ARTICLE

Dominating induced matchings in graphs containing no long claw

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Abstract

An induced matching M in a graph G is dominating if every edge not in M shares exactly one vertex with an edge in M. The DOMINATING INDUCED MATCHING problem (also known as EFFICIENT EDGE DOMINATION) asks whether a graph G contains a dominating induced matching. This problem is generally NP-complete, but polynomial-time solvable for graphs with some special properties. In particular, it is solvable in polynomial time for claw-free graphs. In the present article, we provide a polynomial-time algorithm to solve the DOMINATING INDUCED MATCHING problem for graphs containing no long claw, that is, no induced subgraph obtained from the claw by subdividing each of its edges exactly once.

KEYWORDS

dominating induced matching, graphs containing no long claw, polynomial-time algorithm

1 | INTRODUCTION

In this article, we study the problem that appeared in the literature under various names, such as DOMINATING INDUCED MATCHING [2,6,7,10,11,13] or EFFICIENT EDGE DOMINATION [1,5,9,15,16], and has several equivalent formulations. One of them, which is used in this article, asks whether the vertices of a graph can be partitioned into two subsets B and W so that B induces a graph of vertex degree 1 (also known as an induced matching) and W induces a graph of vertex degree 0 (i.e., an independent set). Throughout the article, we call the vertices of B black and the vertices of W white. This problem finds applications in various fields, such as parallel resource allocation of parallel processing systems [14], encoding theory and network routing [9] and has relations to some other algorithmic graph problems,

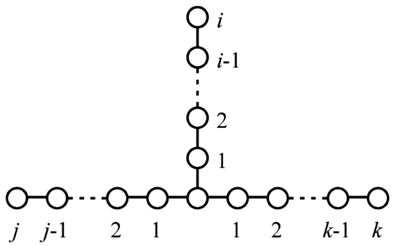

FIGURE 1 The graph $S_{i,j,k}$

such as 3-COLORABILITY and MAXIMUM INDUCED MATCHING. In particular, it is not difficult to see that every graph that can be partitioned into an induced matching and an independent set is 3-colorable. Also, in [5] it was shown that if a graph admits such a partition, then the black vertices form an induced matching of maximum size. Notice that a graph is called polar if its vertex set can be partitioned into a subset $\mathcal{K}$ of disjoint cliques and a subset $\mathcal{I}$ of independent sets with complete links between them [17]. It follows that a graph G has a dominating induced matching if and only if G is a polar graph in which all cliques of $\mathcal{K}$ have size 2 and $\mathcal{I}$ consists of exactly one independent set.

From an algorithmic point of view, the DOMINATING INDUCED MATCHING problem is difficult, that is, it is NP-complete [9]. Moreover, it remains difficult under substantial restrictions, for instance, for planar bipartite graphs [15] or d-regular graphs for arbitrary $d \geq 3$ [5]. Exact nonpolynomial algorithms for general graphs can be found in [12,18]. On the other hand, for some special graph classes, such as hole-free graphs [1], claw-free graphs [6], or P_8-free graphs [3], the problem can be solved in polynomial time.

For classes defined by finitely many forbidden-induced subgraphs, there is an important *necessary* condition for polynomial-time solvability of the problem given implicitly in [6]. To state this condition, we first introduce some basic notions. If a graph G does not contain induced subgraphs isomorphic to a graph H, we say that G is H*-free* and call H a *forbidden -induced subgraph* for G. For a set of graphs M, a graph G is called M*-free* it is H-free for every element $H \in M$. Let us also denote by $\mathcal{S}$ the class of graphs every connected component of which corresponds to a graph $S_{i,j,k}$, $i, j, k \geq 0$, represented in Figure 1.

Theorem 1.1 ([6], see Section 4). *Let M be a finite set of graphs. Unless $P = NP$, the* DOMINATING INDUCED MATCHING *problem is polynomial-time solvable in the class of M-free graphs only if M contains a graph from $\mathcal{S}$.*

We believe that this necessary condition is also sufficient and formally state this as a conjecture.

Conjecture 1. *Let M be a finite set of graphs. Unless $P = NP$, the* DOMINATING INDUCED MATCHING *problem is polynomial-time solvable in the class of M-free graphs if and only if M contains a graph from $\mathcal{S}$.*

Proving (or disproving) this conjecture is a very challenging task. To prove it, one has to show that the problem becomes polynomial-time solvable by forbidding *any* single graph from $\mathcal{S}$. However, so far, the conjecture has only been verified for a few forbidden graphs that belong to $\mathcal{S}$, and only two of these classes are maximal: $S_{1,2,3}$-free graphs[1] [11] and P_8-free graphs [3] (note that $P_8 = S_{0,3,4}$). In the present article, we extend this short list of positive results by one more class where the problem can be solved in polynomial time, namely, the class of $S_{2,2,2}$-free graphs. Since $S_{2,2,2}$ is obtained from the claw ($S_{1,1,1}$) by subdividing each of its edges exactly once, we call $S_{2,2,2}$ a *long claw*.

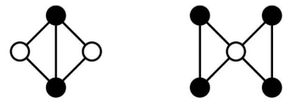

FIGURE 2 A diamond (left) and a butterfly (right)

To solve the problem for graphs containing no long claw, we apply a number of transformations and reductions that eventually reduce the problem to the following question that can be answered in polynomial time: given a graph G and a subset of its vertices, does G contain a matching saturating all vertices of the subset? As a result, we prove that the DOMINATING INDUCED MATCHING problem for graphs containing no long claw can also be solved in polynomial time.

The organization of the article is as follows. In the rest of this section, we introduce basic terminology and notation. In Sections 2, 3, and 4, we describe various tools (reductions and transformations) simplifying the problem. In Section 5, we apply these tools to reduce the problem from an arbitrary $S_{2,2,2}$-free graph G to a graph of particular structure, which we call irreducible. Finally, in Section 6, we show how to solve the problem for irreducible graphs via finding matchings saturating specified vertices. In Section 7, we conclude the article with a number of open problems.

Let $G = (V, E)$ be a graph. If $v \in V$, then $N_G(v)$ is the *neighborhood* of v in G, that is, the set of vertices of G adjacent to v, and $d_G(v)$ is the *degree* of v in G, that is, $d_G(v) = |N_G(v)|$.

An independent set in G is a subset of pairwise nonadjacent vertices. For a subset $U \subseteq V$, we denote by $G[U]$ the subgraph of G induced by vertices of U. As usual, K_n is the complete graph on n vertices, and C_n (resp. P_n) is the chordless cycle (resp. path) on n vertices. A *diamond* and a *butterfly* are two special graphs represented in Figure 2.

2 | PRECOLORING, PROPAGATION RULES AND CLEANING

To solve our problem for a graph G, we will assign either color black or color white to the vertices of G, and the assignment of one of the two colors to each vertex of G is called a *complete coloring* of G. If only some vertices of G have been assigned a color, the coloring is said to be *partial*. A partial coloring is *feasible* if no two adjacent vertices are white and every black vertex has at most one black neighbor. A complete coloring is *feasible* if no two adjacent vertices are white and every black vertex has exactly one black neighbor. Thus, a graph G has a dominating induced matching if and only if G admits a feasible complete coloring. Given a feasible partial coloring γ of G, we say that it is *completable* if it can be extended to a feasible complete coloring of G, the latter one being called a *γ-completion*. Also, for a feasible partial coloring γ, we denote by $\gamma(v)$ the color of vertex v, by B_γ the set of black vertices and by W_γ the set of white vertices.

Let γ be a feasible partial coloring of a graph G, and let G' be the graph obtained from G by removing all white vertices as well as all pairs of adjacent black vertices. The restriction δ of γ to G' is a feasible partial coloring of G' where some of its vertices are forced to be black and form an independent set. Clearly, γ is completable if and only if δ is. The construction of G' and δ is called a *cleaning*.

As shown in the following lemma, there are situations where some vertices of a graph G must have the same color or necessarily have different colors in all feasible complete colorings of G.

Lemma 2.1. *Let γ be a feasible complete coloring of a graph $G = (V, E)$.*

(i) *If G contains C_4 with edge set $\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$, then $\gamma(v_1) = \gamma(v_3) \neq \gamma(v_2) = \gamma(v_4)$.*

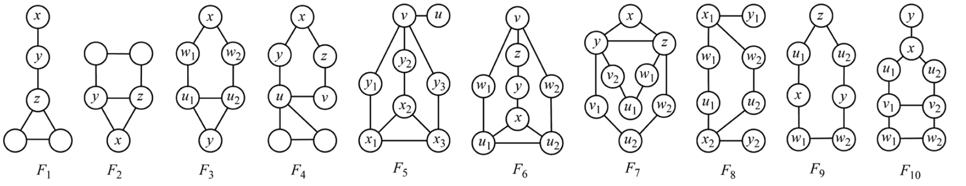

FIGURE 3 The graphs $F_1, \dots, F_{10}$

(ii) *If G contains a triangle with vertex set $\{x, y, z\}$ such that x has a neighbor u that is not adjacent to y, z, then $\gamma(x) \neq \gamma(u)$.*

Proof.

(i) Clearly, at least one of v_1, v_2, v_3, v_4 is white, say v_1. Then both v_2 and v_4 are black, which means that v_3 must be white since it has two black neighbors.

(ii) Clearly, at least one of y, z is black. Hence, if x is black, then u is white. Since x and u cannot be both white, we conclude that $\gamma(x) \neq \gamma(u)$. ■

We now describe several situations where the color of a vertex v can be fixed because if there is a feasible complete coloring of G, then there is at least one in which v has such a color. The graphs F_i, $i = 1, \dots, 10$ we refer to in the following lemma are shown in Figure 3.

Lemma 2.2. *Let γ be a feasible partial coloring of a graph $G = (V, E)$. If γ is completable, then the following rules are valid for obtaining a γ-completion.*

(a) *An isolated vertex must be white, and the neighbor of a vertex of degree 1 must be black.*

(b) *If two nonadjacent vertices in G are in B_γ, then all their common neighbors must be white.*

(c) *If two triangles in G share a single vertex, then this vertex must be white.*

(d) *If two triangles in G share two vertices, then both of these vertices must be black.*

(e) *If a vertex u of G has $k > 1$ neighbors of degree 1, then at least $k - 1$ of these neighbors are not colored black, and color white can be assigned to them.*

(f) *Suppose G contains an induced P_4 with edge set $\{v_1v_2, v_2v_3, v_3v_4\}$. If $d_G(v_3) = 2$ and v_1 is black, then v_4 must be black.*

(g) *Suppose G contains F_1 as an induced subgraph. If $d_G(y) = 2$, then x must be black.*

(h) *If G contains F_2 as an induced subgraph, then x must be black.*

(i) *If G contains F_3 as an induced subgraph, then x must be black. Moreover,*

- *if $d_G(x) = 2$, then y must be black;*
- *if y is black, then all neighbors $z \neq w_1, w_2$ of x must be white;*
- *if $d_G(u_1) = d_G(u_2) = 3$ and $d_G(w_1) = d_G(w_2) = 2$ and if these vertices are not yet colored by γ, then color white can be assigned to w_1.*

(j) *Suppose G contains F_4 as an induced subgraph. If $d_G(y) = d_G(z) = 2$, then x must be black and all neighbors $w \neq y, z$ of x must be white.*

(k) *Suppose G contains F_5 as an induced subgraph. If $d_G(y_1) = d_G(y_2) = d_G(y_3) = 2$, then u must be white.*

(l) *Suppose G contains C_4. If $d_G(v) = 2$ for some vertex v of this C_4, then v must be white.*

(m) *Suppose G contains F_6 as an induced subgraph. If v is black, then x must be white.*

(n) *Suppose G contains F_7 as an induced subgraph. If $d_G(y) = d_G(z) = 4$, $d_G(v_i) = d_G(w_i) = 2$, $i = 1, 2$ and if these vertices are not yet colored by γ, then color white can be assigned to w_1, w_2.*

(p) *If G is $S_{2,2,2}$-free and contains a vertex v such that the subgraph induced by $N(v)$ has three isolated vertices, then v must be black.*

(q) *Suppose G is $S_{2,2,2}$-free. If it contains F_8 as an induced subgraph, and if $d_G(x_1) = d_G(x_2) = 3$, $d_G(w_i) = d_G(u_i) = 2$, $i = 1, 2$, then y_1 and y_2 must be white.*

(r) *Suppose G is butterfly-free and contains a vertex v with four neighbors w_1, w_2, w_3, w_4 such that only two of them are adjacent, say w_1 and w_2. If G does not contain two vertices u_1, u_2 such that $N(u_1) \cap \{v, w_1, w_2, w_3, w_4, u_2\} = \{w_3\}$ and $N(u_2) \cap \{v, w_1, w_2, w_3, w_4, u_1\} = \{w_4\}$, then v must be black.*

(s) *If G contains F_9 as an induced subgraph and x and y are black, then z must be black.*

(t) *If G contains F_{10} as an induced subgraph and x is black, then y must be white.*

Proof.

(a) If $d_G(u) = 0$, then u must be white since u cannot have a black neighbor. If $d_G(u) = 1$, then the neighbor of u cannot be white since otherwise u would need to be black with no black neighbor, a contradiction.

(b) If a common neighbor w of the two non-adjacent black vertices is black, then w has two black neighbors, a contradiction.

(c) If the vertex shared by the two triangles is black, then, since the white vertices form an independent set, it must have at least two black neighbors, one in each triangle, a contradiction.

(d) If one of the two vertices shared by the two triangles is white, then the other one is black and must have at least two black neighbors, one in each triangle, a contradiction.

(e) It follows from (a) that u is black in all γ-completions. Hence, at most one of its neighbors is black. We can therefore impose color white on $k - 1$ of its neighbors of degree 1.

(f) Suppose to the contrary that v_4 is white. Then v_3 is black and since $d_G(v_3) = 2$, it follows that v_2 is black. But now v_2 has two black neighbors, a contradiction.

(g) If x is white, then y is black and it follows from Lemma 2.1 (ii) that z is white, which means that y has no black neighbor, a contradiction.

(h) It follows from Lemma 2.1 (i) that y and z must get different colors, which means that x is necessarily black.

(i) Suppose to the contrary that x is white. Then both w_1, w_2 must be black, and it follows from Lemma 2.1 (ii) that both u_1, u_2 are white, a contradiction. Now,

- if $d_G(x) = 2$, then one of w_1, w_2 must be black, which implies that one of u_1, u_2 must be white. Hence, y must be black;
- if y is black, then one of u_1, u_2 is white, which means that one of w_1, w_2 is the black neighbor of x. Hence, all neighbors $z \neq w_1, w_2$ of x are white;
- if $d_G(u_1) = d_G(u_2) = 3$ and $d_G(w_1) = d_G(w_2) = 2$ and if these vertices are not yet colored by γ, then consider any γ-completion. If w_1 is black, then u_1, w_2 are white and y, u_2 are black. We can easily transform this γ-completion into another by coloring u_1, w_2 black and u_2, w_1 white.

- **(j)** If follows from (g) that x must be black. Suppose that $d_G(y) = d_G(z) = 2$. If y is white, then it follows from Lemma 2.1 (ii) that u is black and v is white, which means that z is black. Hence, either y or z is black, which means that all other neighbors of x must be white.
- **(k)** Clearly, exactly one of x_1, x_2, x_3 must be white, say x_1. Then y_1 and v are black (since $d_G(y_1) = 2$), which means that u must be white.
- **(l)** Suppose to the contrary that v is black. Lemma 2.1 (i) implies that the neighbors of v are white, that is v has no black neighbor, a contradiction.
- **(m)** Suppose to the contrary that x is black. It then follows from Lemma 2.1 (ii) that y is white. Hence, z must be the black neighbor of v, and w_1, w_2 must be white. But then all three vertices of the triangle induced by u_1, u_2, x are black, which is impossible.
- **(n)** Consider any γ-completion. It follows from (g) that u_1 and u_2 are black. By Lemma 2.1 (ii), v_1 has the same color as v_2 and w_1 has the same color as w_2. Note that at least one of these pairs of vertices must be white, otherwise u_1 (and u_2) would have two black neighbors. Assume that w_1 and w_2 are black, then v_1, v_2 are white, x, y are black and z is white. Since $d_G(y) = d_G(z) = 4$, $d_G(v_i) = d_G(w_i) = 2$, $i = 1, 2$, we can easily transform this γ-completion into another by recoloring v_1, v_2, z black and w_1, w_2, y white.
- **(p)** Consider three isolated vertices x, y, z in $N(v)$, and suppose v is white. It follows that x, y, z are black and hence each has a neighbor not in $N(v)$, which must also be black. Since these neighbors must be distinct and nonadjacent, we obtain an induced $S_{2,2,2}$, a contradiction.
- **(q)** It follows from (p) that x_1 and x_2 must be black. Hence, exactly one of w_1, w_2 and exactly one of u_1, u_2 must be black, which means that y_1 and y_2 must be white.
- **(r)** Suppose to the contrary that v is white. Then w_1, w_2, w_3, w_4 are black. Let u_1 the black neighbor of w_3 and u_2 be the black neighbor of w_4 in a γ-completion. Since G is butterfly-free and every black vertex has exactly one black neighbor, we have $N(u_1) \cap \{v, w_1, w_2, w_3, w_4, u_2\} = \{w_3\}$ and $N(u_2) \cap \{v, w_1, w_2, w_3, w_4, u_1\} = \{w_4\}$, a contradiction.
- **(s)** If x and y are black, then one of w_1, w_2 must be black. Hence, one of u_1, u_2 must be white, which implies that z must be black.
- **(t)** If y is black, then u_1, u_2 must be white, v_1, v_2 must be black and w_1, w_2 are then two white adjacent vertices, a contradiction. ■

In addition to the above forcing rules, we will also use the following ones that are clearly valid :

- **(i)** If a vertex v is white, then all its neighbors must be black.
- **(ii)** If two adjacent vertices are black, then all their neighbors must be white.
- **(iii)** If a vertex u is black, and all its neighbors, except v, are white, then v must be black.

If one of the rules (i), (ii), (iii), or one of the rules described in Lemmas 2.1 and 2.2 imposes color black (resp. white) on a vertex that is already forced to be white (resp. black), we conclude that the considered graph does not admit a feasible complete coloring. Applying these rules repeatedly, as often as possible, on a given graph H, we thus either get a proof that H does not admit a feasible complete coloring, or we obtain a feasible partial coloring of H. In the latter case, we can apply a cleaning to obtain a graph G with a feasible partial coloring γ so that $W_\gamma = \emptyset$ and the distance between any two vertices of B_γ is at least 3. Indeed, B_γ is an independent set since adjacent black vertices are removed by a cleaning, and two vertices u, v in B_γ cannot have a common neighbor w since Lemma 2.2 (b) would impose color white on w, and w would therefore be removed by a cleaning. This justifies the following definition.

Definition 2.1. Let γ be a feasible partial coloring of a graph G such that $W_\gamma = \emptyset$ and the distance between any two vertices of B_γ is at least 3. The pair (G, γ) is called CLEAN if none of the forcing rules defined above can color additional vertices.

Remarks 2.1.

- **(a)** *Rules (c) and (d) of Lemma 2.2 show that if (G, γ) is clean, then G does not contain any induced diamond and any induced butterfly.*
- **(b)** *It is easy to see that a graph containing a K_4 cannot admit a feasible complete coloring.*

Hence, it follows from the remarks above that we may suppose that all considered graphs have no induced diamond, no induced butterfly and are K_4-free. Note that if (G, γ) is clean and was obtained from an $S_{2,2,2}$-free graph H by applying the above-mentioned forcing rules followed by a cleaning, then G is an induced subgraph of H and is therefore also $S_{2,2,2}$-free. The following lemma gives additional properties of clean pairs.

Lemma 2.3. *Let (G, γ) be a clean pair. If γ is completable, then the following claims hold.*

- **(a)** *Each vertex of G belongs to at most one triangle.*
- **(b)** *If G is $S_{2,2,2}$-free and contains F_3 as an induced subgraph, then $x \in B_\gamma$, the degree of any neighbor of x is at most two, and x does not belong to any triangle.*
- **(c)** *If G contains F_2 as an induced subgraph, then x is black and x has no other neighbors.*
- **(d)** *Let T_1 and T_2 be two vertex-disjoint triangles in G. Then there are at most two edges between T_1 and T_2. Moreover, if there are exactly two edges between the triangles, then these two edges are not adjacent.*

Proof.

- **(a)** This is a direct consequence of the Remarks 2.1 and the fact that (G, γ) is clean.
- **(b)** Lemma 2.2 (i) implies that $x \in B_\gamma$, which means that no neighbor of x belongs to $B_\gamma \cup W_\gamma$. Assume w_1 has a neighbor z different from u_1 and x. If z is adjacent to x, then (a) and Lemma 2.1 (ii) imply that $w_2 \in W_\gamma$, a contradiction. Hence, no neighbor of w_1 is adjacent to x. Now, if x belongs to a triangle then Lemma 2.1 (ii) implies that $w_1 \in W_\gamma$, a contradiction. Finally, let $v \neq w_1, w_2$ be a neighbor of x. If $d_G(v) \geq 3$, then Lemma 2.1 (ii) and Lemma 2.2 (p) imply that $v \in B_\gamma \cup W_\gamma$, a contradiction.
- **(c)** Lemma 2.2 (h) implies that $x \in B_\gamma$. Assume that x has a neighbor s different from y and z. Then (a) and Lemma 2.1 (ii) imply that $s \in W_\gamma$, a contradiction.
- **(d)** By (a), all edges between T_1 and T_2 are pairwise nonadjacent. Hence, there are at most three edges between the triangles. By Lemma 2.1 (ii) each of the edges connects vertices of different colors. Therefore, there are at most two edges between T_1 and T_2, otherwise one of the triangles would have two white vertices, a contradiction. ■

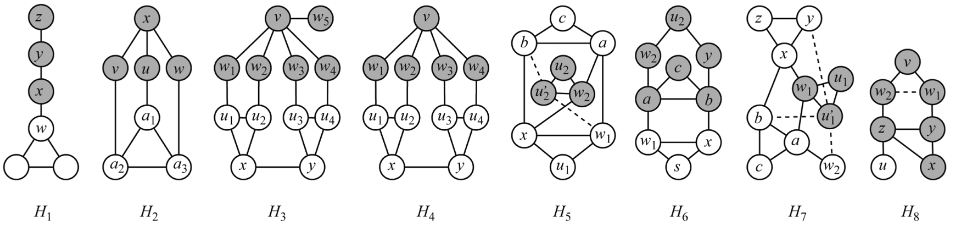

FIGURE 4 Eight reductions

3 | GRAPH REDUCTIONS

Definition 3.1. Let (G, γ) be a clean pair, G' an induced subgraph of G and δ the restriction of γ to G'. The replacement of (G, γ) by (G', δ) is a VALID REDUCTION if either both γ and δ are completable, or none of them is.

In this section, we will present eight valid reductions $\rho_1, \ldots, \rho_8$. Assume $G = (V, E)$ contains one of the graphs H_i $(i = 1, \ldots, 8)$ of Figure 4 as induced subgraph, where each of the dashed edges can be replaced by a true edge or a nonedge. Let $i \in \{1, \ldots, 8\}$. Let U_i be the set of grey vertices in H_i, and assume that no vertex in U_i has other neighbors in G than those in H_i. Finally, let δ_i be the restriction of γ to $G[V \setminus U_i]$. Reduction ρ_i consists in replacing (G, γ) by $(G[V \setminus U_i], \delta_i)$. Note that if G is $S_{2,2,2}$-free, then the graph obtained by applying reduction ρ_i is also $S_{2,2,2}$-free since it is an induced subgraph of G.

Lemma 3.1. *Reductions $\rho_1, \ldots, \rho_8$ are valid.*

Proof. Let $i \in \{1, \ldots, 8\}$. First, observe that $(G[V \setminus U_i], \delta_i)$ is clean since (G, γ) is clean. Now, let S_i be the set of nongrey vertices in H_i, which means that all neighbors of the vertices in U_i belong to S_i. Let $\bar{\gamma}$ be a γ-completion and $\bar{\delta}_i$ the restriction of $\bar{\gamma}$ to $G[V \setminus U_i]$. Consider any two adjacent vertices v_1, v_2 in H_i such that $v_1 \in U_i$ and $v_2 \in S_i$. Note that if v_1 belongs to a triangle in $G[U_i]$, then v_2 is nonadjacent to the other two vertices of that triangle, while if v_2 belongs to a triangle in $G[S_i]$, then v_1 is nonadjacent to the other two vertices of that triangle. It then follows from Lemma 2.1 (ii) that $\bar{\gamma}(v_1) \neq \bar{\gamma}(v_2)$, which implies that $\bar{\delta}_i$ is feasible and thus δ_i is completable.

Let now $\bar{\delta}_i$ be a δ_i-completion. We show how to extend $\bar{\delta}_i$ to a γ-completion.

- $i = 1$. Lemma 2.2 (a) implies that $y \in B_\gamma$. Hence, none of x, z belongs to B_γ. If $\bar{\delta}_1(w) =$ black, we obtain a γ-completion by assigning color black to y, z, and color white to x. If $\bar{\delta}_1(w) =$ white, a γ-completion is obtained by assigning color black to x, y, and color white to z.
- $i = 2$. Lemma 2.2 (p) implies that $x \in B_\gamma$. Hence, none of u, v, w belongs to B_γ. Without loss of generality, we may assume that $\bar{\delta}_2(a_1) =$ white, and we can obtain a γ-completion by assigning color black to x, u, and color white to v, w.
- $i = 3$ or 4. Lemma 2.2 (p) implies that $v \in B_\gamma$. Hence, none of the w_is belong to B_γ. Lemma 2.1 (ii) implies that exactly one of x, y is black in $\bar{\delta}_i$, say x. Then exactly one of u_1, u_2 is white in $\bar{\delta}_i$, say u_1. Hence u_2, u_3, u_4 are black and y is white in $\bar{\delta}_i$. We can then obtain a γ-completion by assigning color black to w_1, v, and color white to w_2, w_3, w_4, and to w_5 if $i = 3$.
- $i = 5$. Lemma 2.1 (ii) implies that $w_2 \notin B_\gamma$ (else x and a would belong to W_γ), and Lemma 2.1 (i) implies $\bar{\delta}_5(x) = \bar{\delta}_5(a) \neq \bar{\delta}_5(b) = \bar{\delta}_5(w_1)$. If $u_2' \in B_\gamma$, then $u_2 \notin B_\gamma$ and b and w_1 are not neighbors of v (else they would belong to W_γ). We can then obtain a γ-completion by assigning color $\bar{\delta}_5(x)$

to u_2, color $\bar{\delta}_5(w_1)$ to w_2 and color black to u_2'. If $u_2' \notin B_\gamma$, we obtain a γ-completion by assigning color $\bar{\delta}_5(x)$ to u_2', color $\bar{\delta}_5(w_1)$ to w_2 and color black to u_2.

- $i = 6$. Lemma 2.1 (ii) implies that $\{a, b, y, w_2\} \cap B_\gamma = \emptyset$ (otherwise two of them would belong to W_γ). Also, Lemma 2.2 (h) implies that $s \in B_\gamma$. Hence $\bar{\delta}_6(x) \neq \bar{\delta}_6(w_1)$, and a γ-completion is obtained by assigning color $\bar{\delta}_6(x)$ to a, y, color $\bar{\delta}_6(w_1)$ to b, w_2 and color black to c, u_2.
- $i = 7$. Lemma 2.2 (h) and Lemma 2.1 (i) imply that $c \in B_\gamma$ and $\bar{\delta}_7(b) = \bar{\delta}_7(w_2) \neq \bar{\delta}_7(a) = \bar{\delta}_7(x)$. If b, y, and w_2 are not adjacent to u_1', then either $u_1 \in B_\gamma$ and a γ-completion is obtained by assigning color black to u_1, color $\bar{\delta}_7(b)$ to w_1 and color $\bar{\delta}_7(a)$ to u_1', or $u_1 \notin B_\gamma$ and a γ-completion is obtained by assigning color black to u_1', color $\bar{\delta}_7(b)$ to w_1 and color $\bar{\delta}_7(a)$ to u_1. So assume at least one of b, y, w_2 is adjacent to u_1'. Then Lemma 2.2 (h) implies that $u_1 \in B_\gamma$. Note that if u_1' is adjacent to y then Lemma 2.2 (h) and Lemma 2.1 (ii) imply that$z \in B_\gamma$ and $\bar{\delta}_7(y) = \bar{\delta}_7(b) \neq \bar{\delta}_7(a)$. Hence, a γ-completion is obtained by assigning color black to u_1, color $\bar{\delta}_7(b)$ to w_1 and color $\bar{\delta}_7(a)$ to u_1'.
- $i = 8$. Lemma 2.1 (ii) implies that $\{w_1, w_2, y, z\} \cap B_\gamma = \emptyset$ (else $W_\gamma \neq \emptyset$). If $\bar{\delta}_8(u)$ =black, then a γ-completion is obtained by assigning color white to z, w_1 and color black to x, y, w_2, v. If $\bar{\delta}_8(u)$ =white, then a γ-completion is obtained by assigning color white to y, w_2 and color black to x, z, w_1, v. ■

4 | GRAPH TRANSFORMATIONS

Let $G = (V, E)$ be a graph, and let γ be a feasible partial coloring of G. Let G' be a graph obtained from G by removing a subset X of its vertices, adding a subset Y of new vertices, adding or/and removing some edges in $G[V \setminus X]$, and finally adding some edges linking pairs of vertices in Y as well as some edges linking some vertices in Y with some vertices in $V \setminus X$. Such an operation is called a *graph transformation*. The restriction δ of γ to G' is defined as the partial coloring of G' obtained by setting $\delta(v) = \gamma(v)$ for all vertices in $V \setminus X$, and by leaving all vertices in Y uncolored.

Definition 4.1. Let (G, γ) be a clean pair where G is $S_{2,2,2}$-free. Let G' be a graph obtained from G by applying some graph transformation, and let δ be the restriction of γ to G'. The replacement of (G, γ) by (G', δ) is a VALID TRANSFORMATION if G' is $S_{2,2,2}$-free and either both γ and δ are completable, or none of them is.

Nine graph transformations $\tau_1, \dots, \tau_9$ are represented in Figure 5. For every transformation, we show on the left an induced subgraph of G while modifications made on G to obtain G' appear on the right. The set X of removed vertices from G and the set Y of added vertices to G' are shown in grey. No vertex in X (resp. Y) has other neighbors in G (resp. G') than those shown in Figure 5. In the following lemmas, we assume that (G, γ) is a clean pair, that G is $S_{2,2,2}$-free, and that δ is the restriction of γ to G'. When constructing a γ-completion $\bar{\gamma}$ from a δ-completion $\bar{\delta}$, or the opposite, we will always assume $\bar{\gamma}(v) = \bar{\delta}(v)$ for all $v \in V \setminus X$, unless otherwise specified.

Lemma 4.1. *Transformation τ_1 is valid.*

Proof. Lemma 2.2 (p) implies that $v \in B_\gamma$, which means that $u_i, w_j \notin B_\gamma$ for $i = 1, \dots, 4$, $j = 1 \dots, 5$, since (G, γ) is a clean pair. Suppose by contradiction that G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, we may assume without loss of generality that x is the vertex of degree 3 in H, and either u_1 or u_2, say u_1 is a neighbor of x in H. In other words, G' contains three vertices z_1, z_2, z_3 such that $x, b, c, u_1, z_1, z_2, z_3$ induce H in G', with $z_1 \in N(u_1)$, and $\{z_3, x\} \subseteq N(z_2)$. Note that Lemma 2.3 (a) implies that u_2 is not adjacent to z_1, z_2. Since $u_1 \notin B_\gamma$ (otherwise $w_1 \in W_\gamma$),

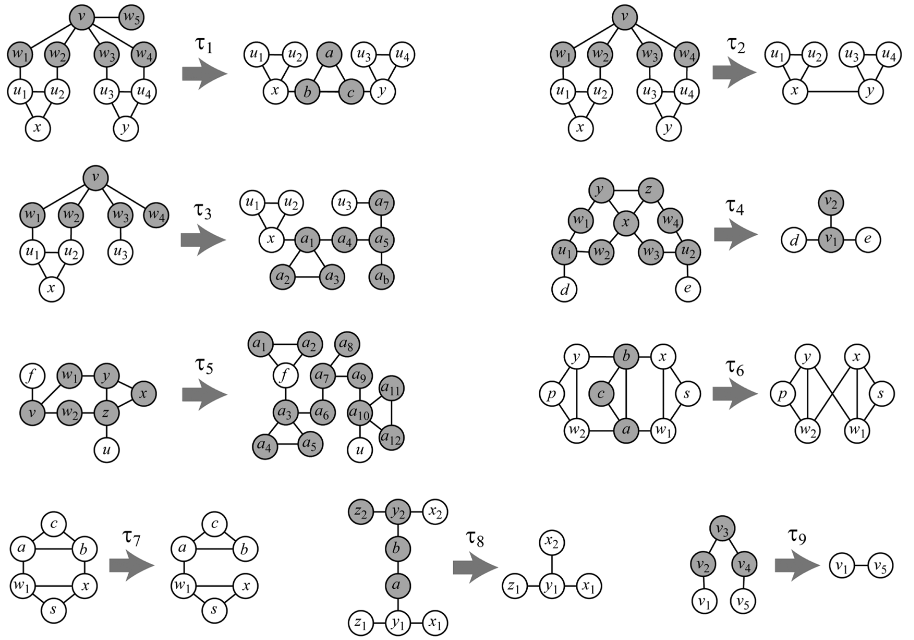

FIGURE 5 Nine graph transformations

Lemma 2.2 (a) implies that $d_G(z_1) > 1$. So let $z_4 \neq u_1$ be another neighbor to z_1. Lemma 2.3 (a) implies that z_4 is not adjacent to u_1, and Lemma 2.3 (a) and (c) imply that z_4 is not adjacent to u_2 and x. Finally, Lemma 2.3 (b) implies that z_4 is not adjacent to z_2, which means that $v, w_1, u_1, z_1, z_4, x, z_2$ induce an $S_{2,2,2}$ in G, a contradiction.

Let now $\bar{\gamma}$ be a γ-completion. If w_5 is black, then w_1, w_2, w_3, w_4, x, y are white, u_1, u_2, u_3, u_4 are black, and we obtain a δ-completion by assigning color black to b, c and color white to a. If w_5 is white, then one of w_1, w_2, w_3, w_4 is black, say w_1, which means that u_2, u_3, u_4, x are black, u_1, y are white, and we obtain a δ-completion by assigning color black to a, c and color white to b.

Finally, let $\bar{\delta}$ be a δ-completion. Note that at least one of x, y is white. Indeed, if x is black, then Lemma 2.1 (ii) implies that b is white, which means that a and c are black and y is white. Hence, at most one of u_1, u_2, u_3, u_4 is white. If none of them is white, we obtain a γ-completion by assigning color black to w_5, v and color white to w_1, w_2, w_3, w_4. If one of u_1, u_2, u_3, u_4 is white, say u_1, we obtain a γ-completion by assigning color black to w_1, v and color white to w_2, w_3, w_4, w_5. ■

Lemma 4.2. *Transformation τ_2 is valid.*

Proof. The proof that G' is $S_{2,2,2}$-free is the same as the one in Lemma 4.1, where b, c are replaced by y and a neighbor $z \neq x$ of y. Hence, we only show that γ is completable if and only if δ is completable. Lemma 2.2 (p) implies that $v \in B_\gamma$, which means that $u_i, w_i \notin B_\gamma$ for $i = 1, \ldots, 4$.

Let $\bar{\gamma}$ be a γ-completion. Exactly one of w_1, w_2, w_3, w_4 is black, which implies that exactly one of u_1, u_2, u_3, u_4 and one of x, y is white. Hence, a δ-completion is obtained by coloring all vertices of G' as in G.

Let now $\bar{\delta}$ be a δ-completion. Lemma 2.1 (ii) implies that x and y have different colors. Hence, exactly one of u_1, u_2, u_3, u_4 is white, say u_1, and we obtain a γ-completion by assigning color black to w_1, v and color white to w_2, w_3, w_4. ■

Lemma 4.3. *Transformation τ_3 is valid.*

Proof. The proof that G' is $S_{2,2,2}$-free is the same as the one in Lemma 4.1, where b, c are replaced by a_1, a_2. Hence, we only show that γ is completable if and only if δ is completable. Lemma 2.2 (a) implies that $v \in B_\gamma$, which means that $u_i, w_j \notin B_\gamma$ for $i = 1, 2, 3$, $j = 1, \dots, 4$.

Let $\bar{\gamma}$ be a γ-completion. Exactly one of w_1, w_2, w_3, w_4 is black. If w_4 is black, then w_1, w_2, w_3, x are white, u_1, u_2, u_3 are black, and we obtain a δ-completion by assigning color black to a_1, a_2, a_5, a_6 and color white to a_3, a_4, a_7. If w_3 is black, then w_1, w_2, w_4, x, u_3 are white, u_1, u_2 are black, and we obtain a δ-completion by assigning color black to a_1, a_2, a_5, a_7 and color white to a_3, a_4, a_6. Finally, if one of w_1, w_2 is black, say w_1, then u_1, w_2, w_3, w_4 are white, u_2, u_3, x are black, and we obtain a δ-completion by assigning color black to a_2, a_3, a_4, a_5 and color white to a_1, a_6, a_7.

Let now $\bar{\delta}$ be a δ-completion. Note that u_3 is black whenever x is black. Indeed, if x is black, then it follows from Lemma 2.1 (ii) that a_1 is white and a_4 is black, which means that a_5 is black, a_7 is white, and u_3 is black. Hence, at most one of u_1, u_2, u_3 is white. If none of them is white, we obtain a γ-completion by assigning color black to w_4, v and color white to w_1, w_2, w_3. If u_3 is white, we obtain a γ-completion by assigning color black to w_3, v and color white to w_1, w_2, w_4. We proceed in a similar way if u_1 or u_2 is white. ■

Lemma 4.4. *Transformation τ_4 is valid.*

Proof. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, we may assume without loss of generality that d and v_1 are vertices in H. We then get an $S_{2,2,2}$ in G as follows: (i) if $d_H(d) = 3$, we replace v_1 and its neighbor in H by u_1, w_1; (ii) if $d_H(d) = 2$, we replace v_1 by u_1; (iii) if $d_H(d) = 1$, we replace d, v_1 by w_3, u_2, a contradiction. Thus, G' is $S_{2,2,2}$-free.

Let $\bar{\gamma}$ be a γ-completion. Lemma 2.2 (p) implies that $u_1, u_2 \in B_\gamma$, which means that none of $w_1, w_2, w_3, w_4, x, y, z, d, e$ belongs to B_γ. At most one of d, e is black, else x, y, z would necessarily be black as well, a contradiction. If d and e are white, a δ-completion is obtained by assigning color black to v_1, v_2. If d and e have different colors, a δ-completion is obtained by assigning color black to v_1 and color white to v_2.

Let now $\bar{\delta}$ be a δ-completion. Lemma 2.2 (a) implies that v_1 is black and hence at most one of d and e is black. If d and e are white, we obtain a γ-completion by assigning color black to u_1, u_2, w_2, w_3, y, z and color white to w_1, x, w_4. If d and e have different colors, say d is black and e is white, we obtain a γ-completion by assigning color black to u_1, u_2, x, y, w_4 and color white to w_1, w_2, w_3, z. ■

Lemma 4.5. *Transformation τ_5 is valid.*

Proof. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free and $d_G(f) \leq 2$ (by Lemma 2.3 (b)), u is the vertex of degree 3 in H, a_{10} is a neighbor of u in H, and without loss of generality, a_9 is the neighbor of a_{10} in H. In other word, G' contains four vertices s_1, s_2, p_1, p_2 such that $u, a_{10}, a_9, s_1, s_2, p_1, p_2$ induce an $S_{2,2,2}$ in G'. But then $u, z, x, s_1, s_2, p_1, p_2$ induce an $S_{2,2,2}$ in G, a contradiction.

Let $\bar{\gamma}$ be a γ-completion. Lemmas 2.1 (ii) and 2.2 (p) imply that w_2, u have the same color, and $v \in B_\gamma$. Hence, at most one of f, u is black, else the black vertex v would have two black neighbors f and w_2. If f is black and u is white , we obtain a δ-completion by assigning color black to $a_1, a_4, a_5, a_6, a_7, a_{10}, a_{12}$ and color white to $a_2, a_3, a_8, a_9, a_{11}$. If f is white and u is black, we obtain a δ-completion by assigning color black to $a_1, a_2, a_3, a_4, a_7, a_9, a_{11}, a_{12}$ and color white to a_5, a_6, a_8, a_{10}. If both f and u are white, we obtain a δ-completion by assigning color black to $a_1, a_2, a_3, a_4, a_7, a_8, a_{10}, a_{11}$ and color white to a_5, a_6, a_9, a_{12}.

Let now $\bar{\delta}$ be a δ-completion. Lemma 2.1 (ii) implies that $\{w_1, w_2, y, z\} \cap B_\gamma = \emptyset$. In G', at most one of f, u can be black. Indeed, if f and u are black, then Lemma 2.1 (ii) implies that a_6 and a_9 are black as well. But this is impossible, since a_7 is black by Lemma 2.2 (p). Now if f is black and u is white, then Lemma 2.2 (i) implies $x \notin B_\gamma$, and we obtain a γ-completion by assigning color black to v, y, z and color white to w_1, w_2, x. If f is white and u is black, a γ-completion is obtained by assigning color black to v, w_2, y, x and color white to w_1, z. Finally, if f and u are both white, a γ-completion is obtained by assigning color black to v, w_1, z, x and color white to w_2, y. ■

Lemma 4.6. *Transformation τ_6 is valid.*

Proof. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, H contains at least one of the new edges yw_1 and xw_2. In fact, H contains exactly one of these edges, because H is C_4-free. Without loss of generality, assume that H contains xw_2. If $d_H(x) = 2$ and $d_H(w_2) = 1$ (resp. $d_H(w_2) = 2$ and $d_H(x) = 1$), then by replacing w_2 (resp. x) in H by b (res. a), we obtain an induced $S_{2,2,2}$ in G, a contradiction. If $d_H(x) = 3$ and $d_H(w_2) = 2$ (resp. $d_H(w_2) = 3$ and $d_H(x) = 2$), then by replacing w_2 (resp. x) and the neighbor of w_2 (resp. x) of degree 1 in H by b and c (resp. a and c), we obtain an induced $S_{2,2,2}$ in G, a contradiction.

Let now $\bar{\gamma}$ be a γ-completion. It follows from Lemma 2.1 (i) that $\bar{\gamma}(y) = \bar{\gamma}(x) \neq \bar{\gamma}(w_1) = \bar{\gamma}(w_2)$. Hence, a δ-completion can be obtained by coloring every vertex of G' as in G.

Finally, let $\bar{\delta}$ be a δ-completion. Lemma 2.1 (ii) implies that $\{a, b\} \cap B_\gamma = \emptyset$, and Lemma 2.1 (i) implies $\bar{\delta}(x) = \bar{\delta}(y) \neq \bar{\delta}(w_1) = \bar{\delta}(w_2)$. We therefore obtain a γ-completion by assigning color $\bar{\delta}(x)$ to a, color $\bar{\delta}(w_1)$ to b, and color black to c. ■

Lemma 4.7. *If $d_G(b) = 3$, then transformation τ_7 is valid.*

Proof. First, notice that Lemma 2.3 (c) implies that $d_G(c) = d_G(s) = 2$. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, H must contain both b and x. Since b and c cannot have degree 2 or 3 in H, we have $d_H(b) = 1$ and $d_H(a) = 2$. But then by replacing b in H with c, we obtain an induced $S_{2,2,2}$ in G, a contradiction.

To show that γ is completable if and only if δ is completable, it is sufficient to prove that all γ-completions and δ-completions assign different colors to b and x. For a γ-completion this is guaranteed by Lemma 2.1 (ii). Now let $\bar{\delta}$ be a δ-completion. Lemma 2.2 (h) implies that $c, s \in B_\gamma$. Hence, $\bar{\delta}(a) \neq \bar{\delta}(b)$, and $\bar{\delta}(x) \neq \bar{\delta}(w_1)$. By Lemma 2.1 (ii), vertices a and w_1 have different colors, and therefore b and x have different colors as well. ■

Lemma 4.8. *If $d_G(z_1) = 1$, then transformation τ_8 is valid.*

Proof. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, both y_1 and x_2 belong to H. If y_1 or x_2 has degree 1 in H, then an $S_{2,2,2}$ in G is obtained by replacing y_1 by y_2 or x_2 by a. Hence, one of y_1, x_2 has degree 2, and the other has degree 3 in H. But then an $S_{2,2,2}$ in G is obtained by replacing y_1 and one of its neighbors different from x_2 by y_2, b (if $d_H(y_1) = 2$) or x_2 and one of its neighbors different from y_1 by a, b (if $d_H(x_2) = 2$), a contradiction.

Let $\bar{\gamma}$ be a γ-completion. Lemma 2.2 (a) implies that $y_1, y_2 \in B_\gamma$, which means that $z_1 \notin B_\gamma$ and exactly one of a, b is black. If a is white or both a and x_2 are black, then a δ-completion $\bar{\delta}$ is obtained by setting $\bar{\delta}(v) = \bar{\gamma}(v)$ for all vertices v in G'. If a is black while x_2 is white, then x_1, z_1, b are white, and z_2 is black. Hence, a δ-completion $\bar{\delta}$ is obtained by changing the color of z_1 to black and setting $\bar{\delta}(v) = \bar{\gamma}(v)$ for all other vertices v in G'.

Let now $\bar{\delta}$ be a δ-completion. Since $y_1, y_2 \in B_\gamma$, we have $\{a, b, z_2\} \cap B_\gamma = \emptyset$ and at most one of x_1, x_2, z_1 is black in $\bar{\delta}$. If x_1 or z_1 is black, or if x_1, x_2, z_1 are white, we obtain a γ-completion by assigning color black to b, y_2 and color white to a, z_2. If x_2 is black, we obtain a γ-completion by assigning color black to a, y_2 and color white to b, z_2. ■

Lemma 4.9. *Transformation τ_9 is valid.*

Proof. Suppose G' contains an induced $S_{2,2,2}$, and denote this $S_{2,2,2}$ by H. Since G is $S_{2,2,2}$-free, both v_1 and v_5 belong to H. If one of them has degree 1 in H, say v_1, then an $S_{2,2,2}$ in G is obtained by replacing v_1 by v_4. Hence one of v_1, v_5 has degree 2 in H, while the other has degree 3, say $d_H(v_1) = 2$ and $d_H(v_5) = 3$. But then an $S_{2,2,2}$ in G is obtained by replacing v_1 and one of its neighbors different from v_5 by v_4, v_3, a contradiction.

Since (G, γ) is clean, at most one among v_2, v_3, v_4 can belong to B_γ, and Lemma 2.2 (f) implies that $v_1 \in B_\gamma$ (resp. $v_2 \in B_\gamma$) if and only if $v_4 \in B_\gamma$ (resp. $v_5 \in B_\gamma$). Hence, if $\{v_1, v_4\} \subseteq B_\gamma$ (resp. $\{v_2, v_5\} \subseteq B_\gamma$), then $\{v_2, v_5\} \cap B_\gamma = \emptyset$ (resp. $\{v_1, v_4\} \cap B_\gamma = \emptyset$).

Let now $\bar{\gamma}$ be a γ-completion. If v_3 is white, then v_1, v_2, v_4, v_5 are black, while if v_3 is black, then exactly one of v_2, v_3 and exactly one of v_1, v_5 is black. In both cases, we obtain a δ-completion $\bar{\delta}$ by setting $\bar{\delta}(v) = \bar{\gamma}(v)$ for all vertices v in G'.

Finally, let $\bar{\delta}$ be a δ-completion. If exactly one of v_1, v_5 is black, say v_1, then $v_2 \notin B_\gamma$ and we obtain a γ-completion by assigning color black to v_3, v_4 and color white to v_2.

Suppose now that both v_1, v_5 are black. We show that $v_3 \notin B_\gamma$. Assume by contradiction that $v_3 \in B_\gamma$. Then none of v_1, v_2 belongs to B_γ and Lemma (2.2) (a) implies $d_G(v_1) > 1$. If $d_G(v_1) = 2$, Lemma (2.2 (f) implies that v_1 has a neighbor $w \neq v_2$ in B_γ, which means that $\bar{\delta}$ is not feasible (since w, v_5 are two black neighbors of v_1), a contradiction. Hence, $d_G(v_1) \geq 3$, and Lemma (2.2) (p) implies that $N(v_1) \setminus \{v_2\}$ contains two adjacent vertices w, w' (else $v_1 \in B_\gamma$). But Lemma (2.1) (ii) then implies that v_1, v_5 have different colors, a contradiction. So $v_3 \notin B_\gamma$, and we obtain a γ-completion by assigning color black to v_2, v_4 and color white to v_3. ■

5 | IRREDUCIBLE GRAPHS

Definition 5.1. We say that a pair (G, γ) is IRREDUCIBLE if it is clean and none of the reductions $\rho_1, \ldots, \rho_8$ and transformations $\tau_1, \ldots, \tau_9$ can be applied to G.

Lemma 5.1. *Let (G, γ) be an irreducible pair. If G is $S_{2,2,2}$-free, then $\Delta(G) \leq 4$ and every vertex of degree 4 belongs to a triangle.*

Proof. Assume $\Delta(G) \geq 4$ and let v be a vertex of maximum degree in G. Lemma 2.3 (a) implies that at most two vertices in $N(v)$ may be adjacent. Hence, at least $d_G(v) - 2$ neighbors of v are isolated vertices in the subgraph induced by $N(v)$. If $v \notin B_\gamma$, then if follows from Lemma 2.2 (p) that $\Delta(G) = 4$ and that v belongs to a triangle. It is therefore sufficient to prove that v cannot belong to B_γ. So assume to the contrary that $v \in B_\gamma$. It then follows from Lemma 2.1 (ii) that v does not belong to a triangle. Consider four neighbors w_1, w_2, w_3, w_4 of v. If a neighbor w of v has degree at least 3, then Lemmas 2.2 (p) and 2.1 (ii) imply that $w \in W_\gamma \cup B_\gamma$, which is a contradiction to the assumption that (G, γ) is clean. Hence, $d_G(w) \leq 2$ for every $w \in N(v)$. Also, we know from Lemma 2.2 (e) that at most one vertex in $N(v)$ has degree 1. Without loss of generality, assume $d_G(w_i) = 2$ for $i = 1, 2, 3$ and $d_G(w_4) \leq 2$. For $i = 1, \ldots, 4$, let u_i be the second neighbor of w_i different from v, if any. Since (G, γ) is clean and $v \in B_\gamma$, we know that $w_i, u_i \notin B_\gamma$ for $i = 1, \ldots 4$. Therefore, Lemma 2.1 (i) implies that no two vertices in $N(v)$ can have a common neighbor w different from v, which means that all u_i are distinct.

Suppose that u_i is adjacent to u_j and u_k, where i, j, k are three distinct indices. Lemma 2.2 (k) implies that u_j is not adjacent to u_k, else at least one vertex in $N(v)$ belongs to W_γ. Since $u_i \notin B_\gamma$, we know from Lemma 2.2 (p) that u_i has a fourth neighbor $y \neq w_i, u_j, u_k$ adjacent to one of u_j, u_k, say u_j. Note that Lemma 2.3 (a) implies that y is not adjacent to both u_j, u_k. But then Lemma 2.2 (j) implies that w_j belongs to W_γ, a contradiction. In summary, every u_i is adjacent to at most one other vertex u_j.

Suppose $d_G(w_4) = 2$. Since G is $S_{2,2,2}$-free, we have $d_G(v) \leq 5$ and the fifth neighbor w_5 of v, if any, has degree 1. Also, without loss of generality, we may assume that u_1 is adjacent to u_2 while u_3 is adjacent to u_4. Since u_1 and u_2 do not belong to B_γ, it follows from Lemma 2.2 (f) that $d_G(u_1), d_G(u_2) \geq 3$. Lemma 2.2 (p) implies then that both u_1 and u_2 have at least two adjacent neighbors. It then follows from Lemma 2.2 (j) that u_1 and u_2 have a common neighbor x, otherwise w_3 and w_4 would belong to W_γ. Similarly, u_3 and u_4 have a common neighbor y. Notice that Lemma 2.3 (a) implies $x \neq y$. Also, Lemma 2.2 (m) implies that x is not adjacent to u_3, u_4, else $x \in W_\gamma$. Similarly, y is not adjacent to u_1, u_2. But this contradict the irreducibility of (G, γ) since ρ_3 or τ_1 can be applied if $d_G(v) = 5$, and ρ_4 or τ_2 can be applied if $d_G(v) = 4$.

We can therefore suppose that $d_G(v) = 4$ and $d_G(w_4) = 1$. Since G is $S_{2,2,2}$-free, without loss of generality, we may assume that u_1 is adjacent to u_2. As was the case when w_4 had two neighbors in G, we know that u_1, u_2 have a common neighbor x that is not adjacent to u_3. But it then follows that (G, γ) is not irreducible since τ_3 can be applied, a contradiction. ■

Lemma 5.2. *Let (G, γ) be an irreducible pair. If G is $S_{2,2,2}$-free, then every vertex of degree 4 belongs to a unique triangle and the two other vertices of the triangle have degree 2.*

Proof. Let a be a vertex of degree 4. It follows from Lemmas 5.1 and 2.3 (a) that a belongs to a exactly one triangle. Denote by b, c the two other vertices of this triangle. Let w_1 and w_2 be the two neighbors of a different from b and c.

Lemma 2.3 (a) implies that w_1, w_2 are nonadjacent to b, c and that w_1 is nonadjacent to w_2. By Lemma 2.1 (ii) w_1 and w_2 have the same color. Moreover, there are vertices u_1 and u_2 such that $N(u_1) \cap \{a, b, c, w_1, w_2, u_2\} = \{w_1\}$ and $N(u_2) \cap \{a, b, c, w_1, w_2, u_1\} = \{w_2\}$, else Lemma 2.2 (r) and 2.1 (ii) would imply that $a \in B_\gamma$ and $\{w_1, w_2\} \subseteq W_\gamma$, and (G, γ) would not be clean.

Suppose to the contrary, that at least one of the vertices b or c has degree at least 3, say b. Let $x \neq a, c$ be a third neighbor of b.

Case 1. $(N(w_1) \cup N(w_2)) \cap (N(b) \cup N(c)) = \{a\}$

Since G is $S_{2,2,2}$-free, x must be adjacent to u_1 or u_2. Lemma 2.3 (b) implies that x is adjacent exactly to one of them, say u_1, that u_1 is black and does not belong to a triangle, and that x, w_1 have no other neighbors. Hence, Lemma 5.1 implies that $d_G(u_1) \leq 3$. Suppose b has a fourth neighbor $y \neq a, c, x$. Similarly to x, vertex y is adjacent to exactly one of the vertices u_1 and u_2. If y is adjacent to u_1, then Lemma 2.1 (i) implies that $\{x, y\} \in W_\gamma$, while if y is adjacent to u_2 then Lemma 2.2 (n) implies $w_1, w_2 \in W_\gamma$, a contradiction. Hence both b and c have at most three neighbors.

Suppose c also has a third neighbor $y \neq a, b$.

1. *Both x and y are adjacent to u_1.* By Lemma 2.3 (b) y has no more neighbors. This contradicts the irreducibility of (G, γ) since ρ_2 can be applied.
2. *x is adjacent to u_1 and y is adjacent to u_2.* As for vertex u_1, Lemma 2.3 (b) and 5.1 imply that $u_1 \in B_\gamma$ and $d_G(u_1) \leq 3$. As (G, γ) is clean, none of the vertices a, b, c, x, y, w_1, w_2 is in B_γ. It then follows from Lemma 2.2 (i) that $d_G(u_1) = 3$, else $c \in B_\gamma$. Similarly, $d_G(u_2) = 3$. So, let $d \neq x, w_1$ be a third neighbor of u_1, and let $e \neq y, w_2$ be a third neighbor of u_2. Since u_1, u_2 both belong to B_γ and since (G, γ) is irreducible, they have no common neighbors, and therefore d and e are different. Moreover, d and e are not adjacent, else Lemma 2.2 (s) implies $a \in B_\gamma$, a contradiction. But now τ_4 can be applied, a contradiction.

Thus, we may assume now that $d_G(c) = 2$. Then $d_G(u_1) \neq 2$, else (G, γ) is not irreducible since ρ_8 can be applied. Hence, u_1 has a third neighbor $f \neq x, w_1$, and $d_G(f) \leq 2$ by Lemma 2.3 (b). Also, Lemma 2.2 (j) implies that f is not adjacent to w_2, else $f \in W_\gamma$. But this contradicts the irreducibility of (G, γ) since τ_5 can be applied.

Case 2. $|\,(N(w_1) \cup N(w_2)) \cap (N(b) \cup N(c))\,| \geq 2$

Assume, without loss of generality, that x is adjacent to w_1. By Lemma 2.3 (c), we have $c \in B_\gamma$ and $d_G(c) = 2$. First, we show that x is nonadjacent to both w_2 and u_2. Lemma 2.3 (b) excludes the case when x is adjacent to u_2, but is not adjacent to w_2. Therefore, suppose x is adjacent to w_2. To avoid a forbidden $S_{2,2,2}$ (induced by $x, b, c, w_1, u_1, w_2, u_2$), x must be adjacent to u_1 or u_2. Lemma 5.1 implies $d_G(x) \leq 4$ and x is therefore adjacent to exactly one of u_1 and u_2. By symmetry, we may assume without loss of generality that x is adjacent to u_1. By Lemma 2.3 (c) u_1 has no neighbors different from x, w_1 and $u_1 \in B_\gamma$. Note that w_2 must belong to a triangle, because otherwise Lemma 2.2 (p) and Lemma 2.1 (ii) would imply $a \in W_\gamma$. Hence there is a vertex u_2' adjacent to both w_2 and u_2. By Lemma 5.1, w_2 has no other neighbors than a, x, u_2, u_2'. Moreover, neither u_2, nor u_2' has a neighbor outside $\{a, b, c, x, w_1, u_1, w_2\}$. Indeed, if say u_2' had such a neighbor z, then $\{w_2, u_2', z, x, u_1, a, c\}$ would induce an $S_{2,2,2}$. Also, it follows from Lemma 2.3 (a) and (c) that at most one of u_2, u_2' can be adjacent to b or w_1. But then ρ_5 can be applied and (G, γ) is therefore not irreducible, a contradiction.

Now Lemma 2.2 (ℓ) implies $d_G(x) \geq 3$, and it follows from Lemma 2.2 (p) and Lemma 2.1 (i) that x belongs to a triangle T_1. Similarly, w_1 belongs to a triangle T_2.

1. $T_1 \neq T_2$. Lemma 2.3 (a) implies that the triangles have no common vertices. Moreover, by Lemma 2.3 (d) there are at most two edges between T_1 and T_2 and if there are exactly two edges, then they are not adjacent. Let x, y, z be the vertices of T_1 and w_1, u_1, u_1' be the vertices of T_2. Denote by M the set of vertices $\{a, b, c, x, y, z, w_1, u_1, u_1', w_2, u_2\}$.
It follows from Lemma 5.1 that x and w_1 have no neighbors outside M. Also, u_1 and u_1' have no neighbors outside M. Indeed, if say u_1 had such a neighbor r, then Lemmas 2.3 (a) and (c) imply that w_1, a, c, u_1, r, x together with y or z induce a $S_{2,2,2}$, a contradiction. Moreover, it follows from Lemmas 2.3 (a) and (b) that y and z are nonadjacent to both b, w_2. It then follows from Lemma 2.3 (a) and (c) that ρ_7 can be applied. Indeed, if $N(u_1') \cap \{b, w_2\} \neq \emptyset$, then $d_G(u_1) = 2$ and u_1' is adjacent to at most one of y, z, while if $N(u_1') \cap \{b, w_2\} = \emptyset$, then at most one of u_1, u_1' is adjacent to at most one of y, z. Hence (G, γ) is not irreducible, a contradiction.

2. $T_1 = T_2$. Let x, w_1, s be the vertices of T_1 (where s may coincide with u_1). Lemma 2.3 (a) and (c) implies that $d_G(s) = 2$ and $s \in B_\gamma$. Now $d_G(b) > 3$, else τ_7 can be applied and (G, γ) is not irreducible. Let $y \neq a, c, x$ be the fourth neighbor of b. Then Lemma 2.3 (a) implies that y is not adjacent to x. Also, y is adjacent to w_1 or w_2. Indeed, if y is nonadjacent to w_1, w_2, then since G is $S_{2,2,2}$-free, it must be adjacent to u_1 or u_2. Lemma 2.3 (b) implies that y cannot be adjacent to u_1. Hence, y is adjacent to u_2. It follows from Lemma 2.3 (b) that $d_G(y) = d_G(w_2) = 2$ and $u_2 \in B_\gamma$. Furthermore, if $d_G(u_2) \geq 3$, say u_2 has a neighbor $t \neq y, w_2$, then it follows from Lemma 2.2 (t) that t must be white, a contradiction. Hence, $d_G(u_2) = 2$ but this contradicts the irreducibility of (G, γ) since ρ_6 can be applied.
If now y is adjacent to w_1, then, similarly to x, we conclude that y and w_1 belong to a same triangle, which is impossible by Lemma 2.3 (a). Hence y is adjacent to w_2, and similarly to x, we know that y and w_2 have a common neighbor p (possibly equal to u_2). Furthermore, $d_G(p) = 2$ by Lemma 2.3 (c). But then τ_6 can be applied, which contradicts the irreducibility of (G, γ). ■

In the remainder of the article $\mathcal{T}$ will denote the subset of vertices that belong to a triangle in G.

Lemma 5.3. *Let (G, γ) be an irreducible pair where $G = (V, E)$ is $S_{2,2,2}$-free. Let P be an induced path in G with edge set $\{v_1v_2, v_2v_3, \dots, v_{\ell-1}v_\ell\}$ ($\ell > 1$) and with $d_G(v_1) \geq 3$, $d_G(v_\ell) \geq 3$ and $d_G(v_i) = 2$ for $i = 2, \dots, \ell - 1$. We have either*

(1) *$\ell = 2$ and both v_1, v_ℓ belong to $\mathcal{T}$, or*

(2) *$\ell = 3$ and exactly one of v_1, v_ℓ belongs to $\mathcal{T}$, or*

(3) *$\ell = 4$ and none of v_1, v_ℓ belongs to $\mathcal{T}$.*

Proof. We necessarily have $\ell \leq 4$, otherwise τ_9 can be applied and hence (G, γ) would not be irreducible.

- If $\ell = 2$, then at least one of v_1, v_2 belongs to $\mathcal{T}$ else Lemma 2.2 (p) implies that the two adjacent vertices v_1, v_2 belong to B_γ. If only one of v_1, v_2 belongs to $\mathcal{T}$, say v_1, then Lemma 2.2 (p) and Lemma 2.1 (ii) imply that $v_2 \in B_\gamma$ and $v_1 \in W_\gamma$, a contradiction.
- Suppose $\ell = 3$. Since v_1 and v_3 are at distance 2, at least one of v_1, v_3 does not belong to B_γ, say v_1. Then Lemma 2.2 (g) and (p) imply that $v_1 \in \mathcal{T}$ and $v_3 \notin \mathcal{T}$.
- Suppose $\ell = 4$ and, without loss of generality, assume $v_1 \in \mathcal{T}$. Then Lemma 2.2 (g) implies $v_3 \in B_\gamma$. Now, either $v_4 \in \mathcal{T}$ and Lemma 2.2 (g) implies that $v_2 \in B_\gamma$, or $v_4 \notin \mathcal{T}$ and Lemma 2.2 (p) implies that $v_4 \in B_\gamma$. Hence G contains two adjacent black vertices, a contradiction. ■

Lemma 5.4. *Let (G, γ) be an irreducible pair where G is $S_{2,2,2}$-free. If $G[V \setminus \mathcal{T}]$ contains an induced cycle, then γ is not completable.*

Proof. Let C be an induced cycle in $G[V \setminus \mathcal{T}]$ with edge set $\{v_1v_2, \cdots, v_{k-1}v_k, v_kv_1\}$. Note that $k \geq 4$ since $G[V \setminus \mathcal{T}]$ contains no triangle. If C is a connected component of G, then $k \leq 5$, otherwise τ_9 can be applied and (G, γ) would not be irreducible. It follows from Lemma 2.2 (ℓ) that $k \neq 4$. Hence, $k = 5$ and it is then easy to observe that C (and thus G) does not admit any feasible complete coloring.

So assume C contains at least one vertex which has degree 3 in G, say v_1. We know from Lemma 5.3 that no vertex in C has a neighbor in $G[\mathcal{T}]$. Hence v_1 has a neighbor $w \neq v_2, v_k$ in $G[V \setminus \mathcal{T}]$. It then follows from Lemma 2.2 (p) that $v_1 \in B_\gamma$. Also, w has no other neighbor on C else this neighbor would also belong to B_γ and would be at distance 2 from v_1. Also, $k \geq 5$, else Lemma 2.1 (i) would imply that $v_2, v_4 \in W_\gamma$. Now, Lemma 2.2 (p) implies $d_G(v_2) = d_G(v_3) = d_G(v_{k-1}) = d_G(v_k) = 2$, else B_γ would contain two vertices at distance at most 2. Since $v_3 \notin B_\gamma$, we know from Lemma 2.2 (f) that $k \geq 6$. Hence $d_G(w) = 1$, otherwise w would have a second neighbor $x \neq v_1$ and vertices $v_1, v_2, v_3, v_{k-1}, v_k, w, x$ would induce an $S_{2,2,2}$ in G. Also, v_4 has a neighbor $w' \neq v_3, v_5$ in $G[V \setminus \mathcal{T}]$, else τ_9 can be applied and (G, γ) would not be irreducible. Using the same arguments as for w, we obtain that $d_G(w') = 1$. Now $k > 6$ by Lemma 2.2 (q). But then τ_8 can be applied, which contradicts the irreducibility of (G, γ). ■

Lemma 5.5. *Let (G, γ) be an irreducible pair where G is $S_{2,2,2}$-free. Then every connected component of $G[V \setminus \mathcal{T}]$ is a claw whose center has exactly one neighbor of degree 1 in G.*

Proof. Let H be a connected component of $G[V \setminus \mathcal{T}]$. By Lemma 5.4, H is a tree. If H contains only one vertex, then it follows from Lemmas 2.2 (a) and 2.1 (ii) that $u \in W_\gamma$, a contradiction. So H contains at least two vertices. Let u be a vertex in H. If $d_H(u) = 0$, it follows from Lemma 5.3 that u has at most one neighbor in $\mathcal{T}$. If $1 \leq d_H(u) \leq 3$, then it follows from Lemmas 2.2 (p) and Lemma 2.1 (i) that u has no neighbor in $\mathcal{T}$.

Claim 1. *If $d_H(u) = 1$ for some vertex u of H, then its neighbor in H is the center of a claw.*

Let v be the neighbor of u in H. We first prove that $v \in B_\gamma$. If $d_G(u) = 1$ or $d_G(v) = 3$, then $v \in B_\gamma$ by Lemma 2.2 (a) and (p). We show that the other cases are impossible. Assume $d_G(u) = 2$ and $d_G(v) \leq 2$. Then $d_G(v) = 2$, else it follows from Lemma 2.2 (a) and (g) that u and v are two adjacent vertices in B_γ. Let $w \neq u$ be the second neighbor of v. We know from Lemma 5.3 that $d_G(w) < 3$. Now, ρ_1 (if $d_G(w) = 1$) or τ_9 (if $d_G(w) = 2$) can be applied, which contradicts the irreducibility of (G, γ).

Let P be a path in H with edge set $\{v_1v_2, \cdots, v_{k-1}v_k\}$, $u = v_1$, $d_G(v_k) \neq 2$ and $d_G(v_i) = 2$, $i = 2, \cdots, k-1$. Since (G, γ) is irreducible, we have $k \leq 4$, otherwise τ_9 can be applied. Also, $\{v_1, v_3, v_4\} \cap B_\gamma = \emptyset$ since $v = v_2 \in B_\gamma$. If $k = 2$, then $d_G(v_2) \neq 1$ otherwise $v_1 \in B_\gamma$ and thus we would have two adjacent vertices in B_γ. Hence, $d_G(v_2) = 3$ and so v_2 is the center of a claw. We finally show that k cannot be equal to 3 or 4. If $k = 4$ then Lemma 2.2 (a) and (p) imply $v_3 \in B_\gamma$ (if $d_G(v_4) = 1$) or $v_4 \in B_\gamma$ (if $d_G(v_4) = 3$), a contradiction. If $k = 3$ then $d_G(v_3) = 1$, else Lemma 2.2 (p) implies $v_3 \in B_\gamma$. But now Lemma 2.2 (a) implies that $v_3 \in B_\gamma$ and hence (G, γ) is not irreducible, and Claim 1 is proven.

Claim 2. *If v is the center of a claw in H, then exactly one of its neighbors has degree 1 in G.*

It follows from Lemmas 5.3 and 2.2 (p) that $v \in B_\gamma$ and no neighbor of v can be in $G[\mathcal{T}]$ or a center of a claw. Hence, all neighbors of v have degree at most 2 in G and none of them belongs to B_γ. We know from Lemma 2.2 (e) that at most one neighbor of v can have degree 1 in G. So assume, by contradiction, that the three neighbors u_1, u_2, u_3 of v are of degree 2 in G, and let w_1, w_2, w_3 be their respective second neighbor. Note that w_1, w_2, w_3 are all distinct by Lemma 2.1 (i). If w_1, w_2, w_3 induce a triangle, then ρ_2 can be applied, and (G, γ) is therefore not irreducible, a contradiction. Since G is $S_{2,2,2}$-free, at least two of w_1, w_2, w_3, say w_1, w_2, are adjacent. Since H is a tree, at least one of w_1, w_2 belongs to $\mathcal{T}$, say w_1. If $w_2 \notin \mathcal{T}$, then $d_G(w_2) = 2$ (otherwise $w_2 \in B_\gamma, w_1 \in W_\gamma$) and Lemma 2.2 (g) implies $u_2 \in B_\gamma$, a contradiction. So, $w_2 \in \mathcal{T}$. If w_1, w_2 belong to distinct triangles, then it follows from Lemma 2.2 (j) that $u_3 \in W_\gamma$. Hence, $\mathcal{T}$ contains a vertex y adjacent to w_1 and w_2. It then follows from Lemma 5.2 that $d_G(w_1) = d_G(w_2) = 3$. Hence, Lemma 2.2 (i) implies that one of u_1, u_2 belongs to W_γ, a contradiction, and Claim 2 is proven.

Let $\{v_1v_2, \cdots, v_{k-1}v_k\}$ be the edge set of a longest path in H. It follows from Claim 1 that $k \geq 3$ and v_2 and v_{k-1} are centers of a claw. If $k > 3$, then we know from Lemma 5.3 that $k \geq 6$ and that v_5 is the center of a claw. Let $v_2' \neq v_1, v_3$ and $v_5' \neq v_4, v_6$ denote the third neighbors of v_2 and v_5, respectively. We know from Claim 1 that one of v_1, v_2' and one of v_6, v_5' has degree 1 in G, which means that τ_8 can be applied, and (G, γ) is therefore not irreducible, a contradiction. So $k = 3$ and H is a claw. ∎

6 | DOMINATING INDUCED MATCHING IN $S_{2,2,2}$-FREE GRAPHS

The main results of the previous sections can be summarized as follows. Let G be an $S_{2,2,2}$-free graph and let (G, γ) be an irreducible pair. Then,

- every vertex in $\mathcal{T} \cap B_\gamma$ has degree 2;
- every maximal clique in G has two or three vertices;
- every vertex of G belongs to at most one triangle;
- every vertex of G has degree at most 4;
- every vertex r of degree 4 belongs to a triangle T, in which the other two vertices p and q have degree 2 (see Fig. 6 a);

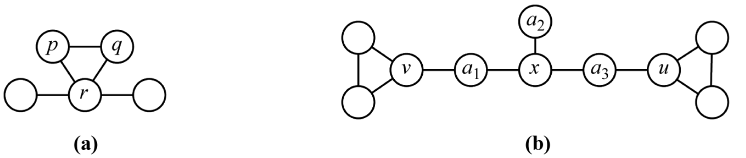

FIGURE 6 Irreducible graph structures

- every connected component of $G[V \setminus \mathcal{T}]$ is a claw. Moreover, if x, a_1, a_2, a_3 are vertices of a connected component C of $G[V \setminus \mathcal{T}]$, such that $d_C(x) = 3$, then $d_G(x) = 3$, $d_G(a_1) = d_G(a_3) = 2$, $d_G(a_2) = 1$ and the neighbors of a_1 and a_3 different from x belong to different triangles in G (see Fig. 6b).

Let (G, γ) be an irreducible pair. Let $C^1, \dots, C^k$ be connected components of $G[V \setminus \mathcal{T}]$ and let x^i, a^i_1, a^i_2, a^i_3 be vertices of C^i, such that $d_G(x^i) = 3$, $d_G(a^i_1) = d_G(a^i_3) = 2$ and $d_G(a^i_2) = 1$. Denote by v^i and u^i the neighbors of a^i_1 and a^i_3 in G, respectively. Also, let $T^1, \dots, T^s$ be the triangles in G which contain a vertex of degree 4 and let r^i, p^i, q^i be the vertices of T^i, such that $d_G(r^i) = 4$ and $d_G(p^i) = d_G(q^i) = 2$. Let $\mathcal{S} = \{p^1, q^1, \dots, p^s, q^s\}$ the set of vertices of degree 2 in triangles $T^1, \dots, T^s$. Let G' be the subgraph of G induced by $V' = \mathcal{T} \setminus (\mathcal{S} \cup B_\gamma)$.

Define a family $\mathcal{K}_{(G,\gamma)}$ of subsets of vertices of G in the following way:

1. $\mathcal{K}_{(G,\gamma)}$ contains every maximal clique of G' of size strictly greater than one;

2. for every connected component C^i of $G[V \setminus \mathcal{T}]$ family $\mathcal{K}_{(G,\gamma)}$ contains $\{v^i, u^i, a^i_2\}$.

Using the definition and the above properties of irreducible pairs (G, γ), it is easy to check that $\mathcal{K}_{(G,\gamma)}$ satisfies the following properties:

(1) every set in $\mathcal{K}_{(G,\gamma)}$ has two or three vertices;

(2) every vertex of G belongs to at most two sets of $\mathcal{K}_{(G,\gamma)}$.

Let $M = V' \cup \{a^1_2, \dots, a^k_2\}$.

Lemma 6.1. *Let (G, γ) be an irreducible pair. Then, γ is completable if and only if there exists a set $H \subseteq M$ such that $|H \cap K| = 1$ for every $K \in \mathcal{K}_{(G,\gamma)}$.*

Proof. Let $\bar{\gamma}$ be a γ-completion of G and let

$$H = (V' \cap W_{\bar{\gamma}}) \cup (\{a^1_2, \dots, a^k_2\} \cap B_{\bar{\gamma}}).$$

Clearly, H is a subset of M. Therefore, we only need to show that every $K \in \mathcal{K}_{(G,\gamma)}$ contains exactly one element in H.

1. *Let* $K = \{x, y\}$. Since K has two elements, the definition of $\mathcal{K}_{(G,\gamma)}$ implies that K is a maximal clique in G'.
If x, y belong to the same triangle T in G, then the third vertex z of T does not belong to G', and therefore $z \in B_\gamma$. It means that $\bar{\gamma}$ assigns color white to exactly one of the vertices x, y and therefore $|H \cap K| = 1$.
If x, y belong to different triangles in G, then by Lemma 2.1 (ii) $\bar{\gamma}$ assigns different colors to x and y and therefore exactly one of them belongs to H.

2. *Let* $K = \{x, y, z\}$.

If K is a maximal clique in G', then K induces a triangle in G. Since $\bar{\gamma}$ is a feasible complete coloring, exactly one of the vertices x, y, z is in $W_{\bar{\gamma}}$ and therefore in H.
If K is not a clique in G', then $K = \{v^i, u^i, a_2^i\}$ for some $i \in \{1, \ldots, k\}$. Note that v^i, u^i cannot both be white, because otherwise a_1^i and a_3^i would be two black neighbors of the black vertex x^i. Thus, either both are black implying that a_2^i is black or one is black and the other one is white implying that a_2^i is white. Hence $|H \cap K| = 1$.

Let now H be a set satisfying the conditions of the lemma and let δ be a coloring of G defined in the following way:

(1) δ assigns color black to every $x \in B_\gamma$;

(2) δ assigns color white to every $x \in V' \cap H$;

(3) δ assigns color black to every $x \in V' \setminus H$;

(4) for every $i = 1, \ldots, s$, define $\delta(p^i)$ and $\delta(q^i)$ (if not yet defined) in such a way that T^i contains exactly one white vertex. (Note that one of the vertices p^i and q^i may already have a color assigned by δ if this vertex belongs to B_γ);

(5) for every $i = 1, \ldots, k$, δ assigns color black to a_2^i and color white to a_1^i and a_3^i, if $a_2^i \in H$; δ assigns color white to a_2^i, a_1^i and color black to a_3^i, if $u^i \in H$; δ assigns color white to a_2^i, a_3^i and color black to a_1^i, if $v^i \in H$. Note that x^i is already assigned color black, since it belongs to B_γ (by Lemma 2.2 (p)).

Clearly, δ is a complete coloring extending γ. We therefore only need to show that δ is feasible. To this end, we prove that W_δ is an independent set and B_δ induces a 1-regular subgraph (i.e., a graph in which all vertices have degree exactly 1) in G.

1. *W_δ is an independent set in G.* The definition of δ implies that every white vertex $x \in V \setminus V'$ has no white neighbors. Let x, y be two white vertices in V'. By items (2) and (3) of the definition of δ, vertices x and y belong to H, and therefore they are not adjacent, since otherwise x, y would belong to a maximal clique of size at least two in G', which contradicts the assumption that H intersects every nontrivial maximal clique of G' in exactly one vertex.

2. *B_δ induces a 1-regular graph in G.* By (5) every black vertex in $V \setminus \mathcal{T}$ has exactly one black neighbor and this neighbor belongs to $V \setminus \mathcal{T}$ as well. Therefore, it is sufficient to show that every black vertex in $\mathcal{T}$ has exactly one black neighbor in $\mathcal{T}$.
First, we show that a black vertex of a triangle T has exactly one black neighbor in T or equivalently that every triangle T has exactly two black vertices. If T is one of the triangles $T^1, \ldots, T^s$ then this is provided by (4). If T does not contain a vertex of degree 4, but has a vertex $x \in B_\gamma$, then $x \notin V'$ and the two other vertices of T form a maximal clique in G' and hence exactly one of them is black. Otherwise, the vertices of T form a maximal clique in G' and exactly two of them are black.
Now let x be a black vertex of a triangle T and suppose x has a neighbor $y \in \mathcal{T}$ outside T. Since every vertex of G belongs to at most one triangle, $\{x, y\}$ forms a maximal clique in G', and therefore y must be white. ■

Lemma 6.1 reduces the Dominating Induced Matching Problem in $S_{2,2,2}$-free graphs to the following.

Problem A. *We are given a finite set S and a family $\mathcal{F} = \{A_i | i \in I\}$ of subsets $A_i \subseteq S$ such that each element of S appears in at most two members of $\mathcal{F}$. We have to find (if it exists) a subset $C \subseteq S$ such that $|C \cap A_i| = 1$ for each $i \in I$.*

We now show how to solve this problem in polynomial time. Let $G = (V, E)$ be a graph and M be a matching in G. We say that M *saturates* a set $U \subseteq V$ if every vertex in U is incident to an edge in M. The vertices in U are called *saturated* (by M) and vertices not incident to any edge of M are *unsaturated*. A matching that saturates all vertices of the graph is called *perfect*.

Without loss of generality, we may assume that in Problem A for any $i, j \in I$ the sets A_i and A_j have at most one common element. Indeed, all elements of $A_i \cap A_j$ belong to exactly the same members in $\mathcal{F}$ and at most one of the elements will appear in C. Therefore, we can remove all but one element of each intersection $A_i \cap A_j$.

With a given instance of Problem A, we may associate a graph $G = (V, E)$, where $V = \{a_i | i \in I\}$ and two different vertices a_i and a_j are adjacent if and only if $A_i \cap A_j \neq \emptyset$. Now let $U \subseteq V$ be the set of vertices a_i of G such that each element in A_i belongs to exactly two members of $\mathcal{F}$. Consider the following problem.

Problem B. *Given a graph $G = (V, E)$ and a subset $U \subseteq V$ find (if it exists) a matching which saturates all vertices in U.*

Lemma 6.2. *Problem A has a solution if and only if Problem B has a solution.*

Proof. Assume we have a solution for Problem A, that is, there exists a subset $C \subseteq S$ satisfying $|C \cap A_i| = 1$ for all $i \in I$. Let M be the set of edges $a_i a_j$ in G, such that the common element of A_i and A_j belongs to C. Clearly, no two edges in M can have a common endpoint a_i, otherwise C would contain at least two elements of A_i, which is a contradiction. In other words, the edges of M form a matching in G. Let now a_i be a vertex of G such that all elements of A_i belong to exactly two members of $\mathcal{F}$. In particular, the element $e \in C \cap A_i$ corresponds to some edge $a_i a_j \in M$. Hence, M is a matching in G which saturates all vertices in U.

Conversely, let M be a matching in G, which saturates all vertices in U. Let $C = \emptyset$. For each edge $a_i a_j \in M$ we add the element $e \in A_i \cap A_j$ to C. This gives us a subset $C \subseteq S$ with $|C \cap A_i| = 1$ for each A_i such that all elements of A_i are in exactly two members of $\mathcal{F}$. Consider the members of $\mathcal{F}$ which have no common element with C. Each such subset A_i contains some element e_i such that $e_i \notin A_j$ for every $j \in I, j \neq i$. By introducing one of these elements into C for every A_i, we obtain a subset with $|C \cap A_i| = 1$ for all $i \in I$. ∎

The following lemma shows that Problem B, and therefore Problem A, can be solved in polynomial time.

Lemma 6.3. *Given a graph $G = (V, E)$ and a subset $U \subseteq V$ of vertices, one can determine in polynomial time whether G has a matching M which saturates all vertices in U.*

Proof. Let H be a graph obtained from G by adding a set A of new vertices such that A is a clique of size $|V|$ and each vertex of A is adjacent to every vertex in $V \setminus U$.

It is not hard to see that G has a matching that saturates U if and only if H has a perfect matching. Indeed, every matching in G saturating U can be extended to a perfect matching in H; and for every perfect matching M' in H, its restriction to G, that is, $M' \cap E$ is a matching in G saturating U.

Therefore, applying the Edmonds' matching algorithm [8] to H, we can determine whether G has a desirable matching, and find such a matching, if any, in polynomial time. ∎

Theorem 6.1. *The dominating induced matching problem can be solved in polynomial time in $S_{2,2,2}$-free graphs. Moreover, if an $S_{2,2,2}$-free graph admits a feasible complete coloring, then such a coloring can be determined in polynomial time.*

Proof. Let $G = (V, E)$ be an $S_{2,2,2}$-free graph. After having applied all forcing rules, propagation rules, all graph reductions and transformations, and all cleanings, three scenarios are possible. The first one is that we get a proof that G does not admit any feasible complete coloring. This occurs if both colors black and white are imposed on the same vertex, or if we get a graph with a connected component isomorphic to C_5 (see Lemma 5.4). Otherwise, we obtain an irreducible pair (H, γ) so that H is $S_{2,2,2}$-free, H being possibly empty. We know from the lemmas of Sections 3 and 4 that γ is completable if and only if G admits a feasible complete coloring.

If H is not empty, we create a new graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows. We create a vertex $a_i \in \mathcal{V}$ for each subset A_i in $\mathcal{K}_{(H,\gamma)}$, and two vertices a_i and a_j are adjacent if and only if the corresponding subsets A_i and A_j have a common element. Let U be the subset of vertices $a_i \in \mathcal{V}$ such that each element of A_i belongs to exactly two members of $\mathcal{K}_{(H,\gamma)}$. We then know from Lemmas 6.1 and 6.2 that γ is completable if and only if $\mathcal{G}$ has a matching that saturates all vertices in U.

Observe that all forcing and propagation rules, graph reductions, graph transformations, and cleanings can be implemented in polynomial time. They are all applied a polynomial number of times. Indeed, define a vertex as *good* if it has less than four neighbors, or if it has degree 4, belongs to a unique triangle and the two other vertices of the triangle have degree 2. The other vertices are called *bad*. We know from Lemma 5.2 that graph H of the irreducible pair (H, γ) does not contain any bad vertex. Note now that none of the graph reductions and transformations increases the number of bad vertices. In fact, transformations $\tau_1, \dots, \tau_6$ strictly decrease the number of bad vertices. Since τ_8, τ_9 as well as reductions $\rho_1, \dots, \rho_8$ strictly decrease the number of vertices, while τ_7 keeps the number of vertices constant but decreases the number of edges, we conclude that the eight graph reductions and nine graph transformations are used a polynomial number of times. Also, we know from Lemma 6.3 that one can determine in polynomial time whether $\mathcal{G}$ has a matching that saturates all vertices in U. Hence, the whole procedure is polynomial.

Note finally that Lemmas 6.1 and 6.2 show how to obtain a feasible complete coloring of H from a matching in $\mathcal{G}$ that saturates all vertices in U, while the lemmas corresponding to graph reductions or transformations show how to obtain a feasible complete coloring of G from a feasible complete coloring of H. ■

7 | CONCLUSION

In this article, we proved that the DOMINATING INDUCED MATCHING problem is polynomial-time solvable in the class of $S_{2,2,2}$-free graphs. This result supports Conjecture 1 about the complexity of the problem in finitely defined hereditary classes. Proving (or disproving) it in its whole generality is a challenging task. As a step toward the complete proof, we suggest analyzing this conjecture under the additional restriction to triangle-free graphs of vertex degree at most 3. This is precisely the class of $(K_3, K_{1,4})$ free graphs, where the problem is NP-complete by Theorem 1.1. One more direction of particular importance in verifying Conjecture 1 is the case of P_k-free graphs. To date, the solution is available only for $k = 8$ [3].

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ENDNOTE

[1] An extension of this result to $S_{1,2,4}$-free graphs was recently announced in [4].