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Mixed graph edge coloring[☆]

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ABSTRACT

We are interested in coloring the edges of a mixed graph, i.e., a graph containing unoriented and oriented edges. This problem is related to a communication problem in job-shop scheduling systems. In this paper we give general bounds on the number of required colors and analyze the complexity status of this problem. In particular, we provide $\mathcal{NP}$-completeness results for the case of outerplanar graphs, as well as for 3-regular bipartite graphs (even when only 3 colors are allowed, or when 5 colors are allowed and the graph is fully oriented). Special cases admitting polynomial-time solutions are also discussed.

1. Introduction

A mixed graph $G_M = (V, U, E)$ is a graph containing unoriented edges (set E) as well as oriented edges (set U), referred to as *arcs*. This notion was first introduced in [18].

1.1. Related work: Vertex coloring of mixed graphs

Vertex coloring problems in mixed graphs have applications in scheduling, where disjunctive and precedence constraints have to be taken into account simultaneously. In particular, two variants of the problem have been given most attention in the literature (see for instance [18,1,2,4–6,8,11,13–17,19]).

In the first problem, simply called *mixed graph vertex coloring*, the goal is to color the vertices of a mixed graph with a given number of colors, such that any two adjacent vertices get different colors, and for any arc (x, y), the color of x must be strictly smaller than the color of y. Notice that a solution only may exist if the oriented part of the mixed graph contains no oriented circuit. Furthermore, the mixed graph vertex coloring problem is a generalization of the usual coloring problem in unoriented graphs, and it has been shown to be $\mathcal{NP}$-complete even in planar cubic bipartite graphs (see [14]). In [4–6] polynomial algorithms are given for the cases of mixed trees and mixed series-parallel graphs. Bounds on the mixed chromatic number (i.e., the smallest integer for which the mixed graph admits a coloring) are presented in [15]. Finally,

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in [16,17] the unit-time job-shop problem is considered via mixed graph coloring, and branch-and-bound algorithms are given and tested on randomly generated mixed graphs.

In the second problem, known as *weak mixed graph vertex coloring*, we have the previous constraints, but we allow vertices linked by an arc to get the same color, i.e., for any arc (x, y) the color of x must be smaller than or equal to the color of y. In general, the previously described mixed graph vertex coloring problem can be treated as a special case of the weak version. Weak mixed graph vertex coloring is also known to be $\mathcal{NP}$-complete in planar cubic bipartite graphs [14]. Bounds on the mixed chromatic number are presented in [15]. In [19] some algorithms for calculating the exact value of the weak mixed chromatic number of graphs of order $n \leq 40$ and upper bounds for mixed graphs of order larger than 40, are presented.

1.2. Problem formulation and motivation: Edge coloring of mixed graphs

In this paper, we shall consider an edge coloring problem in mixed graphs. More precisely, we want to color the edges of a mixed graph $G_M = (V, U, E)$ such that any two adjacent edges (oriented and unoriented) get different colors and for any two adjacent arcs $e, e' \in U$ forming a directed path (e, e'), the color of e must be strictly less than the color of e'. Such a coloring will be called a *mixed graph edge coloring*. If only k colors are available, we call it a *mixed graph edge k-coloring*. The smallest integer k for which a graph G_M admits a mixed graph edge k-coloring will be called the *mixed chromatic index* of G_M and denoted by $q_M(G_M)$. Notice that for a solution to the mixed graph edge coloring problem to exist, the mixed graph must not contain any oriented circuit. Throughout the rest of the paper we shall assume that this is true. To the best of our knowledge, the mixed graph edge coloring problem has not been studied before; some basic properties and the motivation of the problem are discussed below.

Mixed graph edge coloring can be treated as a special case of mixed graph vertex coloring. For a mixed graph $G_M = (V, U, E)$, we define its *mixed line graph* $L(G_M)$ as the mixed graph having vertex set $U \cup E$, arcs (e, e') connecting all pairs of elements $e, e' \in U$ such that arc e ends at the start-vertex of arc e', and unoriented edges connecting all the remaining pairs of elements of $U \cup E$ which share at least one vertex. By analogy to the correspondence between an edge coloring of an undirected graph and a vertex coloring of its line graph, it is evident that a mixed graph edge coloring of G_M is proper if, and only if, the corresponding labeling of the vertices of $L(G_M)$ is a proper mixed graph vertex coloring.

Edge coloring of undirected graphs is often used to model certain job-shop scheduling instances consisting of unit-time tasks assigned to specific pairs of processors [10]. In the case of mixed graphs, it is convenient to look upon an arc from a node u to a node v as a unit-time data transmission process from u to v, requiring the cooperation of processors u and v, which cannot simultaneously perform other tasks. Thus, a correct coloring of the directed arcs of the graph corresponds to a scheduling in which each node first successively receives input data from all incoming arcs, next uses all the collected data for local computations (assumed to be instantaneous), and finally successively sends the output data along all its outgoing arcs. The undirected edges of the mixed graph, which only appear in some considerations, correspond to possibly unrelated two-processor tasks performed in the system, such as mutual self-testing of processors.

1.3. Definitions and notions

Let $G_M = (V, U, E)$ be a mixed graph. We shall denote by $l(G_M)$ the number of oriented edges on a longest directed path in G_M, and by $\Delta(G_M)$ the maximum degree of a vertex in G_M, i.e., the maximum number of edges (unoriented and oriented) incident to a same vertex $v \in V$. The outer degree of a vertex v, denoted by $\deg_{out}(v)$, is defined as the number of oriented edges (arcs) having v as the start-vertex; analogously, the inner degree of v, denoted by $\deg_{in}(v)$, is defined as the number of oriented edges (arcs) with v as the end-vertex. Finally, the *inrank* of a vertex v, denoted by $in(v)$, is the length, i.e., the number of arcs, of a longest directed path ending at v. For all graph theoretical terms not defined here, the reader is referred to [3].

1.4. Contribution and outline of the paper

The rest of the paper is organized as follows:

- In Section 2 we propose lower and upper bounds on the value of the mixed chromatic index of a graph G_M, expressed in terms of $l(G_M)$ and $\Delta(G_M)$. Interestingly, for all cases which are not equivalent to undirected edge coloring ($l(G_M) > 1$), these bounds turn out to be tight, even when G_M is a mixed tree having only oriented edges.
- In Section 3 we study the complexity of the problem *MGEC* of determining if $q_M(G_M) \leq k$ for a mixed graph G_M and integer k given at input. The problem turns out to be $\mathcal{NP}$-complete even if G_M is a bipartite outerplanar graph (Section 3), but it admits a polynomial solution for trees (Section 3.2).
- In Section 4 we consider the edge coloring problem for fully directed graphs (i.e., mixed graphs without unoriented edges, $E = \emptyset$), which appears to be of particular significance from a practical viewpoint. In this case, *MGEC* is shown to be $\mathcal{NP}$-complete even in 3-regular bipartite graphs when allowing 5 colors.
- Finally, we consider the complexity of the *MGEC* problem for bounded values of number of available colors (Section 5). By a generic argument, the problem is then solvable in polynomial time for partial k-trees, but turns out to be $\mathcal{NP}$-complete even in 3-regular planar bipartite mixed graphs when allowing 3 colors.

2. Bounds on the mixed chromatic index

In this section we present some lower and upper bounds on the mixed chromatic index $q_M(G_M)$. We start with a trivial lower bound.

Proposition 2.1. *Let $G_M = (V, U, E)$ be a mixed graph. Then*

$$q_M(G_M) \geq \max\{l(G_M), \Delta(G_M)\}.$$

Proof. Since all arcs on the same path must be assigned different colors, we have $q_M(G_M) \geq l(G_M)$. Furthermore, all edges incident to a same vertex must get different colors. Hence $q_M(G_M) \geq \Delta(G_M)$, and thus $q_M(G_M) \geq \max\{l(G_M), \Delta(G_M)\}$. □

Now, in order to get an upper bound on the mixed chromatic index $q_M(G_M)$, we will give an algorithm which colors the edges of a mixed graph.

Proposition 2.2. *Let $G_M = (V, U, E)$ be a mixed graph. Then*

$$q_M(G_M) \leq \begin{cases} l(G_M)[\Delta(G_M) - 1] + 1 & \text{if } l(G_M) \geq 2, \\ \Delta(G_M) + 1 & \text{if } l(G_M) \leq 1. \end{cases}$$

Proof. If $l(G_M) \leq 1$, then we just consider all oriented edges as unoriented edges, and thus our problem is equivalent to the edge coloring problem of an unoriented graph. By Vizing's theorem (see [20]), at most $\Delta(G_M) + 1$ colors are needed.

Suppose now that $l(G_M) \geq 2$. For each $i \in \{0, 1, \ldots, l(G_M) - 1\}$, let U_i be the set of arcs having a start-vertex v with inrank $in(v) = i$ and an end-vertex u with inrank $in(u) = i + 1$. Consider now the oriented partial graph $G_M^o = (V, \bigcup_{i=0}^{l(G_M)-1} U_i, \emptyset)$ of G_M having arcs connecting vertices of adjacent inrank. G_M^o is clearly bipartite, with vertices of even and odd inrank forming its two bipartite partitions, respectively. Moreover, since the maximum degree in G_M^o is at most $\Delta = \Delta(G_M)$, we conclude by König's theorem [9] that we can color the arcs of G_M^o by using at most Δ colors in such a way that any two adjacent arcs get different colors (at this step, we do not care about the precedence constraints). Let us denote by c this coloring with $c(e) \in \{1, \ldots, \Delta\}$, $\forall e \in \bigcup_{i=0}^{l(G_M)-1} U_i$. Now, the idea is to modify c so as to obtain a coloring c' also respecting the precedence constraints.

The coloring c' is constructed as follows. Each arc $e \in U_i$ gets color $c'(e) = 1 + (\Delta - 1)i + [(c(e) + i - 1) \bmod \Delta]$, where $c(e)$ is the color of e obtained in coloring c and the modulo remainder belongs to the range $\{0, \ldots, \Delta - 1\}$. Notice that for any two adjacent arcs $e \in U_i$ and $f \in U_i$, by the definition of coloring c' and the properties of coloring c, we have

$$\begin{aligned} c'(e) - c'(f) &\equiv (1 + (\Delta - 1)i + c(e) + i - 1) - (1 + (\Delta - 1)i + c(f) + i - 1) \bmod \Delta \\ &\equiv c(e) - c(f) \bmod \Delta \\ &\not\equiv 0 \bmod \Delta \end{aligned}$$

hence $c'(e) \neq c'(f)$. Thus, it remains to be shown that for any two adjacent arcs $e \in U_i$ and $f \in U_{i+1}$ we have $c'(e) < c'(f)$. Observe that $1 + (\Delta - 1)i \leq c'(e) \leq 1 + (\Delta - 1)(i + 1)$ and $1 + (\Delta - 1)(i + 1) \leq c'(f) \leq 1 + (\Delta - 1)(i + 2)$, hence it is enough to show that $c'(e) \neq c'(f)$. Analogously to the previous case, we have:

$$\begin{aligned} c'(e) - c'(f) &\equiv (1 + (\Delta - 1)i + c(e) + i - 1) - (1 + (\Delta - 1)(i + 1) + c(f) + i) \bmod \Delta \\ &\equiv c(e) - c(f) \bmod \Delta \\ &\not\equiv 0 \bmod \Delta \end{aligned}$$

thus $c'(e) \neq c'(f)$. In this way we have shown that c' is a correct edge coloring of G_M^o; it is easy to see that c' uses at most $l(G_M)(\Delta - 1) + 1$ colors.

We will now extend the coloring c' to all arcs from G_M in such a way as to preserve the condition that for each edge e having a start-vertex v with inrank $in(v) = i$ and an end-vertex u with inrank $in(u) = j > i + 1$, $1 + (\Delta - 1)i \leq c'(e) \leq 1 + (\Delta - 1)j$. Notice that this condition immediately implies that precedence constraints are satisfied provided that adjacent arcs receive different colors. Since for any arc (v, u) from $U(G_M) \setminus U(G_M^o)$ with $in(v) = i$ and $in(u) = j$, we have $j \geq i + 2$, the number of available colors is at least $2\Delta - 1$. Since an arc is adjacent to at most $2(\Delta - 1)$ other arcs, colors can be assigned to arcs from $U(G_M) \setminus U(G_M^o)$ in a greedy manner; the obtained coloring still uses no more than $l(G_M)(\Delta - 1) + 1$ colors.

Finally, notice that by a similar argument, the coloring c' can also be extended to all the unoriented edges in E. Indeed, for all $l(G_M) \geq 2$, we have $l(G_M)(\Delta - 1) + 1 > 2(\Delta - 1)$ allowed colors to choose from for each edge. □

It is worth emphasizing that the above bound is tight even for trees: Fig. 1 presents a fully oriented tree $T_M = (V, U, \emptyset)$ with $\Delta(T_M) = 4$ and $l(T_M) = 3$ for which any proper edge coloring satisfying precedence constraints requires at least $l(G_M)[\Delta(G_M) - 1] + 1 = 3(4 - 1) + 1 = 10$ colors. Clearly, this example can be easily extended for arbitrary values of Δ and $l \geq 2$.

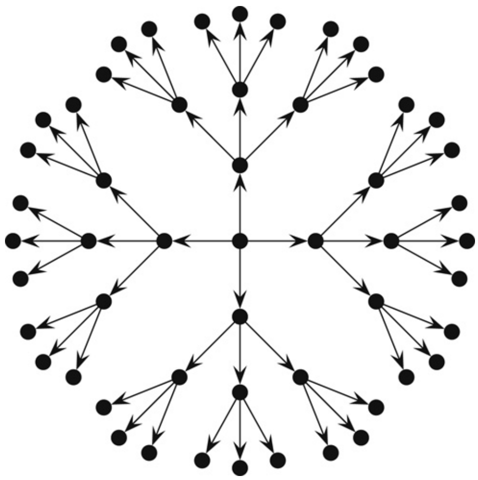

Fig. 1. A mixed tree $T_M = (V, U, \emptyset)$ with $\Delta(T_M) = 4$ and $l(T_M) = 3$ requiring 10 colors.

3. Outerplanar graphs—the complexity results

In this section, we prove that the problem of deciding whether a given mixed graph is edge k-colorable is $\mathcal{NP}$-complete even in the case of mixed bipartite outerplanar graphs. On the other hand, in Section 3.2, we provide a polynomial time algorithm that solves this problem in mixed trees.

3.1. Hardness in outerplanar graphs

Consider the following decision problem.

Mixed Graph Edge Coloring ($MGEC(G_M, k)$):
Given: A mixed graph $G_M = (V, U, E)$ and a positive integer $k > 0$.
Question: Does there exist a mixed edge k-coloring of G_M?

Since edge coloring in an unoriented graph $G = (V, E)$ is a special case of our problem, $MGEC(G_M, k)$ is $\mathcal{NP}$-complete in general mixed graphs. In the following, we shall prove that *MGEC* remains $\mathcal{NP}$-complete even for the case of mixed bipartite outerplanar graphs.

Theorem 3.1. *$MGEC(G_M, \Delta(G_M) = l(G_M))$ is $\mathcal{NP}$-complete if G_M is a bipartite outerplanar mixed graph.*

Proof. The proof is based upon a reduction from the precoloring extension problem on edges, denoted by $PrExtEd(G', p)$, which has been shown to be $\mathcal{NP}$-complete in bipartite outerplanar graphs [12]. Recall that in this problem, some edges of a given graph G' have a preassigned color, and one has to decide whether this precoloring can be extended to a proper edge p-coloring of the graph.

Consider an undirected bipartite outerplanar graph $G' = (V', E')$, and suppose that some of its edges are precolored with colors $1, 2, \ldots, k$. The idea is to transform G' into a bipartite outerplanar mixed graph G_M, with $\Delta(G_M) = l(G_M) = k$, such that the precoloring of G' can be extended to a proper edge k-coloring of the graph if, and only if, edges of G_M can be properly colored with k colors as well.

Let $e \in E'$ be an edge precolored with color i. The replacement of e is depicted in Fig. 2(a–c). We have the following properties:

(A) Fig. 2(a):
- If $i < k$ then in any edge k-coloring of T_i-gadget, edge e_1 can be assigned neither color i nor $i + 1$, $1 \leq i < k$.
- Otherwise, if $i = k$ then in any edge k-coloring of T_k-gadget, edge e_1 cannot be assigned color k.

(B) Fig. 2(b): in any edge k-coloring of an ST_i-gadget, edge e_2 must be assigned color i.

(C) Fig. 2(c): color i of e forces the same color i for both edges e_3 and e_4 in any edge k-coloring.

Replacing all the precolored edges of G' in the above manner results in the mixed graph G_M with $\Delta(G_M) = l(G_M) = k$, and G_M remains bipartite and outerplanar. Next, observe that by the construction, all vertices in G_M that are present in G' are of inner degree 0. Consequently, bearing in mind Property (C), we immediately get that the precoloring of G' can be extended to a proper edge k-coloring of the graph if, and only if, edges of G_M can be colored with $k = \Delta(G_M) = l(G_M)$ colors. $\square$

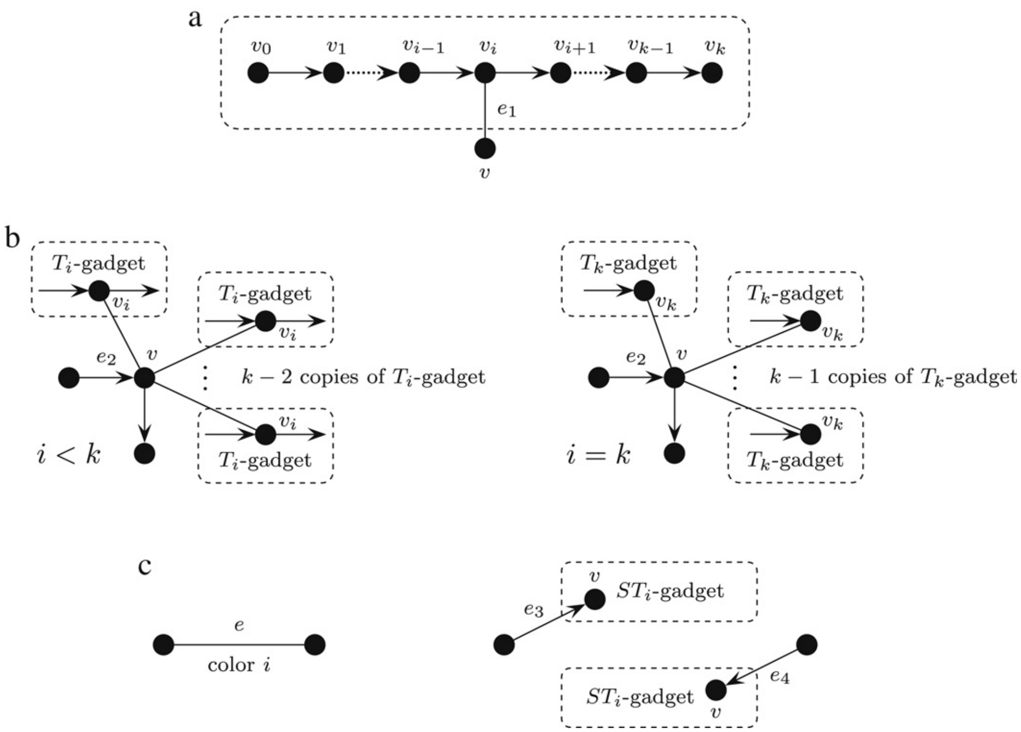

Fig. 2. (a) T_i-gadget. (b) ST_i-gadget: edge e_2 is forced to be assigned color i. (c) Replacement of e preassigned color i with two ST_i-gadgets: edges e_3 and e_4 (which correspond both to an edge e_2) are forced to be assigned the same color i.

3.2. The case of trees

For a vertex $v \in V$ of a mixed graph $G_M = (V, U, E)$, let $U^{\text{in}}_{G_M}(v)$ denote the set of oriented edges from U incident to v and directed towards v; likewise, let $U^{\text{out}}_{G_M}(v)$ denote the set of oriented edges incident to v and directed out of v.

Lemma 3.2. *Consider a mixed star $S_M = (V, U, E)$ together with an assignment $L : (E \cup U) \to 2^{\{1,\ldots,k\}}$ of lists to its edges. It is possible to determine in polynomial time whether S_M admits a mixed edge coloring c with the additional constraint that $c(e) \in L(e)$ for all $e \in E \cup U$.*

Proof. Let v be the vertex of S_m with $d(v) > 1$. Notice that in any mixed coloring of S_M the maximum b of the color values assigned to edges from set $U^{\text{in}}_{S_M}(v)$ must be strictly smaller than all the color values assigned to edges from set $U^{\text{out}}_{S_M}(v)$. Hence, the sought coloring of S_M with lists L exists if, and only if, for some value of parameter $b \in \{1, \ldots, k\}$, there exists a list edge coloring c' of the undirected star $S = (V, U \cup E)$ (we consider all arcs as unoriented edges in S) with lists L' (i.e., $c'(e) \in L'(e)$ for all edges) defined as follows:

$$L'(e) = \begin{cases} L(e) \setminus \{1, \ldots, b\}, & \text{if } e \in U^{\text{out}}_{S_M}(v), \\ L(e) \cap \{1, \ldots, b\}, & \text{if } e \in U^{\text{in}}_{S_M}(v), \\ L(e), & \text{otherwise.} \end{cases}$$

The list edge coloring problem can, however, be solved in polynomial time for undirected stars by reduction to maximum bipartite matching [22]. Thus we conclude that the list edge coloring problem for mixed trees can be solved in polynomial time. $\square$

Theorem 3.3. *MGEC(G_M, k) can be solved in polynomial time if G_M is a mixed tree.*

Proof. Let $T_M = (V, U, E)$ be a mixed tree and let r be the root. For a vertex $v \in V$, $v \neq r$, let $p(v)$ denote the parent of v in the tree, and let e_v be the (possibly directed) edge connecting v and $p(v)$. Next, let $T_M(v)$ be the mixed subtree of T_M having root $p(v)$, which is induced by vertex $p(v)$, vertex v, and all vertices $u \in V$ such that v lies on the chain connecting u and r.

The algorithm proceeds by a standard bottom-up approach. For all vertices $v \in V$, $v \neq r$, we will compute the set $C(e_v) \subseteq \{1, \ldots, k\}$ of all colors c such that edge e_v can receive color c in some mixed edge k-coloring of $T_M(v)$. If v is a leaf, then clearly $C(e_v) = \{1, \ldots, k\}$. Otherwise, let $X(v) \subseteq V$ be the set of children of v, and suppose that all the sets $C(e_u)$, for all $u \in X(v)$, are already given; we need to compute set $C(e_v)$. Observe that for any integer $c \in \{1, \ldots, k\}$, we have that

$c \in C(e_v)$ if, and only if, there exists an edge coloring of the mixed star with center v and leaves $X(v)$, such that each edge e_u of this star, $u \in X(v)$, receives a color from list $L(e_u)$ defined as follows:

$$L(e_u) = \begin{cases} C(e_u) \setminus \{1, \ldots, c\}, & \text{if } e_u \in U^{\text{out}}_{T_M}(v) \quad \text{and} \quad e_v \in U^{\text{in}}_{T_M}(v), \\ C(e_u) \setminus \{c, \ldots, k\}, & \text{if } e_u \in U^{\text{in}}_{T_M}(v) \quad \text{and} \quad e_v \in U^{\text{out}}_{T_M}(v), \\ C(e_u) \setminus \{c\}, & \text{otherwise.} \end{cases}$$

In this way, taking into account Lemma 3.2, the sets $C(e)$ for all edges of the tree can be computed in polynomial time. Finally, the tree T_M admits an edge k-coloring if and only if the mixed star induced by root r and all its children can be list edge colored with lists of available colors given as $L(e) = C(e)$ for all edges e; once again, this condition can be checked in polynomial time by Lemma 3.2. □

4. Fully oriented mixed graphs

In this section we consider the case of fully oriented mixed graph, that is, the case $E = \emptyset$. We prove that $MGEC(G_M, 5)$ is $\mathcal{NP}$-complete even if G_M is cubic and bipartite, $l(G_M) = 3$ and $E = \emptyset$.

First, let us prove a theorem for (2, 3)-regular bipartite graphs which we shall then use to derive $\mathcal{NP}$-completeness for cubic bipartite graphs; we recall that a graph G is (2, 3)-regular if every vertex of G has degree either 2 or 3.

Theorem 4.1. *$MGEC(G_M, 5)$ is $\mathcal{NP}$-complete if G_M is (2, 3)-regular and bipartite, $l(G_M) = 3$, and $E = \emptyset$.*

Proof. We use a reduction from the edge 3-coloring problem, denoted by 3*EdCol*, which has been shown to be $\mathcal{NP}$-complete even for 3-regular graphs (see [7]). Recall that in this problem, given a simple cubic graph, one has to decide whether the edges of the graph can be properly colored with 3 colors.

Consider an undirected 3-regular graph $G' = (V', E')$, and let $G_M = (V, U, E)$ be the graph resulting from G' by applying the following replacements.

1. Every vertex $v \in V'$ is replaced by its three copies v_A, v_B and v_C. Vertex v_A is called the *A-copy of* v; likewise, vertices v_B and v_C are called the *B-* and *C-copies of* v, respectively.
2. Every edge $e = [x, y] \in E'$ is replaced by a T^{ABC}_e-gadget spanned on the relevant vertex copies x_A, x_B, x_C and y_A, y_B, y_C, with three distinguished vertices e_A, e_B and e_C (see Fig. 3(a–c)). Vertex e_A is called the *A-copy of* e; likewise, vertices e_B and e_C are called the *B-* and *C-copies of* e, respectively.

The resulting mixed graph G_M is (2, 3)-regular and bipartite, $l(G_M) = 3$, and $E = \emptyset$. Furthermore, it has the following properties:

(i) The number of all A-, B- and C-copies is the same.
(ii) All A-copies have outer degree 3 in G_M (and thus they have inner degree 0), while all B- and C-copies have outer degree 0 in G_M (and thus they have inner degree 3).

Consider now the T^{ABC}_e-gadget for an edge $e \in E'$. It consists of 5 blocks, called *ABC-blocks* (see Fig. 3(d) and Fig. 5). Observe that each of these five ABC-blocks has exactly one A-, one B-, and one C-copy (of a vertex or of an edge) as leaves, called *connectors*. Moreover, we have the following property.

Claim 1. *In any mixed edge 5-coloring of G_M, there exist only three possible colorings of an ABC-block.*

Let c be an edge 5-coloring of G_M with $c(e) \in \{1, \ldots, 5\}, \forall e \in U \cup E$. Let $v_A \in V$ be the A-copy of some vertex or edge of graph G'. By Property (ii), v_A is incident (serves as a connector) to exactly three ABC-blocks B^1, B^2 and B^3, and the orientation of the arcs in an ABC-gadget forces that the colors of the relevant arcs e^1_1, e^2_1, e^3_1 incident to v_A must be restricted to 1, 2, 3 (see Fig. 4(a)); w.l.o.g. assume that $c(e^i_1) = i, i = 1, 2, 3$. Now, as e^3_1 is assigned color 3, the unique coloring, denoted by α, is forced for B^3: $c(e^3_1) = 3, c(e^3_2) = 5, c(e^3_3) = 4$ and $c(e^3_4) = 5$. (See Fig. 4(b).)

Consider now the coloring of block B^2 by c; recall that we assumed $c(e^2_1) = 2$. As v_A is any A-copy, we obtain from Property (i) and the coloring of block B^3 that color 5 cannot be used anymore by any arc (of an ABC-block) incident to a B- or a C-copy, and thus neither by an arc of B^2. Consequently, the following unique coloring of B^2, denoted by β, is forced: $c(e^2_1) = 2, c(e^2_2) = 4, c(e^2_3) = 3$ and $c(e^2_4) = 4$. (See Fig. 4(c).)

Finally, consider ABC-block B^1. Similarly as above, from Property (i) and the colorings of blocks B^2 and B^3, colors 4, 5 cannot be used by an arc (of an ABC-block) incident to a B- or a C-copy, and thus neither by an arc of B^1. Hence there is only one possible coloring of B^1 by $c : c(e^1_1) = 1, c(e^1_2) = 3, c(e^1_3) = 2$ and $c(e^1_4) = 3$, which we shall denote by γ. (See Fig. 4(d).)

Therefore, by the above discussion, only three colorings α, β, γ are possible for any ABC-block, and they are depicted in Fig. 4(b–d).

So suppose now that $MGEC(G_M, 5)$ has a positive answer. Consider a more general illustration of the relevant replacement of an edge $e = [x, y]$ by T^{ABC}_e-gadget depicted in Fig. 5. By the above claim, a careful look at ABC-blocks shows that in any edge 5-coloring of G_M, blocks B_1, B_4 and B_5 must obtain the same type of coloring $\delta \in \{\alpha, \beta, \gamma\}$. Furthermore, observe that by the definition of colorings α, β, γ and the legality of the edge 5-coloring, there are no two ABC-blocks B' and B'' colored

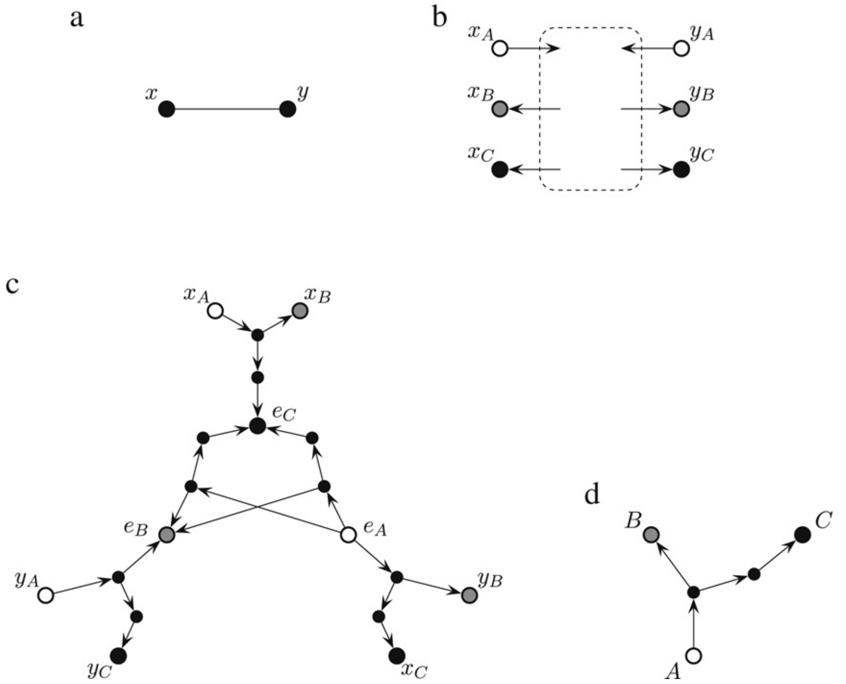

Fig. 3. (a–b) Vertices x and y are replaced by A-, B-, C-copies x_A, x_B, x_C and y_A, y_B, y_C, respectively, and (c) edge $e = [x, y]$ is replaced by a T_e^{ABC}-gadget. (d) An ABC-block.

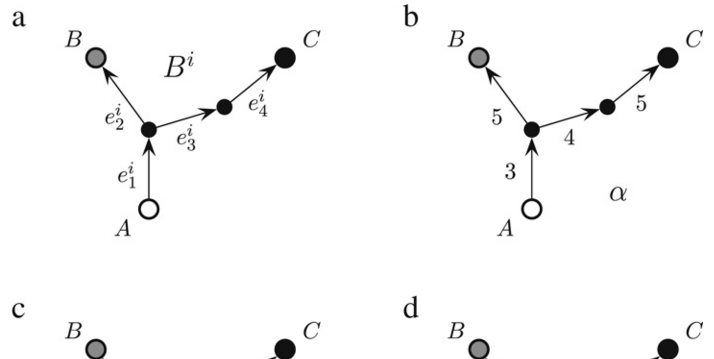

Fig. 4. (a) ABC-block B^i, and (b–d) its three possible colorings in any edge 5-coloring of G_M.

in the same manner (type) and having a vertex (connector) in common. Consequently, by assigning color c_δ to edge e and, according to the same rule, the relevant colors to all the other edges, we obtain a proper edge 3-coloring of G', that is, a positive answer to $3EdCol(G')$.

Conversely, suppose $3EdCol(G')$ has a positive answer. Set $f(0) = \alpha$, $f(1) = \beta$ and $f(2) = \gamma$, and let e be an edge of G' being assigned color $i \in \{0, 1, 2\}$ in an edge 3-coloring of G'. Then in the T_e^{ABC}-gadget, which replaced e, the edges of ABC-blocks B_1, B_4 and B_5 (Ref. Fig. 5) are assigned colors in accordance with coloring $f(i)$, while the edges of blocks B_2 and B_3 are assigned colors according to colorings $f((i + 1) \mod 3)$ and $f((i + 2) \mod 3)$, respectively. Now, bearing in mind the properties of colorings α, β and γ, and the legality of edge 3-coloring of G', one can see that by coloring the $T_{e'}^{ABC}$-gadget according to the same rule for each $e' \in E'$, we obtain a proper edge 5-coloring of G_M which satisfies the precedence constraints. □

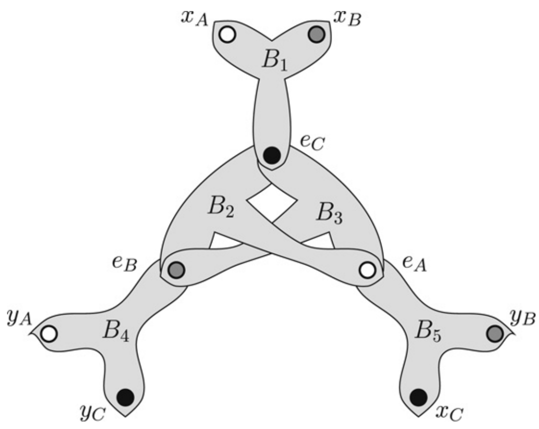

Fig. 5. T_e^{ABC}-gadget in a simplified form.

It is possible to extend the claim of Theorem 4.1 in order to eliminate vertices of degree 2 from the graph; we then obtain the following theorem, whose proof is postponed to the Appendix.

Theorem 4.2. *MGEC*$(G_M, 5)$ *is* $\mathcal{NP}$*-complete if* G_M *is cubic and bipartite,* $l(G_M) = 3$*, and* $E = \emptyset$*.*

5. Final remarks

In Section 3 we have shown that the mixed graph edge coloring problem is $\mathcal{NP}$-complete even for the narrow class of bipartite outerplanar graphs. The proof only holds when the number of allowed colors is unbounded; for the case of a constant number of colors, one can easily provide a polynomial time algorithm for all mixed partial k-trees by adapting standard techniques (cf. e.g. [21]). On the other hand, in Section 4 we see that mixed edge coloring of cubic bipartite graphs is $\mathcal{NP}$-complete even when only 5 colors are allowed, and the graph has oriented edges only. It is interesting to ask about the complexity of the problem when the number of colors is restricted still further. A partial answer is expressed by the following theorem (the proof makes use of a similar reduction from the precoloring extension problem as that of Theorem 3.1 and it is postponed to the Appendix).

Theorem 5.1. *MGEC*$(G_M, 3)$ *is* $\mathcal{NP}$*-complete even when* G_M *is restricted to be cubic planar bipartite,* $l(G_M) = 2$*, and all paths are vertex disjoint.*

However, for fully oriented graphs, the complexity status of edge coloring using 3 or 4 colors, still appears to be open.

Appendix A. proof of Theorem 4.2

Proof. Consider the relevant (2, 3)-regular bipartite graph G_M constructed in the proof of Theorem 4.1. Observe that all its degree 2 vertices are in the same partition, and the number of these vertices is divisible by three. Now, let $G_M^* = (V^*, U^*, E^*)$ be the graph resulting from connecting all the triples of degree 2 vertices in G_M with the gadget depicted in Fig. 6 (triples are chosen in an arbitrary manner). By the construction, G_M^* is cubic and bipartite, with $l(G_M^*) = 3$ and $E^* = \emptyset$. And, taking into account the proof of Theorem 4.1 and the fact that G_M is a subgraph of G_M^*, all we need is to prove that 5-colorability of G_M implies 5-colorability of G_M^*.

Consider a vertex v of degree 2 in G_M. Observe that in any 5-coloring of G_M, none of the arcs incident to v is assigned color 1. This follows from the fact that $in(v) = 2$, and that the outer degree of v is 1 in G_M. Next, consider again the gadget in Fig. 6. It is 4-colorable in a manner that the arcs incident to vertices x, y, and z are assigned color 1 (one of such colorings is depicted in the figure). Consequently, bearing in mind the aforementioned observation for any 5-coloring of edges incident to a degree 2 vertex in G_M, one can easily extend edge 5-coloring of G_M onto arcs of G_M^*: all arcs of G_M^* that are present in G_M are just colored as in G_M, and all the other arcs are colored in the manner depicted in Fig. 6. And thus, edge 5-colorability of G_M implies edge 5-colorability of G_M^*. □

Appendix B. proof of Theorem 5.1

Proof. We use a reduction from the precoloring extension problem on edges (*PrExtEd*), which has been shown to be $\mathcal{NP}$-complete in cubic planar bipartite graphs with 3 colors (see [12]). Consider an undirected cubic planar bipartite graph $G' = (V', E')$ and suppose that some of its edges are precolored with colors 1, 2 and 3. We will transform G' into G_M by making the following replacements:

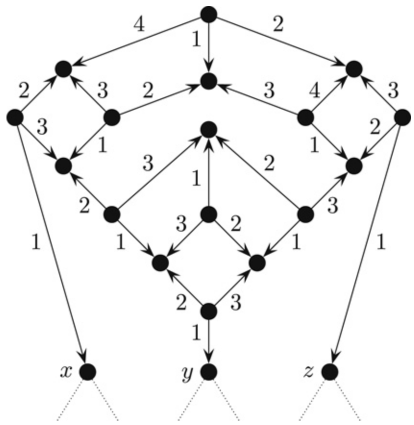

Fig. 6. Connecting vertices of degree 2.

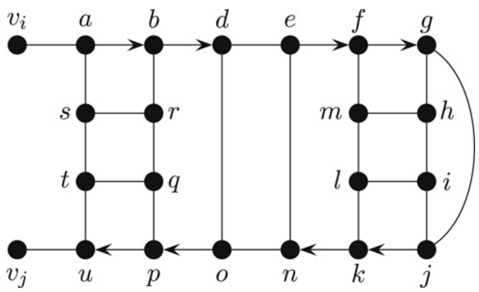

Fig. 7. The 2-gadget replacing an edge $[v_i, v_j]$ precolored with color 2.

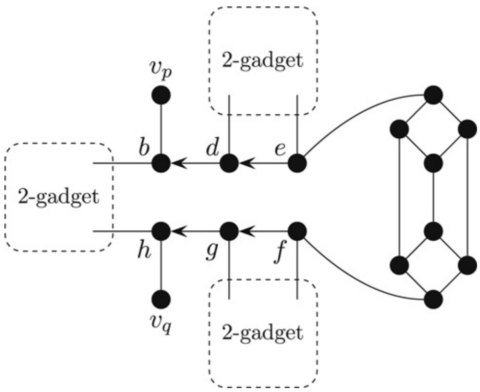

Fig. 8. The 1-gadget replacing an edge $[v_p, v_q]$ precolored with color 1.

(i) if $[v_i, v_j] \in E'$ is precolored with color 2, we replace it by the gadget shown in Fig. 7; this gadget is called the 2-gadget;
(ii) if $[v_p, v_q] \in E'$ is precolored with color 1, we replace it by the gadget shown in Fig. 8, where we attach to the vertices $\{b, h\}$, $\{d, e\}$ and $\{g, f\}$ a 2-gadget (i.e., these pairs of vertices correspond to the vertices $\{v_i, v_j\}$ in the 2-gadget); this gadget is called a 1-gadget;
(iii) if $[v_r, v_s] \in E'$ is precolored with color 3, we replace it by the gadget shown in Fig. 9, where we attach, as before, a 2-gadget to the vertices $\{b, h\}$, $\{d, e\}$ and $\{g, f\}$; this gadget is called a 3-gadget.

The resulting mixed graph $G_M = (V, U, E)$ is clearly cubic planar bipartite and $l(G_M) = 2$. Furthermore, all paths are vertex disjoint.

Claim 2. *In any mixed edge* 3*-coloring of* G_M*, the edges* $[v_i, a]$ *and* $[v_j, u]$ *in a* 2*-gadget must be assigned color* 2*.*

First notice that in any mixed edge 3-coloring of G_M, arcs (b, d) and (k, n) can only get colors 2 or 3, and arcs (e, f) and (o, p) can only get colors 1 or 2. This implies that the cycle $([d, e], [e, n], [n, o], [o, d])$ cannot use color 2, and then the cycle must be colored using twice color 1 and twice color 3. Thus arcs (b, d), (k, n), (e, f) and (o, p) must all get color 2.

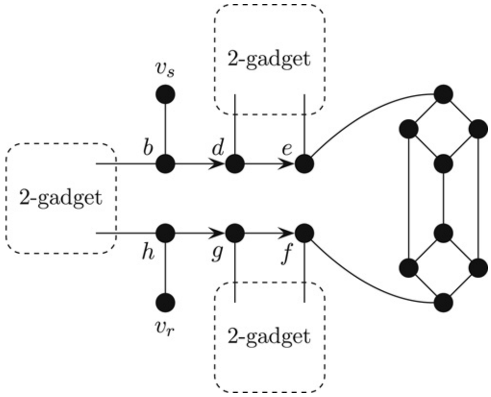

Fig. 9. The 3-gadget replacing an edge $[v_r, v_s]$ precolored with color 3.

Consequently, (a, b) will be assigned color 1, and arc (p, u) will be assigned color 3. So edges $[p, q]$, $[u, t]$ and $[s, r]$ get color 1, edges $[s, t]$ and $[r, q]$ color 2, and edges $[a, s]$, $[q, t]$ and $[b, r]$ color 3. This means that both edges $[v_i, a]$, $[v_j, u]$ are forced to be assigned color 2. Similarly, (f, g), $[h, m]$, $[l, k]$ and $[i, j]$ are forced to be assigned color 3, (j, k), $[l, i]$, $[f, m]$ and $[g, h]$ are forced to be assigned color 1, and finally, edges $[l, m]$, $[h, i]$ and $[j, g]$ are forced to be assigned color 2.

Claim 3. *In any mixed edge 3-coloring of G_M, the edges $[v_p, b]$, $[v_q, h]$ (resp. $[v_s, b]$, $[v_r, h]$) in a 1-gadget (resp. in a 3-gadget) must necessarily get color 1 (resp. color 3).*

It follows from the fact that all the edges linking the vertices b and h, d and e, as well as g and f to the 2-gadgets have color 2. Notice that once these colors are fixed, the remaining yet uncolored edges in a 1-gadget or a 3-gadget are 3-colorable in the unique way.

So suppose now that $MGEC(G_M, 3)$ has a positive answer. Then, as we have just explained above, edges $[v_p, b]$, $[v_q, h]$ have color 1, edges $[v_i, a]$, $[v_j, u]$ have color 2, and edges $[v_s, b]$, $[v_r, h]$ have color 3. Thus by replacing each 1-gadget by the original edge $[v_p, v_q]$ and coloring it with color 1, each 2-gadget by the original edge $[v_i, v_j]$ and coloring it with color 2, and each 3-gadget by the original edge $[v_r, v_s]$ and coloring it with color 3, we get a positive answer for $PrExtEd(G')$.

Conversely, if $PrExtEd(G')$ has a positive answer, then by replacing the precolored edges by the relevant gadgets and by coloring edges $[v_p, b]$, $[v_q, h]$ with color 1, edges $[v_i, a]$, $[v_j, u]$ with color 2 and edges $[v_s, b]$, $[v_r, h]$ with color 3 will give us a positive answer for $MGEC(G_M, 3)$, since the remaining uncolored edges can be colored in the unique way as explained above. □

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