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On Split B_1-EPG Graphs

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Abstract. In this paper, we are interested in edge intersection graphs of paths in a grid, such that each such path has at most one bend. These graphs were introduced in [12] and they are called B_1-EPG graphs. In particular, we focus on split graphs and characterise those that are B_1-EPG. This characterisation allows us to disprove a conjecture of Cameron et al. [7]. The existence of polynomial-time recognition algorithm for this graph class is still unknown. We furthermore investigate inclusion relationships among subclasses of split graphs that are B_1-EPG.

1 Introduction

Golumbic et al. introduced in [12] the notion of *edge intersection graphs of paths in a grid* (referred to as *EPG graphs*). An undirected graph $G = (V, E)$ is called an *EPG graph*, if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-edge. The authors showed in [12] that every graph is in fact an EPG graph. Therefore, they introduced additional restrictions on the paths by limiting the number of *bends* (a bend is a 90° turn of a path at a grid-point) that a path can have. An undirected graph $G = (V, E)$ is then called a *B_k-EPG graph*, for some integer $k \geq 0$, if one can associate with each vertex a path with at most k bends in a rectangular grid such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-edge.

One motivation for introducing these graphs comes from chip manufacturing. Indeed, each wire on a chip can be seen as a path on a rectangular grid. Since each wire bend requires a so-called transition hole, and since a large number of such holes increase the layout area as well as the overall cost of the chip, it is of interest to limit the total number of holes respectively to limit the number of bends per wire. Another motivation comes from the fact that B_k-EPG graphs generalize the well-known class of interval graphs. From its definition, it is easy to see that the class of B_0-EPG graphs is indeed equivalent to the class of interval graphs.

Since the introduction of the notion of B_k-EPG graphs, there has been a lot of research done on these graphs from several points of view (see for instance [1–8,10,11,13–15]). Since B_0-EPG graphs coincide with the class of interval graphs, particular attention has been paid to the class of B_1-EPG graphs. The authors in [13] showed that recognizing B_1-EPG graphs is an NP-complete problem; the same holds for B_2-EPG graphs as recently shown in [15]. In any representation of a B_1-EPG graph, each path can only have one of the following four shapes: $\llcorner$, $\urcorner$, $\ulcorner$, $\lrcorner$ (a path with only a horizontal part or only a vertical part can be considered as a degenerate path of one of the four shapes mentioned before). In [7], the authors analysed B_1-EPG graphs for which the number of different shapes is restricted to a subset of the set above. They showed that testing membership to each of these restricted classes is also NP-complete. Furthermore, they focused on chordal graphs that are B_1-EPG with the additional restriction that only one particular shape (namely $\llcorner$) is allowed for all paths. In particular, they state a conjecture concerning the characterisation of split graphs that are B_1-EPG and where only paths with an $\llcorner$ shape are allowed, by a family of forbidden induced subgraphs. Indeed, they present a list of nine forbidden induced subgraphs and conjecture that these are the only ones. In this paper, we disprove this conjecture by providing an additional forbidden induced subgraph. However, giving a complete list of forbidden induced subgraphs or deciding whether such a finite list exists remains open. Furthermore, we provide a characterisation of split graphs that are B_1-EPG and where only paths with an $\llcorner$ shape are allowed. Notice that this characterisation does not imply a polynomial-time recognition algorithm. In addition, for any subset P of the four possible shapes mentioned above, we investigate inclusion relationships among subclasses of split graphs that are B_1-EPG and where only shapes from P are allowed.

Our paper is organised as follows. In Sect. 2, we present definitions and notations as well as some preliminary results and observations that we will use throughout the paper. Section 3 deals with the inclusion relationships among subclasses of split graphs that are B_1-EPG. In Sect. 4, we present a characterisation of split B_1-EPG graphs and disprove the conjecture of Cameron et al. [7]. We finish with Sect. 5 in which we also mention further results that we obtained and suggest some further research directions.

2 Preliminaries

We only consider finite, undirected graphs that have no self-loops and no multiple edges. We refer to [9] or [16] for undefined terminology. Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \in E \mid u, v \in S\}$. We write $H \subseteq_i G$ if a graph H is an induced subgraph of G. Moreover, for a vertex $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$, we write $G - V' = G[V \setminus V']$. The set of vertices adjacent to some vertex u is called the *neighborhood of* u and will be denoted by $N(u)$. The *closed neighborhood of* u is defined as $N[u] = N(u) \cup \{u\}$. A vertex u *dominates* some adjacent (resp. non-adjacent) vertex v if $N[v] \subseteq N[u]$

(resp. if $N(v) \subseteq N(u)$). Two vertices u, v in G are said to be *comparable* if u dominates v or v dominates u. Two vertices that are not comparable are said to be *incomparable*. A *split graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into a clique K (i.e., a set of pairwise adjacent vertices) and a stable set S (i.e., a set of pairwise non-adjacent vertices). We say that (K, S) is a *split partition of* G. The vertices in S will be called the *S-vertices*.

Let $\mathcal{G}$ be a rectangular grid of size $m \times m'$. The horizontal grid lines will be referred to as *rows* and denoted by $x_0, x_1, \cdots, x_{m-1}$ and the vertical grid lines will be referred to as *columns* and denoted by $y_0, y_1, \cdots, y_{m'-1}$. As already mentioned above, in any representation of a B_1-EPG graph, each path can only have one of the following four possible shapes: ⌞, ⌝, ⌜, ⌟. A path with an ⌞-shape will be called an *⌞-path*. In a similar way we define ⌝-path, ⌜-path and ⌟-path. For any subset P of the four possible shapes, we denote by $[P]$ the class of B_1-EPG graphs which admit a representation in which each path has one of the shapes in P. In particular, we denote by $[P]_s$ the class of B_1-EPG split graphs which admit a representation in which each path has one of the shapes in P. For simplicity, if P contains all four shapes, we write B_1-EPG$_s$. A representation of a B_1-EPG graph containing only paths with a shape in P is called a $[P]$-representation.

Let $G = (V, E)$ be a B_1-EPG graph and let $v \in V$. We denote by P_v the path representing v in a B_1-EPG representation of G. Consider a clique K (resp. a stable set S) in G. Any path representing a vertex in K (resp. in S) will simply be referred to as a *path of* K (resp. *path of* S). Concerning cliques, the following useful lemma has been shown in [12].

Lemma 1. *Let $G = (V, E)$ be a B_1-EPG graph. In any B_1-EPG representation of G, a clique K of G is represented either as an edge-clique or as a claw-clique (see Fig. 1).*

Notice that in an edge-clique, all paths share a common grid-edge, called the *base of the clique*, while in a claw-clique, all paths share a common grid-point, called the *center of the clique*.

A *gem* is a graph with vertex set $\{c_1, c_2, c_3, s_1, s_2\}$ and edge set $\{s_1c_1, s_1c_2, c_1c_2, c_2c_3, c_1c_3, s_2c_2, s_2c_3\}$ (see Fig. 2(a)). It is easy to see that a gem, as an induced subgraph of a split graph $G = (V, E)$ with split partition (K, S), must satisfy $c_1, c_2, c_3 \in K$ and $s_1, s_2 \in S$. A *bull* is a graph with vertex set $\{c_1, c_2, s_1, s_2, s_3\}$ and edge set $\{c_1c_2, c_1s_2, c_2s_2, c_1s_1, c_2s_3\}$ (see Fig. 2(b)). Again, it is easy to see that a bull, as an induced subgraph of a split graph $G = (V, E)$ with split partition (K, S), must satisfy $c_1, c_2 \in K$ and $s_1, s_3 \in S$. In the case

(a) (b)

Fig. 1. An edge-clique (a) and a claw-clique (b).

where $s_2 \in S$ as well, the bull is called an *S-bull*. Gems and S-bulls have played an important role in [7]. As we will see, they are also crucial in our results.

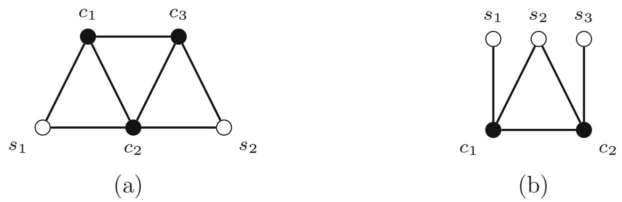


Fig. 2. (a) A gem. (b) An S-bull.

The following definitions have been introduced in [7]. Let $G = (V, E)$ be in $[\llcorner]_s$ with split partition (K, S). Consider an $[\llcorner]_s$-representation of G. Clearly, the clique K must be represented as an edge-clique. This grid-edge is called the *base*. Without loss of generality, we may assume that the base is vertical. The horizontal parts of the paths representing vertices in K are called *branches*. Let F be the vertical line-segment which is the union of the vertical parts of all paths representing vertices in K. The part of F below the base is called the *trunk*. The part of F above trunk is called the *crown* (see Fig. 3).

The following three observations have been made in [7]. As we will see, they will be very helpful in the proof of our main results.

Observation 1 *([7]). Let $G = (V, E)$ be a split graph in $[\llcorner]_s$. Then, the S-vertices whose paths lie on the same branch (or on the crown) are pairwise comparable. Furthermore, an S-vertex whose path lies on the trunk dominates all S-vertices whose paths lie below it in the representation.*

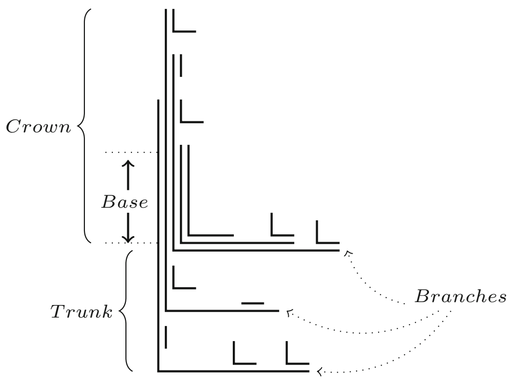


Fig. 3. An $[\llcorner]$-representation of a split graph with the notions of crown, base, branches and trunk.

Observation 2 *([7]). Let $G = (V, E)$ be a split graph in $[\llcorner]_s$. If G contains a gem, then exactly one of the gem's S-vertices has its path lying on the crown of the representation.*

Observation 3 *([7]). Let $G = (V, E)$ be a split graph in $[\llcorner]_s$. If G contains an S-bull, then some S-vertices of this bull have their paths lying on either the crown or trunk of the representation.*

It is easy to see that we can generalize Observation 1 in the following way.

Observation 4. *Let $G = (V, E)$ be a split graph in B_1-EPG_s with split partition (K, S).*

Assume that K is represented as an edge-clique with base going from (x_i, y_j) to (x_{i+1}, y_j) (see Fig. 4(a)). Then, the S-vertices whose paths use column y_j above (x_{i+1}, y_j), say between rows x_{i+1} and x_{i+k} (resp. below (x_i, y_j), say between rows x_{i-k} and x_i) and the S-vertices whose paths use some row $x_{i+\ell}$, $\ell \geq k$ (resp. $x_{i-\ell}$, $\ell \geq k$) on a same side of y_j (right or left) are pairwise comparable.

Similarly, assume that K is represented as a claw-clique with center (x_i, y_j) and assume that no path of K uses the grid-edge going from (x_i, y_{j-1}) to (x_i, y_j) (see Fig. 4(b)). Then, the S-vertices whose paths use column y_j above (x_i, y_j), say between rows x_{i+1} and x_{i+k} (resp. below (x_i, y_j), say between rows x_{i-k} and x_{i-1}) and the S-vertices whose paths use some row $x_{i+\ell}$, $\ell \geq k$ (resp. $x_{i-\ell}$, $\ell \geq k$) on a same side of y_j are pairwise comparable. Furthermore, the S-vertices whose paths use row x_i to the right of (x_i, y_j) are also pairwise comparable.

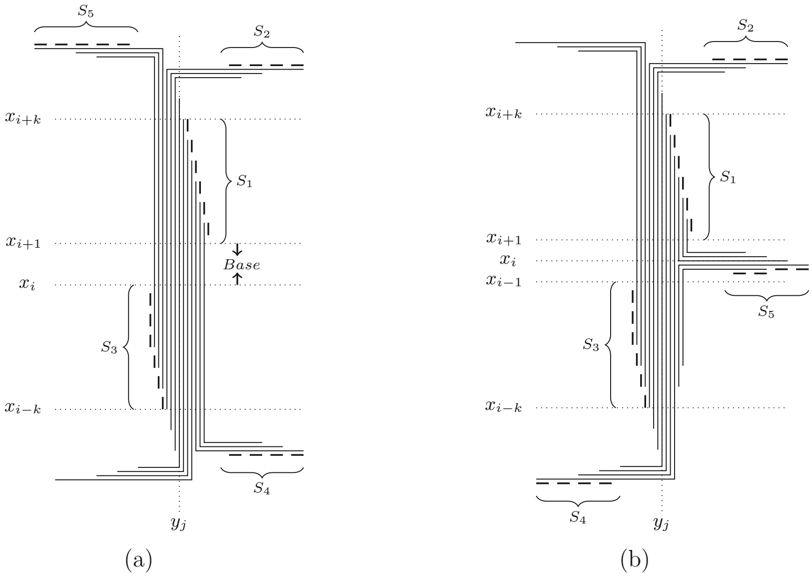


Fig. 4. (a) Vertices in $S_1 \cup S_2$ (resp. $S_1 \cup S_5$, $S_3 \cup S_4$) are pairwise comparable. (b) Vertices in $S_1 \cup S_2$ (resp. $S_3 \cup S_4$) are pairwise comparable; also the vertices of S_5 are pairwise comparable.

3 Subclasses of B_1-EPG_s

In [7], the authors showed that $[\llcorner] \subsetneq [\llcorner, \urcorner], [\llcorner, \ulcorner] \subsetneq [\llcorner, \ulcorner, \urcorner] \subsetneq B_1\text{-}EPG$ and that the two classes $[\llcorner, \urcorner], [\llcorner, \ulcorner]$ are incomparable. Here, we obtain a similar result when restricted to split graphs.

Theorem 1. $[\llcorner]_s \subsetneq [\llcorner, \urcorner]_s \subsetneq [\llcorner, \ulcorner]_s \subsetneq [\llcorner, \ulcorner, \urcorner]_s \subsetneq B_1\text{-}EPG_s$.

Notice that for split graphs we have $[\llcorner, \urcorner]_s \subsetneq [\llcorner, \ulcorner]_s$. We will prove Theorem 1 by a series of four lemmas (Lemmas 2, 3, 4 and 5). We first start with a useful proposition.

Proposition 1. *Consider a B_1-EPG representation of a gem (see Fig. 2(a)). Let $K = \{c_1, c_2, c_3\}$ and $S = \{s_1, s_2\}$. If K is represented as an edge-clique with base going from (x_i, y_j) to (x_{i+1}, y_j) or if K is represented as a claw-clique with center (x_i, y_j) and no path of K uses the grid-edge going from (x_i, y_{j-1}) to (x_i, y_j), then at least one of P_{s_1}, P_{s_2} intersects paths of K on column y_j.*

Proof. Consider a B_1-EPG representation of a gem with $K = \{c_1, c_2, c_3\}$ and $S = \{s_1, s_2\}$. Suppose that K is represented as an edge-clique with base going from (x_i, y_j) to (x_{i+1}, y_j) or K is represented as a claw-clique with center (x_i, y_j) and no path of K uses the grid-edge going from (x_i, y_{j-1}) to (x_i, y_j). By contradiction assume that both P_{s_1}, P_{s_2} do not intersect paths of K on column y_j. Since all paths have at most one bend, it follows that both P_{s_1}, P_{s_2} intersect paths of K on rows. Since s_1, s_2 have a common neighbour, P_{s_1}, P_{s_2} must intersect paths of K on a same row x_k either both to the right of y_j or both to the left of y_j. But this is not possible since s_1, s_2 are incomparable (see Observation 4).

Lemma 2. $[\llcorner]_s \subsetneq [\llcorner, \urcorner]_s$.

Proof. We clearly have $[\llcorner]_s \subseteq [\llcorner, \urcorner]_s$. Consider the graph G_4 in Fig. 5(a). We know from [7] that G_4 is not in $[\llcorner]_s$. But it is easy to see that G_4 is in $[\llcorner, \urcorner]_s$ (see Fig. 5(b)). Thus, $[\llcorner]_s \subsetneq [\llcorner, \urcorner]_s$.

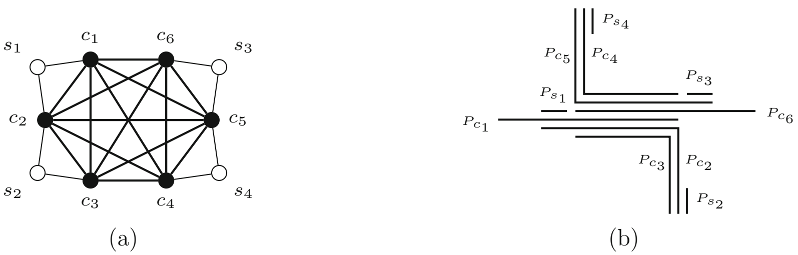


Fig. 5. The graph G_4 and a $[\llcorner, \urcorner]$-EPG representation of it.

Since G_4 is a minimal forbidden induced subgraph for the class $[\llcorner]_s$, it is minimal under inclusion with the property that it belongs to $[\llcorner, \urcorner]_s \backslash [\llcorner]_s$. However, there may exist other examples with fewer vertices.

Fig. 6. The graph G_1 and a $[\llcorner,\ulcorner]$-EPG representation of it.

Lemma 3. $[\llcorner,\urcorner]_s \subsetneq [\llcorner,\ulcorner]_s$.

Proof. Consider the graph G_1, also called the 3-*sun*, in Fig. 6(a). It is clearly a split graph, and it has been shown in [7] to belong to $[\llcorner,\ulcorner] \backslash [\llcorner,\urcorner]$ (see Fig. 6(b) for a $[\llcorner,\ulcorner]$-EPG representation of it).

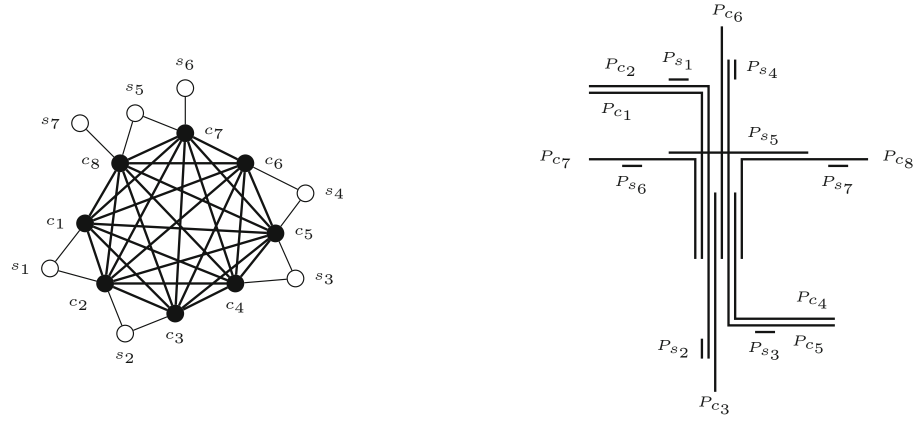
So we conclude that $[\llcorner,\urcorner]_s \neq [\llcorner,\ulcorner]_s$. It remains to show that $[\llcorner,\urcorner]_s \subset [\llcorner,\ulcorner]_s$. Consider a split graph G in $[\llcorner,\urcorner]_s$ with split partition (K, S). It follows from Lemma 1, that K must be represented as an edge-clique. Without loss of generality, we may assume that the base of K is vertical and goes from (x_i, y_j) to (x_{i+1}, y_j). Notice that, since only $\llcorner$-paths and $\urcorner$-paths are allowed, we may assume that each path of S intersects paths of K either with its vertical part or with its horizontal part, but never with both and thus, it is a degenerate path. Notice that no $\urcorner$-path has its horizontal part below (x_{i+1}, y_j), and no $\llcorner$-path has its horizontal part above (x_i, y_j). We may therefore transform the part above (x_{i+1}, y_j) of the whole representation by a symmetry with respect to column y_j, resulting in a $[\llcorner,\ulcorner]$ representation of G. Thus, $[\llcorner,\urcorner]_s \subsetneq [\llcorner,\ulcorner]_s$.

Notice that the symmetry used in the proof of Lemma 3 could not be used if one wanted to show that $[\llcorner,\ulcorner]_s \subsetneq [\llcorner,\urcorner]_s$, since the graph may have its clique represented by a claw-clique. Therefore, $[\llcorner,\ulcorner]_s \backslash [\llcorner,\urcorner]_s$ is exactly the set of those $[\llcorner,\ulcorner]_s$-EPG graphs admitting no split partition (K, S) such that K can be represented by an edge-clique. Notice also that since G_1 is the smallest graph not in $[\llcorner]_s$ (see [7]), it is also the smallest graph in $[\llcorner,\ulcorner]_s \backslash [\llcorner,\urcorner]_s$.

Lemma 4. $[\llcorner,\ulcorner]_s \subsetneq [\llcorner,\ulcorner,\urcorner]_s$.

Proof. We clearly have $[\llcorner,\ulcorner]_s \subseteq [\llcorner,\ulcorner,\urcorner]_s$. Let us consider the graph G_5 which belongs to $[\llcorner,\ulcorner,\urcorner]_s$ (see Fig. 7(a) and (b)). We will show that G_5 does not belong to $[\llcorner,\ulcorner]_s$.

By contradiction, assume that G_5 belongs to $[\llcorner,\ulcorner]_s$. We will distinguish two cases. First, suppose that the clique K induced by $\{c_1, \cdots, c_8\}$ is represented as an edge-clique. Without loss of generality, we may assume that the base of K is vertical and goes from (x_i, y_j) to (x_{i+1}, y_j). Since the vertex set $\{c_1, c_2, c_3, s_1, s_2\}$ induces a gem, it follows from Proposition 1 that at least one of P_{s_1}, P_{s_2} intersects paths of K on column y_j, say P_{s_1}. Also, we may assume, without loss



(a) (b)

Fig. 7. The graph G_5 and a $[\llcorner, \ulcorner, \urcorner]$-EPG representation of it.

of generality, that it intersects paths of K above row x_{i+1}, say above row x_{r_1}, $r_1 \geq i + 1$. Using the same argument for the gem induced by $\{c_4, c_5, c_6, s_3, s_4\}$, we may assume that P_{s_3} intersects path of K on column y_j. Since s_1, s_3 are incomparable, it follows that P_{s_3} lies below row x_i, say below row x_{r_3}, $r_3 \leq i$ (see Observation 4). Now consider the S-bull induced by $\{c_7, c_8, s_5, s_6, s_7\}$. Since c_7, c_8 are non-adjacent to s_1, s_3, their paths do neither go above row x_{r_1}, nor below row x_{r_3}. Since s_5, s_6, s_7 are non-adjacent to $c_1, \cdots, c_6$, it follows that $P_{s_5}, P_{s_6}, P_{s_7}$ cannot intersect paths of K on column y_j. Therefore, they must intersect paths of K on some same row x_r (since s_5 is adjacent to both c_7, c_8) and these intersections are all to the right of y_j (since we only allow $\llcorner$-paths and $\ulcorner$-paths. But this is clearly impossible, since s_6 and s_7 are incomparable. So we conclude that K cannot be represented as an edge-clique.

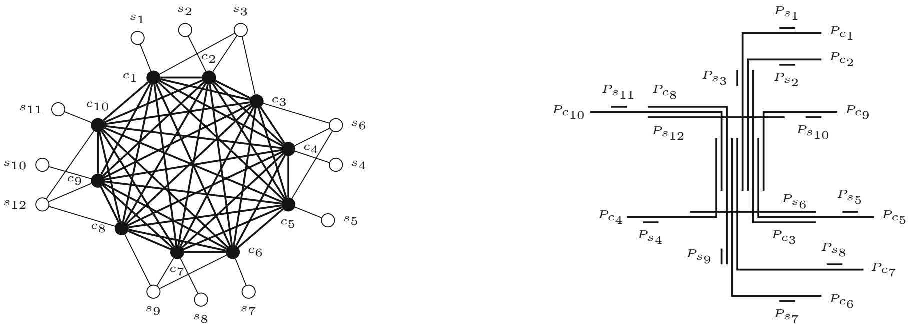
So we may assume now that K is represented as a claw-clique with center (x_i, y_j). Without loss of generality, we may assume that only paths of K use the grid edges going from (x_{i-1}, y_j) to (x_{i+1}, y_j). Clearly, all paths of S intersecting paths of K on row x_i must be pairwise comparable (see Observation 4). Hence, we immediately see that there are at most two such paths of S.

First assume there are exactly two paths of S intersecting paths of K on row x_i. Then, these must be paths P_{s_5}, P_{s_6} (resp. P_{s_5}, P_{s_7}). Without loss of generality, we may assume that c_7 is represented as a $\ulcorner$-path and c_8 is represented as an $\llcorner$-path. Notice that every other path of K with bend point (x_i, y_j) can be transformed into a vertical path (by deleting its horizontal part and extending it to (x_{i-1}, y_j) if it is an $\llcorner$-path and to (x_{i+1}, y_j) if it is a $\ulcorner$-path), since it does not intersect any path on row x_i. Hence, P_{c_7}, P_{c_8} are the only paths of K with bend point (x_i, y_j). Now, since c_7 (resp. c_8) has only two neighbours in S, namely s_5, s_6 (resp. s_5, s_7), it follows that it does not intersect any path of S with its vertical part. So we may transform P_{c_7} (resp. P_{c_8}) into an $\llcorner$-path (resp. a $\ulcorner$-path) with vertical part going from (x_i, y_j) to (x_{i+1}, y_j) (resp. to (x_{i-1}, y_j)). Hence, K is representable as an edge-clique. But we know from the above that this is not possible.

So let us now assume, that exactly one path of S is intersecting paths of K on row x_i. As before, notice that every path of K with bend point (x_i, y_j) not intersecting any path of S on row x_i can be transformed into a vertical path (by deleting its horizontal part and extending it to (x_{i-1}, y_j) if it is an $\llcorner$-path and to (x_{i+1}, y_j) if it is a $\ulcorner$-path). So this unique path of S intersecting paths of K on row x_i represents a vertex of degree at least two, hence one of $s_1, \cdots, s_5$ (otherwise we obtain again the case where K is represented as an edge-clique). First assume it represents s_1 (the cases when it represents s_2, s_3 or s_4 can be handled similarly). In other words, P_{c_1}, P_{c_2} are the only paths of K using row x_i. We may assume, without loss of generality, that c_1 is represented by a $\ulcorner$-path and c_2 by an $\llcorner$-path. Since c_1 does not have any neighbour in S except s_1, it follows that it does not intersect any path of S with its vertical part. Thus, as in the previous case, we can transform P_{c_1} into an $\llcorner$-path with vertical part going from (x_i, y_j) to (x_{i+1}, y_j). But then K is again represented as an edge-clique, a contradiction. So we may assume now that this unique path of S intersecting paths of K is s_5. Using the same arguments as above, we may assume that c_7 is represented by a $\ulcorner$-path and c_8 by an $\llcorner$-path and P_{c_7}, P_{c_8} are the only paths of K using row x_i. We immediately conclude that P_{s_6} must intersect P_{c_7} on column y_j below row x_{i-1}, and P_{s_7} must intersect P_{c_8} on column y_j above row x_{i+1}. It follows from Proposition 1, that at least one of P_{s_1}, P_{s_2} intersects paths of K on column y_j, since $\{c_1, c_2, c_3, s_1, s_2\}$ induces a gem. But this is not possible since neither of them is comparable with one of s_6, s_7 (see Observation 4). Thus, G_5 does not belong to $[\llcorner, \ulcorner]_s$.

Lemma 5. $[\llcorner, \ulcorner, \urcorner]_s \subsetneq B_1\text{-}EPG_s$.

Proof. We clearly have $[\llcorner, \ulcorner, \urcorner]_s \subseteq B_1\text{-}EPG_s$. Consider the graph G_8 which belongs to $B_1\text{-}EPG_s$ (see Fig. 8(a) and (b)). We will show that G_8 does not belong to $[\llcorner, \ulcorner, \urcorner]_s$. By contradiction suppose that $G_8 \in [\llcorner, \ulcorner, \urcorner]_s$. Assume first there exists a $[\llcorner, \ulcorner, \urcorner]$-representation of G_8, where the clique K induced by $\{c_1, \cdots, c_{10}\}$ is an edge-clique. Without loss of generality, we may assume that the base of K is vertical, say it goes from (x_i, y_j) to (x_{i+1}, y_j). Since $\{s_3, s_6, c_2, c_3, c_4\}$ induces a gem, it follows from Proposition 1 that at least one of P_{s_3}, P_{s_6} intersects paths of K on column y_j, say P_{s_3}. Similarly, since $\{s_9, s_{12}, c_7, c_8, c_9\}$ also induces a gem, we may assume that P_{s_9} intersects paths of K on column y_j. Since s_3 and s_9 are incomparable, one of these paths will be above row x_{i+1}, say P_{s_3}, and the other will be below row x_i, say P_{s_9} (see Observation 4). Now consider $P_{s_{12}}$. Since s_9 and s_{12} are incomparable and have a common neighbor, it follows from Observation 4 that $P_{s_{12}}$ must be above row x_{i+1}. But s_{12} has no common neighbour with s_3, so it follows from Observation 4 that $P_{s_{12}}$ must intersect all three paths P_{c_8}, P_{c_9} and $P_{c_{10}}$ on a same row, say row x_k, $k > i$, below P_{s_3}. Next consider s_{10} and s_{11}. Since they have each only one neighbour in K, and it is a common neighbour with s_{12}, it follows from the above that they must intersect this neighbour on the same row x_k. But since s_{10}, s_{11} are incomparable, they cannot intersect their neighbours on the same side of column y_j. So one will be to the right and the other to the left of column y_j (see Fig. 8(b)). Thus one of P_{c_9} and $P_{c_{10}}$ must be a $\urcorner$-path and one must be a $\ulcorner$-path. The same reasoning can be done for s_6, s_4 and s_5 with the conclusion that one of P_{c_4} and P_{c_5} must be an $\llcorner$-path and one must be a $\lrcorner$-path. But this contradicts the fact that $G_8 \in [\llcorner, \ulcorner, \urcorner]$.



(a) (b)

Fig. 8. The graph G_8 and a B_1-EPG representation of it.

Now assume there exists a $[\llcorner, \ulcorner, \urcorner]$-representation of G_8 where the clique K is a claw-clique with center (x_i, y_j). Without loss of generality, we may assume that only paths of K use the grid-edges going from (x_{i-1}, y_j) to (x_{i+1}, y_j). Also, we may assume that all paths of K have a part lying on column y_j. It follows from Observation 4 that the S-vertices whose paths intersect paths of K on row x_i are pairwise comparable. Thus, we conclude that there can be at most two such vertices. First assume there are exactly two. Without loss of generality, we may assume that P_{s_1} and P_{s_3} intersect paths of K on row x_i (the proof is the same if two other paths of comparable S-vertices intersect paths of K on row x_i). Since s_2 and s_6 have both a common neighbour with s_3, their paths intersect paths of K on column y_j. Furthermore, s_2, s_6 are not comparable, so one of the paths P_{s_2}, P_{s_6} uses column y_j above (x_{i+1}, y_j) and the other uses column y_j below (x_{i-1}, y_j). Now $\{s_9, s_{12}, c_7, c_8, c_9\}$ induces a gem, thus it follows from Proposition 1 that at least one of the paths $P_{s_9}, P_{s_{12}}$ intersects paths of K on the column y_j. But this implies that s_9 or s_{12} is comparable with one of s_2 and s_6, a contradiction. So we may assume now that there is exactly one S-vertex whose path intersects paths of K on row x_i. We will distinguish two cases: this path represents an S-vertex of degree 3, or this paths represents an S-vertex of degree 1. First assume it is an S-vertex of degree 3, say, without loss of generality, s_3. Since s_1, s_2 and s_6 have a common neighbour with s_3, their paths must intersect paths of K on column y_j. But these three vertices are pairwise incomparable, a contradiction with Observation 4. Now assume, without loss of generality, that the path P_{s_1} is the unique path representing an S-vertex which intersects paths of K on row x_i. Similar to the proof of Lemma 4, every path of K with bend point (x_i, y_j) not intersecting any path of S on row x_i can be transformed into a vertical path (by deleting its horizontal part and extending it to (x_{i-1}, y_j) if it is an $\llcorner$-path and to (x_{i+1}, y_j) if it is a $\ulcorner$-path). Since s_1 has degree 1, it follows that exactly one path of K uses row x_i. Hence K is represented as an edge-clique, but this is impossible due to the above. Thus, G_8 is not in $[\llcorner, \ulcorner, \urcorner]_s$.

As for G_4 in the proof of Lemma 2, we can say that G_5 and G_8 are inclusion-wise minimal examples to show strictness of class inclusion for Lemmas 4 and 5, respectively. However, in both cases, we do not know whether there exist other examples with fewer vertices.

4 Split Graphs as $[\llcorner]$-Graphs

In this section, we characterise those split graphs that are in $[\llcorner]$. As already noticed in [7], gems and S-bulls play an important role with respect to the characterisation of split $[\llcorner]$-graphs.

Theorem 2. *Let G be a split graph with split partition (K, S). Then $G \in [\llcorner]$ if and only if there exist $S_1, S_2 \subseteq S$ such that:*

(a) each S_i for $i \in \{1, 2\}$ is a set of pairwise comparable vertices;
(b) for every gem in G with vertex set $\{c_1, s_1, c_2, s_2, c_3\}$ (see Fig. 2(a)), either $s_1 \in S_1$ or $s_2 \in S_1$;
(c) for every S-bull in G with vertex set $\{s_1, c_1, s_2, c_2, s_3\}$ (see Fig. 2(b)), at least one of s_1, s_2, s_3 belongs to S_1 or $s_2 \in S_2$.

Proof. Let G be a split graph with split partition (K, S). Assume that $G \in [\llcorner]$, and consider an $[\llcorner]$-representation of G. We define S_1 and S_2 as follows:

- S_1 is the set of vertices whose corresponding paths belong to the crown;
- S_2 is the set of vertices whose corresponding paths belong to the trunk.

It immediately follows from Observation 1 that each $S_i, i \in \{1, 2\}$ as defined above is a set of pairwise comparable vertices. Furthermore, it follows from Observation 2 that (b) is satisfied. Finally, (c) is an immediate consequence of Observation 3.

Conversely, let $G = (V, E)$ be a split graph with split partition (K, S), and assume that there exist $S_1, S_2 \subseteq S$ satisfying (a), (b) and (c). In addition, let us assume that we choose S_2 maximal with these properties. Let $S' = S \setminus (S_1 \cup S_2)$. Consider a partition $S'_1, S'_2, ..., S'_k$ of S' into non-empty sets such that $\forall\ i \neq j, N(S'_i) \cap N(S'_j) = \emptyset$ and k is maximal.

Claim 1: The vertices in S'_i, $i \in \{1, ..., k\}$, are pairwise comparable.

Let $s, s' \in S'_i$, for some $i \in \{1, ..., k\}$. Suppose that s, s' are not comparable. Denote by S''_i the vertices in S'_i that have a common neighbour with s. Then each vertex in S''_i is comparable to s. Indeed, let $u \in S''_i$. If u and s are not comparable, then there exist $c, c' \in K$ such that $sc, uc' \in E$ and $sc', uc \notin E$.

Since $u \in S''_i$, it follows that there exists c'' such that sc'', $uc'' \in E$. But then, $\{u, s, c, c', c''\}$ induces a gem, and hence (b) is not satisfied, a contradiction. So we conclude that $s' \notin S''_i$, since s, s' are incomparable. Now, assume there exist a vertex $u \in S''_i$ and a vertex $v \in S'_i \setminus S''_i$, $v \neq s$, that have a common neighbour c_1. Since $v \notin S''_i$, it follows that $sc_1 \notin E$. Then $\{s, u, v, c_1, c_2\}$ induces an S-bull, where c_2 is a common neighbour of s and u, and hence (c) is not satisfied, a contradiction. It follows from the above that we may partition S'_i into two sets, $S''_i \cup \{s\}$ and $S'_i \setminus (S''_i \cup \{s\})$ such that $N(S''_i \cup \{s\}) \cap N(S'_i \setminus (S''_i \cup \{s\})) = \emptyset$. But this contradicts the maximality of k. Therefore, s, s' are comparable. This proves Claim 1.

Let $S_2 = \{u_1, ..., u_\ell\}$ such that $N(u_\ell) \subseteq N(u_{\ell-1}) \subseteq ... \subseteq N(u_2) \subseteq N(u_1)$. Furthermore, let $A_0 = K \setminus N(u_1)$, for all $i \in \{1, ..., \ell - 1\}$ $A_i = N(u_i) \setminus N(u_{i+1})$ and $A_\ell = N(u_\ell)$.

Claim 2: There exists no set S'_i, $i \in \{1, ..., k\}$, such that $N(S'_i) \cap A_{j_1} \neq \emptyset$ and $N(S'_i) \cap A_{j_2} \neq \emptyset$, for $j_1 \neq j_2$ and $j_1, j_2 \in \{0, 1, \ldots, \ell\}$.

Let S'_i be such that $x \in N(S'_i) \cap A_{j_1}$ and $y \in N(S'_i) \cap A_{j_2}$, for $j_1 \neq j_2$ and $j_1, j_2 \in \{0, 1, \ldots, \ell\}$. Without loss of generality, we may assume that $j_1 < j_2$ and that j_1 is chosen smallest with the property that $N(S'_i) \cap A_{j_1} \neq \emptyset$. Let u be a dominant vertex in S'_i, i.e. $N(u) = N(S'_i)$. Consider vertex $u_{j_1+1} \in S_2$. Notice that x and u_{j_1+1} are not adjacent. Hence, if there exists a vertex $z \in K$ which is adjacent to u_{j_1+1} and non-adjacent to u, then $\{u, u_{j_1+1}, x, y, z\}$ induces a gem, and hence (b) is not satisfied, a contradiction. Thus, u dominates u_{j_1+1}. Since u_{j_1+1} dominates u_j, for $j = j_1 + 2, \ldots, \ell$, we conclude that u actually dominates u_j, for $j = j_1 + 1, \ldots, \ell$. If $j_1 = 0$, we obtain that u dominates all vertices in S_2, and thus we may add u to S_2 (and (a), (b), (c) would still be satisfied), which contradicts the maximality of S_2. So we may assume that $j_1 > 0$. Notice that u is dominated by every vertex u_j, with $j \in \{1, \ldots, j_1\}$, since j_1 is chosen smallest with the property that $N(S'_i) \cap A_{j_1} \neq \emptyset$. Hence, we may again add u to S_2 (and (a), (b), (c) would still be satisfied), which contradicts the maximality of S_2. This proves Claim 2.

We will construct an $[\llcorner]$-representation of G as follows. We start with the base, which, without loss of generality, we may assume vertical. Next, we extend the paths of the base and add all vertices of S_1 in the crown and all vertices of S_2 in the trunk (see Fig. 9(a)). This is possible since the vertices in S_1 (resp. S_2) are pairwise comparable. Notice that currently each path P_c, for $c \in A_j$, $j \in \{1, \ldots, \ell\}$, has its lower endpoint below P_{u_j} and above $P_{u_{j+1}}$, and each path P_c, for $c \in A_0$, has its lower endpoint above P_{u_1}. Consider a set A_j, $j \in \{0, 1, \ldots, \ell\}$ as well as all sets among $S'_1, \ldots, S'_k$ which have neighbours in A_j, say $S'_{i_1}, \ldots, S'_{i_r}$. It follows from Claim 2 and the fact that $N(S'_i) \cap N(S'_l) = \emptyset$ for all $i, l \in \{1, \ldots, k\}$, $i \neq l$, that we may partition the vertices of A_j into sets $A_j^{i_1}, \ldots, A_j^{i_r}, A'_l$ such that $N(S'_{i_s}) = A_j^{i_s}$, for $s = 1, \ldots, r$ and the vertices of A'_j have no neighbours in S'. Since each set S'_i for $i \in \{1, ..., k\}$ contains pairwise comparable vertices (see Claim 1), we may now represent the vertices of each set

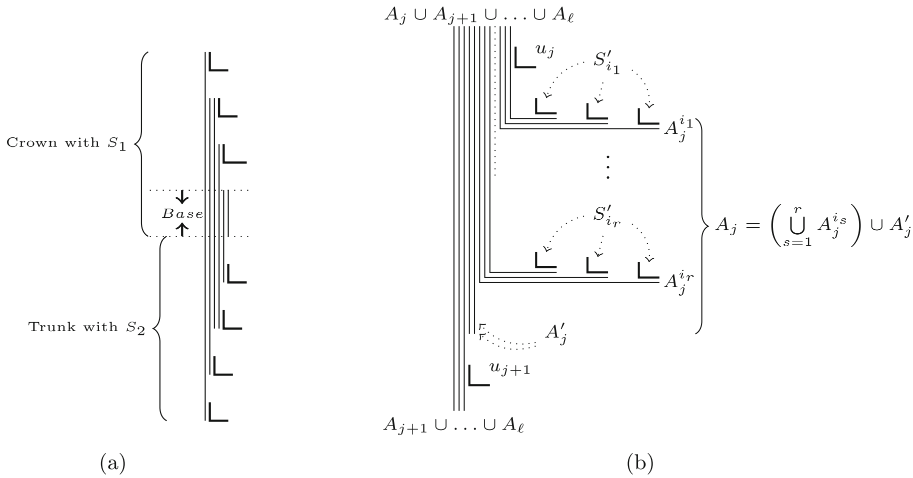


Fig. 9. Illustration of Theorem 2.

S'_{i_s} on a separate branch formed by the horizontal parts of the paths P_c, with $c \in A_j^{i_s}$, for $s = 1, \ldots, r$ (see Fig. 9(b)).

Notice that the previous characterisation does not imply that graphs in $[\llcorner]_s$ can be recognised in polynomial time. This still remains open.

In [7], the authors state a conjecture concerning the characterisation of the class $[\llcorner]_s$ by a family of forbidden induced subgraphs. Here, we will show that the conjecture is wrong by presenting an additional forbidden induced subgraph that was not mentioned in their list (and which is not contained in any of their forbidden graphs as induced subgraph), using Theorem 2. Consider the graph H shown in Fig. 10. We obtain the following.

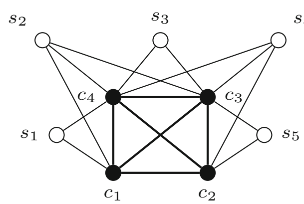


Fig. 10. The graph H.

Lemma 6. *The graph H is not in $[\llcorner]_s$.*

Proof. Suppose by contradiction that $H \in [\llcorner]_s$. Since $\{c_1, c_3, c_4, s_1, s_3\}$ induces a gem, it follows from Theorem 2, that either s_1 or s_3 belong to S_1.

First assume that $s_1 \in S_1$. Hence $s_3 \notin S_1$. Now $\{c_2, c_3, c_4, s_3, s_5\}$ also induces a gem. It follows again from Theorem 2 and the fact that $s_3 \notin S_1$ that $s_5 \in S_1$. So both s_1, s_5 belong to S_1. But they are incomparable, a contradiction.

So we may assume now that $s_3 \in S_1$ and $s_1 \notin S_1$. The vertex set $\{c_1, c_3, c_4, s_1, s_4\}$ induces a gem. Thus, it follows from Theorem 2 and the fact that $s_1 \notin S_1$ that $s_4 \in S_1$. Similarly, the vertices $\{c_2, c_3, c_4, s_2, s_5\}$ induces a gem. So according to Theorem 2, either s_2 or s_5 belongs to S_1. But s_2, s_4 are incomparable and s_5, s_3 are incomparable. Thus, we obtain again a contradiction since S_1 is a set of pairwise comparable vertices.

Note that one can easily check that the graph H is a minimal forbidden induced subgraph by removing each vertex separately and applying Theorem 2.

5 Conclusion

In this paper, we were interested in split graphs as edge intersection graphs of single bend paths on a grid. We presented a characterisation of this graph class using the notions of gems and S-bulls. Our characterisation allowed us to disprove a conjecture by Cameron et al. stating that this class can be characterised by a list of 9 forbidden induced subgraphs [7]. Notice that, even though we only gave here a single additional forbidden induced subgraph, we actually managed to detect 20 new ones so far. Furthermore, we investigated some subclasses of split B_1-EPG graphs for which only a subset of the four possible shapes are allowed. We presented the complete set of inclusion relationships between these graph families.

Our characterisation mentioned above does not immediately lead to a polynomial-time recognition algorithm. Thus, it is still open whether split B_1-EPG graphs can be recognised in polynomial time or not. Furthermore, it would be interesting to obtain a characterisation of chordal B_1-EPG graphs.

In [7], the authors present a characterisation of gem-free (resp. S-bull-free) graphs that are in $[\llcorner]_s$. We managed to generalise these results to gem-free (resp. S-bull-free) graphs that are in $[P]_s$, for any subset P of $\{\llcorner, \urcorner, \ulcorner, \lrcorner\}$. All these graph classes can be recognised in polynomial time. Due to space constraints, we were not able to include these results in the present paper.

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