithm for transversals can be adapted to d-blockers and even generalized to the case of graphs with bounded treewidth.

5.1. Complexity of the algorithm

The main purpose of this section is to show that the proposed algorithm runs in polynomial time. Therefore, a rather conservative but simple complexity analysis will be presented. We begin by discussing the complexity of determining $\tau(G)$ for a tree $G = (V, E)$ satisfying $P_0(G) = P_1(G) = \emptyset$. In a second step the case of arbitrary trees G is discussed.

Notice that every set of the type $Q^v_{a,b}(k)$ contains at most $|E^v_{a,b}| + 1$ elements since $Q^v_{a,b}(k) \subseteq \{0, \dots, |E^v_{a,b}|\}^2$ and every set $A \subset \{0, \dots, |E^v_{a,b}|\}^2$ with $\mathcal{E}(A) = A$ satisfies $|A| \le |E^v_{a,b}| + 1$. Therefore the size of each set $Q^v_{a,b}(k)$ can be bounded by $O(|V|)$.

Operation 1 is called once for every edge, i.e., $|V| - 1$ times. By Proposition 5.6 we have that for some fixed k, $Q^v_{p,p}(k)$ can be determined from $Q^{v_p}(k-1)$ and $Q^{v_p}(k)$ in $O(|V|)$ time. Since Operation 1 determines $Q^v_{p,p}(k)$ for all $k \in \{0, \dots, |E^v_{p,p}|\}$, it can be performed in $O(|V|^2)$ time. Therefore, the total time needed for all calls of Operation 1 can be bounded by $O(|V|^3)$.

For every vertex v, Operation 2 is called $|D(v)| - 2$ times. Therefore the total number of calls of Operation 2 is bounded by $O(|V|)$. By Proposition 5.7 we have that for some fixed k, $Q^v_{1,p+1}(k)$ can be determined from $Q^v_{1,p}$ and $Q^v_{p+1,p+1}$ in $O(|V|^3)$ time. Since Operation 2 determines $Q^v_{1,p+1}(k)$ for all $k \in \{0, \dots, |E^v_{1,p+1}|\}$, it can be performed in $O(|V|^4)$ time. Therefore, the total time needed for all calls of Operation 2 can be bounded by $O(|V|^5)$.

To determine whether $e \in P_0(G)$, we remove the vertices incident to e from G. If the matching number of the remaining graph G' decreases by only one, then $e \notin P_0(G)$ (we can add e to a maximum matching of G' and get a maximum matching of G). Otherwise, $e \in P_0(G)$ (if $e \in M$ for some maximum matching M of G then $M \setminus \{e\}$ would be a maximum matching of G'). Since finding a maximum matching in a tree over n vertices can be done in linear time, the time needed for determining $P_0(G)$ is bounded by $O(|V|^2)$.

Let $G_1, \dots, G_q$ be the connected components of the graph $G \setminus P_0(G)$. We denote by $n_1, \dots, n_q$ their sizes. By the above discussion we can determine for every component G_i, the vector $\tau(G_i)$ in $O(n_i^5)$ time. Finally we have to combine the components to get $\tau(G)$. By Property 5.1, combining two components whose sizes are bounded by n can be done in $O(n^2)$ time. Therefore, the total time needed to find $\tau(G)$ from $\tau(G_1), \dots, \tau(G_q)$ can be done in $O(|V|^3)$ time.

The total time needed for our algorithm to find all minimum d-transversals of a given tree G can therefore be bounded by $O(|V|^5)$. In this analysis Operation 2 is the bottleneck. A more elaborate analysis of the complexity of Operation 2 shows that the time needed for all calls of Operation 2 can be bounded by $O(|V|^4)$ instead of $O(|V|^5)$. However, we expect that the bound can be further sharpened by improving the way Operation 2 is performed.

6. Conclusion

We have determined closed formulas for the minimum size of d-transversals and d-blockers in grid graphs; it may also be interesting to enumerate those subsets in grid graphs.

For the case of trees we have shown how a minimum d-transversal can be found in polynomial time by using dynamic programming. The proposed approach can easily be adapted for finding minimum d-blockers.

An interesting direction of research could be the study of blockers and transversals on planar graphs. For this class of graphs, neither a hardness result nor an efficient algorithm is known to determine blockers and transversals. The hardness result for blockers and transversals that was presented in [10] uses a reduction that typically leads to non-planar graphs. Considering planar graphs is furthermore motivated by the fact that some other hard removal problems of a similar nature can be solved efficiently on planar graphs [9].

Another interesting question would be to study the unicity of such minimum d-blockers and d-transversals. Besides this, one may replace matchings by 2-matchings or more generally by k-matchings (subset of edges such that every vertex is adjacent to at most k edges of the subset). Also in an even more general framework one may consider transversals and blockers of stable sets instead of matchings. Such studies would certainly be motivated by a variety of potential applications in operations research.

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