

Detecting strong cliques

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A B S T R A C T

A strong clique in a graph is a clique intersecting every maximal independent set. We study the computational complexity of six algorithmic decision problems related to strong cliques in graphs and almost completely determine their complexity in the classes of chordal graphs, weakly chordal graphs, line graphs and their complements, and graphs of maximum degree at most three. Our results rely on connections with matchings and relate to several graph properties studied in the literature, including well-covered graphs, localizable graphs, and general partition graphs.

1. Introduction

1.1. Background

A *strong clique* in a graph G is a clique intersecting every maximal independent set in G . Replacing the graph G with its complement \bar{G} maps the strong cliques of G into the equivalent concept of *strong independent sets* of \bar{G} , that is, independent sets intersecting every maximal clique in \bar{G} . The notions of strong cliques and strong independent sets in graphs played an important role in the study of perfect graphs and their subclasses (see, e.g., [4,5,10,21,31]). Moreover, various other graph classes studied in the literature can be defined in terms of properties involving strong cliques (see, e.g., [7,8,15,24,25,30]). In some cases, strong cliques can be seen as a generalization of perfect matchings: transforming any regular triangle-free graph to the complement of its line graph maps any perfect matching into a strong clique (see [6,33] for applications of this observation).

In this paper we report on advance regarding the computational complexity of several problems related to strong cliques in graphs. We pay particular attention to the class of graphs whose vertex set can be partitioned into strong cliques. Such graphs were called *localizable* by Yamashita and Kameda in 1999 [41] and studied further by Hujdurović et al. in 2018 [24]. Localizable graphs form a rich class of *well-covered graphs*, that is, graphs in which all maximal independent sets are of the same size. Moreover, localizable graphs and well-covered graphs coincide within the class of perfect graphs (see, e.g., [24]). Well-covered graphs were introduced by Plummer in 1970 [34] and have been studied extensively in the literature (see, e.g., the surveys by Plummer [35] and Hartnell [20]).

Relatively few complexity results regarding problems involving strong cliques are available in the literature. Zang [42] showed that it is co-NP-complete to test whether a given clique in a graph is strong and Hoàng [22] established NP-hardness of testing whether a given graph contains a strong clique. Hujdurović et al. [24] showed that recognizing

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localizable graphs is NP-hard, even in the class of well-covered graphs whose complements are localizable. Polynomial-time recognition algorithms for localizable graphs are known within several classes of graphs, including triangle-free graphs, C_4 -free graphs, and line graphs (see [24, Section 4.1] for an overview). Results of Hoàng [21] and Burlet and Fonlupt [10] imply the existence of a polynomial-time algorithm for determining if in every induced subgraph of a given graph G , each vertex belongs to a strong clique. On the other hand, to the best of our knowledge, the complexity status of recognizing several graph classes related to strong cliques is in general still open. In particular, this is the case for: (i) the problem of recognizing *strongly perfect graphs* [4,5], introduced by Berge as graphs every induced subgraph of which has a strong independent set, (ii) the problem of determining whether every edge of a given graph is contained in a strong clique or, equivalently, the problem of recognizing *general partition graphs* (see, e.g., [25,30,32]), and (iii) the problem of recognizing *CIS graphs*, defined as graphs in which every maximal clique is strong, or, equivalently, as graphs in which every maximal independent set is strong (see, e.g., [7,9,15,19,40]).

In this paper we report on several advances regarding the computational complexity of six decision problems related to strong cliques. The problems are formally stated as follows.

STRONG CLIQUE

Input: A graph G and a clique C in G .
Question: Is C strong?

STRONG CLIQUE EXISTENCE

Input: A graph G .
Question: Does G have a strong clique?

STRONG CLIQUE VERTEX COVER

Input: A graph G .
Question: Is every vertex of G contained in a strong clique?

STRONG CLIQUE EDGE COVER (a.k.a. GENERAL PARTITIONABILITY)

Input: A graph G .
Question: Is every edge of G contained in a strong clique?

STRONG CLIQUE PARTITION

Input: A graph G and a partition of its vertex set into cliques.
Question: Is every clique in the partition strong?

STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY)

Input: A graph G .
Question: Can the vertex set of G be partitioned into strong cliques?

In the rest of the paper, we will refer to the above six problems simply as the ‘six strong clique problems’.

1.2. Motivation and overview of results

Our study has many sources of motivation. First, as mentioned above, problems STRONG CLIQUE, STRONG CLIQUE EXISTENCE, and STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) are known to be NP-hard (see [22,42], and [24], respectively). On the other hand, to the best of our knowledge, the complexity of the remaining three problems has not been determined in the literature. One of the aims of this paper is to partially close this gap and, more generally, to obtain further complexity and algorithmic results on the six strong clique problems in specific classes of graphs.

Our work is also motivated by the general quest for finding further applications of connections between strong cliques and other graph theoretic concepts, in particular: (i) of the fact that localizable graphs and well-covered graphs coincide within the class of perfect graphs, as observed in [24], and (ii) of connections between strong cliques and matchings – which naturally appear in the study of line graphs and their complements [6,8,24,27,33].

Regarding (i), we further elaborate on the approach used by Chvátal and Slater [14] and Sankaranarayana and Stewart [39] for proving hardness of recognizing well-covered graphs within the class of weakly chordal graphs (which are perfect), to derive similar results for STRONG CLIQUE VERTEX COVER and STRONG CLIQUE PARTITION, two of the three problems from the above list whose complexity status was unknown prior to this work. These results are presented in Section 3.

Many of our results motivated by (ii) are based on the study of STRONG CLIQUE EXTENSION, an auxiliary problem which we introduce in Section 4. We show that in any class of graphs, each of the first five of the six strong clique problems (as listed above) polynomially reduces to STRONG CLIQUE EXTENSION. Results motivated by (ii) can be roughly divided into two parts.

Table 1

Summary of complexity results for the six strong clique problems in various graph classes. SC stands for STRONG CLIQUE, P for polynomial-time solvable, and coNPc for co-NP-complete. Unreferenced entries mark results obtained in this paper. Entries for chordal graphs marked with an asterisk follow from [Corollary 4.3](#). Results for weakly chordal graphs are presented in [Section 3](#). Results for line graphs and their complements are presented in [Section 5](#). Results for subcubic graphs are presented in [Section 6](#).

	All graphs	Chordal graphs	Weakly chordal graphs	Line graphs	Complements of line graphs	Subcubic graphs
SC	coNPc [42]	P*	coNPc [42]	P	P	P
SC EXISTENCE	NP-hard [22]	P*	NP-hard	P	P	P
SC VERTEX COVER	NP-hard	P*	NP-hard	P	P	P
SC EDGE COVER	?	P* [2]	?	P [27]	P	P
SC PARTITION	coNPc	P*	coNPc	P	P	P
SC PARTITION EXISTENCE	coNPc [24]	P [24,36]	coNPc [24]	P [24]	NP-complete	P

- We show that STRONG CLIQUE EXTENSION in either line graphs or their complements polynomially reduces to the maximum-weight matching problem. This implies polynomial-time solvability of the first five strong clique problems from the above list in the classes of line graphs and their complements. This is a significant generalization of the known fact that one of these problems, STRONG CLIQUE EDGE COVER (a.k.a. GENERAL PARTITIONABILITY), is polynomial-time solvable in the class of line graphs, as proved by Levit and Milanič [27], and in the class of complements of line graphs of bipartite graphs, which follows from results of Boros et al. [6]. Exploiting further the connection with matchings, we show that STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is NP-complete in the class of complements of line graphs. Together with a result from Hujdurović et al. [24], our results completely determine the computational complexity of the six strong clique problems from the above list in the classes of line graphs and their complements. These results are presented in [Section 5](#).
- We show that when restricted to subcubic graphs, STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) can be reduced to testing the existence of a perfect matching in a derived graph. We further complement this result, presented in [Section 6](#), with a polynomial-time algorithm for STRONG CLIQUE EXTENSION, and consequently for each of the remaining five strong clique problems, in any class of graphs of bounded maximum degree, or, more generally, of bounded clique number.

Together with some previous results, our approach using the STRONG CLIQUE EXTENSION problem also leads to a complete classification of the complexity of the six strong clique problems in the class of chordal graphs, a well-known subclass of weakly chordal graphs. In contrast with the hardness results for the class of weakly chordal graphs, all the six strong clique problems are solvable in polynomial time in the class of chordal graphs (see [Section 4](#)).

Finally, our work is motivated by the question about characterizations of localizable graphs raised by Yamashita and Kameda [41]. One of our results ([Theorem 5.9](#)) shows that the property of localizability is difficult to recognize not only in the classes of weakly chordal graphs and well-covered graphs, as shown in [24], but also in the class of complements of line graphs. Furthermore, as a corollary of our approach presented in [Section 5](#), we show that STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is NP-complete in the class of graphs of independence number k , for every fixed $k \geq 3$. On the positive side, we enhance our polynomial-time algorithm for testing localizability in the class of subcubic graphs by showing that within cubic graphs, the property coincides with the condition that every vertex is in a strong clique and classify graphs with these properties ([Theorem 6.8](#)).

In [Table 1](#) we summarize the known and newly obtained complexity results for the six strong clique problems in general graphs and in special graph classes studied in this paper.

2. Preliminaries

In this section we recall some definitions, fix the notation, and state some known results on line graphs and on localizable graphs. We consider only finite, simple, and undirected graphs. Given a graph $G = (V, E)$, its complement \bar{G} is the graph with vertex set V in which two distinct vertices are adjacent if and only if they are non-adjacent in G . By K_n , P_n , and C_n we denote as usual the n -vertex complete graph, path, and cycle, respectively, and by $K_{m,n}$ the complete bipartite graph with parts of sizes m and n . The *degree* of a vertex v in a graph G is denoted by $d_G(v)$, its neighborhood by $N_G(v)$ (or simply by $N(v)$ if the graph is clear from the context), and its closed neighborhood by $N_G[v]$ (or simply by $N[v]$). For a set of vertices $X \subseteq V(G)$, we denote by $N_G(X)$ (or $N(X)$) the set of all vertices in $V(G) \setminus X$ having a neighbor in X . The fact that two graphs G and H are isomorphic will be denoted by $G \cong H$. The minimum (resp. maximum) degree of a vertex in a graph G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A graph is k -*regular* if $\delta(G) = \Delta(G) = k$, and *regular* if it is k -regular for some k . A graph is *cubic* if it is 3-regular and *subcubic* if it is of maximum degree at most 3. The disjoint union of two vertex-disjoint graphs G and H , denoted by $G + H$, is defined as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

For a graph H , we say that a graph G is H -*free* if G has no induced subgraph isomorphic to H . A graph is said to be *chordal* if it does not contain any induced cycle of length at least four. The *line graph* of a graph $G = (V, E)$ is the graph

$L(G)$ with vertex set E in which two distinct vertices are adjacent if and only if they have a common endpoint as edges of G . The following result about line graphs will be useful in Section 5.

Theorem 2.1 (Roussopoulos [38] and Lehot [26]). *There exists a linear-time algorithm that tests if a given graph G is (isomorphic to) the line graph of some graph H and, if this the case, computes a graph H such that $G \cong L(H)$.*

For a positive integer k , a k -coloring of G is a mapping $c : V \rightarrow \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The *chromatic number* $\chi(G)$ is the smallest integer k for which G has a k -coloring. A subset of edges $M \subseteq E$ is called a *matching* if no two edges of M share a common end-vertex. The *chromatic index* of a graph G is denoted by $\chi'(G)$ and defined as the smallest number of matchings of G the union of which is E ; clearly, $\chi'(G) = \chi(L(G))$. A matching M in a graph G is *perfect* if every vertex of G is an endpoint of an edge in M . For undefined terms related to matchings, we refer the reader to [29].

Let $G = (V, E)$ be a graph. A subset $C \subseteq V$ is called a *clique* in G if any two vertices in C are adjacent to each other. The *clique number* $\omega(G)$ is the number of vertices in a maximum clique of G . A *triangle* is the graph K_3 . We will often identify a triangle with the set of its edges; whether we consider a triangle as a set of vertices or as a set of edges will always be clear from the context. A subset $I \subseteq V$ is called an *independent set* in G if any two vertices in I are non-adjacent to each other. The *independence number* $\alpha(G)$ is the number of vertices in a maximum independent set of G . Given a clique C in G and a set $S \subseteq V(G) \setminus C$, we say that clique C is *dominated by* S if $C \subseteq N_C(S)$. The following is an easy but useful characterization of strong cliques (see, e.g., remarks following [24, Theorem 2.3]).

Lemma 2.2. *A clique C in a graph G is not strong if and only if it is dominated by an independent set $I \subseteq V(G) \setminus C$.*

A set $D \subseteq V$ is called *dominating* if every vertex in $V \setminus D$ has at least one neighbor in D . If D is in addition an independent set, it is called an *independent dominating set*. We denote by $i(G)$ the *independent domination number* of G , that is, the minimum size of an independent dominating set in G . Furthermore, we denote by $\theta(G)$ the *clique cover number* of G , that is, the minimum number of cliques that partition its vertex set. It is well-known that every graph G has $\alpha(G) = \omega(\bar{G})$, $\theta(G) = \chi(\bar{G})$, $\chi(G) \geq \omega(G)$, and $\theta(G) \geq \alpha(G)$. It follows that every graph G satisfies the following chain of inequalities:

$$i(G) \leq \alpha(G) \leq \theta(G).$$

Clearly, a graph G is well-covered if and only if $i(G) = \alpha(G)$.

For a positive integer k , we say that a graph G is k -localizable if it admits a partition of its vertex set into k strong cliques. A graph G is said to be *semi-perfect* if $\theta(G) = \alpha(G)$, that is, if there exists a collection of $\alpha(G)$ cliques partitioning its vertex set. We will refer to such a collection as an α -clique cover of G . Thus, G is semi-perfect if and only if it has an α -clique cover, and G is $\alpha(G)$ -localizable if and only if it has a α -clique cover in which every clique is strong.

In the following theorem we recall several equivalent formulations of localizability.

Theorem 2.3 (Hujdurović et al. [24]). *For every graph G , the following statements are equivalent.*

- (a) G is localizable.
- (b) G is $\alpha(G)$ -localizable.
- (c) G has an α -clique cover and every clique in every α -clique cover of G is strong.
- (d) G is well-covered and semi-perfect.
- (e) $i(G) = \theta(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G . Since the complementary graph \bar{G} is perfect whenever G is perfect [28], every perfect graph G is also semi-perfect. The following consequence of Theorem 2.3 will be useful.

Corollary 2.4. *A perfect graph is well-covered if and only if it is localizable.*

3. A hardness proof and its implications

An important subclass of perfect graphs generalizing the class of chordal graphs is the class of weakly chordal graphs. A graph is *weakly chordal* if neither G nor its complement contain an induced cycle of length at least 5. Chvátal and Slater [14] and Sankaranarayana and Stewart [39] proved the following hardness result.

Theorem 3.1. *The problem of recognizing well-covered graphs is co-NP-complete, even for weakly chordal graphs.*

Since weakly chordal graphs are perfect, Corollary 2.4 implies that a weakly chordal graph is well-covered if and only if it is localizable. Thus, Theorem 3.1 yields the following.

Theorem 3.2 (Hujdurović et al. [24]). **STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY)** is co-NP-complete in the class of weakly chordal graphs.

Recall that, as noted in Table 1, it follows from results of Prisner et al. [36] that localizable graphs can be recognized in polynomial time within the class of chordal graphs (and, more generally, also in the class of C_4 -free graphs [24]).

Both proofs of Theorem 3.1 from [14,39] are based on a reduction from the 3-SAT problem. The input to the 3-SAT problem is a set $X = \{x_1, \dots, x_n\}$ of Boolean variables and a collection $\mathcal{C} = \{C_1, \dots, C_m\}$ of clauses of size 3 over X . Each clause is a disjunction of exactly three *literals*, where a literal is either one of the variables (say x_i) or its negation (denoted by \bar{x}_i). The task is to determine whether the formula $\varphi = \bigwedge_{i=1}^m C_i$ is satisfiable, that is, whether there exists a truth assignment to the n variables which makes all the clauses simultaneously evaluate to true. The reduction produces from a given 3-SAT instance I a weakly chordal graph $G(I)$ such that $G(I)$ is not well-covered if and only if the formula is satisfiable. More precisely, given an input (X, \mathcal{C}) to the 3-SAT problem, representing a formula φ , a graph $G = G(X, \mathcal{C})$ is constructed with vertex set $V(G) = C \cup L$, where $C = \{c_1, \dots, c_m\}$ forms a clique, $L = \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$, for each $i \in \{1, \dots, n\}$, vertices x_i and \bar{x}_i are adjacent, for each $c_i \in C$ and each literal $\ell \in L$, $c_i \ell \in E(G)$ if and only if clause C_i contains literal ℓ , and there are no other edges. Then, φ is satisfiable if and only if G is not well-covered [14,39].

A similar construction, using the complement of G instead of G , was used in a reduction from the satisfiability problem by Zang [42] to show that the problem of testing whether a given independent set in a graph is strong is co-NP-complete. Since the class of weakly chordal graphs is closed under taking complements, Zang's proof shows the following.

Theorem 3.3. *STRONG CLIQUE is co-NP-complete in the class of weakly chordal graphs.*

We now show that suitable modifications of the above reduction can be used to establish NP-hardness of three of the six strong clique problems listed in Section 1.1, even in the class of weakly chordal graphs.

Theorem 3.4. *STRONG CLIQUE EXISTENCE and STRONG CLIQUE VERTEX COVER are NP-hard, even in the class of weakly chordal graphs. STRONG CLIQUE PARTITION is co-NP-complete, both in general and for weakly chordal graphs.*

Proof. We use a modification of the above reduction. First, note that the 3-SAT problem remains NP-complete on instances satisfying the following assumptions:

- (i) *No clause contains a pair of the form $\{x_i, \bar{x}_i\}$.* Indeed, if there exists a clause containing both x_i and \bar{x}_i , then such a clause is always satisfied and can be removed from the instance to obtain an equivalent one.
- (ii) *Each literal appears in at least one clause.* Indeed, if some literal ℓ does not appear in any clause, then we can set that literal to false and remove each clause containing its negation to obtain an equivalent instance.
- (iii) *No variable x_i is such that every clause contains either x_i or \bar{x}_i .* Indeed, instances having a variable x_i such that every clause contains either x_i or \bar{x}_i can be solved in polynomial time, by solving two instances of the 2-SAT problem corresponding to setting x_i to true (resp., to false) and obtained by eliminating all clauses containing x_i (resp., \bar{x}_i) and deleting \bar{x}_i (resp., x_i) from all clauses containing it. The 2-SAT problem is defined similarly as the 3-SAT problem, except that every clause is of size two. The problem is solvable in linear time [3,18].

We may thus assume that we are given a 3-SAT instance I satisfying conditions (i)–(iii). Let $X, \mathcal{C}, \varphi, G, C = \{c_1, \dots, c_m\}$, and $L = \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$ be as above.

Condition (i) implies that none of the cliques $\{x_i, \bar{x}_i\}$ in G is contained in the neighborhood of a vertex in C . Since $N_G(\{x_i, \bar{x}_i\})$ is a clique, it follows that each clique $\{x_i, \bar{x}_i\}$ is strong in G .

Rephrasing the observation of Zang [42] in our setting, we claim that formula φ is satisfiable if and only if clique C is not strong. Indeed, on the one hand, literals in L that are set to true in a satisfying assignment form an independent set disjoint from C that dominates C , and Lemma 2.2 applies. Conversely, if C is not strong then any maximal independent set I disjoint from C necessarily consists of exactly one literal from each pair $\{x_i, \bar{x}_i\}$; setting all literals in I to true yields a satisfying assignment.

It follows that each of the cliques in the partition $\{C, \{x_1, \bar{x}_1\}, \dots, \{x_n, \bar{x}_n\}\}$ of $V(G)$ is strong if and only if C is strong, if and only if the formula is not satisfiable. Consequently, STRONG CLIQUE PARTITION is NP-hard in the class of weakly chordal graphs. Membership in co-NP (in any class of graphs) follows from the observation that a short certificate of a no instance consists of a pair (K, I) where K is one of the cliques in the partition and I is a maximal independent set disjoint from K .

Next, we prove that STRONG CLIQUE VERTEX COVER is NP-hard in the class of weakly chordal graphs. Observe that each vertex $\ell \in L$ is contained in a strong clique (namely the K_2 containing the literal ℓ and its negation). Moreover, we will now show that no clique of G containing both a vertex from L and a vertex from C is strong. Let K be such a clique, with, say, $\ell \in L \cap K$ and $c_i \in C \cap K$. Since by assumption (i) on the 3-SAT instance, no clause contains a literal and its negation, it follows that c_i is not adjacent to $\bar{\ell}$. Therefore $\bar{\ell} \notin K$ and hence $K \cap L = \{\ell\}$. By assumption (iii) on the 3-SAT instance, there is a clause c_j that contains neither ℓ nor $\bar{\ell}$. Then $c_j \neq c_i$ and the set $I = \{\ell, c_j\}$ is an independent set disjoint from K that dominates K . By Lemma 2.2, K is not strong.

We know that every vertex $\ell \in L$ is contained in a strong clique and, since no clique of G containing both a vertex from L and a vertex from C is strong, a vertex in C is contained in a strong clique if and only if C is strong. (Note that it follows from the definition of strong cliques that every strong clique is maximal, hence no proper subclique of C can be strong.) Therefore, every vertex of G is contained in a strong clique if and only if C is strong, which, as argued above, is equivalent to non-satisfiability of φ . The claimed NP-hardness follows.

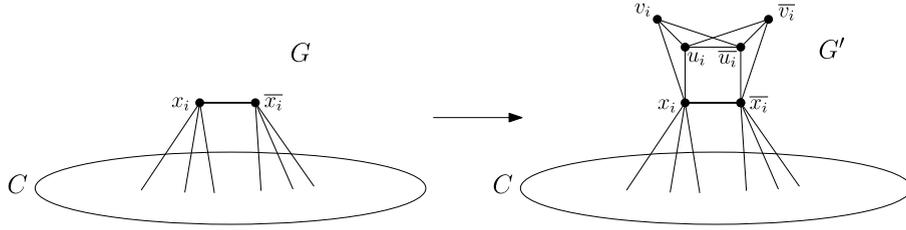


Fig. 1. Transforming G into G' .

It remains to show that STRONG CLIQUE EXISTENCE is NP-hard in the class of weakly chordal graphs. To this end, take the graph G constructed as above and modify it to a graph G' by adding, for each $i \in \{1, \dots, n\}$, four new vertices $u_i, v_i, \bar{u}_i, \bar{v}_i$ and edges $x_i u_i, x_i v_i, u_i v_i, u_i \bar{u}_i, u_i \bar{v}_i, v_i \bar{u}_i, \bar{u}_i \bar{v}_i, \bar{x}_i \bar{u}_i$, and $\bar{x}_i \bar{v}_i$ (see Fig. 1).

It is not difficult to see that the resulting graph G' is weakly chordal. Since none of the newly added vertices is adjacent to any vertex in C , the fact that C is not strong if and only if formula φ is satisfiable remains valid for G' . Moreover, using Lemma 2.2 it can be verified that every maximal clique of G' other than C is not strong:

- The fact that no clique of G containing both a vertex from L and a vertex from C is strong (in G) implies the same property for G' .
- For each $i \in \{1, \dots, n\}$, the clique $\{x_i, \bar{x}_i\}$ is not strong since it is dominated by the independent set $\{v_i, \bar{v}_i\}$.
- For each $i \in \{1, \dots, n\}$, the clique $\{x_i, u_i, v_i\}$ is not strong since it is dominated by the independent set $\{c_j, \bar{u}_i\}$, where C_j is a clause containing x_i (note that such a clause exists by assumption (ii)). A symmetric argument shows that the clique $\{\bar{x}_i, \bar{u}_i, \bar{v}_i\}$ is not strong.
- For each $i \in \{1, \dots, n\}$, the clique $\{u_i, \bar{u}_i, \bar{v}_i\}$ is not strong since it is dominated by the independent set $\{v_i, \bar{x}_i\}$. A symmetric argument shows that the clique $\{u_i, \bar{u}_i, v_i\}$ is not strong.

It follows that G' contains a strong clique if and only if C is strong, which is equivalent to non-satisfiability of φ . This establishes the claimed NP-hardness of STRONG CLIQUE EXISTENCE in the class of weakly chordal graphs. \square

Theorem 3.4 implies that the obvious certificate for yes instances of the problem of recognizing localizable graphs is most likely not verifiable in polynomial time, leaving the membership status of STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) in NP (or in co-NP) unclear.

4. A useful auxiliary problem

We introduce one more problem related to strong cliques.

STRONG CLIQUE EXTENSION

Input: A graph G and a clique C in G .

Question: Does G contain a strong clique C' such that $C \subseteq C'$?

As the next lemma shows, STRONG CLIQUE EXTENSION in some sense generalizes the first five of the six strong clique problems listed in Section 1.1.

Lemma 4.1. *Let \mathcal{G} be a class of graphs such that STRONG CLIQUE EXTENSION is solvable in polynomial time for graphs in \mathcal{G} . Then, each of the problems STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION is solvable in polynomial time for graphs in \mathcal{G} .*

Proof. Let G be a graph from \mathcal{G} and let C be a clique in G . It is clear that C is a strong clique in G if and only if C is a maximal clique and G contains a strong clique C' such that $C \subseteq C'$. It follows that testing if C is strong can be done in polynomial time by testing if C is maximal and if C extends to a strong clique. The STRONG CLIQUE EXISTENCE problem is equivalent to determining if the empty clique extends to some strong clique. Similarly, the STRONG CLIQUE VERTEX COVER (resp., STRONG CLIQUE EDGE COVER) problem can be solved in polynomial time simply by testing if every clique of size one (resp., two) extends to a strong clique. Finally, since STRONG CLIQUE is polynomial-time solvable for graphs in \mathcal{G} , STRONG CLIQUE PARTITION can also be solved in polynomial time for graphs in \mathcal{G} by testing whether each of the cliques in the given partition is strong. \square

As an immediate application of Lemma 4.1, we observe that together with a result from [24], the lemma implies that all the above problems are polynomial-time solvable in the class of chordal graphs, and even more generally, in the class of C_4 -free graphs.

Proposition 4.2. STRONG CLIQUE EXTENSION is solvable in polynomial time in the class of C_4 -free graphs.

Proof. As shown in [24, Lemma 4.4], a clique in a C_4 -free graph is strong if and only if it is simplicial, that is, if it consists of a vertex and all its neighbors. Therefore, in the class of C_4 -free graphs STRONG CLIQUE EXTENSION reduces to the problem of testing if a given clique is contained in a simplicial clique. Since the simplicial cliques of a graph can be listed in polynomial time, polynomial-time solvability of STRONG CLIQUE EXTENSION in the class of C_4 -free graphs follows. \square

Corollary 4.3. Each of the problems STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION is solvable in polynomial time in the class of C_4 -free graphs.

Further applications of Lemma 4.1 will be given in Sections 5 and 6.

5. Connections with matchings: line graphs and their complements

Relationships between strong cliques and matchings have already been used in a number of works studying properties of line graphs and their complements [6,8,24,27,33]. We now further elaborate on these connections in order to obtain new algorithmic and complexity results on the six strong cliques problems in the classes of line graphs and their complements. Two cases have already been settled in the literature, both giving polynomial-time algorithms for the class of line graphs: one for STRONG CLIQUE EDGE COVER (a.k.a. GENERAL PARTITIONABILITY), due to Levit and Milanič [27], and one for STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY), due to Hujdurović et al. [24]. A related result is also a result of Boros et al. [8] giving a polynomial-time algorithm to test whether a given line graph is CIS, that is, whether every maximal clique in a line graph is strong. Since the class of CIS graphs is closed under complementation, this result also yields a polynomial-time algorithm for the same problem in the class of complements of line graphs.

In this section, we show that STRONG CLIQUE EXTENSION is solvable in polynomial time in the class of line graphs and their complements. Together with the above-mentioned result from [24], this implies that all six strong clique problems listed in Section 1.1 are polynomial-time solvable in the class of line graphs. For the class of complements of line graphs, however, we show that STRONG CLIQUE PARTITION EXISTENCE remains NP-complete.

We first introduce some notation and translate the relevant concepts from the line graph or its complement to the root graph.

Definition 5.1. Given a graph G , we denote by \mathcal{T}_G the set $\{E(T) \mid T \text{ is a triangle in } G\}$, by \mathcal{S}_G the set of all *stars* of G , that is, sets of edges of the form $E(v) = \{e \in E(G) \mid v \text{ is an endpoint of } e\}$ for $v \in V(G)$, and by \mathcal{M}_G the set of all inclusion-maximal sets in the family $\mathcal{T}_G \cup \mathcal{S}_G$.

The following observation is an immediate consequence of the definitions.

Observation 5.2. For every graph G without isolated vertices, the following holds:

1. e is an edge of $G \iff e$ is a vertex of $L(G) \iff e$ is a vertex of $\overline{L(G)}$.
2. For every $F \subseteq E(G)$, we have:
 - F is a clique in $L(G) \iff F \subseteq E(v)$ for some $v \in V(G)$ or $F \in \mathcal{T}_G$.
 - F is a maximal clique in $L(G) \iff F$ is a maximal independent set in $\overline{L(G)} \iff F \in \mathcal{M}_G$.
 - F is an independent set in $L(G) \iff F$ is a clique in $\overline{L(G)} \iff F$ is a matching in G .
 - F is a maximal independent set in $L(G) \iff F$ is a maximal clique in $\overline{L(G)} \iff F$ is a maximal matching in G .
 - F is a strong clique in $L(G) \iff F$ is a set in \mathcal{M}_G that intersects every maximal matching in G .
 - F is a strong clique in $\overline{L(G)} \iff F$ is a maximal matching in G that intersects every set in \mathcal{M}_G .

Theorem 5.3. STRONG CLIQUE EXTENSION is solvable in polynomial time in the class of line graphs.

Proof. Let G be a line graph and let C be a clique in G . Using Theorem 2.1, we can compute in linear time a graph H such that $G \cong L(H)$. We also compute, in polynomial time, the set \mathcal{M}_H . Since every strong clique in a graph is a maximal clique and the maximal cliques of G correspond exactly to the elements of \mathcal{M}_H , it suffices to show that there is a polynomial-time algorithm to check if a given $C' \in \mathcal{M}_H$ corresponds to a strong clique C'_G in G . Then, to check if C is contained in a strong clique it will suffice to check if any $C' \in \mathcal{M}_H$ such that $C \subseteq C'_G$ corresponds to a strong clique.

Let $C' \in \mathcal{M}_H$. Thus, C' is a set of edges in H and we need to check if C' intersects every maximal matching in H . Suppose first that $C' \in \mathcal{T}_H$. Then C' is the edge set of a triangle T in H . It is not difficult to see that H contains a maximal matching disjoint from C' if and only if $H - C'$ contains a matching saturating at least two vertices of T . This is a local property that can be easily checked in polynomial time, for example by examining all pairs of disjoint edges in $H - C'$ connecting $V(T)$ with the rest of the graph. Suppose now that $C' \in \mathcal{S}_H$. Then there exists a vertex $v \in V(H)$ such that $C' = E(v)$, and $E(v)$ intersects every maximal matching in H if and only if every maximal matching in H saturates v . This fails if and only if $H - v$ contains a matching saturating all vertices in $N_H(v)$. There is a standard reduction of this

problem to an instance of the maximum-weight matching problem in $H - v$: to each edge e of $H - v$ assign the weight $w(e) = |e \cap N_H(v)|$ and verify if $H - v$ has a matching of weight at least $|N_H(v)|$. Since the maximum-weight matching problem is polynomial-time solvable (e.g., by Edmonds [16]), the theorem follows. \square

Lemma 4.1 and Theorem 5.3 imply the following.

Corollary 5.4. STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION problems are solvable in polynomial time in the class of line graphs.

We now turn to complements of line graphs. For this class of graphs, let us first note that Theorem 2.1 together with results of Boros et al. [6, Proposition 4 and Corollary 5] imply that STRONG CLIQUE EDGE COVER is solvable in polynomial time in the class of complements of line graphs of bipartite graphs. This result is significantly strengthened in the following theorem and its corollary.

Theorem 5.5. STRONG CLIQUE EXTENSION is solvable in polynomial time in the class of complements of line graphs.

Proof. Let G be the complement of a line graph and let C be a clique in G . We first compute the complement of G and then, using Theorem 2.1, a graph H such that $G \cong L(H)$. Thus C corresponds to a matching C_H in H and by Observation 5.2, it suffices to check if H contains a matching M such that $C_H \subseteq M$ and M intersects every set in \mathcal{M}_H , where \mathcal{M}_H is the set of all inclusion-maximal sets in the family of triangles and stars of H . We reduce this problem to an instance of the maximum-weight matching problem in H' where H' is the graph obtained from H by deleting from it all edges not in C_H that share an endpoint with an edge in C_H (as no such edge can be part of a matching containing C_H).

Before explaining the reduction, we show that for every matching M in H' that intersects every set in \mathcal{M}_H , we have $C_H \subseteq M$. Indeed, if $e \in C_H$ then there exists a set $S \in \mathcal{M}_H$ such that $e \in S$; moreover, the construction of H' implies that $S \cap E(H') = \{e\}$ and hence M can only intersect S in e ; this implies $e \in M$. Since $e \in C_H$ was arbitrary, we have $C_H \subseteq M$, as claimed.

The reduction is as follows. We compute the set \mathcal{M}_H and assign to each edge $e \in E(H')$ weight $w(e)$, which denotes the number of sets in \mathcal{M}_H containing e . Note that no set in \mathcal{M}_H contains two edges from the same matching. It follows that for any matching M in H' , the total number of sets in \mathcal{M}_H intersected by some edge in M is equal to the total weight of M (with respect to w). Consequently, H' contains a matching containing C_H that intersects every set in \mathcal{M}_H , if and only if H' contains a matching that intersects every set in \mathcal{M}_H , if and only if H' contains a matching of total w -weight at least (equivalently, exactly) k , where k is the cardinality of \mathcal{M}_H . The weight function $w(e)$ can be computed in polynomial time and the maximum-weight matching problem is also polynomial-time solvable, hence the theorem is proved. \square

Lemma 4.1 and Theorem 5.5 imply the following.

Corollary 5.6. STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION problems are solvable in polynomial time in the class of complements of line graphs.

An immediate consequence of the fact that STRONG CLIQUE PARTITION is solvable in polynomial time in the class of complements of line graphs is the following.

Corollary 5.7. STRONG CLIQUE PARTITION EXISTENCE restricted to the class of complements of line graphs is in NP.

In contrast to Corollary 5.6, we show next that STRONG CLIQUE PARTITION EXISTENCE is NP-complete in the class of complements of line graphs. The proof is based on the following.

Lemma 5.8. Let $k \geq 2$ be an integer and let G be a triangle-free k -regular graph. Then G is k -edge-colorable if and only if $\overline{L(G)}$ is localizable.

Proof. First, note that since G is k -regular, we have $\chi'(G) \geq k$. In particular, G is k -edge-colorable if and only if $\chi'(G) = k$. Since G has no vertices of degree 0 or 1, no two stars of G are comparable with respect to set inclusion, which implies that the maximal cliques of $L(G)$ are exactly the stars of G . Since G is k -regular, all maximal cliques of $L(G)$ are of size k and consequently $L(G)$ is a well-covered graph of independence number k . It now follows from Theorem 2.3 that $L(G)$ is localizable if and only if $\theta(L(G)) = k$. The statement of the lemma now follows from the fact that $\theta(L(G)) = \chi(L(G)) = \chi'(G)$ is the chromatic index of G . \square

Theorem 5.9. For every integer $k \geq 3$, STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is NP-complete in the class of complements of line graphs of triangle-free k -regular graphs.

Proof. The fact that the problem is in NP follows from Corollary 5.7. Cai and Ellis showed in [11] that for every $k \geq 3$, it is NP-complete to determine whether a given k -regular triangle-free graph is k -edge-colorable. Thus, if $k \geq 3$ and G is a given k -regular triangle-free graph, Lemma 5.8 implies that G is k -edge-colorable if and only if the complement of its line graph is localizable. The theorem follows. \square

We conclude the section with some remarks related to [Theorem 5.9](#). The fact that testing if a given complement of a line graph is localizable is NP-complete is in contrast with the fact that the more general property of well-coveredness can be tested in polynomial time in the class of complements of line graphs. Indeed, in order to verify whether $\overline{L(G)}$ is well-covered, it suffices to compute G (using [Theorem 2.1](#)), compute \mathcal{M}_G (see [Definition 5.1](#)), and test if all sets in \mathcal{M}_G are of the same size.

[Theorem 5.9](#) also has the following consequence.¹

Theorem 5.10. *For every integer $k \geq 3$, STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is NP-complete in the class of graphs of independence number k .*

Proof. As noted in the proof of [Lemma 5.8](#), if G is a k -regular triangle-free graph, then $\overline{L(G)}$ has independence number k . Hence, NP-hardness of the problem follows from [Theorem 5.9](#). Membership in NP follows from the fact that for every fixed k , STRONG CLIQUE is polynomial-time solvable in the class of graphs of independence number k . Indeed, to test if a given clique C in such a graph G is strong, one only needs to enumerate all $\mathcal{O}(|V(G)|^k)$ maximal independent sets of G and test if C intersects each of them. Consequently, a partition of the vertex set of G into k strong cliques is a polynomially verifiable certificate. \square

The statement of [Theorem 5.10](#) is sharp in the sense that localizable graphs of independence number at most two are recognizable in polynomial time. Indeed, using [Theorem 2.3](#) it is not difficult to see that a graph of independence number two is localizable if and only if its complement is a bipartite graph without isolated vertices. Finally, we remark that the result of [Theorem 5.10](#) contrasts with the fact that for every fixed k , testing if a given graph G with $\alpha(G) \leq k$ is well-covered can be done in polynomial time, simply by enumerating all the $\mathcal{O}(|V(G)|^k)$ independent sets and comparing any pair of maximal ones with respect to their cardinality. In particular, unless $P = NP$, one cannot obtain a simultaneous strengthening of [Theorems 3.2](#) and [5.10](#) that would establish hardness of recognizing localizable graphs within the class of weakly chordal graphs of bounded independence number. Indeed, the above observation together with [Corollary 2.4](#) implies that this problem is solvable in polynomial time.

6. Graphs of small maximum degree

In this section, we study the six strong clique problems in graphs of small maximum degree. For the first five problems from the list, we adopt a similar approach as in [Section 5](#), by showing that STRONG CLIQUE EXTENSION, the auxiliary problem introduced in [Section 4](#), is solvable in polynomial time in any class of graphs of bounded degree and, more generally, in any class of graphs of bounded clique number. Determining the complexity status of the ‘last’ of the six strong clique problems, STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY), seems to be more challenging. For this problem, we show polynomial-time solvability in the class of subcubic graphs.

Proposition 6.1. *For every positive integer k , STRONG CLIQUE EXTENSION is solvable in polynomial time in the class of graphs of clique number at most k .*

Proof. Fix a positive integer k and let G be a given graph with clique number at most k . Since k is a constant, the $\mathcal{O}(|V(G)|^k)$ maximal cliques of G can be enumerated in polynomial time. Thus, it suffices to show that STRONG CLIQUE can be solved in polynomial time. Indeed, if this is the case, then an efficient algorithm for testing if a given clique can be extended to a strong one can be obtained by checking if any of the maximal cliques of G that contains the given clique is strong.

By [Lemma 2.2](#), a clique C in G is not strong if and only if it is dominated by an independent set $I \subseteq V(G) \setminus C$. However, if this is the case, then any minimal set $I' \subseteq I$ that dominates C is contained in $N_G(C)$ and contains at most k vertices. Consequently, to test if C is strong, it suffices to enumerate all $\mathcal{O}(|V(G)|^k)$ subsets of $N_G(C)$ of size at most k and test whether any of them is independent and dominates C . This completes the proof. \square

[Proposition 6.1](#) and [Lemma 4.1](#) imply the following.

Corollary 6.2. *For every positive integer k , each of the problems STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION is solvable in polynomial time in the class of graphs of clique number at most k .*

Since every graph of maximum degree at most k has clique number at most $k + 1$, [Proposition 6.1](#) and [Corollary 6.2](#) also have the following consequence.

Corollary 6.3. *For every non-negative integer k , each of the problems STRONG CLIQUE EXTENSION, STRONG CLIQUE, STRONG CLIQUE EXISTENCE, STRONG CLIQUE VERTEX COVER, STRONG CLIQUE EDGE COVER, and STRONG CLIQUE PARTITION is solvable in polynomial time in the class of graphs of maximum degree at most k .*

¹ [Theorem 5.10](#) can also be derived using a reduction from k -colorability of triangle-free graphs, following the approach used in the proof of [[24](#), [Theorem 3.3](#)].

We now turn to the more challenging STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) problem. Determining the complexity status of this problem in the class of K_4 -free graphs, or more generally, in any class of graphs of bounded clique number, was stated as an open question in [24]. We are also not aware of any complexity results for this problem in classes of graphs of bounded maximum degree. This is in contrast with the fact that there is a linear-time algorithm for recognizing well-covered graphs of bounded maximum degree, due to Caro et al. [13]. A characterization of well-covered subcubic graphs was given by Ramey [37].

The structure of graphs of maximum degree at most two is so restricted that a polynomial-time algorithm for STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) in this graph class is straightforward. In what follows, we prove that the problem is also solvable in polynomial time in the more general class of subcubic graphs. We begin with a lemma.

Lemma 6.4. *Let $G = (V, E)$ be a connected subcubic graph admitting a strong clique C of size 3. If there exists a strong clique C' in G such that $C \neq C'$ and $C \cap C' \neq \emptyset$, then G is localizable if and only if $G \cong \overline{P_2 + P_3}$.*

Proof. Let G, C and C' be as in the statement. It is straightforward to verify that G is localizable if $G \cong \overline{P_2 + P_3}$. Suppose that G is localizable, let \mathcal{C} be a partition of V into strong cliques, and let $C = \{a, b, c\}$. Suppose first that $|C'| = 3$. If $|C \cap C'| = 1$, then the vertex in $C \cap C'$ would have degree at least 4, contrary to the assumption that G is subcubic. Therefore, $|C \cap C'| = 2$. We may, without loss of generality, suppose that $C \cap C' = \{a, b\}$. Let d be the remaining vertex of C' . If there are no other vertices in G , then G is isomorphic to the diamond graph (K_4 minus an edge), which is not localizable. Since G is connected with $\Delta(G) = 3$, it follows that one of c and d has a neighbor distinct from a and b . If c and d are adjacent then $G \cong K_4$ contradicting the assumption that C is a strong clique. Without loss of generality, we may assume that x is the neighbor of c different from a and b . If x and d are non-adjacent, then we may extend $\{d, x\}$ to a maximal independent set I in G such that $I \cap C = \emptyset$, contrary to the assumption that C is strong clique. We conclude that x and d are adjacent. One of the cliques C and C' must be contained in \mathcal{C} , since they are the only strong cliques containing vertices a and b . It is now clear that at least one of the cliques $\{c, x\}$ and $\{d, x\}$ must belong to \mathcal{C} . If G has more than 5 vertices, then since G is connected with $\Delta(G) = 3$, it follows that x has a neighbor y different from c and d . It is clear that y and a are non-adjacent, since a already has degree 3. But now, any maximal independent set in G containing $\{a, y\}$ is disjoint from both cliques $\{c, x\}$ and $\{d, x\}$, contrary to the assumption that one of them is in \mathcal{C} and hence is strong. Thus, G has exactly 5 vertices, and it is obvious that $G \cong \overline{P_2 + P_3}$.

Suppose now that $|C'| = 2$. Without loss of generality we may assume that $C \cap C' = \{a\}$ and $C' = \{a, d\}$. We claim that d has degree at least 2. Suppose on contrary that d has degree 1. Then the only strong clique containing d is $\{a, d\}$, hence it must be contained in \mathcal{C} . This implies that $C \notin \mathcal{C}$. Therefore, vertices b and c must be contained in some strong cliques different from C . By the first part of the proof, it follows that b and c are contained in strong cliques of size 2 (otherwise we would have two strong triangles with non-empty intersection). Let $\{b, x\}$ and $\{c, y\}$ be strong cliques containing b and c , respectively. Since C is a strong clique in G , it follows that x and y are adjacent. However, a maximal independent set containing a and x cannot intersect $\{c, y\}$, contradicting the assumption that $\{c, y\}$ is a strong clique. The obtained contradiction establishes our claim that d has degree at least 2.

Observe that d cannot be adjacent to b or c , since $\{a, d\}$ is strong clique. Let x be a neighbor of d different from a . Since $\{a, d\}$ is a strong clique in G , it follows that x is adjacent to both b and c . If d has a third neighbor y , then y would have to be adjacent to both b and c as well, contrary to the assumption that $\Delta(G) \leq 3$. Hence, d has no other neighbor, and since G is connected, it follows that G is again isomorphic to $\overline{P_2 + P_3}$. \square

We are now ready to state the main result of this section.

Theorem 6.5. STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is solvable in polynomial time in the class of subcubic graphs.

Proof. Let $G = (V, E)$ be a subcubic graph. Since a graph is localizable if and only if each of its connected components is localizable, we may assume without loss of generality that G is connected and non-trivial. Since the maximum degree of G is at most 3, we can find all strong cliques of G in polynomial time (we need to check all subsets of size at most 3, and the subgraphs induced by their neighborhoods). If there exists a strong clique of size 4, then $G \cong K_4$, hence localizable. Otherwise, all strong cliques in G are of sizes 2 or 3. If there exists a strong clique of size 3 with non-empty intersection with another strong clique in G , then by Lemma 6.4, G is localizable if and only if $G \cong \overline{P_2 + P_3}$.

Hence we may assume from now on that every strong clique of size 3 is disjoint with every other strong clique in G . Therefore, if G is localizable, then every strong clique of size 3 is contained in every partition of V into strong cliques. We now form a new graph G' formed by vertices of G that do not belong to strong cliques of size 3. Two vertices of G' are adjacent if they form a strong clique of size 2 in G . If there exists a perfect matching M in G' , then this matching, together with all strong cliques of G of size 3 is a partition of V into strong cliques. Similarly, if there exists a partition of V into strong cliques, then it is clear that the set of the cliques of size 2 in this partition induces a perfect matching in G' . Hence G is localizable if and only if G' admits a perfect matching. Verifying if G' has a perfect matching can be done in polynomial time (see for instance [17]). \square

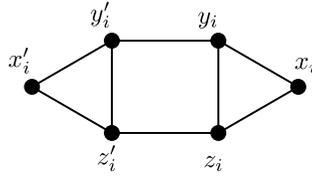


Fig. 2. The graph F .

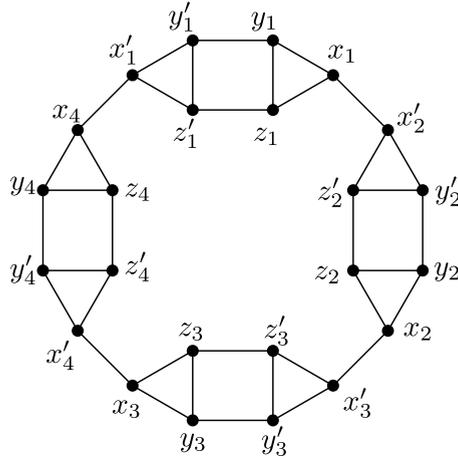


Fig. 3. The graph F_4 .

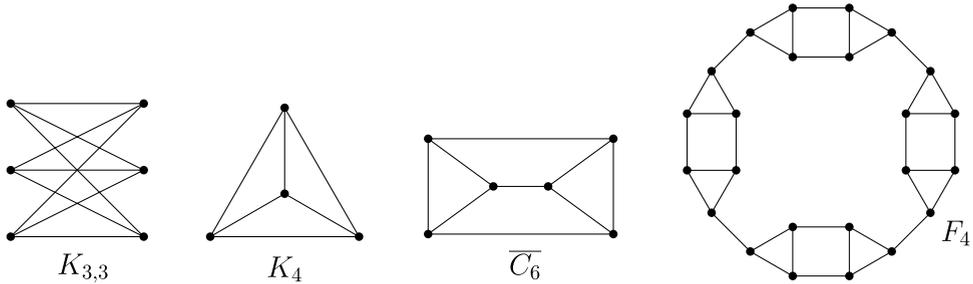


Fig. 4. Cubic localizable graphs: $K_{3,3}$, K_4 , $\overline{C_6}$, and an infinite family $\{F_n \mid n \geq 2\}$.

In the case of cubic (that is, 3-regular) graphs, the result of [Theorem 6.5](#) can be strengthened to a complete classification. This can be derived using the classification of well-covered cubic graphs due to Campbell et al. [[12](#)]. The list consists of three infinite families together with six exceptional graphs. By testing each of the graphs in the list for localizability, the classification of localizable cubic graphs given by [Theorem 6.6](#) can be derived. For an integer $n \geq 2$, let F_n denote the graph obtained as follows: take a cycle $v_1 v_2 \dots v_n v_1$ of length n (if $n = 2$, then $v_1 v_2 v_1$ is a cycle of length two with two parallel edges), replace every vertex v_i of the cycle by vertices $x_i, x'_i, y_i, y'_i, z_i, z'_i$ inducing the graph F (see [Fig. 2](#)); replace each edge $v_i v_{i+1}$ ($i = 1, \dots, n - 1$) of the cycle by an edge $x_i x'_{i+1}$, finally replace the edge $v_n v_1$ by the edge $x_n x'_1$ (see [Fig. 3](#) for an example).

Theorem 6.6. *Let G be a connected cubic graph. Then, G is localizable if and only if G is isomorphic to one of the graphs in the set $\{K_{3,3}, K_4, \overline{C_6}\} \cup \{F_n \mid n \geq 2\}$ (see [Fig. 4](#)).*

We now give here a short direct proof of [Theorem 6.6](#) and strengthen it further by adding to the list the property that every vertex of G is contained in a strong clique. First we develop a property of strong cliques in general regular graphs.

Lemma 6.7. *Let $r \geq 1$ and let G be a connected r -regular graph. Then G has a strong clique of size two if and only if $G \cong K_{r,r}$.*

Proof. Suppose that G is a connected r -regular graph, and let $\{u, v\}$ be a strong clique in G . Since the clique $\{u, v\}$ is strong, it is a maximal clique and therefore it is not contained in any triangle. Let A denote the set of neighbors of u

different from v , and let B denote the set of neighbors of v different from u . By definition, we have $A \cap B = \emptyset$. We claim that every vertex in A is adjacent to every vertex in B . Suppose for a contradiction that some vertex $x \in A$ is not adjacent to some vertex $y \in B$. Then extending the set $\{x, y\}$ to a maximal independent set of G results in a maximal independent not containing any of u, v , contradicting the assumption that $\{u, v\}$ is a strong clique. Therefore, every vertex in A is adjacent to every vertex in B , as claimed.

Since G is r -regular, $N(u) = A \cup \{v\}$, and $N(v) = B \cup \{u\}$, we have $|A| = |B| = r - 1$. The connectedness of G and the fact that G is r -regular implies that $G \cong K_{r,r}$. The converse direction is immediate. \square

Theorem 6.8. *Let G be a connected cubic graph. Then, the following statements are equivalent:*

1. G is localizable.
2. Every vertex of G is contained in a strong clique.
3. G is isomorphic to one of the graphs in the set $\{K_{3,3}, K_4, \overline{C_6}\} \cup \{F_n \mid n \geq 2\}$ (see Fig. 4).

Proof. It immediately follows from the definition of localizable graphs that every vertex of a localizable graph is contained in a strong clique. Therefore, statement 1 implies statement 2.

It is also easy to check that each of the graphs in the set $\{K_{3,3}, K_4, \overline{C_6}\} \cup \{F_n \mid n \geq 2\}$ is localizable. Indeed, each edge of $K_{3,3}$ is a strong clique; in K_4 the whole vertex set is a strong clique; in each of the remaining graphs each vertex is contained in a unique triangle and all these triangles are strong. Therefore, statement 3 implies statement 1.

Finally, we show that statement 2 implies statement 3. Since G is cubic, it cannot have a clique of size 5 or a strong clique of size 1. If there exists a strong clique of size 2 in G , then by Lemma 6.7, $G \cong K_{3,3}$. If there exists a strong clique of size 4 in G , then $G \cong K_4$ since G is cubic.

Suppose now that every vertex of G belongs to a strong clique of size 3. Let $C_1 = \{x_1, y_1, z_1\}$ be a strong clique in G . All three vertices of C_1 cannot have a common neighbor, otherwise C_1 would not be a strong clique. Suppose that two vertices of C_1 , say x_1 and y_1 , have a common neighbor, say w , outside C_1 . Let $w' \neq z_1$ be the remaining neighbor of w . Since the only clique of size 3 that contains w is $\{x_1, y_1, w\}$, it follows that $\{x_1, y_1, w\}$ is a strong clique. If w' is not adjacent to z_1 , then a maximal independent set in G containing $\{w', z_1\}$ would be disjoint from $\{x_1, y_1, w\}$, contradicting the fact that $\{x_1, y_1, w\}$ is a strong clique. This implies that w' is adjacent to z_1 . However, in this case, vertex w' does not belong to any triangle in G , which contradicts the assumption that every vertex of G belongs to a strong clique of size 3. This shows that no two vertices of C_1 have a common neighbor outside of C_1 .

If the neighbors of x_1, y_1, z_1 outside of C_1 form an independent set S , then any maximal independent set in G containing S would be disjoint from C_1 , a contradiction. Suppose now that y'_1 and z'_1 are the remaining neighbors of y_1 and z_1 , and that y'_1 is adjacent to z'_1 . Since every vertex belongs to a triangle, it follows that y'_1 and z'_1 have a common neighbor, say x'_1 . If x_1 is adjacent to x'_1 then $G \cong \overline{C_6}$. So we may assume that x_1 and x'_1 are not adjacent. It follows from the above that for any strong clique $C = \{u, v, w\}$ in G , there exist vertices u', v', w' such that $vv', ww' \in E$, $\{u', v', w'\}$ is a strong clique and u is not adjacent to u' .

Let x'_2 be the remaining neighbor of x_1 . Since x'_2 belongs to a strong clique, it follows that there exist vertices y'_2 and z'_2 such that $C_2 = \{x'_2, y'_2, z'_2\}$ is a strong clique. Note that the fact that G is cubic implies that $\{y'_2, z'_2\} \cap \{x_1, y_1, z_1, x'_1, y'_1, z'_1\} = \emptyset$. Let y_2 and z_2 be the remaining neighbors of y'_2 and z'_2 respectively. Then using the same argument as above, replacing C with C_2 one can easily see that y_2 and z_2 are adjacent, and that they have a common neighbor, say x_2 . If x_2 is adjacent with x'_1 , then $G \cong F_2$. If x_2 and x'_1 are not adjacent, then let x'_3 be the remaining neighbor of x_2 and repeat the same argument as before. Since the graph is finite and connected, it follows that for some n , we will have that x_n is adjacent with x'_1 , hence $G \cong F_n$. \square

We conclude the section by noting that, as observed in [23], the equivalence between statements 1 and 2 in Theorem 6.8 cannot be generalized to arbitrary regular graphs. A small 8-regular counterexample is given by the line graph of K_6 . Furthermore, in [23] non-localizable regular graphs are constructed in which every maximal clique is strong.

7. Conclusion

In this work we performed a detailed study of the complexity of six algorithmic decision problems related to strong cliques in graphs, with a focus on the classes of weakly chordal graphs, line graphs and their complements, and subcubic graphs. Our results, summarized in Table 1, leave open the complexity status of STRONG CLIQUE EDGE COVER (a.k.a. GENERAL PARTITIONABILITY) not only for general graphs (which is a known open problem, see, e.g., [1,8,25]) but also for the class of weakly chordal graphs. As a corollary of our matching-based study of strong cliques in the class of complements of line graphs, we showed that STRONG CLIQUE PARTITION EXISTENCE (a.k.a. LOCALIZABILITY) is NP-complete in the class of graphs of independence number k , for every fixed $k \geq 3$. As a natural question for future research, it would be interesting to perform a more detailed complexity study of problems related to strong cliques in graphs of bounded independence number.

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Conflict of interest

None.

Declaration of competing interest

The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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