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Regularity of quasi-n-harmonic mappings into NPC spaces

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Abstract
We prove local Hölder continuity of quasi-n-harmonic mappings from Euclidean domains into metric spaces with non-positive curvature in the sense of Alexandrov. We also obtain global Hölder continuity of such mappings from bounded Lipschitz domains.

Keywords n-Harmonic mappings · Quasi-n-harmonic mappings · NPC spaces · Regularity · Reverse Hölder inequalities

Mathematics Subject Classification 49N60 · 58E20

1 Introduction and main results

1.1 Background

Given a mapping $u\colon M \to N$ between two Riemannian manifolds with $\dim M = n$ and $1 < p < \infty$, there is a natural concept of p-energy associated with u. Minimizers (or more generally, critical points) of such energy functionals are referred to as p-harmonic mappings and harmonic mappings in case $p = 2$. The research on harmonic mappings has a long and distinguished history, making it one of the most central topics in geometric analysis on manifolds [31,40]. In his pioneering work, Morrey [35] proved the Hölder continuity of an energy minimizing map when $n = 2$ (and smooth if M and N smooth). The breakthrough in higher-dimensional theory of harmonic mappings was made by Eells and Sampson [5],

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where they proved that every homotopy class of maps from a closed manifold M into N has a smooth representative, with N having non-positive curvature. See more results in, e.g., Hartman [16] and Hamilton [14]. The regularity theory for harmonic mappings into general target Riemannian manifolds has later been developed by Schoen and Uhlenbeck in the seminal paper [38], and which obtained extension by Hardt and Lin [15] to general p-harmonic mappings ($1 < p < \infty$).

Inspired by the celebrated work of Gromov and Schoen [11], where the authors proposed a variational approach for the theory of harmonic mappings to the setting of mappings into singular metric spaces and successfully applied to important rigidity problems for certain discrete groups, and by the fundamental work [28], where the authors established existence, uniqueness and local Lipschitz regularity theory for harmonic mappings from compact smooth Riemannian manifolds to singular metric spaces, harmonic mappings into or between singular metric spaces have received considerable amount of growing interest during the last twenty years, with a particular emphasis on metric spaces of non-positive curvature in the sense of Alexandrov (NPC); see, for instance, [1–4,12,13,20,21,24,25,33,34,43–45,47]. In particular, in the research monograph of Eells-Fuglede [4], the authors extended the theory of harmonic mappings $u : \Omega \to X$ to the setting where Ω is an admissible Riemannian polyhedron and X an NPC space. Gregori [10] further extended the existence and uniqueness theory of harmonic mappings to the setting where X is a Lipschitz Riemannian manifold. Capogna and Lin [1] extended part of the harmonic mapping theories to the setting of mappings from Euclidean spaces to the Heisenberg groups. Sturm [43–45] developed a theory of harmonic mappings between singular metric spaces via a probabilistic theory and (generalized) Dirichlet forms. In the recent remarkable work of Zhang and Zhu [47], the authors proved the important interior Lipschitz regularity of harmonic mappings from certain Euclidean domains to NPC spaces. Parallel to the mapping case, the theory of harmonic functions on singular metric spaces also gained growing interest in the last twenty years; see, for instance, [23,27,29,42] and the references therein.

Besides the harmonic case ($p = 2$) and the intermediate case ($1 < p < n$), the borderline case $p = n$ also received special attention as they often enjoy better property than general p-harmonic mappings. For instance, among other results, Hardt and Lin [15] showed that minimizing n-harmonic mappings from an n-dimensional compact Riemannian manifold into a C^2 Riemannian manifold are locally $C^{1,\alpha}$ for some $0 < \alpha < 1$; Wang [46] proved that n-harmonic mappings into Riemannian manifolds (without boundary) enjoy nice compactness properties; Mou and Yang [36] obtained that n-harmonic mappings are everywhere regular in the interior, continuous up to the boundary (of a bounded smooth domain), and have removable isolated singularities; see also [37] for a recent improvement of this result. When the target metric space is the real line $\mathbb{R}$, n-harmonic functions play a particularly important role in the theory of quasiconformal mappings and quasiregular mappings; see, for instance, [17,22] and the various references therein.

1.2 Main results

We first recall the following definition of quasi-n-harmonic mappings.

Definition 1.1 (*quasi-n-harmonic mappings*) Let $\Omega \subset \mathbb{R}^n$ be an open domain and X a metric space. A mapping $u \in W^{1,n}(\Omega, X)$ is said to be Q-quasi-n-harmonic, $Q \geq 1$, if

$$E_n(u|_{\Omega'}) \leq Q \cdot E_n(v|_{\Omega'})$$

for every relatively compact domain $\Omega' \subset \Omega$ and every $v \in W^{1,n}(\Omega, X)$ with $u = v$ almost everywhere in $\Omega \backslash \Omega'$, where $E_n(u)$ is the n-energy of u defined as in Sect. 2.1.

Note that 1-quasi-n-harmonic mappings are also called n-harmonic mappings in the literature (see, e.g., Hardt and Lin [15]). When $n = 2$, we recover the class of quasiharmonic or harmonic mappings. When X is a proper metric space, the existence of n-harmonic mappings $u : \Omega \to X$ follows from the classical direct method in the calculus of variations by noticing the compactness of Sobolev spaces and the lower semi-continuity of the energy E_n; see, e.g., [33, Theorem 2.3]. When X is a (possibly infinite dimensional) NPC space, considered as in this paper, the existence of n-harmonic mappings is a special case of [13, Theorem 1.4].

Another typical non-trivial example in higher dimensions is given by the class of quasiregular mappings (see Giaquinta and Giusti [8, Theorem 2.4]) between higher-dimensional Euclidean domains, i.e., a mapping $u : \Omega \to \mathbb{R}^n$ satisfying $|Du|^n \leq K \det(Du)$. In particular, when u is a homeomorphism, we recover the well-known class of quasiconformal mappings. Quasiconformal mappings are natural higher-dimensional extensions of the classical planar conformal mappings, and they are closely related to the so-called $\mathcal{U}_n$*-harmonic morphism* in the sense of Heinonen–Kilpeläinen–Martio [17, Chapters 13 and 14]. Recall that a mapping $u : \Omega \to \Omega'$ is an $\mathcal{U}_n$-harmonic morphism if $v \circ u$ is an $\mathcal{A}'$-harmonic function of order n on $\Omega = u^{-1}(\Omega')$ whenever v is an $\mathcal{A}$-harmonic function of order n on Ω' (see [17, Chapter 3] for the precise definition of $\mathcal{A}$-harmonic functions). Quasiconformal mappings are precisely those homeomorphic $\mathcal{U}_n$-harmonic morphisms (see, e.g., [17, Section 14.35]). More generally, quasi-n-harmonic mappings are a special case of quasiminima that was initially studied by Giaquinta and Giusti [8] in Euclidean spaces.

Based on the recent solution of Plateau's problem in proper metric spaces [32], Lytchak and Wenger [33] considered the interior regularity of quasiharmonic mappings from two-dimensional Euclidean domains to proper metric spaces. They proved that each quasiharmonic mapping $u : \Omega \to X$ from a planar Euclidean domain to a large class of proper metric spaces has a locally Hölder continuous representative.

Motivated by the above work of Lytchak and Wenger [33] and also by the recent development of harmonic mappings in singular metric spaces, in this short note, we study interior and boundary regularity of quasi-n-harmonic mappings from Euclidean domains to NPC spaces.

Our first main result can be viewed as a natural partial extension of the interior regularity result of Lytchak and Wenger [33] to higher dimensions.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be an open domain and X an NPC space. Then each Q-quasi-n-harmonic mapping $u : \Omega \to X$ has a locally α-Hölder continuous representative for some α depending only on Q and n.*

We would like to point out the Hölder continuity in Theorem 1.2 is best possible even when $X = \mathbb{R}$; see [27,42]. As a corollary of Theorem 1.2, we obtain that each quasiharmonic mapping from planar Euclidean domains to NPC spaces has a locally Hölder continuous representative.

Corollary 1.3 *Let $\Omega \subset \mathbb{R}^2$ be an open domain and X an NPC space. Then each Q-quasiharmonic mapping $u : \Omega \to X$ has a locally α-Hölder continuous representative for some α depending only on Q.*

Corollary 1.3 is not really new, and in fact, it follows from the proof of [33, Theorem 1.3]. Indeed, the main ingredients in their arguments are *solvability of Plateau problem in proper metric spaces* and *an energy filling inequality* (i.e., [33, Theorem 1.5]). The solvability of Plateau problem is well known in the context of NPC spaces (see, e.g., [13]), and the authors also pointed out the energy filling inequality holds for general NPC spaces. Thus Corollary 1.3 follows from the proof of Theorem 1.3 there. However, our proof of Corollary 1.3 is more elementary and simpler, comparing with the more general proof there. On the other hand, our proof relies heavily on the special structure of NPC spaces and hence seems hard to be extended to the more general setting as considered in [33]. As to quasiharmonic mappings on higher-dimensional Euclidean domains ($n \geq 3$), there is no hope to derive Hölder continuity in this respect, even for quasiharmonic mappings into Euclidean domains (in this case, quasiharmonic mappings are also named quasiminima). As pointed out in Giaquinta [7, p. 253], there exists a quasiminima for some Dirichlet integral which is singular in a dense set.

Our second main result concerns boundary regularity of quasi-n-harmonic mappings from bounded Lipschitz domain to NPC spaces, which can be viewed as a natural partial extension of [33, Theorem 1.4]. There have been extensive contributions for boundary regularity on harmonic mappings in the literature which is impossible to list completely. We only mention a few that are most related to our work. For mappings from bounded smooth Euclidean domain to Euclidean spaces, we refer to Jost and Meier [26] in which more general result was obtained. That is, bounded minimum of certain quadratic functional is proven to be Hölder continuous in a neighborhood of the boundary with sharp Hölder exponent. For mappings from a compact Riemannian manifold with boundary or from bounded smooth domain of Riemannian manifolds to smooth Riemannian manifolds, we refer to Schoen and Uhlenbeck [39] for harmonic mappings and Hardt and Lin [15] for p-harmonic mappings, respectively. As for boundary regularity results on harmonic mappings from compact Riemannian domains to NPC, we would like to refer to the work of Serbinowski [41]. Our result is new in the setting of quasi-n-harmonic mappings.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with a Lipschitz boundary $\partial\Omega$ and X an NPC space. Let $u\colon \Omega \to X$ be a Q-quasi-n-harmonic mapping whose trace coincides with the trace of $h \in W^{1,p}(\Omega, X)$ with $p > n$. Then u is α-Hölder continuous in a neighborhood of $\partial\Omega$ for some α depending only on Q, n and p.*

Above, the trace of $u \in W^{1,n}(\Omega, X)$ coincides with the trace of $g \in W^{1,p}(\Omega, X)$ is equivalent to the requirement that $d(u, g) \in W_0^{1,n}(\Omega)$ (see, e.g., [28, Section 1.12]). We would like to point out that in [33, Theorem 1.4], the trace of u was required to be Lipschitz continuous, which is a little bit stronger than what we have assumed in Theorem 1.4.

1.3 Outline of proof

Our main tool to prove Theorems 1.2 and 1.4 is the reverse Hölder inequality, which was discovered by Gehring [6] in his celebrated work on higher regularity of quasiconformal mappings and was later developed by Giaquinta and Modica [9] (see also [7]) and many others in the theory of elliptic partial differential equations. In this note, we will use the following local-type reverse Hölder inequality; see Proposition 5.1 of [9] or Proposition 1.1 of Chapter V of [7].

Denote by $Q_R \subset \mathbb{R}^n$ a cube with side length R and let $q > 1$. Let $g \in L^q_{\mathrm{loc}}(Q_1)$ and $f \in L^r_{\mathrm{loc}}(Q_1)$ $(r > q)$ be two nonnegative functions. Suppose there exist constants $b > 1$ and $\theta \in [0, 1)$, such that for every $x_0 \in Q_1$ and $2R <$ dist $(x_0, \partial Q_1)$ the following estimate holds

$$⨍_{Q_R(x_0)} g^q \mathrm{d}x \leq b\left\{\left(⨍_{Q_{2R}(x_0)} g \mathrm{d}x\right)^q + ⨍_{Q_{2R}(x_0)} f^q \mathrm{d}x\right\} + \theta ⨍_{Q_{2R}(x_0)} g^q \mathrm{d}x, \tag{1.1}$$

where $\fint_A f\mathrm{d}x := |A|^{-1}\int_A f\mathrm{d}x$. Then, there exist $\epsilon > 0$ and $C > 0$, depending only on θ, b, q, n, such that $g \in L^p_{\mathrm{loc}}(Q)$ for $p \in [q, q+\epsilon)$ and

$$\left(\fint_{Q_{1/2}} g^p \mathrm{d}x\right)^{1/p} \le C\left\{\left(\fint_Q g^q \mathrm{d}x\right)^{1/q} + \left(\fint_Q f^p \mathrm{d}x\right)^{1/p}\right\}. \tag{1.2}$$

In our arguments, to derive estimates of type (1.1) with suitable choices g and f, we will borrow the idea from [8]. More precisely, we will compare u with "$u_\eta = (1-\eta)u + \eta p_0$" for $0 \le \eta \le 1$, which is well defined in NPC spaces. The main new ingredient in our proof is certain new estimates on pullback tensors.

Our notations are rather standard. We will use C or c to denote various constants that may be different from line to line.

2 Preliminaries

2.1 Sobolev mappings with value in metric spaces

Let Ω be a domain in $\mathbb{R}^n$ and (X, d) a metric space. We follow Korevaar and Schoen [28] to define Sobolev mappings $u\colon \Omega \to X$. Given $p > 1$, $\epsilon > 0$, we define

$$e^u_{p,\epsilon}(x) = \fint_{B_\epsilon(x)} \frac{d^p(u(x), u(y))}{\epsilon^p}\mathrm{d}y$$

for all $x \in \Omega_\varepsilon := \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$ and $e^u_{p,\epsilon}(x) = 0$ for all $x \in \Omega \backslash \Omega_\varepsilon$. For each $u \in L^p(\Omega, X)$, we define the approximate energy

$$E^u_{p,\epsilon}(f) = c(n, p)\int_\Omega f(x)e^u_{p,\epsilon}(x)\mathrm{d}x, \qquad f \in C_c(\Omega),$$

where $C_c(\Omega)$ consists of continuous functions in Ω with compact support and $c(n, p) > 0$ is a normalization constant. Then, u is said to have finite p-energy, written as $u \in W^{1,p}(\Omega, X)$, if

$$E_p(u) \equiv \sup_{f\in C_c(\Omega), 0\le f\le 1} \limsup_{\epsilon\to 0} E^u_{p,\epsilon}(f) < \infty.$$

By [28, Theorem 1.5.1], if $u \in W^{1,p}(\Omega, X)$, then the measures $e^u_{p,\varepsilon}dx$ converges weakly as $\varepsilon \to 0$ to an energy density measure de^u_p with total measure $E_p(u)$. Remark that Ω was assumed to be bounded in Korevaar and Schoen [28]. This assumption can be removed via exhaustion by relatively compact smooth subdomains and then apply the argument of [28], ensuring that no mass from the limit measure de^u_p accumulates on the boundary. Moreover, by [28, Theorem 1.10], de^u_p is absolutely continuous with respect to the Lebesgue measure. In particular, there exists $|\nabla u|_p \in L^1(\Omega, \mathbb{R})$ such that

$$de^u_p = |\nabla u|_p dx.$$

When $p = 2$, we write $|\nabla u|^2$ instead of $|\nabla u|_2$. Note that in general

$$|\nabla u|_p \ne \left(|\nabla u|^2\right)^{p/2}$$

but they are comparable up to a uniform constant.

There are also many other equivalent definitions of metric-valued Sobolev spaces, and we recommend the interested readers to [19] for more information. As a special consequence of the equivalence with Newtonian–Sobolev spaces, we have the following Sobolev–Poincaré inequality for Sobolev mappings:

Lemma 2.1 *For $1 < p < n$, there exists a positive constant $c(n, q, p)$ such that for each $p < q < p^* = \frac{np}{n-p}$*

$$\inf_{a\in X}\Big(\fint_B d^q(u,a)\mathrm{d}x\Big)^{1/q} \le c(n,p,q)\operatorname{diam} B\Big(\fint_B |\nabla u|_p \mathrm{d}x\Big)^{1/p} \tag{2.1}$$

holds for every $u \in W^{1,p}(\Omega, X)$ and every ball B with $4B \subset\subset \Omega$.

Proof The proof is essentially the same as [30, Proof of Theorem 3.6] (whereas the idea dates back to [18]). For the convenience of the readers, we include the main steps here. Embed X isometrically in the Banach space $\mathbb{V} = l^\infty(X)$ such that $X \subset \mathbb{V} = (\mathbb{V}, \|\cdot\|)$. Let $\Lambda \in V^*$ be such that $L := \|\Lambda\|_{V^*} \le 1$. Then $\Lambda\colon \mathbb{V} \to \mathbb{R}$ is an L-Lipschitz map and $f := \Lambda \circ u \in W^{1,p}(\Omega)$. Moreover, the standard Sobolev–Poincaré inequality for f implies that for each $q < p^*$,

$$\fint_B |f - f_B|^q \mathrm{d}x \le c(n,p,q)L^q\big(\operatorname{diam} B\big)^q\Big(\fint_B |\nabla u|_p \mathrm{d}x\Big)^{q/p},$$

where we have also used the fact that

$$|\nabla f|^p \le \big(\operatorname{Lip}\Lambda\big)^p \cdot g_u \le c(n,p)L^p|\nabla u|_p$$

holds almost everywhere on B, where g_u is the minimal p-weak upper gradient of u (see [19] for precise definition) (and the last inequality comes from the equivalence of Newtonian–Sobolev spaces [19, Section 10.4]).

If x, y are Lebesgue points of f, then letting $B_0 = B(x, 2|x-y|)$, $B_i = B(x, 2^{-i}|x-y|)$, $B_{-i} = B(y, 2^{-i}|x-y|)$ and using the standard telescoping argument, we obtain the following useful pointwise inequality for f:

$$\begin{aligned}|\Lambda\circ u(x) - \Lambda\circ u(y)| &\le \sum_{i\in\mathbb{Z}} |f_{B_i} - f_{B_{i+1}}| \\ &\le c(n,p)L|x-y|\Big(M_{2|x-y|}|\nabla u|_p(x)^{1/p} + M_{2|x-y|}|\nabla u|_p(y)^{1/p}\Big),\end{aligned}$$

where

$$M_R|\nabla u|_p(x) = \sup_{0<r<R}\fint_{B(x,r)} |\nabla u|_p(z)\mathrm{d}z$$

is the standard restricted maximal function of $|\nabla u|_p$.

The next step is to show that for almost every $x, y \in B$ we have

$$\|u(x) - u(y)\| \le c(n,p)|x-y|\Big(M_{2|x-y|}|\nabla u|_p(x)^{1/p} + M_{2|x-y|}|\nabla u|_p(y)^{1/p}\Big). \tag{2.2}$$

In this step, one can follow the arguments used in [30, the last paragraph in the proof of Theorem 3.6] word by word. In fact, only the previous pointwise inequality for f is needed.

The final step is to show that the pointwise inequality (2.2) implies the following Sobolev–Poincaré inequality: for each $q < p^*$,

$$\inf_{a\in Y}\fint_B d(u(x),a)^q \mathrm{d}x \le c(n,p,q)\big(\operatorname{diam} B\big)^q\Big(\fint_B |\nabla u|_p \mathrm{d}x\Big)^{q/p}. \tag{2.3}$$

The proof of this is more or less well known (see [30, Proof of Proposition 3.12] or the monograph [19, Section 9.1]), and we include a sketch here for the convenience of the readers. First, note that (2.2) together with Hölder's inequality implies that

$$\fint_B \|u(x) - u_B\| \mathrm{d}x \le c(n, p, q) \operatorname{diam} B \Big(\fint_B M_{2 \operatorname{diam} B} |\nabla u|_p \mathrm{d}x \Big)^{1/p}. \tag{2.4}$$

Fix $0 < \varepsilon < 1$ (to be determined later), and, for $t > 0$, let

$$A_t = \big\{ x \in B : \|u(x) - u_B\| > t \big\}.$$

For each $i \in \mathbb{N}$, set $B_i = \{z \in \Omega : |x - z| < 2^{-i} \operatorname{diam} B\}$. It is clear that $B_i \subset 2B$. At each Lebesgue point $x \in A_t$ of the map $u \colon X \to Y \subset \mathbb{V}$, we have

$$\begin{aligned} C(\varepsilon) t \sum_{i \in \mathbb{N}} 2^{-i(1-\varepsilon)} = t < \|u(x) - u_B\| &\le \sum_{i \in \mathbb{N}} \|u_{B_{i+1}} - u_{B_i}\| \\ &\le c(n, p, q) \fint_{B_i} \|u(z) - u_{B_i}\| \mathrm{d}z \\ &\le c(n, p, q) \operatorname{diam} B \sum_{i \in \mathbb{N}} 2^{-i} \Big(\fint_{B_i} M_{2 \operatorname{diam} B_i} |\nabla u|_p \mathrm{d}x \Big)^{1/p}. \end{aligned}$$

Hence, there exists a positive integer i_x such that

$$C(\varepsilon) t 2^{-i_x(1-\varepsilon)} \le c(n, p, q) \operatorname{diam} B \cdot 2^{-i_x} \Big(\fint_{B_{i_x}} M_{2 \operatorname{diam} B_{i_x}} |\nabla u|_p \mathrm{d}x \Big)^{1/p},$$

or equivalently,

$$\mathcal{L}^n(B_{i_x}) \le c(n, p, q)^s \left(\frac{\operatorname{diam} B}{t} \right)^{ps} \frac{\Big(\int_{B_{i_x}} M_{2 \operatorname{diam} B_{i_x}} |\nabla u|_p \mathrm{d}x \Big)^s}{\mathcal{L}^n(B)^{s-1}},$$

where $s = \frac{n}{n - p\varepsilon} > 1$ and $\varepsilon < 1$ is a fixed small number. Using the $(5B)$-covering lemma, we easily obtain that

$$\mathcal{L}^n(A_t) \le c(n, p, q)^s \left(\frac{\operatorname{diam} B}{t} \right)^{ps} \mathcal{L}^n(B)^{1-s} \left(\int_{4B} M_{4 \operatorname{diam} B} |\nabla u|_p \mathrm{d}x \right)^s.$$

Set

$$C_0 := c(n, p, q)^s \big(\operatorname{diam} B \big)^{ps} \left(\int_{4B} M_{4 \operatorname{diam} B} |\nabla u|_p \mathrm{d}x \right)^s.$$

Then $\mathcal{L}^n(A_t) \le C_0 \frac{\mathcal{L}^n(B)^{1-s}}{t^{ps}}$ and an easy application of the Cavalier's principle (see [30, Proof of Lemma 3.23]) gives

$$\int_B \|u(x) - u_B\|^q \mathrm{d}x \le C_0^{\frac{q}{ps}} \left(\frac{q}{q - ps} \right)^{q/(ps)} \mathcal{L}^n(B)^{1 - \frac{q}{p}},$$

which reduces to the desired inequality (2.3) upon noticing the L^p-boundedness of the maximal operator. ◻

Remark 2.2 In the case $1 < p < n$, Lemma (2.1) holds with $q = p^*$ as well. This can be proved by a truncation argument due to Hajlasz and Koskela. Since this stronger case is not needed for the current paper, we do not include the proof here.

In the borderline case $p = n$, one can similarly prove that

$$\fint_B \exp\Big(\Big(\frac{d(u, u_B)}{c_1(n)\operatorname{diam} B(\fint_B |\nabla u|_n \mathrm{d}x)^{1/n}}\Big)^{n/(n-1)}\Big) \le c_2(n)$$

for $u \in W^{1,n}(\Omega, X)$.

In the case $p > n$, one can similarly prove that each $u \in W^{1,p}(\Omega, X)$ has a locally Hölder continuous representative (see also [32, Proposition 3.3]).

2.2 Metric spaces with non-positive curvature in the sense of Alexandrov

Definition 2.3 (*NPC spaces*) A complete metric space (X, d) (possibly infinite dimensional) is said to be non-positively curved in the sense of Alexandrov (NPC) if the following two conditions are satisfied:

- (X, d) is a length space, that is, for any two points P, Q in X, the distance $d(P, Q)$ is realized as the length of a rectifiable curve connecting P to Q. (We call such distance-realizing curves geodesics.)
- For any three points P, Q, R in X and choices of geodesics γ_{PQ} (of length r), γ_{QR} (of length p), and γ_{RP} (of length q) connecting the respective points, the following comparison property is to hold: For any $0 < \lambda < 1$, write Q_λ for the point on γ_{QR} which is a fraction λ of the distance from Q to R. That is,

  $$d(Q_\lambda, Q) = \lambda p, \quad d(Q_\lambda, R) = (1-\lambda)p.$$

  On the (possibly degenerate) Euclidean triangle of side lengths p, q, r and opposite vertices $\bar{P}$, $\bar{Q}$, $\bar{R}$, there is a corresponding point

  $$\bar{Q}_\lambda = \bar{Q} + \lambda(\bar{R} - \bar{Q}).$$

  The NPC hypothesis is that the metric distance $d(P, Q_\lambda)$ (from Q_λ to the opposite vertex P) is bounded above by the Euclidean distance $|\bar{P} - \bar{Q}_\lambda|$. This inequality can be written precisely as

  $$d^2(P, Q_\lambda) \le (1-\lambda)d^2(P, Q) + \lambda d^2(P, R) - \lambda(1-\lambda)d^2(Q, R).$$

In an NPC space X, geodesics connecting each pair of points are unique and so one can define the t-fraction mapping u_t of two mapping $u_0, u_1 \colon \Omega \to X$ as $u_t =$ "$(1-t)u_0 + tu_1''$, that is, for each x, $u_t(x)$ is the unique point P on the geodesic connecting $u_0(x)$ and $u_1(x)$ such that $d(P, u_0(x)) = td(u_0(x), u_1(x))$ and $d(P, u_1(x)) = (1-t)d(u_0(x), u_1(x))$. We refer the interested readers to [28, Section 2.1] for more discussions on NPC spaces.

2.3 Pullback tensors

Let $\Omega \subset \mathbb{R}^n$ be a domain and X an NPC space. For each $u \in W^{1,2}(\Omega, X)$ and for any Lipschitz vector fields Z, W on $\overline{\Omega}$, u induces an integrable directional energy functional $|u_*(Z)|^2$, and moreover, Korevaar and Schoen [28, Lemma 2.3.1] proved the following important parallelogram identity

$$|u_*(Z+W)|^2 + |u_*(Z-W)|^2 = 2|u_*(Z)|^2 + 2|u_*(W)|^2.$$

This property induces a pullback tensor $\pi = \pi_u$ over the Lipschitz vector fields on $\overline{\Omega}$ by setting

$$\pi(Z, W) = \frac{1}{4}|u_*(Z+W)|^2 - \frac{1}{4}|u_*(Z-W)|^2.$$

It was proved in [28, Theorem 2.3.2]) that π is continuous, symmetric, bilinear, nonnegative and tensorial. The pullback tensor generalizes the classical pullback metric u^*h for mappings into Riemannian manifolds (N, h) and plays a fundamental role in understanding the structure of harmonic mappings to NPC spaces in [28].

3 Proof of main results

First, we derive the interior Hölder continuity for quasi-n-harmonic mappings.

Proof of Theorem 1.2 Let $B_R \subset\subset \Omega$ be a ball of radius R and let $\frac{R}{2} < t < s < R$. Let $\eta \in C_0^\infty(B_R)$ be such that $0 < \eta < 1$ on $B_s \backslash \bar{B}_t$, $\eta \equiv 0$ outside B_s, $\eta \equiv 1$ on $\bar{B}_t$, $|D\eta| \leq c(s-t)^{-1}$, $\{\eta = \frac{1}{2}\}$ has zero Lebesgue measure and $\int_{B_R} \frac{1}{(1-2\eta(x))^4}\mathrm{d}x < \infty$. We will compare the n-energy of u with the function $u_\eta :=$ "$(1-\eta(x))u(x) + \eta(x)p_0''$, where p_0 is chosen such that

$$\left(\fint_{B_R} d^n(u(x), p_0)\mathrm{d}x\right)^{1/n} \leq 2 \inf_{a\in X} \left(\fint_{B_R} d^n(u(x), a)\mathrm{d}x\right)^{1/n}$$

We now show that for any smooth function $\eta \in C_0^\infty(\Omega)$ which satisfies either $0 \leq \eta < \frac{1}{2}$ or $\frac{1}{2} < \eta \leq 1$, it holds

$$\begin{aligned} \pi_{u_\eta} &\leq \pi_u - \mathcal{C}(u_0, p_0, \eta) - \nabla\eta \otimes \nabla d^2(u, p_0) + Q(\eta, \nabla\eta) - \pi_{u_{1-\eta}} \\ &\leq (1-\eta)\pi_u + C|\nabla\eta| d(u, p_0)|\nabla u|_1 - \nabla\eta \otimes \nabla d^2(u, p_0) + Q(\eta, \nabla\eta), \end{aligned} \tag{3.1}$$

where

$$\mathcal{C}(u, p_0, \eta) = \pi_u - P(u, p_0, \eta) - P(u, p_0, 1-\eta)$$

is the auxiliary tensor defined in (2.4xiv) of [28] and

$$Q(\eta, \nabla\eta) = C\frac{|\nabla\eta(x)|^2 d^2(u(x), p_0)}{(1-2\eta(x))^2}$$

is quadratic in terms of η and $\nabla\eta$ defined as in [28, Lemma 2.4.2], for some constant C which may be different from line to line. For this, we first prove the following estimate:

$$\pi_{u_{1-\eta}} \geq \eta\pi_u - \mathcal{C}(u, p_0, \eta) - C|\nabla\eta| d(u, p_0)|\nabla u|_1 - Q(\eta, \nabla\eta). \tag{3.2}$$

We first consider the case $0 \leq \eta < \frac{1}{2}$. In this case, by [28, (2.4xvii)], on $\{\eta > 0\}$ we have

$$\pi_{u_{1-\eta}} \geq P(u, p_0, 1-\eta) - C|\nabla\eta| d(u, p_0)|\nabla u|_1 - Q(\eta, \nabla\eta),$$

where $C > 0$ depends on n and $P(u, p_0, 1-\eta)$ is a symmetric bilinear integrable tensor defined in [28, Lemma 2.4.4]. Moreover, by [28, (2.4xvi)], we have

$$\eta\pi_u - \mathcal{C}(u, p_0, \eta) \leq P(u, p_0, 1-\eta),$$

from which (3.2) follows. The desired Eq. (3.1) follows by combining (3.2) with [28, (2.4xv)]. The case $\frac{1}{2} < \eta \leq 1$ can be proved similarly. Indeed, the only difference in this case would be the equation (2.4vii) of [28], where one needs to replace the negative term $1 - 2\eta(y)$ by the positive term $2\eta(y) - 1$. Then the equations (2.4xv), (2.4xvi) and (2.4xvi) hold (with the same proof as in [28, Lemma 2.4.5]). Consequently, (3.1) holds by the same arguments as in the first case. Since $u \in W^{1,n}(B_R, X)$, we have $d(u, p_0) \in L^{2n}(B_R)$ (indeed, in any L^p with $p < \infty$), this together with our choice that $\frac{1}{(1-2\eta)^2} \in L^2(B_R)$, implies that $Q(\eta, \nabla\eta)^{\frac{n}{2}} \in L^s(B_R)$ for some $s = s(n) > 1$ (by the Hölder's inequality and by the proof of Lemma 2.4.2 in [28]).

Set $\mu = n/2$. Taking trace on both sides of Eq. (3.1) (for simplicity and without any confusion, we denote the trace of π by the same symbol), and then taking the μ-th power on both sides, we get

$$\pi_{u_\eta}^\mu \le c(n)\Big((1-\eta)^\mu \pi_u^\mu + |\nabla\eta|^\mu d^\mu(u, p_0)|\nabla u|_\mu - (\nabla\eta \otimes \nabla d^2(u, p_0))^\mu + Q(\eta, \nabla\eta)^\mu\Big).$$

Write $g = Q(\eta, \nabla\eta)^\mu$. Then, $g \in L^s(B_R)$ for some $s > 1$. Since u is Q-quasi-n-harmonic, we obtain from the above inequality that

$$\begin{aligned}
\frac{1}{c(n)} \int_{B_s} (|\nabla u|^2)^\mu \mathrm{d}x &\le \int_{B_s} |\nabla u|_n \mathrm{d}x \le Q \int_{B_s} |\nabla u_\eta|_n \mathrm{d}x \le C \int_{B_s} \big(|\nabla u_\eta|^2\big)^\mu \mathrm{d}x \\
&\le C \bigg(\int_{B_s} (1-\eta)^\mu (|\nabla u|^2)^\mu \mathrm{d}x + \int_{B_s} |\nabla\eta|^\mu \big(d^\mu(u, p_0)|\nabla u|_\mu \\
&\quad + |\nabla d^2(u, p_0)|^\mu\big)\, \mathrm{d}x + \int_{B_s} g \mathrm{d}x \bigg) \\
&\le C \bigg(\int_{B_s - B_t} (|\nabla u|^2)^\mu \mathrm{d}x + \frac{1}{(s-t)^\mu} \int_{B_R} \big(d^\mu(u, p_0)|\nabla u|_\mu \\
&\quad + |\nabla d^2(u, p_0)|^\mu\big)\, \mathrm{d}x + \int_{B_R} g \mathrm{d}x \bigg) \\
&\le C \bigg(\int_{B_s - B_t} (|\nabla u|^2)^\mu \mathrm{d}x + \frac{1}{(s-t)^\mu} \int_{B_R} d^\mu(u, p_0)|\nabla u|_\mu \mathrm{d}x + \int_{B_R} g \mathrm{d}x \bigg)
\end{aligned}$$

for some constant $C > 0$. By Hölder's inequality and Young's inequality, we have for any $\epsilon > 0$

$$\frac{1}{(s-t)^\mu} \int_{B_R} d^\mu(u, p_0)|\nabla u|_\mu \mathrm{d}x \le \frac{C_\epsilon}{(s-t)^n} \int_{B_s} d^n(u, p_0)\mathrm{d}x + \epsilon \int_{B_s} (|\nabla u|^2)^\mu \mathrm{d}x.$$

Thus, by taking ϵ suitably small, we derive from the above that

$$\int_{B_s} (|\nabla u|^2)^\mu \mathrm{d}x \le C_0 \int_{B_s - B_t} (|\nabla u|^2)^\mu \mathrm{d}x + C_0 \left\{ \frac{1}{(s-t)^n} \int_{B_s} d^n(u, p_0)\mathrm{d}x + \int_{B_R} g \mathrm{d}x \right\}$$

holds for some $C_0 > 0$. Adding $C_0 \int_{B_t} (|\nabla u|^2)^\mu \mathrm{d}x$ on both sides of the above inequality and then dividing by $1 + C_0$, we deduce

$$\int_{B_t} (|\nabla u|^2)^\mu \mathrm{d}x \le \theta \int_{B_s} (|\nabla u|^2)^\mu \mathrm{d}x + C \left\{ \frac{1}{(s-t)^n} \int_{B_s} d^n(u, p_0)\mathrm{d}x + \int_{B_R} g \mathrm{d}x \right\}$$

for some $C > 0$, where $\theta = C_0/(1 + C_0) < 1$.

Applying Lemma 3.2 of [8], we obtain

$$\int_{B_{R/2}} (|\nabla u|^2)^\mu \mathrm{d}x \le C\Big(R^{-n} \int_{B_R} d^n(u, p_0)\mathrm{d}x + \int_{B_R} g \mathrm{d}x\Big). \tag{3.3}$$

By the Sobolev–Poincaré inequality (Lemma 2.1), for $\frac{n}{2} < q < n$, we get

$$R^{-n}\int_{B_R} d^n(u(x), p_0)\mathrm{d}x \le C\Big(\int_{B_R}(|\nabla u|^2)^{\frac{q}{2}}\mathrm{d}x\Big)^{n/q}|B_R|^{1-\frac{n}{q}}.$$

Combining the above estimate and (3.3), we infer that

$$\int_{B_{R/2}}(|\nabla u|^2)^{\mu}\mathrm{d}x \le C\Big\{\Big(\int_{B_R}(|\nabla u|^2)^{\frac{q}{2}}\mathrm{d}x\Big)^{n/q}|B_R|^{(q-n)/q} + \int_{B_R} g\mathrm{d}x\Big\}.$$

Now, set $w = \big((|\nabla u|^2)^{\mu}\big)^{q/n}$, $\frac{n}{2} < q < n$. It follows, for any $B_R \subset\subset \Omega$, that

$$\fint_{B_{R/2}} w^{\frac{n}{q}}\mathrm{d}x \le C\left(\left(\fint_{B_R} w\mathrm{d}x\right)^{\frac{n}{q}} + \fint_{B_R} g\mathrm{d}x\right). \tag{3.4}$$

As commented in Sect. 1.3, the conclusion of Theorem 1.2 follows from the local-type reverse Hölder inequality (1.2). The proof is complete. □

Next we derive the boundary regularity for quasi-n-harmonic mappings. In [26, Lemma 1], using boundary-type reverse Hölder's inequality, Jost and Meier improved the integrability of gradients of local minima for certain quadratic functionals $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ close to the boundary. We will generalize the arguments in our setting.

Proof of Theorem 1.4 Let $x_0 \in \partial\Omega$ and V an open neighborhood of x_0. Since Ω is Lipschitz and quasi-n-harmonic mappings are stable under bi-Lipschitz transformations, we may perform a local bi-Lipschitz coordinate transformation such that x_0, $V \cap \Omega$ and $V \cap \partial\Omega$ get mapped onto 0, B_1^+, and Γ_1, respectively. Here we denote by B_R^+ the open half ball $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| < R, x_n > 0\}$ and $\Gamma_R = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| < R, x_n = 0\}$. It suffices to show that $u \in W^{1,q}(B_{1/2}^+, X)$ for some $n < q = q(Q, n, p) \le p$.

Fix any $R < 1$. We will show that if $x \in B_R^+ \cup \Gamma_R$ and $r < 1 - R$, then

$$\left(\fint_{B_{r/2}(x)\cap B_R^+}|\nabla u|_{\tilde p}\mathrm{d}x\right)^{1/\tilde p} \le c\left(\fint_{B_r(x)\cap B_R^+}|\nabla u|_n\mathrm{d}x\right)^{1/n} + c\left(\fint_{B_r(x)\cap B_R^+}(|\nabla h|_p + g)\mathrm{d}x\right) \tag{3.5}$$

for some $\tilde p$ with $p > \tilde p > n$.

The argument is quite similar with that of Theorem 1.2 and so we only point out the differences. Fix $x_0 \in B_R$ and s, t, r with $0 < t < s \le r < 1 - R$. Let $\eta \in C_0^{\infty}(\Omega)$ be such that $0 < \eta < 1$ on $B_s(x_0)\backslash \bar B_t(x_0)$, $\eta \equiv 0$ outside $B_s(x_0)$, $\eta \equiv 1$ on $\bar B_t(x_0)$, $|D\eta| \le c(s-t)^{-1}$, $\{\eta = \frac{1}{2}\}$ has zero Lebesgue measure and $\int_{B_1}\frac{1}{(1-2\eta(x))^4}\mathrm{d}x < \infty$. The only difference with the proof of Theorem 1.2 is that we will compare the mapping u with the mapping $u_\eta = (1-\eta(x))u(x) + \eta(x)h(x)$.

By similar computation, we obtain that for any smooth function $\eta \in C_0^{\infty}(\Omega)$ which satisfies either $0 \le \eta < \frac{1}{2}$ or $\frac{1}{2} < \eta \le 1$, it holds

$$\pi_{u_\eta} \le (1-\eta)\pi_u + \eta\pi_h + C|\nabla\eta| d(u,h)(|\nabla u|_1 + |\nabla h|_1) - \nabla\eta\otimes\nabla d^2(u,h) + Q(\eta, \nabla\eta). \tag{3.6}$$

Still set $\mu = n/2$. Taking the trace on both sides of (3.6) and then the μ-th power on both sides, we get

$$\begin{aligned}\pi_{u_\eta}^{\mu} \le c(n)\Big(&(1-\eta)^{\mu}\pi_u^{\mu} + \eta^{\mu}|\nabla h|^{\mu} + |\nabla\eta|^{\mu} d^{\mu}(u,h)(\nabla u|_{\mu} + |\nabla h|_{\mu})\\ &- (\nabla\eta\otimes\nabla d^2(u,h))^{\mu} + Q(\eta,\nabla\eta)^{\mu}\Big).\end{aligned}$$

Using the Q-quasi-n-harmonic condition as in the previous proof, we obtain

$$\int_{B_s(x_0)} (|\nabla u|^2)^{\mu} \mathrm{d}x \leq C \left(\int_{B_s(x_0)-B_t(x_0)} (|\nabla u|^2)^{\mu} \mathrm{d}x + \int_{B_s(x_0)} |\nabla h|^n \mathrm{d}x \right. \\ \left. + \frac{1}{(s-t)^{\mu}} \int_{B_s(x_0)} d^{\mu}(u,h)(|\nabla u|_{\mu} + |\nabla h|_{\mu}) \mathrm{d}x + \int_{B_s(x_0)} g \mathrm{d}x \right).$$

Applying the Young's inequality as before, we deduce

$$\int_{B_t(x_0)} (|\nabla u|^2)^{\mu} \mathrm{d}x \leq \theta \int_{B_s(x_0)} (|\nabla u|^2)^{\mu} \mathrm{d}x + C \int_{B_s(x_0)} |\nabla h|_n \mathrm{d}x \\ + C \left\{ \frac{1}{(s-t)^n} \int_{B_s(x_0)} d^n(u,h) \mathrm{d}x + \int_{B_s(x_0)} g \mathrm{d}x \right\}$$

for some $C > 1$ and $\theta < 1$. Then it follows that

$$\int_{B_{r/2}(x_0)\cap B_R^+} (|\nabla u|^2)^{\mu} \mathrm{d}x \leq C \left(r^{-n} \int_{B_r(x_0)\cap B_R^+} d^n(u,h) \mathrm{d}x + \int_{B_r(x_0)\cap B_R^+} (g + |\nabla h|_n) \mathrm{d}x \right). \tag{3.7}$$

We first assume that the n-th component of x_0 is no bigger than $3r/4$. In this case, $d(u,h) = 0$ in $B_r(x_0)\backslash B_R^+$ and $\mathcal{L}^n\big(B_r(x_0)\backslash B_R^+\big) \geq c\mathcal{L}^n\big(B_r(x_0)\big)$ for some $c = c(n) > 0$. By the Sobolev–Poincaré inequality and Hölder's inequality, for $n/2 < \hat{q} < n$, we have

$$⨍_{B_r(x_0)\cap B_R^+} d^n(u,h) \mathrm{d}x \leq C r^n \left(⨍_{B_r(x_0)} \left((|\nabla u|^2)^{\frac{\hat{q}}{2}} + (|\nabla h|^2)^{\frac{\hat{q}}{2}} \right) \mathrm{d}x \right)^{\frac{n}{\hat{q}}} \\ \leq C r^n \left\{ \left(⨍_{B_r(x_0)} (|\nabla u|^2)^{\frac{\hat{q}}{2}} \mathrm{d}x \right)^{\frac{n}{\hat{q}}} + ⨍_{B_r(x_0)} |\nabla h|_n \mathrm{d}x \right\}.$$

From now on, the remaining proof for this case has no much difference from that of Theorem 1.2 and so we omit the details.

If the n-th component of x_0 is no less than $3r/4$, we directly apply the proof of Theorem 1.2 to obtain interior regularity estimate of type (3.4) in which the balls $B_{R/2}$ and B_R are replaced by $B_{r/2}(x_0)$ and $B_{3r/4}(x_0)$, respectively. This again implies (3.5). The proof is complete. □

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