

# Characterising Chordal Contact $B_0$ -VPG Graphs

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**Abstract.** A graph  $G$  is a  $B_0$ -VPG graph if it is the vertex intersection graph of horizontal and vertical paths on a grid. A graph  $G$  is a *contact  $B_0$ -VPG graph* if the vertices can be represented by interiorly disjoint horizontal or vertical paths on a grid and two vertices are adjacent if and only if the corresponding paths touch. In this paper, we present a minimal forbidden induced subgraph characterisation of contact  $B_0$ -VPG graphs within the class of chordal graphs and provide a polynomial-time algorithm for recognising these graphs.

**Keywords:** Vertex intersection graphs · Contact  $B_0$ -VPG graphs  
Forbidden induced subgraphs · Chordal graphs  
Polynomial-time algorithm

## 1 Introduction

Golumbic et al. introduced in [2] the concept of *vertex intersection graphs of paths on a grid* (referred to as *VPG graphs*). An undirected graph  $G = (V, E)$  is called a VPG graph if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect at at least one grid-point. It is not difficult to see that VPG graphs are equivalent to the well known class of string graphs, i.e., intersection graphs of curves in the plane (see [2]).

A particular attention was paid to the case where the paths have a limited number of *bends* (a bend is a 90 degrees turn of a path at a grid-point). An undirected graph  $G = (V, E)$  is then called a  $B_k$ -VPG graph, for some integer

$k \geq 0$ , if one can associate a path with at most  $k$  bends on a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect at at least one grid-point. Since their introduction in 2012,  $B_k$ -VPG graphs,  $k \geq 0$ , have been studied by many researchers and the community of people working on these graph classes is still growing (see [1, 2, 4–8, 11, 12]).

These classes are shown to have many connections to other, more traditional, graphs classes such as interval graphs (which are clearly  $B_0$ -VPG graphs), planar graphs (recently shown to be  $B_1$ -VPG graphs (see [12])), string graphs (as mentioned above equivalent to VPG graphs), circle graphs (shown to be  $B_1$ -VPG graphs (see [2])) and grid intersection graphs (GIG) (equivalent to bipartite  $B_0$ -VPG graphs (see [2])). Unfortunately, due to these connections, many natural problems are hard for  $B_k$ -VPG graphs. For instance, colouring is NP-hard even for  $B_0$ -VPG graphs and recognition is NP-hard for both VPG and  $B_0$ -VPG graphs [2]. However, there exists a polynomial-time algorithm for deciding whether a given chordal graph is  $B_0$ -VPG (see [4]).

A related notion to intersection graphs are *contact graphs*. Such graphs can be seen as a special type of intersection graphs of geometrical objects in which objects are not allowed to cross but only to touch each other. In the context of VPG graphs, we obtain the following definition. A graph  $G = (V, E)$  is called a *contact VPG graph* if the vertices can be represented by interiorly disjoint paths (i.e., if an intersection occurs between two paths, then it occurs at one of their endpoints) on a grid and two vertices are adjacent if and only if the corresponding paths touch. If we limit again the number of bends per path, we obtain *contact  $B_k$ -VPG graphs*. These graphs have also been considered in the literature (see for instance [5, 9, 13]). It is shown in [9] that every planar bipartite graph is a contact  $B_0$ -VPG graph. Later, in [5], the authors show that every  $K_3$ -free planar graph is a contact  $B_1$ -VPG graph. The authors in [13] consider the special case in which whenever two paths touch on a grid point, this grid point has to be the endpoint of one of the paths and belong to the interior of the other path. It is not difficult to see that in this case, the considered graphs must all be planar.

In this paper, we will consider contact  $B_0$ -VPG graphs and we will present a minimal forbidden induced subgraph characterisation of contact  $B_0$ -VPG graphs restricted to chordal graphs. This characterisation allows us to derive a polynomial-time recognition algorithm for the class of chordal contact  $B_0$ -VPG graphs. Recall that chordal  $B_0$ -VPG graphs can also be recognised in polynomial time (see [4]), even though no structural characterisation of them is known so far. Our results can be considered as a first step to obtain a better understanding of contact  $B_0$ -VPG graphs and their structure.

Our paper is organised as follows. In Sect. 2, we give definitions and notations that we will use throughout the paper. We also present some first observations and results that will be useful in the remaining of the paper. In Sect. 3, we consider chordal graphs and characterise those that are contact  $B_0$ -VPG by minimal forbidden induced subgraphs. Section 4 presents a polynomial-time algorithm for recognising chordal contact  $B_0$ -VPG graphs based on the characterisation mentioned before. Finally, in Sect. 5, we present conclusions and future work.

## 2 Preliminaries

For concepts and notations not defined here we refer the reader to [3]. All graphs that we consider here are simple (i.e., without loops or multiple edges). Let  $G = (V, E)$  be a graph. If  $u, v \in V$  and  $uv \notin E$ ,  $uv$  is called a *nonedge* of  $G$ . We write  $G - v$  for the subgraph obtained by deleting vertex  $v$  and all the edges incident to  $v$ . Similarly, we write  $G - e$  for the subgraph obtained by deleting edge  $e$  without deleting its endpoints.

Given a subset  $A \subseteq V$ ,  $G[A]$  stands for the *subgraph of  $G$  induced by  $A$* , and  $G \setminus A$  denotes the induced subgraph  $G[V \setminus A]$ .

For each vertex  $v$  of  $G$ ,  $N_G(v)$  denotes the *neighbourhood* of  $v$  in  $G$  and  $N_G[v]$  denotes *closed neighbourhood*  $N_G(v) \cup \{v\}$ .

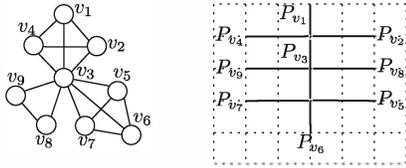
A *clique* is a set of pairwise adjacent vertices. A vertex  $v$  is *simplicial* if  $N_G(v)$  is a clique. A *stable set* is a set of vertices no two of which are adjacent. The *complete graph* on  $n$  vertices corresponds to a clique on  $n$  vertices and is denoted by  $K_n$ .  $nK_1$  stands for a stable set on  $n$  vertices.  $K_4 - e$  stands for the graph obtained from  $K_4$  by deleting exactly one edge.

Given a graph  $H$ , we say that  $G$  *contains no induced  $H$*  if  $G$  contains no induced subgraph isomorphic to  $H$ . If  $\mathcal{H}$  is a family of graphs, a graph  $G$  is said to be  *$\mathcal{H}$ -free* if  $G$  contains no induced subgraph isomorphic to some graph belonging to  $\mathcal{H}$ .

Let  $\mathcal{G}$  be a class of graphs. A graph belonging to  $\mathcal{G}$  is called a  *$\mathcal{G}$ -graph*. If  $G \in \mathcal{G}$  implies that every induced subgraph of  $G$  is a  $\mathcal{G}$ -graph,  $\mathcal{G}$  is said to be *hereditary*. If  $\mathcal{G}$  is a hereditary class, a graph  $H$  is a *minimal forbidden induced subgraph of  $\mathcal{G}$* , or more briefly, *minimally non- $\mathcal{G}$* , if  $H$  does not belong to  $\mathcal{G}$  but every proper induced subgraph of  $H$  is a  $\mathcal{G}$ -graph.

We denote as usual by  $C_n$ ,  $n \geq 3$ , the *chordless cycle* on  $n$  vertices and by  $P_n$  the *chordless path* or *induced path* on  $n$  vertices. A graph is called *chordal* if it does not contain any chordless cycle of length at least four. A *block graph* is a chordal graph which is  $\{K_4 - e\}$ -free.

An undirected graph  $G = (V, E)$  is called a  *$B_k$ -VPG graph*, for some integer  $k \geq 0$ , if one can associate a path with at most  $k$  *bends* (a bend is a  $90^\circ$  turn of a path at a grid-point) on a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect at at least one grid-point. Such a representation is called a  *$B_k$ -VPG representation*. The horizontal grid lines will be referred to as *rows* and denoted by  $x_0, x_1, \dots$  and the vertical grid lines will be referred to as *columns* and denoted by  $y_0, y_1, \dots$ . We are interested in a subclass of  $B_0$ -VPG graphs called contact  $B_0$ -VPG. A *contact  $B_0$ -VPG representation*  $\mathcal{R}(G)$  of  $G$  is a  $B_0$ -VPG representation in which each path in the representation is either a horizontal path or a vertical path on the grid, such that two vertices are adjacent if and only if the corresponding paths intersect at at least one grid-point without crossing each other and without sharing an edge of the grid. A graph is a *contact  $B_0$ -VPG graph* if it has a contact  $B_0$ -VPG representation. For every vertex  $v \in V(G)$ , we denote by  $P_v$  the corresponding path in  $\mathcal{R}(G)$  (see Fig. 1). Consider a clique  $K$  in  $G$ . A path  $P_v$  representing a vertex  $v \in K$  is called a *path of the clique  $K$* .



**Fig. 1.** A graph  $G$  and a contact  $B_0$ -VPG representation.

Let us start with some easy but very helpful observations.

**Observation 1.** *Let  $G$  be a contact  $B_0$ -VPG graph. Then the size of a biggest clique in  $G$  is at most 4, i.e.,  $G$  is  $K_5$ -free.*

Let  $G$  be a contact  $B_0$ -VPG graph, and  $K$  be a clique in  $G$ . A vertex  $v \in K$  is called an *end* in a contact  $B_0$ -VPG representation of  $K$ , if the grid-point representing the intersection of the paths of the clique  $K$  corresponds to an endpoint of  $P_v$ .

**Observation 2.** *Let  $G$  be a contact  $B_0$ -VPG graph, and  $K$  be a clique in  $G$  of size four. Then, every vertex in  $K$  is an end in any contact  $B_0$ -VPG representation of  $K$ .*

Next we will show certain graphs that are not contact  $B_0$ -VPG graphs and that will be part of our characterisation. Let  $H_0$  denote the graph composed of three  $K_4$ 's that share a common vertex and such that there are no other edges (see Fig. 3).

**Lemma 1.** *If  $G$  is a contact  $B_0$ -VPG graph, then  $G$  is  $\{K_5, H_0, K_4-e\}$ -free.*

*Proof.* Let  $G$  be a contact  $B_0$ -VPG graph. It immediately follows from Observation 1 that  $G$  is  $K_5$ -free.

Now let  $v, w$  be two adjacent vertices in  $G$ . Then, in any contact  $B_0$ -VPG representation of  $G$ ,  $P_v$  and  $P_w$  intersect at a grid-point  $P$ . Clearly, every common neighbour of  $v$  and  $w$  must also contain  $P$ . Hence,  $v$  and  $w$  cannot have two common neighbours that are non-adjacent. So,  $G$  is  $\{K_4-e\}$ -free.

Finally, consider the graph  $H_0$  which consists of three cliques of size four, say  $A$ ,  $B$  and  $C$ , with a common vertex  $x$ . Suppose that  $H_0$  is contact  $B_0$ -VPG. Then, it follows from Observation 2 that every vertex in  $H_0$  is an end in any contact  $B_0$ -VPG representation of  $H_0$ . In particular, vertex  $x$  is an end in any contact  $B_0$ -VPG representation of  $A$ ,  $B$  and  $C$ . In other words, the grid-point representing the intersection of the paths of each of these three cliques corresponds to an endpoint of  $P_x$ . Since these cliques have only vertex  $x$  in common, these grid-points are all distinct. But this is a contradiction, since  $P_x$  has only two endpoints. So we conclude that  $H_0$  is not contact  $B_0$ -VPG, and hence the result follows.  $\square$

### 3 Chordal Graphs

In this section, we will consider chordal graphs and characterise those that are contact  $B_0$ -VPG. First, let us point out the following important observation.

**Observation 3.** *A chordal contact  $B_0$ -VPG graph is a block graph.*

This follows directly from Lemma 1 and the definition of block graphs.

The following lemma states an important property of minimal chordal non contact  $B_0$ -VPG graphs that contain neither  $K_5$  nor  $K_4-e$ .

**Lemma 2.** *Let  $G$  be a chordal  $\{K_5, K_4-e\}$ -free graph. If  $G$  is a minimal non contact  $B_0$ -VPG graph, then every simplicial vertex of  $G$  has degree exactly three.*

*Proof.* Since  $G$  is  $K_5$ -free, every clique in  $G$  has size at most four. Therefore, every simplicial vertex has degree at most three. Let  $v$  be a simplicial vertex of  $G$ . Assume first that  $v$  has degree one and consider a contact  $B_0$ -VPG representation of  $G - v$  (which exists since  $G$  is minimal non contact  $B_0$ -VPG). Let  $w$  be the unique neighbour of  $v$  in  $G$ . Without loss of generality, we may assume that the path  $P_w$  lies on some row of the grid. Now clearly, we can add one extra column to the grid between any two consecutive vertices of the grid belonging to  $P_w$  and adapt all paths without changing the intersections (if the new column is added between column  $y_i$  and  $y_{i+1}$ , we extend all paths containing a grid-edge with endpoints in column  $y_i$  and  $y_{i+1}$  in such a way that they contain the new edges in the same row and between column  $y_i$  and  $y_{i+2}$  of the new grid, and any other path remains the same). But then we may add a path representing  $v$  on this column which only intersects  $P_w$  (adding a row to the grid and adapting the paths again, if necessary) and thus, we obtain a contact  $B_0$ -VPG representation of  $G$ , a contradiction. So suppose now that  $v$  has degree two, and again consider a contact  $B_0$ -VPG representation of  $G - v$ . Let  $w_1, w_2$  be the two neighbours of  $v$  in  $G$ . Then,  $w_1, w_2$  do not have any other common neighbour since  $G$  is  $\{K_4-e\}$ -free. Let  $P$  be the grid-point corresponding to the intersection of the paths  $P_{w_1}$  and  $P_{w_2}$ . Since these paths do not cross and since  $w_1, w_2$  do not have any other common neighbour (except  $v$ ), there is at least one grid-edge having  $P$  as one of its endpoints and which is not used by any path of the representation. But then we may add a path representing  $v$  by using only this particular grid-edge (or adding a row/column to the grid that subdivides this edge and adapting the paths, if the other endpoint of the grid-edge belongs to a path in the representation). Thus, we obtain a contact  $B_0$ -VPG representation of  $G$ , a contradiction. We conclude therefore that  $v$  has degree exactly three.  $\square$

Let  $v$  be a vertex of a contact  $B_0$ -VPG graph  $G$ . An endpoint of its corresponding path  $P_v$  is *free* in a contact  $B_0$ -VPG representation of  $G$ , if  $P_v$  does not intersect any other path at that endpoint;  $v$  is called *internal* if there exists no representation of  $G$  in which  $P_v$  has a free endpoint. If in a representation of  $G$  a path  $P_v$  intersects a path  $P_w$  but not at an endpoint of  $P_w$ ,  $v$  is called a *middle neighbour* of  $w$ .

In the following two lemmas, we associate the fact of being or not an internal vertex of  $G$  with the contact  $B_0$ -VPG representation of  $G$ .

Due to lack of space, the proof of the following lemma is omitted.

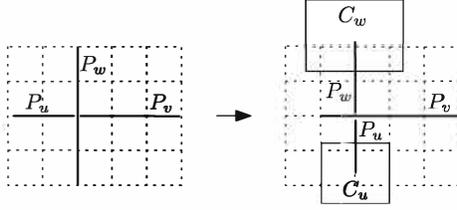
**Lemma 3.** *Let  $G$  be a chordal contact  $B_0$ -VPG graph and let  $v$  be a non internal vertex in  $G$ . Then, there exists a contact  $B_0$ -VPG representation of  $G$  in which all the paths representing vertices in  $G - v$  lie to the left of a free endpoint of  $P_v$  (by considering  $P_v$  as a horizontal path).*

**Lemma 4.** *Let  $G$  be a chordal contact  $B_0$ -VPG graph. A vertex  $v$  in  $G$  is internal if and only if in every contact  $B_0$ -VPG representation of  $G$ , each endpoint of the path  $P_v$  either corresponds to the intersection of a representation of  $K_4$  or intersects a path  $P_w$ , which represents an internal vertex  $w$ , but not at an endpoint of  $P_w$ .*

*Proof.* The if part is trivial. Assume now that  $v$  is an internal vertex of  $G$  and consider an arbitrary contact  $B_0$ -VPG representation of  $G$ . Let  $P$  be an endpoint of the path  $P_v$  and  $K$  the maximal clique corresponding to all the paths containing the point  $P$ . Notice that clearly  $v$  is an end in  $K$  by definition of  $K$ . First, suppose there is a vertex  $w$  in  $K$  which is not an end. Then, it follows from Observation 2 that the size of  $K$  is at most three. Without loss of generality, we may assume that  $P_v$  lies on some row and  $P_w$  on some column. If  $w$  is an internal vertex, we are done. So we may assume now that  $w$  is not an internal vertex in  $G$ . Consider  $G \setminus (K \setminus \{w\})$ , and let  $C_w$  be the connected component of  $G \setminus (K \setminus \{w\})$  containing  $w$ . Notice that  $w$  is not an internal vertex in  $C_w$  either. By Lemma 3, there exists a contact  $B_0$ -VPG representation of  $C_w$  with all the paths lying to the left of a free endpoint of  $P_w$ . Now, replace the old representation of  $C_w$  by the new one such that  $P$  corresponds to the free endpoint of  $P_w$  in the representation of  $C_w$  (it might be necessary to refine –by adding rows and/or columns– the grid to ensure that there are no unwanted intersections) and  $P_w$  uses the same column as before. Finally, if  $K$  had size three, say it contains some vertex  $u$  in addition to  $v$  and  $w$ , then we proceed as follows. Similar to the above, there exists a contact  $B_0$ -VPG representation of  $C_u$ , the connected component of  $G \setminus (K \setminus \{u\})$  containing  $u$ , with all the paths lying to the left of a free endpoint of  $P_u$ , since  $u$  is clearly not internal in  $C_u$ . We then replace the old representation of  $C_u$  by the new one such that the endpoint of  $P_u$  that intersected  $P_w$  previously corresponds to the grid-point  $P$  and  $P_u$  lies on the same column as  $P_w$  (again, we may have to refine the grid). This clearly gives us a contact  $B_0$ -VPG representation of  $G$ . But now we may extend  $P_v$  such that it strictly contains the grid-point  $P$  and thus,  $P_v$  has a free endpoint, a contradiction (see Fig. 2). So  $w$  must be an internal vertex.

Now, assume that all vertices in  $K$  are ends. If  $|K| = 4$ , we are done. So we may assume that  $|K| \leq 3$ . Hence, there is at least one grid-edge containing  $P$ , which is not used by any paths of the representation. Without loss of generality, we may assume that this grid-edge belongs to some row  $x_i$ . If  $P_v$  is horizontal, we may extend it such that it strictly contains  $P$ . But then  $v$  is not internal anymore,

a contradiction. If  $P_v$  is vertical, then we may extend  $P_w$ , where  $w \in K$  is such that  $P_w$  is a horizontal path. But now we are again in the first case discussed above.  $\square$



**Fig. 2.** Figure illustrating Lemma 4.

In other words, Lemma 4 tells us that a vertex  $v$  is an internal vertex in a chordal contact  $B_0$ -VPG graph if and only if we are in one of the following situations:

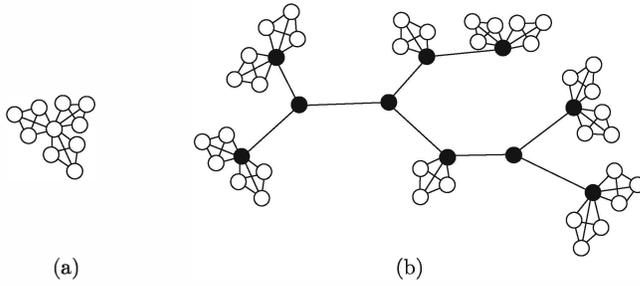
- $v$  is the intersection of two cliques of size four (we say that  $v$  is of type 1);
- $v$  belongs to exactly one clique of size four and in every contact  $B_0$ -VPG representation,  $v$  is a middle neighbour of some internal vertex (we say that  $v$  is of type 2);
- $v$  does not belong to any clique of size four and in every contact  $B_0$ -VPG representation,  $v$  is a middle neighbour of two internal vertices (we say that  $v$  is of type 3).

Notice that two internal vertices of type 1 cannot be adjacent (except when they belong to a same  $K_4$ ). Furthermore, an internal vertex of type 1 cannot be the middle-neighbour of some other vertex.

Let  $\mathcal{T}$  be the family of graphs defined as follows.  $\mathcal{T}$  contains  $H_0$  (see Fig. 3) as well as all graphs constructed in the following way: start with a tree of maximum degree at most three and containing at least two vertices; this tree is called the *base tree*; add to every leaf  $v$  in the tree two copies of  $K_4$  (sharing vertex  $v$ ), and to every vertex  $w$  of degree two one copy of  $K_4$  containing vertex  $w$  (see Fig. 3). Notice that all graphs in  $\mathcal{T}$  are chordal.

**Lemma 5.** *The graphs in  $\mathcal{T}$  are not contact  $B_0$ -VPG.*

*Proof.* By Lemma 1, the graph  $H_0$  is not contact  $B_0$ -VPG. Consider now a graph  $T \in \mathcal{T}$ ,  $T \neq H_0$ . Suppose that  $T$  is contact  $B_0$ -VPG. Denote by  $B(T)$  the base tree of  $T$  and consider an arbitrary contact  $B_0$ -VPG representation of  $T$ . Consider the base tree  $B(T)$  and direct an edge  $uv$  of it from  $u$  to  $v$  if the path  $P_v$  contains an endpoint of the path  $P_u$  (this way some edges might be directed both ways). If a vertex  $v$  has degree  $d_B(v)$  in  $B(T)$ , then by definition of the family  $\mathcal{T}$ ,  $v$  belongs to  $3 - d_B(v)$   $K_4$ 's in  $T$ . Notice that  $P_v$  spends one endpoint in each of



**Fig. 3.** (a) The graph  $H_0$ . (b) An example of a graph in  $\mathcal{T}$ ; the bold vertices belong to the base tree.

these  $K_4$ 's. Thus, any vertex  $v$  in  $B(T)$  has at most  $2 - (3 - d_B(v)) = d_B(v) - 1$  outgoing edges. This implies that the sum of out-degrees in  $B(T)$  is at most  $\sum_{v \in B(T)} (d_B(v) - 1) = n - 2$ , where  $n$  is the number of vertices in  $B(T)$ . But this is clearly impossible since there are  $n - 1$  edges in  $B(T)$  and all edges are directed.  $\square$

Using Lemmas 2–5, we are able to prove the following theorem, which provides a minimal forbidden induced subgraph characterisation of chordal contact  $B_0$ -VPG graphs.

**Theorem 4.** *Let  $G$  be a chordal graph. Let  $\mathcal{F} = \mathcal{T} \cup \{K_5, K_4-e\}$ . Then,  $G$  is a contact  $B_0$ -VPG graph if and only if  $G$  is  $\mathcal{F}$ -free.*

*Proof.* Suppose that  $G$  is a chordal contact  $B_0$ -VPG graph. It follows from Lemmas 1 and 5 that  $G$  is  $\mathcal{T}$ -free and contains neither a  $K_4-e$  nor a  $K_5$ .

Conversely, suppose now that  $G$  is chordal and  $\mathcal{F}$ -free. By contradiction, suppose that  $G$  is not contact  $B_0$ -VPG and assume furthermore that  $G$  is a minimal non contact  $B_0$ -VPG graph. Let  $v$  be a simplicial vertex of  $G$  ( $v$  exists since  $G$  is chordal). By Lemma 2, it follows that  $v$  has degree three. Consider a contact  $B_0$ -VPG representation of  $G - v$  and let  $K = \{v_1, v_2, v_3\}$  be the set of neighbours of  $v$  in  $G$ . Since  $G$  is  $\{K_4-e\}$ -free, it follows that any two neighbours of  $v$  cannot have a common neighbour which is not in  $K$ . First suppose that all the vertices in  $K$  are ends in the representation of  $G - v$ . Thus, there exists a grid-edge not used by any path and which has one endpoint corresponding to the intersection of the paths  $P_{v_1}, P_{v_2}, P_{v_3}$ . But now we may add the path  $P_v$  using exactly this grid-edge (we may have to add a row/column to the grid that subdivides this grid-edge and adapt the paths, if the other endpoint of the grid-edge belongs to a path in the representation). Hence, we obtain a contact  $B_0$ -VPG representation of  $G$ , a contradiction.

Thus, we may assume now that there exists a vertex in  $K$  which is not an end, say  $v_1$ . Notice that  $v_1$  must be an internal vertex. If not, there is a contact  $B_0$ -VPG representation of  $G - v$  in which  $v_1$  has a free end. Then, using similar arguments as in the proof of Lemma 4, we may obtain a representation

of  $G - v$  in which all vertices of  $K$  are ends. As described previously, we can add  $P_v$  to obtain a contact  $B_0$ -VPG representation of  $G$ , a contradiction. Now, by Lemma 4,  $v_1$  must be of type 1, 2 or 3. Let us first assume that  $v_1$  is of type 1. But then  $v_1$  is the intersection of three cliques of size 4 and thus,  $G$  contains  $H_0$ , a contradiction. So  $v_1$  is of type 2 or 3. But this necessarily implies that  $G$  contains a graph  $T \in \mathcal{T}$ . Indeed, if  $v_1$  is of type 2, then  $v_1$  corresponds to a leaf in  $B(T)$  (remember that  $v_1$  already belongs to a  $K_4$  containing  $v$  in  $G$ ); if  $v_1$  is of type 3, then  $v_1$  corresponds to a vertex of degree two in the base tree of  $T$ . Now, use similar arguments for an internal vertex  $w$  adjacent to  $v_1$  and for which  $v_1$  is a middle neighbour: if  $w$  is of type 2, then it corresponds to a vertex of degree two in  $B(T)$ ; if  $w$  is of type 3, then it corresponds to a vertex of degree three in  $B(T)$ ; if  $w$  is of type 1, it corresponds to a leaf of  $B(T)$ . In this last case, we stop. In the other two cases, we simply repeat the arguments for an internal vertex adjacent to  $w$  and for which  $w$  is a middle neighbour. We continue this process until we find an internal vertex of type 1 in the procedure which then gives us, when all vertices of type 1 are reached, a graph  $T \in \mathcal{T}$ . Since  $G$  is finite, we are sure to find such a graph  $T$ .  $\square$

Interval graphs form a subclass of chordal graphs. They are defined as being chordal graphs not containing any asteroidal triple, i.e., not containing any three pairwise non adjacent vertices such that there exists a path between any two of them avoiding the neighbourhood of the third one. Clearly, any graph in  $\mathcal{T}$  for which the base tree has maximum degree three contains an asteroidal triple. On the other hand,  $H_0$  and every graph in  $\mathcal{T}$  obtained from a base tree of maximum degree at most two are clearly interval graphs. Denote by  $\mathcal{T}'$  the family consisting of  $H_0$  and the graphs of  $\mathcal{T}$  whose base tree has maximum degree at most two. We obtain the following corollary which provides a minimal forbidden induced subgraph characterisation of contact  $B_0$ -VPG graphs restricted to interval graphs.

**Corollary 1.** *Let  $G$  be an interval graph and  $\mathcal{F}' = \mathcal{T}' \cup \{K_5, K_4-e\}$ . Then,  $G$  is a contact  $B_0$ -VPG graph if and only if  $G$  is  $\mathcal{F}'$ -free.*

## 4 Recognition Algorithm

In this section, we will provide a polynomial-time recognition algorithm for chordal contact  $B_0$ -VPG graphs which is based on the characterisation given in Sect. 3. This algorithm takes a chordal graph as input and returns YES if the graph is contact  $B_0$ -VPG and, if not, it returns NO as well as a forbidden induced subgraph. We will first give the pseudo-code of our algorithm and then explain the different steps.

**Input:** a chordal graph  $G = (V, E)$ ;

**Output:** YES, if  $G$  is contact  $B_0$ -VPG; NO and a forbidden induced subgraph, if  $G$  is not contact  $B_0$ -VPG.

1. list all maximal cliques in  $G$ ;
2. if some edge belongs to two maximal cliques, return NO and  $K_4 - e$ ;
3. if a maximal clique contains at least five vertices, return NO and  $K_5$ ;
4. label the vertices such that  $l(v) =$  number of  $K_4$ 's that  $v$  belongs to;
5. if for some vertex  $v$ ,  $l(v) \geq 3$ , return NO and  $H_0$ ;
6. if  $l(v) \leq 1 \forall v \in V \setminus \{w\}$  and  $l(w) \leq 2$ , return YES;
7. while there exists an unmarked vertex  $v$  with  $2 - l(v)$  outgoing arcs incident to it, do
  - 7.1 mark  $v$  as internal;
  - 7.2 direct the edges that are currently undirected, uncoloured, not belonging to a  $K_4$ , and incident to  $v$  towards  $v$ ;
  - 7.3 for any two incoming arcs  $wv, w'v$  such that  $ww' \in E$ , colour  $ww'$ ;
8. if there exists some vertex  $v$  with more than  $2 - l(v)$  outgoing arcs, return NO and find  $T \in \mathcal{T}$ ; else return YES.

Steps 1–5 can clearly be done in polynomial time (see for example [10] for listing all maximal cliques in a chordal graph). Furthermore, it is obvious to see how to find the forbidden induced subgraph in steps 2, 3 and 5. Notice that if the algorithm has not returned NO after step 5, we know that  $G$  is  $\{K_4 - e, K_5, H_0\}$ -free. So we are left with checking whether  $G$  contains some graph  $T \in \mathcal{T}$ ,  $T \neq H_0$ . Since each graph  $T \in \mathcal{T}$  contains at least two vertices belonging to two  $K_4$ 's, it follows that if at most one vertex has label 2,  $G$  is  $\mathcal{T}$ -free (step 6), and thus we conclude by Theorem 4 that  $G$  is contact  $B_0$ -VPG.

During step 7, we detect those vertices in  $G$  that, in case  $G$  is contact  $B_0$ -VPG, must be internal vertices (and mark them as such) and those vertices  $w$  that are middle neighbours of internal vertices  $v$  (we direct the edges  $wv$  from  $w$  to  $v$ ). Furthermore, we colour those edges whose endpoints are middle neighbours of a same internal vertex.

Consider a vertex  $v$  with  $2 - l(v)$  outgoing arcs. If a vertex  $v$  has  $l(v) = 2$ , then, in case  $G$  is contact  $B_0$ -VPG,  $v$  must be an internal vertex (see Lemma 4). This implies that any neighbour of  $v$ , which does not belong to a same  $K_4$  as  $v$ , must be a middle neighbour of  $v$ . If  $l(v) = 1$ , this means that  $v$  belongs to one  $K_4$  and is a middle neighbour of some internal vertex. Thus, by Lemma 4 we know that  $v$  is internal. Similarly, if  $l(v) = 0$ , this means that  $v$  is a middle neighbour of two distinct internal vertices. Again, by Lemma 4 we conclude that  $v$  is internal. Clearly, step 7 can be run in polynomial time.

So we are left with step 8, i.e., we need to show that  $G$  is contact  $B_0$ -VPG if and only if there exists no vertex with more than  $2 - l(v)$  outgoing arcs. First notice that only vertices marked as internal have incoming arcs. Furthermore, notice that every maximal clique of size three containing an internal vertex has two directed edges of the form  $wv, w'v$  and the third edge is coloured, where  $v$  is the first of the three vertices that was marked as internal. This is because the graph is  $(K_4 - e)$ -free and the edges of a  $K_4$  are neither directed nor coloured.

Thanks to the marking process described in step 7 and the fact that only vertices marked as internal have incoming arcs, we can make the following observation.

**Observation 5.** *Every vertex marked as internal in step 7 has either label 2 or is the root of a directed induced tree (directed from the root to the leaves) where the root  $w$  has degree  $2 - l(w)$  and every other vertex  $v$  has degree  $3 - l(v)$  in that tree, namely one incoming arc and  $2 - l(v)$  outgoing arcs.*

Let us show that the tree mentioned in the previous observation is necessarily induced. Suppose there is an edge not in the tree that joins two vertices of the tree. Since the graph is a block graph, the vertices in the resulting cycle induce a clique, so in particular there is a triangle formed by two edges of the tree and an edge not in the tree. But, as observed above, in every triangle of  $G$  having two directed edges, the edges point to the same vertex (and the third edge is coloured, not directed). Since no vertex in the tree has in-degree more than one, this is impossible.

Based on the observation, it is clear now that if a vertex has more than  $2 - l(v)$  outgoing arcs, then that vertex is the root of a directed induced tree (directed from the root to the leaves), where every vertex  $v$  has degree  $3 - l(v)$ , i.e., a tree that is the base tree  $B(T)$  of a graph  $T \in \mathcal{T}$ . Indeed, notice that every vertex  $v$  in a base tree has degree  $3 - l(v)$ . The fact that tree is induced can be proved the same way as above. This base tree can be found by a breadth-first search from a vertex having out-degree at least  $3 - l(v)$ , using the directed edges. Thanks to the labels, representing the number of  $K_4$ 's a vertex belongs to, it is then possible to extend the  $B(T)$  to an induced subgraph  $T \in \mathcal{T}$ . This can clearly be implemented to run in polynomial time.

To finish the proof that our algorithm is correct, it remains to show that if  $G$  contains an induced subgraph in  $\mathcal{T}$ , then the algorithm will find a vertex with at least  $3 - l(v)$  outgoing arcs. This, along with Theorem 4, says that if the algorithm outputs YES then the graph is contact  $B_0$ -VPG (given that the detection of  $K_5$ ,  $K_4-e$  and  $H_0$  is clear). Recall that we know that  $G$  is a block graph after step 2. Notice that if a block of size 2 in a graph of  $\mathcal{T}$  is replaced by a block of size 4, we obtain either  $H_0$  or a smaller graph in  $\mathcal{T}$  as an induced subgraph. Moreover, adding an edge to a graph of  $\mathcal{T}$  in such a way that now contains a triangle, then we obtain a smaller induced graph in  $\mathcal{T}$ . Let  $G$  be a block graph with no induced  $K_5$  or  $H_0$ . By the observation above, if  $G$  contains a graph in  $\mathcal{T}$  as induced subgraph, then  $G$  contains one, say  $T$ , such that no edge of the base tree  $B(T)$  is contained in a  $K_4$  in  $G$ , and no triangle of  $G$  contains two edges of  $B(T)$ . So, all the edges of  $B(T)$  are candidates to be directed or coloured.

In fact, by step 7 of the algorithm, every vertex of  $B(T)$  is eventually marked as internal, and every edge incident with it is either directed or coloured, unless the algorithm ends with answer NO before. Notice that by the observation about the maximal cliques of size three and the fact that no triangle of  $G$  contains two edges of  $B(T)$ , if an edge  $vw$  of  $B(T)$  is coloured, then both  $v$  and  $w$  have an

outgoing arc not belonging to  $B(T)$ . So, in order to obtain a lower bound on the out-degrees of the vertices of  $B(T)$  in  $G$ , we can consider only the arcs of  $B(T)$  and we can consider the coloured edges as bidirected edges. With an argument similar to the one in the proof of Lemma 5, at least one vertex has out-degree at least  $3 - l(v)$ .

## 5 Conclusions and Future Work

In this paper, we presented a minimal forbidden induced subgraph characterisation of chordal contact  $B_0$ -VPG graphs and provide a polynomial-time recognition algorithm based on that characterisation. In order to obtain a better understanding of what contact  $B_0$ -VPG graphs look like, the study of contact  $B_0$ -VPG graphs within other graph classes is needed. It would also be interesting to investigate contact  $B_0$ -VPG graph from an algorithmic point of view and analyse for instance the complexity of the colouring problem or the stable set problem in that graph class.

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