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Binomial edge ideals of bipartite graphs

Davide Bolognini [a], Antonio Macchia [b], Francesco Strazzanti [c]

[a] *Département de Mathématiques, Université de Fribourg, Chemin du Musée 23, CH-1700 Fribourg, Switzerland*
[b] *Dipartimento di Matematica, Università degli Studi di Bari "Aldo Moro", Via Orabona 4, 70125 Bari, Italy*
[c] *Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, Avda. Reina Mercedes s/n, 41080 Sevilla, Spain*

Binomial edge ideals are a noteworthy class of binomial ideals that can be associated with graphs, generalizing the ideals of 2-minors. For bipartite graphs we prove the converse of Hartshorne's Connectedness Theorem, according to which if an ideal is Cohen–Macaulay, then its dual graph is connected. This allows us to classify Cohen–Macaulay binomial edge ideals of bipartite graphs, giving an explicit and recursive construction in graph-theoretical terms. This result represents a binomial analogue of the celebrated characterization of (monomial) edge ideals of bipartite graphs due to Herzog and Hibi (2005).

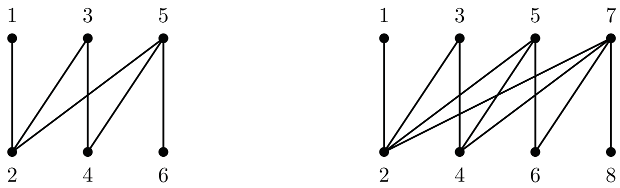
1. Introduction

Binomial edge ideals were introduced independently in [10] and [17]. They are a natural generalization of the ideals of 2-minors of a $(2 \times n)$-generic matrix [3]: their generators are those 2-minors whose column indices correspond to the edges of a graph. In this perspective, the ideals of 2-minors are binomial edge ideals of complete graphs. On the other hand, binomial edge ideals arise naturally in Algebraic Statistics, in the context of conditional independence ideals, see [10, Section 4].

More precisely, given a finite simple graph G on the vertex set $[n] = \{1, \dots, n\}$, the *binomial edge ideal* associated with G is the ideal

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \text{ is an edge of } G) \subset R = K[x_i, y_i : i \in [n]].$$

E-mail addresses: davide.bolognini@unifr.ch (D. Bolognini), antonio.macchia@uniba.it (A. Macchia), fstrazzanti@us.es (F. Strazzanti).



(a) The graph F_3.

(b) The graph F_4.

Fig. 1. The basic blocks.

Binomial edge ideals have been extensively studied, see e.g. [1,5,6,13–15,18,19]. Yet a number of interesting questions is still unanswered. In particular, many authors have studied classes of Cohen–Macaulay binomial edge ideals in terms of the associated graph, see e.g. [1,5,13,18,19]. Some of these results concern a class of chordal graphs, the so-called *closed graphs*, introduced in [10], and their generalizations, such as block and generalized block graphs [13].

In the context of squarefree monomial ideals, any graph can be associated with the so-called *edge ideal*, whose generators are monomials of degree 2 corresponding to the edges of the graph. Herzog and Hibi, in [9, Theorem 3.4], classified Cohen–Macaulay edge ideals of bipartite graphs in purely combinatorial terms. In the same spirit, we provide a combinatorial classification of Cohen–Macaulay binomial edge ideals of bipartite graphs.

To this aim we exploit the dual graph of an ideal. Following the notation used in [2], we recall this notion and some facts.

Let I be an ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of I. The *dual graph* $\mathcal{D}(I)$ is a graph with vertex set $[r]$ and edge set

$$\{\{i, j\} : \mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_j) - 1 = \mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{p}_j) = \mathrm{ht}(I)\}.$$

This notion was originally studied by Hartshorne in [8] in terms of *connectedness in codimension one*. By [8, Corollary 2.4], if A/I is Cohen–Macaulay, then the algebraic variety defined by I is connected in codimension one, hence I is unmixed by [8, Remark 2.4.1]. The connectedness of the dual graph translates in combinatorial terms the notion of connectedness in codimension one, see [8, Proposition 1.1]. Thus, if A/I is Cohen–Macaulay, then $\mathcal{D}(I)$ is connected. The converse does not hold in general, see for instance Remark 5.1. We will show that for binomial edge ideals of connected bipartite graphs this is indeed an equivalence. In geometric terms, this means that the algebraic variety defined by J_G is Cohen–Macaulay if and only if it is connected in codimension one.

We now describe the explicit structure of the bipartite graphs in the classification. For the terminology about graphs we refer to [4]. First we present a family of bipartite graphs F_m whose binomial edge ideal is Cohen–Macaulay, and we prove that, if G is connected and bipartite, then J_G is Cohen–Macaulay if and only if G can be obtained recursively by gluing a finite number of graphs of the form F_m via two operations.

Basic blocks: For every $m \geq 1$, let F_m be the graph (see Fig. 1) on the vertex set $[2m]$ and with edge set

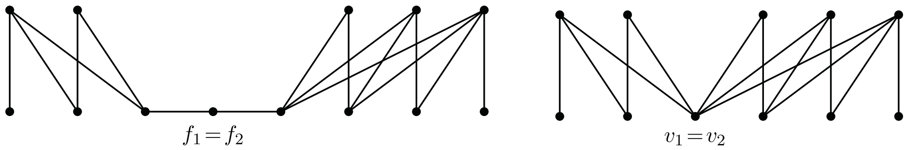
$$E(F_m) = \{\{2i, 2j - 1\} : i = 1, \dots, m, j = i, \dots, m\}.$$

Notice that F_1 is the single edge $\{1, 2\}$ and F_2 is the path of length 3.

Operation $*$: For $i = 1, 2$, let G_i be a graph with at least one vertex f_i of degree one, i.e., a *leaf* of G_i. We denote the graph G obtained by identifying f_1 and f_2 by $G = (G_1, f_1) * (G_2, f_2)$, see Fig. 2(a). This is a particular case of an operation studied by Rauf and Rinaldo in [18, Section 2].

Operation $\circ$: For $i = 1, 2$, let G_i be a graph with at least one leaf f_i, v_i its neighbour and assume $\deg_{G_i}(v_i) \geq 3$. We define $G = (G_1, f_1) \circ (G_2, f_2)$ to be the graph obtained from G_1 and G_2 by removing the leaves f_1, f_2 and identifying v_1 and v_2, see Fig. 2(b).

For both operations, if it is not important to specify the vertices f_i or it is clear from the context, we simply write $G_1 * G_2$ or $G_1 \circ G_2$.



(a) The graph $F_3 * F_4$.

(b) The graph $F_3 \circ F_4$.

Fig. 2. The operations $*$ and $\circ$.

Notice that, we may assume G connected with at least two vertices since the ideal J_G is Cohen–Macaulay if and only if the binomial edge ideal of each connected component of G is Cohen–Macaulay.

Before stating the main result, we recall the notion of cut set, which is central in the study of binomial edge ideals. In fact, there is a bijection between the cut sets of a graph G and the minimal prime ideals of J_G, see [10, Section 3]. For a subset $S \subseteq [n]$, let $c_G(S)$ be the number of connected components of the induced subgraph $G_{[n]\setminus S}$. The set S is called *cut set* of G if $S = \emptyset$ or $S \neq \emptyset$ and $c_G(S \setminus \{i\}) < c_G(S)$ for every $i \in S$. Moreover, we call *cut vertex* a cut set of cardinality one. We denote by $\mathcal{M}(G)$ the set of cut sets of G.

We are now ready to state our main result.

Theorem 6.1. *Let G be a connected bipartite graph. The following properties are equivalent:*

(a) *J_G is Cohen–Macaulay;*
(b) *the dual graph $\mathcal{D}(J_G)$ is connected;*
(c) *$G = A_1 * A_2 * \cdots * A_k$, where $A_i = F_m$ or $A_i = F_{m_1} \circ \cdots \circ F_{m_r}$, for some $m \geq 1$ and $m_j \geq 3$;*
(d) *J_G is unmixed and for every non-empty $S \in \mathcal{M}(G)$, there exists $s \in S$ such that $S \setminus \{s\} \in \mathcal{M}(G)$.*

The paper is structured as follows. In Section 2 we study unmixed binomial edge ideals of bipartite graphs. A combinatorial characterization of unmixedness was already proved in [10] (see also [18, Lemma 2.5]), in terms of the cut sets of the underlying graph.

A first distinguishing fact about bipartite graphs with J_G unmixed is that they have exactly two leaves (Proposition 2.3). This, in particular, means that G has at least two cut vertices. In Proposition 2.8, we present a construction that is useful in the study of the basic blocks and to produce new examples of unmixed binomial edge ideals, which are not Cohen–Macaulay.

In Section 3 we prove that the ideals J_{F_m}, associated with the basic blocks of our construction, are Cohen–Macaulay, see Proposition 3.3. In Section 4 we study the operations $*$ and $\circ$. In [18, Theorem 2.7], Rauf and Rinaldo proved that $J_{G_1 * G_2}$ is Cohen–Macaulay if and only if so are J_{G_1} and J_{G_2}. In Theorem 4.9, we show that J_G is Cohen–Macaulay if $G = F_{m_1} \circ \cdots \circ F_{m_k}$, for every $k \geq 2$ and $m_i \geq 3$. Using these results, we prove the implication (c) $\Rightarrow$ (a) of Theorem 6.1.

Section 5 is devoted to the study of the dual graph of binomial edge ideals. This is one of the main tools in the proof of Theorem 6.1. First of all, given a (not necessarily bipartite) graph G with J_G unmixed, in Theorem 5.2 we provide an explicit description of the edges of the dual graph $\mathcal{D}(J_G)$ in terms of the cut sets of G. This allows us to show infinite families of bipartite graphs whose binomial edge ideal is unmixed and not Cohen–Macaulay, see Examples 2.2 and 5.4.

A crucial result concerns a basic, yet elusive, property of cut sets of unmixed binomial edge ideals. In Lemma 5.5, we show that, mostly for bipartite graphs and under some assumption, the intersection of any two cut sets is a cut set. This leads to the proof of the equivalence (b) $\Leftrightarrow$ (d) in Theorem 6.1, see Theorem 5.7. On the other hand, if $G = G_1 * G_2$ or $G = G_1 \circ G_2$ is bipartite and $\mathcal{D}(J_G)$ is connected, then the dual graphs of G_1 and G_2 are connected, see Theorem 5.8. Thus, we may reduce to consider bipartite graphs with exactly two cut vertices and prove the implication (b) $\Rightarrow$ (c) of Theorem 6.1. This also shows that the converse of Hartshorne's Connectedness Theorem holds for these ideals.

It is worth noting that, the main theorem gives also a classification of other classes of Cohen–Macaulay binomial ideals associated with bipartite graphs, Corollary 6.2: *Lovász–Saks–Schrijver ideals* [11], *permanental edge ideals* [11, Section 3] and *parity binomial edge ideals* [12].

As an application of the main result, in Corollary 6.3, we show that Cohen–Macaulay binomial edge ideals of bipartite graphs are Hirsch, meaning that the diameter of the dual graph of J_G is bounded above by the height of J_G, verifying [2, Conjecture 1.6].

All the results presented in this paper are independent of the field.

2. Unmixed binomial edge ideals of bipartite graphs

In this paper all graphs are *finite* and *simple* (without loops and multiple edges). In what follows, unless otherwise stated, we assume that G is a connected graph with at least two vertices. Given a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. If G is a *bipartite graph*, we denote by $V(G) = V_1 \sqcup V_2$ the *bipartition* of the vertex set and call V_1, V_2 the *bipartition sets* of G.

For a subset $S \subseteq V(G)$, we denote by G_S the *subgraph induced* in G by S, which is the graph with vertex set S and edge set consisting of all the edges of G with both endpoints in S.

We recall some definitions and results from [10]. Let G be a graph with vertex set $[n]$. We denote by $R = K[x_i, y_i : i \in [n]]$ the polynomial ring in which the ideal J_G is defined and, if $S \subseteq [n]$, we set $\overline{S} = [n] \setminus S$. Let $c_G(S)$, or simply $c(S)$, be the number of connected components of the induced subgraph $G_{\overline{S}}$ and let $G_1, \dots, G_{c_G(S)}$ be the connected components of $G_{\overline{S}}$. For each G_i, denote by $\widetilde{G}_i$ the complete graph on $V(G_i)$ and define the ideal

$$P_S(G) = \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\widetilde{G}_1}, \dots, J_{\widetilde{G}_{c_G(S)}} \right).$$

In [10, Section 3], it is shown that $P_S(G)$ is a prime ideal for every $S \subseteq [n]$, $\mathrm{ht}(P_S(G)) = n + |S| - c_G(S)$ and $J_G = \bigcap_{S \subseteq [n]} P_S(G)$. Moreover, $P_S(G)$ is a minimal prime ideal of J_G if and only if $S = \emptyset$ or $S \neq \emptyset$ and $c_G(S \setminus \{i\}) < c_G(S)$ for every $i \in S$. In simple terms the last condition means that, adding a vertex of S to $G_{\overline{S}}$, we connect at least two connected components of $G_{\overline{S}}$. We set

$$\begin{aligned} \mathcal{M}(G) &= \{S \subset [n] : P_S(G) \text{ is a minimal prime ideal of } J_G\} \\ &= \{\emptyset\} \cup \{S \subset [n] : S \neq \emptyset,\ c_G(S \setminus \{i\}) < c_G(S) \text{ for every } i \in S\}, \end{aligned}$$

and we call *cut sets* of G the elements of $\mathcal{M}(G)$. If $\{v\} \in \mathcal{M}(G)$, we say that v is a *cut vertex* of G.

We further recall that a *clique* of a graph G is a subset $C \subseteq V(G)$ such that G_C is complete. A *free vertex* of G is a vertex that belongs to exactly one maximal clique of G. A vertex of degree 1 in G, which in particular is a free vertex, is called a *leaf* of G.

Remark 2.1. Notice that a vertex v is free in a graph G if and only if $v \notin S$ for every $S \in \mathcal{M}(G)$, see [18, Proposition 2.1].

Recall that an ideal is *unmixed* if all its minimal primes have the same height. By [18, Lemma 2.5], J_G is unmixed if and only if for every $S \in \mathcal{M}(G)$,

$$c_G(S) = |S| + 1. \tag{1}$$

This follows from the equality $\mathrm{ht}(P_\emptyset(G)) = n - 1 = \mathrm{ht}(P_S(G)) = n + |S| - c_G(S)$.

Moreover, for every graph G, with J_G unmixed, we have that $\dim(R/J_G) = |V(G)| + c$, where c is the number of connected components of G, see [10, Corollary 3.3].

In this section, we study some properties of unmixed binomial edge ideals of bipartite graphs. It is well-known that if J_G is Cohen–Macaulay, then J_G is unmixed. The converse is, in general, not true, also for binomial edge ideals of bipartite graphs. In fact, in the following example we show two classes of bipartite graphs whose binomial edge ideals are unmixed but not Cohen–Macaulay.

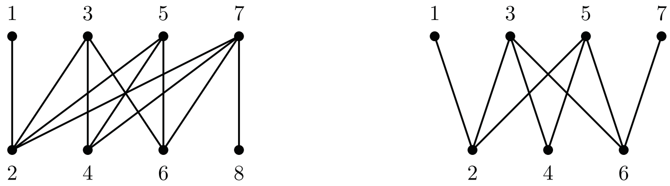
Example 2.2. For every $k \geq 4$, let $M_{k,k}$ be the graph with vertex set $[2k]$ and edge set

$$E(M_{k,k}) = \{\{1, 2\}, \{2k - 1, 2k\}\} \cup \{\{2i, 2j - 1\} : i = 1, \dots, k - 1, j = 2, \dots, k\},$$

see Fig. 3(a), and let $M_{k-1,k}$ be the graph with vertex set $[2k - 1]$ and edge set

$$E(M_{k-1,k}) = \{\{1, 2\}, \{2k - 2, 2k - 1\}\} \cup \{\{2i, 2j - 1\} : i = 1, \dots, k - 1, j = 2, \dots, k - 1\},$$

see Fig. 3(b).



(a) The graph $M_{4,4}$.

(b) The graph $M_{3,4}$.

Fig. 3. The graphs $M_{h,k}$.

Notice that the graphs $M_{k,k}$ and $M_{k-1,k}$ are obtained by adding two *whiskers* to some complete bipartite graph. Recall that adding a whisker to a graph G means adding a new vertex and connect it to one of the vertices of G.

Let $V_1 \sqcup V_2$ be the bipartition of $M_{k,k}$ and of $M_{k-1,k}$ such that V_1 contains the odd labelled vertices and V_2 contains the even labelled vertices. We claim that

$\mathcal{M}(M_{k,k}) = \{\emptyset, \{2\}, \{2k-1\}, \{2, 2k-1\}, V_1 \setminus \{1\}, V_2 \setminus \{2k\}\}$ and
$\mathcal{M}(M_{k-1,k}) = \{\emptyset, \{2\}, \{2k-2\}, \{2, 2k-2\}, V_1 \setminus \{1, 2k-1\}, V_2\}$.

The inclusion $\supseteq$ is clear. We prove the other inclusion for $M_{k,k}$, the proof is similar for $M_{k-1,k}$. Let $S \in \mathcal{M}(M_{k,k})$. If $S \subseteq \{2, 2k-1\}$, there is nothing to prove. If there exists $v \in S \setminus \{2, 2k-1\}$, then $S = V_1 \setminus \{1\}$ or $S = V_2 \setminus \{2k\}$. In fact, if $v \in V_1 \setminus \{1\}$ and there is $w \in (V_1 \setminus \{1\}) \setminus S$, then $c(S \setminus \{v\}) = c(S)$, a contradiction. Hence, $V_1 \setminus \{1\} \subseteq S$. On the other hand, if $w \in V_2 \setminus \{2k\}$, then $w \notin S$. This shows that $S = V_1 \setminus \{1\}$ The other case is similar.

Moreover, it is easy to check that $J_{M_{k,k}}$ and $J_{M_{k-1,k}}$ are unmixed. In Example 5.4 we will show that these ideals are not Cohen–Macaulay.

A first nice fact about bipartite graphs with unmixed binomial edge ideal is that they have at least two cut vertices.

Proposition 2.3. *Let G be a bipartite graph such that J_G is unmixed. Then G has exactly 2 leaves.*

Proof. Let $V(G) = V_1 \sqcup V_2$ be the bipartition of G, with $m_1 = |V_1| \geq 1$ and $m_2 = |V_2| \geq 1$. Assume that G has exactly h leaves, $f_1, \ldots, f_h$, in V_1 and k leaves, $g_1, \ldots, g_k$, in V_2. We claim that $S_1 = V_1 \setminus \{f_1, \ldots, f_h\}$ and $S_2 = V_2 \setminus \{g_1, \ldots, g_k\}$ are cut sets of G. Notice that $c_G(S_1) = |V_2| = m_2$ and $c_G(S_1 \setminus \{v\}) < c_G(S_1)$ since the vertex v joins at least two connected components of $G_{\overline{S}_1}$. By symmetry, the claim is true for S_2 and, in particular $c_G(S_2) = |V_1| = m_1$. From the unmixedness of J_G it follows that $\mathrm{ht}(P_\emptyset(G)) = \mathrm{ht}(P_{S_1}(G))$ and $\mathrm{ht}(P_\emptyset(G)) = \mathrm{ht}(P_{S_2}(G))$. Thus $n-1 = n+|S_1|-c_G(S_1) = n+m_1-h-m_2$ and $n-1 = n+|S_2|-c_G(S_2) = n+m_2-k-m_1$. Hence $h = m_1-m_2+1$ and $k = m_2-m_1+1$. The sum of the two equations yields $h+k = 2$. $\square$

Remark 2.4. Assume that G is bipartite and J_G is unmixed. The proof of Proposition 2.3 implies that:

(i) either $h = 2$ and $k = 0$, i.e., the two leaves are in the same bipartition set and in this case $m_1 = m_2 + 1$, or $h = 1$ and $k = 1$, i.e., each bipartition set contains exactly one leaf and in this case $m_1 = m_2$;
(ii) if G has at least 4 vertices, then the leaves cannot be attached to the same vertex v, otherwise $c_G(\{v\}) \geq 3 > 2 = |\{v\}| + 1$, against the unmixedness of J_G, see (1). Hence G has at least two distinct cut vertices, which are the neighbours of the leaves.

Remark 2.5. Notice that Proposition 2.3 does not hold if G is not bipartite. In fact, there are non-bipartite graphs G with an arbitrary number of leaves and such that J_G is Cohen–Macaulay. For $n \geq 2$ the binomial edge ideal J_{K_n} of the complete graph K_n is Cohen–Macaulay, since it is the ideal of 2-minors of a generic $(2 \times n)$-matrix (see [3, Corollary 2.8]). Moreover, for $n \geq 3$, K_n has 0 leaves. Let

$W \subseteq [n]$, with $|W| = k \geq 1$. Adding a whisker to a vertex of W, the resulting graph H has 1 leaf and J_H is Cohen–Macaulay by [18, Theorem 2.7]. Applying the same argument to all vertices of W, we obtain a graph H' with k leaves such that $J_{H'}$ is Cohen–Macaulay.

In the remaining part of the section we present a construction, Proposition 2.8, that produces new examples of unmixed binomial edge ideals. It will also be important in the proof of the main theorem.

If X is a subset of $V(G)$, we define the set of neighbours of the elements of X, denoted $N_G(X)$, or simply $N(X)$, as the set

$$N_G(X) = \{y \in V(G) : \{x, y\} \in E(G) \text{ for some } x \in X\}.$$

Lemma 2.6. *Let G be a bipartite graph with bipartition $V_1 \sqcup V_2$, J_G unmixed and let v_1 and v_2 be the neighbours of the leaves.*

(a) *If $X \subseteq V_i$, for some i, and $v_1, v_2 \notin X$, then $N(X)$ is a cut set of G and $|N(X)| \geq |X|$.*
(b) *If $\{v_1, v_2\} \in E(G)$, then $m = |V_1| = |V_2|$ and v_i has degree m, for $i = 1, 2$. Moreover, v_1 and v_2 are the only cut vertices of G.*

Proof. (a) We remark that it does not matter in which bipartition sets v_1, v_2 are. First notice that $N(X)$ is a cut set. In fact, every element of X is isolated in $G_{\overline{N(X)}}$. Let $v \in N(X)$. Then $\deg(v) \geq 2$, since $v_1, v_2 \notin X$. Adding v to $G_{\overline{N(X)}}$, it connects at least a vertex of X with some other connected component.

Now, suppose by contradiction that $|N(X)| < |X|$. Then $G_{\overline{N(X)}}$ has at least $|X|$ isolated vertices and another connected component containing a leaf, because $v_1, v_2 \notin X$. Hence, $c_G(N(X)) \geq |X| + 1 > |N(X)| + 1$, against the unmixedness of J_G.

(b) Assume that $v_1 \in V_1$. Then $v_2 \in V_2$, since $\{v_1, v_2\} \in E(G)$. By Remark 2.4(i), it follows that $m = |V_1| = |V_2|$. Define $X = \{w \in V_2 : \{v_1, w\} \notin E(G)\}$ and assume that $X \neq \emptyset$. Since $\{v_1, v_2\} \in E(G)$, $v_2 \notin X$, hence $N(X)$ is a cut set and $|N(X)| \geq |X|$ by (a). We claim that the inequality is strict. Assume $|N(X)| = |X|$. Let f be the leaf of G adjacent to v_1, then $S = V_2 \setminus (X \cup \{f\})$ is a cut set of G and $|S| = m - |X| - 1$. In fact, in $G_{\overline{S}}$ all vertices of $V_1 \setminus N(X)$ are isolated, except for v_1 that is connected only to f. Moreover, by definition of X, if we add an element of S to $G_{\overline{S}}$, we join the connected component of v_1 with some other connected component of $G_{\overline{S}}$. Thus, S is a cut set and $G_{\overline{S}}$ consists of at least $|V_1| - |N(X)| - 1 = m - |X| - 1$ isolated vertices, the single edge $\{v_1, f\}$, and the connected component containing the vertices of X and $N(X)$. Hence, $c_G(S) \geq m - |X| + 1 > |S| + 1$, a contradiction since J_G is unmixed. This shows that $|N(X)| > |X|$.

Now, the vertices of X are isolated in $G_{\overline{N(X)}}$. Moreover, the remaining vertices belong to the same connected component, because, by definition of X, $\{v_1, w\} \in E(G)$ for every $w \in V_2 \setminus X$ and all vertices in $V_1 \setminus N(X)$ are adjacent to vertices of $\overline{X}$. Hence, $c_G(N(X)) = |X| + 1 < |N(X)| + 1$, which again contradicts the unmixedness of J_G. Hence, $X = \emptyset$ and v_1 has degree m. In the same way it follows that v_2 has degree m.

For the last part of the claim, notice that if $v \in V(G) \setminus \{v_1, v_2\}$, the first part implies that every vertex of $G_{\overline{\{v\}}}$ is adjacent to either v_1 or v_2. Hence, $G_{\overline{\{v\}}}$ is connected and, thus, v is not a cut vertex of G. $\square$

Remark 2.7. Let G be a bipartite graph such that J_G is unmixed. If G has exactly two cut vertices, they are not necessarily adjacent. Thus, the converse of the last part of Lemma 2.6(b) does not hold. In fact, if $|V_1| = |V_2| + 1$, then v_1 and v_2 belong to the same bipartition set, hence $\{v_1, v_2\} \notin E(G)$. On the other hand, if $|V_1| = |V_2|$, let G be the graph in Fig. 4. One can check with `Macaulay2` [7] that the ideal J_G is unmixed, and we notice that the vertices 2 and 11 are the only cut vertices, but $\{2, 11\} \notin E(G)$.

Proposition 2.8. *Let H be a bipartite graph with bipartition $V_1 \sqcup V_2$ and $|V_1| = |V_2|$. Let v and f be two new vertices and let G be the bipartite graph with $V(G) = V(H) \cup \{v, f\}$ and $E(G) = E(H) \cup \{\{v, x\} : x \in V_1 \cup \{f\}\}$. If J_H is unmixed and the neighbours of the leaves of H are adjacent, then J_G is unmixed and*

$$\mathcal{M}(G) = \{\emptyset, V_1\} \cup \{S \cup \{v\} : S \in \mathcal{M}(H)\} \cup \{T \subset V_1 : T \in \mathcal{M}(H)\}.$$

Moreover, the converse holds if there exists $w \in V_1$ such that $\deg_G(w) = 2$.

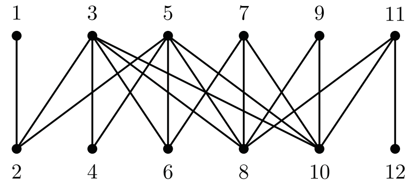


Fig. 4. A graph G with J_G unmixed, cut vertices $2, 11$ and $\{2, 11\} \notin E(G)$.

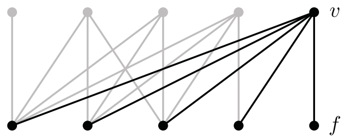


Fig. 5. The construction in Proposition 2.8.

Proof. Assume that J_H is unmixed and the neighbours of the leaves of H are adjacent. Clearly, $\emptyset, V_1 \in \mathcal{M}(G)$. If $S \in \mathcal{M}(H)$, then adding v to $G_{\overline{S\cup\{v\}}}$ we join f with some other connected component of $H_{\overline{S}}$. Moreover, if $w \in S$, adding w to $G_{\overline{S\cup\{v\}}}$ we join at least two connected components of $H_{\overline{S}}$ (since $S \in \mathcal{M}(H)$), which are different components of $G_{\overline{S\cup\{v\}}}$. Finally, let $T \in \mathcal{M}(H)$, $T \subset V_1$. By Lemma 2.6(b), in H there exists a unique cut vertex $v_2 \in V_2$ and $N_H(v_2) = V_1$. Hence, adding $w \in T$ to $G_{\overline{T}}$, we join at least two components since $N_G(v) = V_1 \cup \{f\}$ and $T \in \mathcal{M}(H)$.

Conversely, let $S \in \mathcal{M}(G)$ and suppose first that $v \in S$. Then $G_{\overline{S}} = H_{\overline{S\setminus\{v\}}} \sqcup \{f\}$ and this implies that $S \setminus \{v\}$ is a cut set of H, since every element of $S \setminus \{v\}$ has to join some connected components that only contain vertices of $H_{\overline{S\setminus\{v\}}}$. Therefore $c_G(S) = c_H(S \setminus \{v\}) + 1 = |S| + 1$.

Suppose now that $v \notin S$. Let w be the leaf of H adjacent to v_2, that is also adjacent to v in G. First of all, notice that $S \subset V_1$. Indeed, in $G_{\overline{S}}$ every vertex of $V_1 \setminus S$ is in the same connected component of v. Thus, a vertex of V_2 cannot join different connected components. Since w is adjacent only to v and v_2, if $w \in S$, then v and v_2 cannot be in the same connected component of $G_{\overline{S}}$. This means that $V_1 \subset S$, because all the vertices of V_1 are adjacent to v and v_2, by Lemma 2.6(b). Thus $S = V_1$ and $c_G(S) = |V_2| + 1 = |S| + 1$. Hence, we may assume that $w \notin S$. We claim that, in this case, $S \in \mathcal{M}(H)$. In fact, it is clear that v_2, w, v and f are in the same connected component C of $G_{\overline{S}}$, which also contains all vertices of $V_1 \setminus S$, since they are adjacent to v. Then, the connected components of $G_{\overline{S}}$ and $H_{\overline{S}}$ are the same except for C, that in $H_{\overline{S}}$ is $C_{\overline{\{v,f\}}} \neq \emptyset$. Therefore, if $x \in S$ joins two connected components of $G_{\overline{S}}$, it also joins the same connected components of $H_{\overline{S}}$ (or $C_{\overline{\{v,f\}}}$, if it joins C), hence S is a cut set of H. Moreover, $c_G(S) = c_H(S) = |S| + 1$.

Conversely, assume that J_G is unmixed and let $S \in \mathcal{M}(H)$. Notice that w is a leaf of H, hence $w \notin S$, by Remark 2.1. We prove that $T = S \cup \{v\}$ is a cut set of G. As before, $G_{\overline{T}} = H_{\overline{S}} \cup \{f\}$. Thus the elements of S join different connected components also in $G_{\overline{T}}$ and v connects the isolated vertex f with the connected component of w. Hence, $T \in \mathcal{M}(G)$ and $c_H(S) = c_G(T) - 1 = |T| + 1 - 1 = |S| + 1$.

Finally, let v_i be the cut vertex of H in V_i for $i = 1, 2$. Since $\{v, v_1\} \in E(G)$, it follows, from Lemma 2.6(b), that $\{v_1, v_2\} \in E(G)$. Then v_1 and v_2 are adjacent also in H. $\square$

In Fig. 5, we show an example of the above construction. The ideal J_G is unmixed by Proposition 2.8, since $H = M_{4,4}$ and J_H is unmixed by Example 2.2. Moreover, it will follow from Example 5.4 and Proposition 5.14 that J_G is not Cohen–Macaulay.

In Proposition 2.8, the existence of a vertex $w \in V_1$ such that $\deg_G(w) = 2$ means that w is a leaf of H. This is not true in general, see for instance the graph $M_{k,k}$ in Example 2.2 for $k \geq 4$. However, if J_H is unmixed, this always holds:

Corollary 2.9. *Let H be a bipartite graph with bipartition $V_1 \sqcup V_2$, $|V_1| = |V_2|$ and such that J_H is unmixed. Let G be the graph in Proposition 2.8. Then J_G is unmixed if and only if the neighbours of the leaves of H are adjacent.*

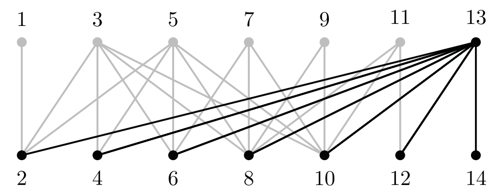


Fig. 6. A graph G with J_G not unmixed by Corollary 2.9.

Example 2.10. The graph H of Fig. 4 is such that J_H is unmixed, but the two cut vertices 2 and 11 are not adjacent. The graph in Fig. 6 is the graph G obtained from H with the construction in Proposition 2.8. According to Corollary 2.9, J_G is not unmixed: in fact $S = N(11) = \{8, 10, 12\}$ is a cut set and $c_G(S) = 3 \neq |S| + 1$.

3. Basic blocks

In this section we study the basic blocks F_m of our classification, proving that J_{F_m} is Cohen–Macaulay.

In what follows we will use several times the following argument.

Remark 3.1. Let G be a graph, v be a vertex of G, $H' = G \setminus \{v\}$ and assume that $\mathcal{M}(H') = \{S \setminus \{v\} : S \in \mathcal{M}(G),\, v \in S\}$, in particular v is a cut vertex of G since $\emptyset \in \mathcal{M}(H')$. Let $J_G = \bigcap_{S\in\mathcal{M}(G)} P_S(G)$ be the primary decomposition of J_G and set $A = \bigcap_{S\in\mathcal{M}(G), v\notin S} P_S(G)$ and $B = \bigcap_{S\in\mathcal{M}(G), v\in S} P_S(G)$. Then $J_G = A \cap B$ and we have the short exact sequence

$$0 \longrightarrow R/J_G \longrightarrow R/A \oplus R/B \longrightarrow R/(A+B) \longrightarrow 0. \tag{2}$$

Notice that

(i) $A = J_H$, where H is the graph obtained from G by adding all possible edges between the vertices of $N_G(v)$. In other words, $V(H) = V(G)$ and $E(H) = E(G) \cup \{\{k, \ell\} : k, \ell \in N_G(v), k \neq \ell\}$. In fact, notice that $v \notin S$ for every $S \in \mathcal{M}(H)$ by Remark 2.1 and all cut sets of G not containing v are cut sets of H as well. Thus, $\mathcal{M}(H) = \{S \in \mathcal{M}(G) : v \notin S\}$. Moreover, for every $S \in \mathcal{M}(H)$, the connected components of $G_{\overline{S}}$ and $H_{\overline{S}}$ are the same, except for the component containing v, which is G_i in $G_{\overline{S}}$ and H_i in $H_{\overline{S}}$. Nevertheless, $\widetilde{G}_i = \widetilde{H}_i$, hence $P_S(G) = P_S(H)$ for every $S \in \mathcal{M}(H)$.

(ii) $B = (x_v, y_v) + J_{H'}$, where $H' = G \setminus \{v\}$. In fact, if $S \in \mathcal{M}(G)$ with $v \in S$, then $S \setminus \{v\} \in \mathcal{M}(H')$ by assumption and we have that $P_S(G) = (x_v, y_v) + P_{S\setminus\{v\}}(H')$. Thus,

$$B = (x_v, y_v) + \bigcap_{S\in\mathcal{M}(G), v\in S} P_{S\setminus\{v\}}(H') = (x_v, y_v) + \bigcap_{T\in\mathcal{M}(H')} P_T(H') = (x_v, y_v) + J_{H'}.$$

(iii) $A + B = (x_v, y_v) + J_{H''}$, where $H'' = H \setminus \{v\}$.

We now describe a new family of Cohen–Macaulay binomial edge ideals associated with non-bipartite graphs, which will be useful in what follows. Let K_n be the complete graph on the vertex set $[n]$ and $W = \{v_1, \ldots, v_r\} \subseteq [n]$. Let H be the graph obtained from K_n by attaching, for every $i = 1, \ldots, r$, a complete graph K_{h_i} to K_n in such a way that $V(K_n) \cap V(K_{h_i}) = \{v_1, \ldots, v_i\}$, for some $h_i > i$. We say that the graph H is obtained by *adding a fan to* K_n *on the set* W. For example, Fig. 7 shows the result of adding a fan to K_6 on a set W of three vertices.

Lemma 3.2. *Let K_n be the complete graph on $[n]$ and $W_1 \sqcup \cdots \sqcup W_k$ be a partition of a subset $W \subseteq [n]$. Let G be the graph obtained from K_n by adding a fan on each set W_i. Then J_G is Cohen–Macaulay.*

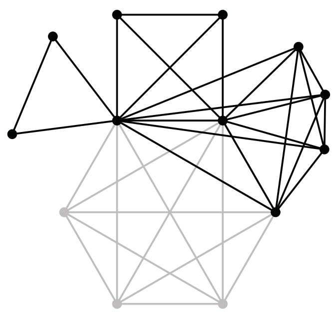


Fig. 7. Adding a fan to K_6 on three vertices.

Proof. First we show that J_G is unmixed. For every $i = 1, \ldots, k$, set $W_i = \{v_{i,1}, \ldots, v_{i,r_i}\}$ and $\mathcal{M}_i = \{\emptyset\} \cup \{\{v_{i,1}, \ldots, v_{i,h}\} : 1 \le h \le r_i\}$. We claim that

$$\mathcal{M}(G) = \{T_1 \cup \cdots \cup T_k : T_i \in \mathcal{M}_i, T_1 \cup \cdots \cup T_k \subsetneq [n]\}. \tag{3}$$

Let $T = T_1 \cup \cdots \cup T_k \neq \emptyset$, with $T_i \in \mathcal{M}_i$ for $i = 1, \ldots, k$, and $T \subsetneq [n]$. Let $v \in T$. Then $v \in T_j$ for some j, say $v = v_{j,\ell}$, with $1 \le \ell \le r_j$. Hence, if we add v to the graph $G_{\overline{T}}$, it joins the connected component containing $K_n \setminus T$ (which is non-empty since $T \subsetneq [n]$) with $K_{h_{j,\ell}} \setminus T$, where $V(K_{h_{j,\ell}}) \cap V(G) = \{v_{j,1}, \ldots, v_{j,\ell}\}$. This shows that $c_G(T) > c_G(T \setminus \{v\})$ for every $v \in T$, thus $T \in \mathcal{M}(G)$.

Conversely, let $T \in \mathcal{M}(G)$. First notice that $T \neq [n]$, since $c_G([n]) = c_G([n] \setminus \{v_{i,r_i}\})$ for every i. Moreover, T does not contain any vertex $v \in V(G) \setminus \bigcup_{i=1}^k W_i$, otherwise v belongs to exactly one maximal clique of G, see Remark 2.1. Then $c_G(T) = c_G(T \setminus \{v\})$. Hence $T \subseteq \bigcup_{i=1}^k W_i$ and $T \subsetneq [n]$. Let $T = \bigcup_{i=1}^k T_i$, where $T_i \subseteq W_i$. We want to show that, if $v_{i,j} \in T_i \subseteq T$, then $v_{i,h} \in T$ for every $1 \le h < j$. Assume $v_{i,h} \notin T_i$ for some $h < j$. Then $c_G(T) = c_G(T \setminus \{v_{i,j}\})$ because all maximal cliques of G containing $v_{i,j}$ contain $v_{i,h}$ as well, since $h < j$. This shows that $T_i \in \mathcal{M}_i$ for every i.

Finally, for every $T \in \mathcal{M}(G)$, since $G_{\overline{T}}$ consists of $|T|$ connected components that are complete graphs ($K_{h_{j,\ell}} \setminus T$ for every $j = 1, \ldots, k$ and $\ell = 1, \ldots, |T_j|$) and a graph obtained from $K_n \setminus T$ by adding a fan on each $W_i \setminus T_i$, it follows that $c_G(T) = |T| + 1$. This means that J_G is unmixed and $\dim(R/J_G) = |V(G)| + 1$.

In order to prove that J_G is Cohen–Macaulay, we proceed by induction on $k \ge 1$ and $|W_k| \ge 1$. Let $k = 1$ and set $W_1 = \{1, \ldots, r\}$. If $|W_1| = 1$, then the claim follows by [18, Theorem 2.7]. Assume that $|W_1| = r \ge 2$ and the claim true for $r - 1$. Notice that $G = \operatorname{cone}(1, G_1 \sqcup G_2)$, where $G_1 \cong K_{h_1 - 1}$ (graph isomorphism) and G_2 is the graph obtained from $K_n \setminus \{1\}$ by adding a fan on the clique $\{2, \ldots, r\}$. We know that J_{G_1} is Cohen–Macaulay by [3, Corollary 2.8] and J_{G_2} is Cohen–Macaulay by induction. Hence, the claim follows by [18, Theorem 3.8].

Now, let $k \ge 2$ and assume the claim true for $k - 1$. Again, if $|W_k| = 1$, the claim follows by induction and by [18, Theorem 2.7]. Assume that $|W_k| = r_k \ge 2$ and the claim true for $r_k - 1$. For simplicity, let $W_k = \{1, \ldots, r_k\}$. Let $J_G = \bigcap_{S \in \mathcal{M}(G)} P_S(G)$ be the primary decomposition of J_G and set $A = \bigcap_{S \in \mathcal{M}(G), 1 \notin S} P_S(G)$ and $B = \bigcap_{S \in \mathcal{M}(G), 1 \in S} P_S(G)$. Then $J_G = A \cap B$.

By Remark 3.1, $A = J_H$, where H is a complete graph on the vertices of $\{1\} \cup N_G(1)$ to which we add a fan on the cliques $W_1, \ldots, W_{k-1}$. Hence R/A is Cohen–Macaulay by induction on k and $\operatorname{depth}(R/A) = |V(G)| + 1$.

Notice that $H' = G \setminus \{1\}$ is the disjoint union of a complete graph and a graph K', which is obtained by adding a fan to $K_n \setminus \{1\} \cong K_{n-1}$ on the cliques $W_1, \ldots, W_{k-1}$ and $W_k \setminus \{1\}$. From (3), it follows that $\mathcal{M}(H') = \{S \setminus \{1\} : S \in \mathcal{M}(G), 1 \in S\}$, thus $B = (x_1, y_1) + J_{H'}$ by Remark 3.1. By induction on $|W_k|$, $J_{K'}$ is Cohen–Macaulay, hence $J_{H'}$ is Cohen–Macaulay since it is the sum of Cohen–Macaulay ideals on disjoint sets of variables. In particular, $\operatorname{depth}(R/B) = |V(H')| + 2 = |V(G)| + 1$ (it follows from the formula for the dimension [10, Corollary 3.4]).

Finally, by Remark 3.1, $A + B = (x_1, y_1) + J_{H''}$, where $H'' = H \setminus \{1\}$. Hence $R/(A + B)$ is Cohen–Macaulay by induction on k and $\operatorname{depth}(R/(A + B)) = |V(G)|$.

The Depth Lemma [20, Lemma 3.1.4] applied to the short exact sequence (2) yields $\operatorname{depth}(R/J_G) = |V(G)| + 1$. The claim follows from the first part, since $\dim(R/J_G) = |V(G)| + 1$. □

Notice that the graphs produced by Lemma 3.2 are not generalized block graphs (see [13]) nor closed graphs if $k \geq 2$ (studied in [5]). Hence they form a new family of non-bipartite graphs whose binomial edge ideal is Cohen–Macaulay.

Now we prove that the binomial edge ideals of the graphs F_m (see Fig. 1) are Cohen–Macaulay. The graphs F_m are the basic blocks in our classification, Theorem 6.1.

Recall that, for every $m \geq 1$, if $n = 2m$, F_m is the graph on the vertex set $[n]$ and with edge set

$$E(F_m) = \{\{2i, 2j - 1\} : i = 1, \dots, m, j = i, \dots, m\}.$$

Notice that F_m, with $m \geq 2$, can be obtained from F_{m-1} using the construction of Proposition 2.8.

Proposition 3.3. *For every $m \geq 1$, J_{F_m} is Cohen–Macaulay.*

Proof. First we show that J_{F_m} is unmixed. We proceed by induction on $m \geq 1$. If $m = 1$, then J_{F_1} is a principal ideal, hence it is prime and unmixed of height 1. Let $m \geq 2$ and assume the claim true for $m - 1$. Then F_m is obtained from F_{m-1} by adding the vertices $n - 1$ and n and connecting $n - 1$ to the vertices $2, 4, \dots, n$. Since $J_{F_{m-1}}$ is unmixed by induction and $\{2, n - 3\} \in E(F_{m-1})$, by Proposition 2.8, it follows that J_{F_m} is unmixed and

$$\mathcal{M}(F_m) = \{\emptyset\} \cup \{\{2, 4, \dots, 2i\} : 1 \leq i \leq m - 1\} \cup \{\{n - 1\} \cup S : S \in \mathcal{M}(F_{m-1})\}. \tag{4}$$

Now we prove that J_{F_m} is Cohen–Macaulay by induction on $m \geq 1$. The graphs F_1 and F_2 are paths, hence the ideals J_{F_1} and J_{F_2} are complete intersections, by [5, Corollary 1.2], thus Cohen–Macaulay.

Let $m \geq 3$ and assume that $J_{F_{m-1}}$ is Cohen–Macaulay. Let $J_{F_m} = \bigcap_{S \in \mathcal{M}(F_m)} P_S(F_m)$ be the primary decomposition of J_{F_m} and define $A = \bigcap_{S \in \mathcal{M}(F_m), n-1 \notin S} P_S(F_m)$ and $B = \bigcap_{S \in \mathcal{M}(F_m), n-1 \in S} P_S(F_m)$. Then $J_{F_m} = A \cap B$.

By Remark 3.1, $A = J_H$, where H is obtained by adding a fan to the complete graph with vertex set $N_{F_m}(n - 1) = \{2, 4, \dots, n\}$ on the set $N_{F_m}(n - 1)$, hence it is Cohen–Macaulay by Lemma 3.2 and $\operatorname{depth}(R/A) = n + 1$.

Since $F_m \setminus \{n - 1\} = F_{m-1} \sqcup \{n\}$, by (4), $\mathcal{M}(F_{m-1} \sqcup \{n\}) = \{S \setminus \{n - 1\} : S \in \mathcal{M}(F_m), n - 1 \in S\}$. Thus, $B = (x_{n-1}, y_{n-1}) + J_{F_{m-1} \sqcup \{n\}} = (x_{n-1}, y_{n-1}) + J_{F_{m-1}}$, hence it is Cohen–Macaulay by induction and $\operatorname{depth}(R/B) = n + 1$.

Finally, $A + B = (x_{n-1}, y_{n-1}) + J_{H''}$, where $H'' = H \setminus \{n - 1\}$, which is Cohen–Macaulay again by Lemma 3.2 and $\operatorname{depth}(R/(A + B)) = n$.

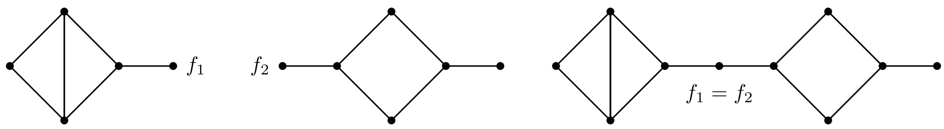
The Depth Lemma applied to the exact sequence (2) yields $\operatorname{depth}(R/J_{F_m}) = n + 1$. Moreover, since J_{F_m} is unmixed, it follows that $\dim(R/J_{F_m}) = n + 1$ and, therefore, J_{F_m} is Cohen–Macaulay. □

4. Gluing graphs: operations $*$ and $\circ$

In this section we consider two operations that, together with the graphs F_m, are the main ingredients of Theorem 6.1. Given two (not necessarily bipartite) graphs G_1 and G_2, we glue them to obtain a new graph G. If G_1 and G_2 are bipartite, both constructions preserve the Cohen–Macaulayness of the associated binomial edge ideal. The first operation is a particular case of the one studied by Rauf and Rinaldo in [18, Section 2].

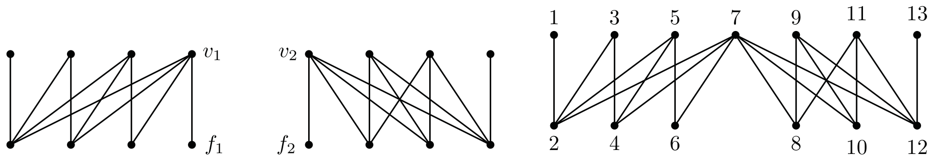
Definition 4.1. For $i = 1, 2$, let G_i be a graph with at least one leaf f_i. We define the graph $G = (G_1, f_1) * (G_2, f_2)$ obtained by identifying f_1 and f_2 (see Fig. 8). If it is not important to specify the vertices f_i or it is clear from the context, we simply write $G_1 * G_2$.

In the next Theorem we recall some results about the operation $*$, see [18, Lemma 2.3, Proposition 2.6, Theorem 2.7].



(a) A graph G_1. (b) A graph G_2. (c) The graph $(G_1, f_1) * (G_2, f_2)$.

Fig. 8. The operation $*$.



(a) The graph G_1. (b) The graph G_2. (c) The graph $G = G_1 \circ G_2$.

Fig. 9. The operation $\circ$.

Theorem 4.2. *For $i = 1, 2$, consider a graph G_i with at least one leaf f_i and $G = (G_1, f_1) * (G_2, f_2)$. Let v_1 and v_2 be the neighbours of the leaves and let v be the vertex obtained by identifying f_1 and f_2. If*

$$\mathcal{A} = \{S_1 \cup S_2 : S_i \in \mathcal{M}(G_i),\ i = 1, 2\} \text{ and}$$
$$\mathcal{B} = \{S_1 \cup S_2 \cup \{v\} : S_i \in \mathcal{M}(G_i) \text{ and } v_i \notin S_i,\ i = 1, 2\},$$

the following properties hold:

(a) $\mathcal{M}(G) = \mathcal{A} \cup \mathcal{B}$;
(b) *J_G is unmixed if and only if J_{G_1} and J_{G_2} are unmixed;*
(c) *J_G is Cohen–Macaulay if and only if J_{G_1} and J_{G_2} are Cohen–Macaulay.*

We now introduce the second operation.

Definition 4.3. For $i = 1, 2$, let G_i be a graph with at least one leaf f_i, v_i its neighbour and assume $\deg_{G_i}(v_i) \geq 3$. We define $G = (G_1, f_1) \circ (G_2, f_2)$ to be the graph obtained from G_1 and G_2 by removing the leaves f_1, f_2 and identifying v_1 and v_2 (see Fig. 9). If it is not important to specify the leaves f_i or it is clear from the context, then we simply write $G_1 \circ G_2$.

We denote by v the vertex of G resulting from the identification of v_1 and v_2 and, with abuse of notation, we write $V(G_1) \cap V(G_2) = \{v\}$.

Notice that, if $\deg_{G_i}(v_i) = 2$ for $i = 1, 2$, then $(G_1, f_1) \circ (G_2, f_2) = (G_1 \setminus \{f_1\}, v_1) * (G_2 \setminus \{f_2\}, v_2)$. On the other hand, we do not allow $\deg_{G_1}(v_1) = 2$ and $\deg_{G_2}(v_2) \geq 3$ (or vice versa), since in this case the operation $\circ$ does not preserve unmixedness, see Remark 4.7(ii).

Remark 4.4. Unlike the operation $*$ (cf. Theorem 4.2), if one of J_{G_1} and J_{G_2} is not Cohen–Macaulay, then $J_{G_1 \circ G_2}$ may not be unmixed, even if G_1 and G_2 are bipartite. For example, let G_1 and G_2 be the graphs in Figs. 9(a) and 9(b). Then $J_{G_1 \circ G_2}$ is not unmixed even if $J_{G_1} = J_{F_4}$ is Cohen–Macaulay (by Proposition 3.3) and $J_{G_2} = J_{M_{4,4}}$ is unmixed (by Example 2.2). In fact, $S = \{5, 7, 8, 10, 12\} \in \mathcal{M}(G)$, but $c_G(S) = 5 \neq |S| + 1$.

We describe the structure of the cut sets of $G_1 \circ G_2$ under some extra assumption on G_1 and G_2. In this case, $\circ$ preserves unmixedness.

Theorem 4.5. *Let $G = G_1 \circ G_2$ and set $V(G_1) \cap V(G_2) = \{v\}$, where $\deg_{G_i}(v) \geq 3$ for $i = 1, 2$. If for $i = 1, 2$ there exists $u_i \in N_{G_i}(v)$ with $\deg_{G_i}(u_i) = 2$, then*

$$\mathcal{M}(G) = \mathcal{A} \cup \mathcal{B}, \tag{5}$$

where

$$\mathcal{A} = \{S_1 \cup S_2 : S_i \in \mathcal{M}(G_i), i = 1, 2, v \notin S_1 \cup S_2\} \text{ and}$$
$$\mathcal{B} = \{S_1 \cup S_2 : S_i \in \mathcal{M}(G_i), i = 1, 2, S_1 \cap S_2 = \{v\}\}.$$

If J_{G_1} and J_{G_2} are unmixed and for $i = 1, 2$ there exists $u_i \in N_{G_i}(v)$ with $\deg_{G_i}(u_i) = 2$, then J_G is unmixed. The converse holds if G is bipartite. In particular, if G is bipartite and J_G is unmixed, the cut sets of G are described in (5).

Proof. Let $S = S_1 \cup S_2 \subset V(G)$, where $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Notice that

$$c_G(S) = c_{G_1}(S_1) + c_{G_2}(S_2) - 1, \text{ if } v \notin S, \tag{6}$$

$$c_G(S) = c_{G_1}(S_1) + c_{G_2}(S_2) - 2, \text{ if } v \in S. \tag{7}$$

In fact, if $v \notin S$, the connected components of $G_{\overline{S}}$ are those of $(G_1)_{\overline{S}_1}$ and $(G_2)_{\overline{S}_2}$, where the component containing v is counted once. On the other hand, if $v \in S$, clearly $v \in S_1 \cap S_2$ and the connected components of $G_{\overline{S}}$ are those of $(G_1)_{\overline{S}_1}$ and $(G_2)_{\overline{S}_2}$, except for the two leaves f_1 and f_2.

In order to prove (5), we show the two inclusions.

$\subseteq$: Let $S \in \mathcal{M}(G)$ and define S_1 and S_2 as before. Suppose by contradiction that $S_1 \notin \mathcal{M}(G_1)$, i.e., there exists $w \in S_1$ such that $c_{G_1}(S_1) = c_{G_1}(S_1 \setminus \{w\})$. If $v \notin S$, then by (6)

$$c_G(S \setminus \{w\}) = c_{G_1}(S_1 \setminus \{w\}) + c_{G_2}(S_2) - 1 = c_{G_1}(S_1) + c_{G_2}(S_2) - 1 = c_G(S),$$

a contradiction. On the other hand, if $v \in S$ and $w \neq v$, by (7) we have

$$c_G(S \setminus \{w\}) = c_{G_1}(S_1 \setminus \{w\}) + c_{G_2}(S_2) - 2 = c_{G_1}(S_1) + c_{G_2}(S_2) - 2 = c_G(S),$$

again a contradiction. We show that the case $w = v$ cannot occur. In fact, by assumption, there exists $u_1 \in N_{G_1}(v)$ such that $\deg_{G_1}(u_1) = 2$. Since $v \in S$, we have that $c_G(S) = c_G(S \setminus \{u_1\})$, hence $u_1 \notin S$. Thus $c_{G_1}(S_1) > c_{G_1}(S_1 \setminus \{v\})$, because by adding v to $(G_1)_{\overline{S}_1}$, we join the connected component of u_1 and the isolated vertex f_1, which is a leaf in G_1. Hence $w \neq v$. The same argument also shows that $S_2 \in \mathcal{M}(G_2)$.

$\supseteq$: Let $S = S_1 \cup S_2$, with $S_i \in \mathcal{M}(G_i)$, for $i = 1, 2$. Assume first $S_1 \cap S_2 = \{v\}$. By the equalities (6) and (7) we have

$$c_G(S \setminus \{v\}) = c_{G_1}(S_1 \setminus \{v\}) + c_{G_2}(S_2 \setminus \{v\}) - 1 \leq c_{G_1}(S_1) + c_{G_2}(S_2) - 3 = c_G(S) - 1 < c_G(S).$$

Let $w \in S$, $w \neq v$. Without loss of generality, we may assume $w \in S_1$. Then

$$c_G(S \setminus \{w\}) = c_{G_1}(S_1 \setminus \{w\}) + c_{G_2}(S_2) - 2 \leq c_{G_1}(S_1) + c_{G_2}(S_2) - 3 = c_G(S) - 1 < c_G(S).$$

Assume now that $v \notin S_1 \cup S_2$. Let $w \in S$, and without loss of generality $w \in S_1$. Then

$$c_G(S \setminus \{w\}) = c_{G_1}(S_1 \setminus \{w\}) + c_{G_2}(S_2) - 1 \leq c_{G_1}(S_1) + c_{G_2}(S_2) - 2 = c_G(S) - 1 < c_G(S).$$

Let now J_{G_1} and J_{G_2} be unmixed and for $i = 1, 2$ there exists $u_i \in N_{G_i}(v)$ with $\deg_{G_i}(u_i) = 2$. By the last assumption, the cut sets of G are described in (5). Let $S \in \mathcal{M}(G)$ and $S_i = S \cap V(G_i)$ for $i = 1, 2$. Thus, by (6) and (7),

(i) if $v \notin S$, $c_G(S) = c_{G_1}(S_1) + c_{G_2}(S_2) - 1 = |S_1| + 1 + |S_2| + 1 - 1 = |S_1| + |S_2| + 1 = |S| + 1$,
(ii) if $v \in S$, $c_G(S) = c_{G_1}(S_1) + c_{G_2}(S_2) - 2 = |S_1| + 1 + |S_2| + 1 - 2 = |S_1| + |S_2| = |S| + 1$.

It follows that J_G is unmixed.

Conversely, let J_G be unmixed and G bipartite. If S is a cut set of G_1, then it is also a cut set of G and clearly $c_{G_1}(S) = c_G(S)$; therefore J_{G_1} is unmixed and the same holds for J_{G_2}. By Proposition 2.3, the graphs G, G_1 and G_2 have exactly two leaves. Let f_i be the leaf of G_i adjacent to v and g_i be the other leaf of G_i. Thus, g_1 and g_2 are the leaves of G.

By symmetry, it is enough to prove that there exists $u_1 \in N_{G_1}(v)$ such that $\deg_{G_1}(u_1) = 2$. For $i = 1, 2$, let $V(G_i) = V_i \cup W_i$ and assume $|V_1| \leq |W_1|$. By Remark 2.4, we have one of the following two cases:

(a) if $|V_1| = |W_1|$, we may assume $f_1 \in W_1$ and $g_1 \in V_1$. Set $S = (W_1 \setminus \{f_1\}) \cup \{v\}$. Hence, $c_{G_1}(S) = |V_1| = |W_1| = |S|$.
(b) If $|W_1| = |V_1| + 1$, then $f_1, g_1 \in W_1$. Hence $v \in V_1$. Set $S = (W_1 \setminus \{f_1, g_1\}) \cup \{v\}$. Thus, $c_{G_1}(S) = |V_1| = |W_1| - 1 = |S|$.

First suppose $|V(G_2)|$ even and assume $f_2 \in W_2$. Hence, $v, g_2 \in V_2$ and $T = V_2 \setminus \{g_2\}$ is a cut set of G_2.

Now, let $|V(G_2)|$ be odd and assume $f_2 \in W_2$. Hence, $g_2 \in W_2$, $v \in V_2$ and $|W_2| = |V_2| + 1$. Then $T = V_2$ is a cut set of G_2.

In both cases, notice that $S \cup T$ is not a cut set of G, since $S \cap T = \{v\}$ and, by (7),

$$c_G(S \cup T) = c_{G_1}(S) + c_{G_2}(T) - 2 = |S| + |T| - 1 = |S \cup T|,$$

which contradicts the unmixedness of J_G. Let $u \in S \cup T$ such that $c_G((S \cup T) \setminus \{u\}) = c_G(S \cup T) = |S \cup T|$.

We show that $u \in S$ and $u \neq v$. If $u \notin S$, then $u \in T$ and $u \neq v$. By (7),

$$\begin{aligned} c_G((S \cup T) \setminus \{u\}) &= c_{G_1}(S) + c_{G_2}(T \setminus \{u\}) - 2 < |S| + c_{G_2}(T) - 2 = |S| + |T| - 1 \\ &= |S \cup T| = c_G(S \cup T), \end{aligned}$$

against our assumption (the inequality holds since T is a cut set of G_2 and the second equality follows from the unmixedness of J_{G_2}). Thus, $u \in S$. Moreover, in both cases $c_{G_1}(S \setminus \{v\}) = c_{G_1}(S) = |S|$ (since v is a leaf of $(G_1)_{\overline{S \setminus \{v\}}}$) and, by (6),

$$\begin{aligned} c_G((S \cup T) \setminus \{v\}) &= c_{G_1}(S \setminus \{v\}) + c_{G_2}(T \setminus \{v\}) - 1 = |S| + |T| - |N_{G_2}(v)| + 2 - 1 < |S| + |T| - 1 \\ &= |S \cup T| = c_G(S \cup T), \end{aligned}$$

where the inequality holds since $\deg_{G_2}(v) \geq 3$. This contradicts our assumption, thus $u \neq v$.

We conclude that $u \in S \setminus \{v\}$. Since $u \neq f_1, g_1$, we have $\deg_{G_1}(u) \geq 2$. On the other hand, since $c_G((S \cup T) \setminus \{u\}) = c_G(S \cup T)$, it follows that $u \in N_{G_1}(v)$ and $\deg_{G_1}(u) = 2$. $\square$

Corollary 4.6. *Let $G = F_{m_1} \circ \cdots \circ F_{m_k}$, where $m_i \geq 3$ for $i = 1, \ldots, k$. Then J_G is unmixed.*

Proof. Set $G_1 = F_{m_1} \circ \cdots \circ F_{m_{k-1}}$, $G_2 = F_{m_k}$ and let v be the only vertex of $V(G_1) \cap V(G_2)$. We proceed by induction on $k \geq 2$. If $k = 2$, the claim follows by Theorem 4.5, because J_{G_1} and J_{G_2} are unmixed by Proposition 3.3 and for $i = 1, 2$, there exists $u_i \in N_{G_i}(v)$ such that $\deg_{G_i}(u_i) = 2$, by definition of F_{m_i}.

Now let $k > 2$ and assume the claim true for $k - 1$. By induction, J_{G_1} is unmixed. Since $m_{k-1} \geq 3$, there exists $u_1 \in N_{G_1}(v)$ such that $\deg_{G_1}(u_1) = 2$. The claim follows again by Theorem 4.5. $\square$

Remark 4.7. In Corollary 4.6 the condition $m_i \geq 3$, for $i = 2, \ldots, k - 1$, cannot be omitted. For instance, the binomial edge ideal $J_{F_3 \circ F_2 \circ F_3}$ is not unmixed: in fact $S = \{3, 5, 6, 8\}$ is a cut set and $c_{F_3 \circ F_2 \circ F_3}(S) = 4 \neq |S| + 1$, see Fig. 10.

On the other hand, we may allow $m_1 = m_k = 2$, since, in this case, the graph $G = F_{m_1} \circ \cdots \circ F_{m_k} = F_1 * F_{m_2} \circ \cdots \circ F_{m_{k-1}} * F_1$. Hence, J_G is unmixed by Theorem 4.2 and Corollary 4.6.

Let $n \geq 3$, $W_1 \sqcup \cdots \sqcup W_k$ be a partition of a subset of $[n]$ and $W_i = \{v_{i,1}, \ldots, v_{i,r_i}\}$ for some $r_i \geq 1$ and $i = 1, \ldots, k$. Let E be the graph obtained from K_n by adding a fan on each set W_i in such a way that we attach a complete graph K_{h+1} to K_n, with $V(K_n) \cap V(K_{h+1}) = \{v_{i,1}, \ldots, v_{i,h}\}$, for $i = 1, \ldots, k$ and $h = 1, \ldots, r_i$, see Fig. 11 (cf. Fig. 7). By Lemma 3.2, J_E is Cohen–Macaulay.

Lemma 4.8. *Let $G = F_{m_1} \circ \cdots \circ F_{m_k} \circ E$, where E is the graph defined above, $m_i \geq 3$ for every $i = 2, \ldots, k$ and $V(F_{m_1} \circ \cdots \circ F_{m_k}) \cap V(E) = \{v\}$. Assume that $v \in W_1$ and $|W_1| \geq 2$. Then J_G is unmixed.*

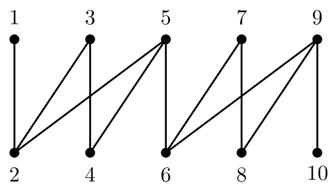


Fig. 10. The graph $F_3 \circ F_2 \circ F_3$.

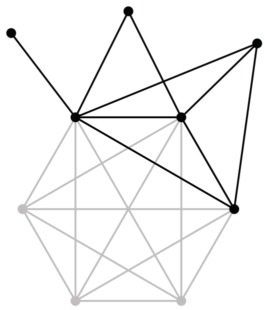


Fig. 11. The graph E.

Proof. Set $G_1 = F_{m_1} \circ \cdots \circ F_{m_k}$ and $G_2 = E$. Then J_{G_1} is unmixed by Corollary 4.6 and J_{G_2} is Cohen–Macaulay by Lemma 3.2, hence it is unmixed.

Notice that, since $m_k \geq 3$, there exists $u_1 \in N_{G_1}(v)$ such that $\deg_{G_1}(u_1) = 2$. Moreover, since $|W_1| \geq 2$ and by definition of $G_2 = E$, we attach K_3 to K_n in such a way that $|V(K_n) \cap V(K_3)| = 2$ and $v \in V(K_n) \cap V(K_3)$. Thus, there exists $u_2 \in K_3$, hence $u_2 \in N_{G_2}(v)$, such that $\deg_{G_2}(u_2) = 2$. The statement follows by Theorem 4.5. $\square$

In Lemma 4.8 we assume $|W_1| \geq 2$, since this is the only case we need in the following theorem. Moreover, in the next statement the case $F = E$ is useful to prove that the binomial edge ideal associated with $F_{m_1} \circ \cdots \circ F_{m_k} \circ F_n$ is Cohen–Macaulay.

Theorem 4.9. *Let $G = F_{m_1} \circ \cdots \circ F_{m_k} \circ F$, where $m_i \geq 3$ for every $i = 1, \dots, k$ and $F = F_n$ for some $n \geq 3$ or $F = E$ is the same graph of Lemma 4.8. Then J_G is Cohen–Macaulay.*

Proof. Let $V(F_{m_1} \circ \cdots \circ F_{m_k}) \cap V(F) = \{w\}$ and call f_k and f the leaves that we remove from $F_{m_1} \circ \cdots \circ F_{m_k}$ and F. Let $J_G = \bigcap_{S \in \mathcal{M}(G)} P_S(G)$ be the primary decomposition of J_G and set $A = \bigcap_{S \in \mathcal{M}(G), w \notin S} P_S(G)$ and $B = \bigcap_{S \in \mathcal{M}(G), w \in S} P_S(G)$.

We proceed by induction on $k \geq 1$. First assume $k = 1$ and, for simplicity, let $m = m_1$. By Remark 3.1, the ideal A is the binomial edge ideal of the graph H, obtained by adding a fan to the complete graph with vertex set $\{w\} \cup N_G(w)$ on the sets $N_{F_m}(w) \setminus \{f_k\}$ and $N_F(w) \setminus \{f\}$. Hence R/A is Cohen–Macaulay and $\operatorname{depth}(R/A) = |V(G)| + 1$ by Lemma 3.2.

Notice that $G \setminus \{w\} = (F_m \setminus \{w, f_k\}) \sqcup (F \setminus \{w, f\})$. By Theorem 4.5 and Remark 3.1, $B = (x_w, y_w) + J_{F_m \setminus \{w, f_k\}} + J_{F \setminus \{w, f\}}$, where $F_m \setminus \{w, f_k\} \cong F_{m-1}$. Moreover, if $F = E$, $E \setminus \{w, f\}$ is of the same form as E, otherwise $F = F_n$ and $F_n \setminus \{w, f\} \cong F_{n-1}$. In any case, $J_{F_m \setminus \{w, f_k\}}$ and $J_{F \setminus \{w, f\}}$ are Cohen–Macaulay (by Lemma 3.2 and Proposition 3.3), hence B is Cohen–Macaulay since it is the sum of Cohen–Macaulay ideals on disjoint sets of variables. In particular, it follows from the formula for the dimension [10, Corollary 3.4] that $\operatorname{depth}(R/B) = |V(F_{m-1})| + 1 + |V(F \setminus \{w, f\})| + 1 = |V(G)| + 1$.

Finally, $A + B = (x_w, y_w) + J_{H''}$, where $H'' = H \setminus \{w\}$ is the binomial edge ideal of the graph obtained by adding a fan to the complete graph with vertex set $N_G(w)$ on the sets $N_{F_m}(w) \setminus \{f_k\}$ and $N_F(w) \setminus \{f\}$. Hence $R/(A + B)$ is Cohen–Macaulay and $\operatorname{depth}(R/(A + B)) = |V(G)|$ by Lemma 3.2.

The Depth Lemma applied to the short exact sequence (2) yields $\operatorname{depth}(R/J_G) = |V(G)| + 1$. The claim follows by Lemma 4.8 (resp. Corollary 4.6), since $\dim(R/J_G) = |V(G)| + 1$.

Now let $k > 1$ and assume the claim true for $k-1$. By Remark 3.1, the ideal A is the binomial edge ideal of the graph $H = F_1 \circ \cdots \circ F_{m_{k-1}} \circ F'$, where F' is obtained by adding a fan to the complete graph with vertex set $\{w\} \cup N_G(w)$ on the sets $N_{F_{m_k}}(w) \setminus \{f_k\}$ and $N_F(w) \setminus \{f\}$. Notice that, since $m_k \geq 3$, $|N_{F_{m_k}}(w) \setminus \{f_k\}| \geq 2$ and we are in the assumption of Lemma 4.8. Hence, R/A is Cohen–Macaulay by induction and $\operatorname{depth}(R/A) = |V(G)| + 1$.

Similarly to the case $k = 1$, the ideal B equals $(x_w, y_w) + J_{(F_{m_1} \circ \cdots \circ F_{m_k}) \setminus \{w, f_k\}} + J_{F \setminus \{w,f\}}$, where $(F_{m_1} \circ \cdots \circ F_{m_k}) \setminus \{w, f_k\} \cong F_{m_1} \circ \cdots \circ F_{m_k-1}$ and $J_{F_{m_1} \circ \cdots \circ F_{m_k-1}}$ is Cohen–Macaulay by induction (notice that, if $m_k = 3$, then $F_{m_1} \circ \cdots \circ F_{m_k} = F_{m_1} \circ \cdots \circ F_{m_{k-1}} * F_1$ and the corresponding binomial edge ideal is Cohen–Macaulay by induction and Theorem 4.2). Moreover, if $F = E$, then $E \setminus \{w\}$ is of the same form as E, otherwise $F = F_n$ and $F_n \setminus \{w, f\} \cong F_{n-1}$. Thus $J_{F \setminus \{w,f\}}$ is Cohen–Macaulay (by Lemma 3.2 and Proposition 3.3), hence B is Cohen–Macaulay since it is the sum of Cohen–Macaulay ideals on disjoint sets of variables. In particular, $\operatorname{depth}(R/B) = |V(F_{m-1})| + 1 + |V(F \setminus \{w, f\})| + 1 = |V(G)| + 1$ (it follows from the formula for the dimension [10, Corollary 3.4]).

Finally, $A + B = (x_w, y_w) + J_{H''}$, where $H'' = H \setminus \{w\}$ (again, since $m_k \geq 3$, we have $|N_{F_{m_k}}(w) \setminus \{f_k\}| \geq 2$). Hence $R/(A + B)$ is Cohen–Macaulay by induction and $\operatorname{depth}(R/(A + B)) = |V(G)|$.

The Depth Lemma applied to the short exact sequence (2) yields $\operatorname{depth}(R/J_G) = |V(G)| + 1$. Notice that, if $F = E$, the ideal J_G is unmixed by Lemma 4.8, whereas, if $F = F_n$, it is unmixed by Corollary 4.6. This implies that $\dim(R/J_G) = |V(G)| + 1$ and the claim follows. $\square$

5. The dual graph of binomial edge ideals

In this section we study the dual graph of binomial edge ideals. This is one of the main tools to prove that, if G is bipartite and J_G is Cohen–Macaulay, then G can be obtained recursively via a sequence of operations $*$ and $\circ$ on a finite set of graphs of the form F_m, Theorem 6.1(c).

Let I be an ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of I. Following [2], the *dual graph* $\mathcal{D}(I)$ of I is a graph with vertex set $\{1, \dots, r\}$ and edge set

$$\{\{i, j\} : \operatorname{ht}(\mathfrak{p}_i + \mathfrak{p}_j) - 1 = \operatorname{ht}(\mathfrak{p}_i) = \operatorname{ht}(\mathfrak{p}_j) = \operatorname{ht}(I)\}.$$

Notice that, if $\mathcal{D}(I)$ is connected, then I is unmixed. In [8], Hartshorne proved that if A/I is Cohen–Macaulay, then $\mathcal{D}(I)$ is connected. We will show that this is indeed an equivalence for binomial edge ideals of bipartite graphs. Nevertheless, this does not hold when G is not bipartite, see Remark 5.1.

To ease the notation, we denote by $\mathcal{D}(G)$ the dual graph of the binomial edge ideal J_G of a graph G. Moreover, we denote by $P_S(G)$ or P_S both the minimal primes of J_G and the vertices of $\mathcal{D}(G)$.

Remark 5.1. The dual graph of the non-bipartite graph G in Fig. 12(a) is connected, see Fig. 12(b), but using `Macaulay2` [7] one can check that J_G is not Cohen–Macaulay.

We now describe the edges of the dual graph of J_G, when J_G is unmixed. This result holds for non-bipartite graphs as well.

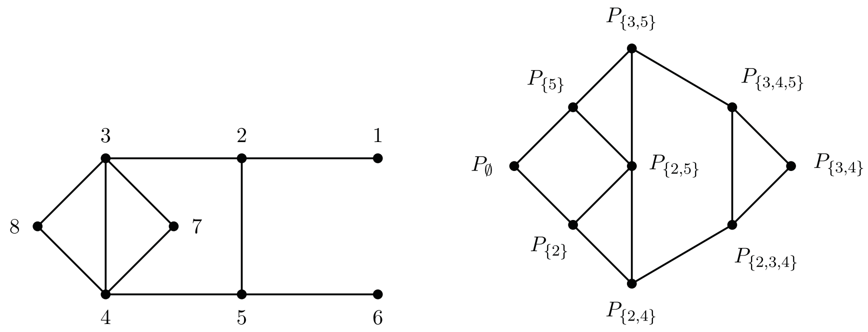
Theorem 5.2. *Let G be a graph such that J_G is unmixed and let $S, T \in \mathcal{M}(G)$, with $|T| \geq |S|$. Denote by P_S the minimal primes of J_G. Then the following properties hold:*

(a) *if $|T \setminus S| > 1$, then $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$;*
(b) *if $|T \setminus S| = 1$ and $S \subset T$, then $\{P_S, P_T\}$ is an edge of $\mathcal{D}(G)$;*
(c) *if $T \setminus S = \{t\}$ and $S \not\subseteq T$, then $\{P_S, P_T\}$ is an edge of $\mathcal{D}(G)$ if and only if t is not a cut vertex of $G_{\overline{S}}$.*

Proof. Let $E_1, E_2, \dots, E_{c(S)}$ be the connected components of $G_{\overline{S}}$.

(a) Let $v, w \in T \setminus S$. Then $P_S + P_T \supseteq P_S + (x_v, x_w, y_v, y_w)$. If E_j and E_k are the connected components of $G_{\overline{S}}$ containing v and w respectively (possibly $j = k$), it follows that

$$P_S + (x_v, x_w, y_v, y_w) = \left(\bigcup_{i \in S \cup \{v,w\}} \{x_i, y_i\}, J_{\widetilde{E}_1}, \dots, J_{(\widetilde{E}_j)_{\overline{\{v\}}}}, \dots, J_{(\widetilde{E}_k)_{\overline{\{w\}}}} \dots, J_{\widetilde{E}_{c(S)}} \right).$$



(a) The graph G.

(b) The dual graph of G.

Fig. 12. A non-bipartite graph G with $\mathcal{D}(G)$ connected and J_G not Cohen–Macaulay.

Thus, $\mathrm{ht}(P_S + P_T) \geq \mathrm{ht}(P_S + (x_v, x_w, y_v, y_w)) = \mathrm{ht}(P_S) + 4 - 2 = \mathrm{ht}(P_S) + 2$. Hence, $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$.

(b) Let $T \setminus S = \{t\}$ and let E_j be the connected component of $G_{\overline{S}}$ containing t. Then

$$P_S + P_T = \left(\bigcup_{i\in S} \{x_i, y_i\}, (x_t, y_t), J_{\widetilde{E}_1}, \ldots, J_{(\widetilde{E}_j)_{\overline{\{t\}}}}, \ldots, J_{\widetilde{E}_{c(S)}} \right).$$

Thus, $\mathrm{ht}(P_S + P_T) = \mathrm{ht}(P_S) + 2 - 1 = \mathrm{ht}(P_S) + 1$. Hence, $\{P_S, P_T\}$ is an edge of $\mathcal{D}(G)$.

(c) Let $G_1, G_2, \ldots, G_r$ be the connected components of $G_{\overline{S\cap T}}$. Let also $S \setminus T = \{s\}$, $T \setminus S = \{t\}$ and assume that $s \in G_j$ and $t \in G_k$. Since $S, T \in \mathcal{M}(G)$, it follows that s and t are cut vertices of G_j and G_k, respectively.

If $j \neq k$, then t is a cut vertex of $G_{\overline{S}}$. Moreover, if $V(G_j) = V(E_1 \cup \cdots \cup E_h \cup \{s\})$, where $h \geq 2$ and $G_k = E_{h+1}$, then

$$P_S + P_T = \left(\bigcup_{i\in S\cup\{t\}} \{x_i, y_i\}, J_{(\widetilde{G}_j)_{\overline{\{s\}}}}, J_{(\widetilde{E}_{h+1})_{\overline{\{t\}}}}, J_{\widetilde{E}_{h+2}}, \ldots, J_{\widetilde{E}_{c(S)}} \right).$$

It follows that $\mathrm{ht}(P_S+P_T) = \mathrm{ht}(P_S)+2+|V(G_j)|-2-\sum_{i=1}^{h}(|V(E_i)|-1)-1 = \mathrm{ht}(P_S)+2+1-2+h-1 = \mathrm{ht}(P_S) + h > \mathrm{ht}(P_S) + 1$. Thus, $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$.

Assume now that $j = k$ and let $j = 1$ for simplicity. Denote by $H_1, \ldots, H_i$ the connected components of $(G_1)_{\overline{\{s\}}}$ and by $K_1, \ldots, K_i$ the connected components of $(G_1)_{\overline{\{t\}}}$ (note that the number of components is the same because $S, T \in \mathcal{M}(G)$ and J_G is unmixed). Suppose also that $t \in H_1$ and $s \in K_1$. If there exists $v \in H_p \cap K_q$ with $p, q \neq 1$, then, since $v \in H_p$, there exists a path from v to s that does not involve t. This is a contradiction because $v \in K_q$ and $s \in K_1$. Hence, $K_q \subseteq H_1$ and $H_p \subseteq K_1$ for all $p, q = 2, \ldots, i$. In particular, the connected components of $G_{\overline{S\cup T}}$ are $H_2, \ldots, H_i, K_2, \ldots, K_i, G_2, \ldots, G_r$ and the connected components of $H_1 \cap K_1$, if it is not empty.

Suppose first that $H_1 \cap K_1 = \emptyset$. Hence, $V(H_1) = V(K_2 \cup \cdots \cup K_i \cup \{t\})$ and $V(K_1) = V(H_2 \cup \cdots \cup H_i \cup \{s\})$. If $i \geq 3$, then t is a cut vertex of H_1, hence a cut vertex of $G_{\overline{S}}$. It follows that

$$P_S = \left(\bigcup_{h\in S} \{x_h, y_h\}, J_{\widetilde{H}_1}, J_{\widetilde{H}_2}, \ldots, J_{\widetilde{H}_i}, J_{\widetilde{G}_2}, \ldots, J_{\widetilde{G}_r} \right) \text{ and}$$

$$P_S + P_T = \left(\bigcup_{h\in S\cup\{t\}} \{x_h, y_h\}, J_{(\widetilde{H}_1)_{\overline{\{t\}}}}, J_{(\widetilde{K}_1)_{\overline{\{s\}}}}, J_{\widetilde{G}_2}, \ldots, J_{\widetilde{G}_r} \right).$$

Therefore, $\mathrm{ht}(P_S+P_T) = \mathrm{ht}(P_S)+2-1-\sum_{h=2}^{i}(|V(H_h)|-1)+|V(K_1)|-2 = \mathrm{ht}(P_S)+1-\sum_{h=2}^{i}|V(H_h)|+(i-1)+(\sum_{h=2}^{i}|V(H_h)|+1)-2 = \mathrm{ht}(P_S)+i-1 > \mathrm{ht}(P_S)+1$, since $i \geq 3$. Thus, $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$.

On the other hand, if $i = 2$, then t is not a cut vertex of H_1, since K_2 is connected. Therefore, t is not a cut vertex of $G_{\overline{S}}$. It follows that

$$P_S + P_T = \left(\bigcup_{h \in S \cup \{t\}} \{x_h, y_h\}, J_{(\widetilde{H}_1)_{\overline{\{t\}}}}, J_{\widetilde{H}_2}, J_{\widetilde{G}_2}, \dots, J_{\widetilde{G}_r} \right).$$

Hence, $\mathrm{ht}(P_S + P_T) = \mathrm{ht}(P_S) + 2 - 1 = \mathrm{ht}(P_S) + 1$ and $\{P_S, P_T\}$ is an edge of $\mathcal{D}(G)$.

Let now $H_1 \cap K_1 \neq \emptyset$. It follows that

$$V(H_1) = V(K_2 \cup \cdots \cup K_i \cup (H_1 \cap K_1) \cup \{t\}) \quad \text{and} \quad V(K_1) = V(H_2 \cup \cdots \cup H_i \cup (H_1 \cap K_1) \cup \{s\})$$

and in this case t is a cut vertex of $G_{\overline{S}}$. Moreover,

$$P_S + P_T = \left(\bigcup_{h \in S \cup \{t\}} \{x_h, y_h\}, J_{(\widetilde{H}_1)_{\overline{\{t\}}}}, J_{(\widetilde{K}_1)_{\overline{\{s\}}}}, J_{\widetilde{G}_2} \dots, J_{\widetilde{G}_r} \right).$$

In fact, $J_{\widetilde{H}_h} \subseteq J_{(\widetilde{K}_1)_{\overline{\{s\}}}}$ and $J_{\widetilde{K}_h} \subseteq J_{(\widetilde{H}_1)_{\overline{\{t\}}}}$ for all $h = 2, \dots, i$. We now compute the height of $J = J_{(\widetilde{H}_1)_{\overline{\{t\}}}} + J_{(\widetilde{K}_1)_{\overline{\{s\}}}}$. Setting $W_1 = H_2 \cup \cdots \cup H_i$ and $W_2 = K_2 \cup \cdots \cup K_i$, the ideal J is the binomial edge ideal of the graph F obtained from $\widetilde{W}_1 \cup \widetilde{W}_2 \cup (\widetilde{H}_1 \cap \widetilde{K}_1)$ by adding the edges $\{\{v, w\} : v \in \widetilde{H}_1 \cap \widetilde{K}_1, w \in \widetilde{W}_1 \cup \widetilde{W}_2\}$. It is easy to check that the only cut sets of F are $\emptyset$ and $\widetilde{H}_1 \cap \widetilde{K}_1$. Moreover,

$$\begin{aligned} \mathrm{ht}(P_{\widetilde{H}_1 \cap \widetilde{K}_1}(F)) &= |V(F)| + |V(\widetilde{H}_1 \cap \widetilde{K}_1)| - 2 \geq |V(F)| - 1 = \mathrm{ht}(P_{\emptyset}(F)) \\ &= |V(\widetilde{W}_1)| + |V(\widetilde{W}_2)| + |V(\widetilde{H}_1 \cap \widetilde{K}_1)| - 1. \end{aligned}$$

Thus $\mathrm{ht}(J) = |V(F)| - 1 = \sum_{h=1}^{i}|V(H_h)| - 2$. Since $i \geq 2$, we get

$$\mathrm{ht}(P_S + P_T) = \mathrm{ht}(P_S) + 2 - \sum_{h=1}^{i}(|V(H_h)| - 1) + \sum_{h=1}^{i}|V(H_h)| - 2 = \mathrm{ht}(P_S) + i > \mathrm{ht}(P_S) + 1.$$

Hence, $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$. □

Remark 5.3. Let G be a connected graph such that $\mathcal{D}(G)$ is connected. If G is not a complete graph, then G has at least one cut vertex. In fact, if G does not have cut vertices, by Theorem 5.2(a), it follows that $P_\emptyset$ is an isolated vertex of the dual graph $\mathcal{D}(G)$, a contradiction. Notice that, if G is bipartite, by Proposition 2.3, it is enough to require J_G unmixed. Nevertheless, in the non-bipartite case we need to assume that $\mathcal{D}(G)$ is connected. In fact, the graph G in Fig. 13 does not have cut vertices, J_G is unmixed and $\mathcal{D}(G)$ consists of two isolated vertices.

We also observe that the above statement generalizes [1, Proposition 3.10], since having a connected dual graph is weaker than the Serre's condition S_2, see [8, Corollary 2.4]. In particular, if J_G is Cohen–Macaulay, then G has at least one cut vertex.

Example 5.4. For every $k \geq 4$, let $M_{k,k}$ and $M_{k-1,k}$ be the graphs defined in Example 2.2. With the same notation used there and by Theorem 5.2, their dual graphs are represented in Fig. 14.

Thus, $J_{M_{k,k}}$ and $J_{M_{k-1,k}}$ are not Cohen–Macaulay by Hartshorne's Theorem [8]. Notice that, $M_{3,4}$ is the bipartite graph with the smallest number of vertices whose binomial edge ideal is unmixed and not Cohen–Macaulay.

The following technical result has several crucial consequences, see Theorems 5.7 and 5.8. We show that, under some assumption on the graph, the intersection of two cut sets, which differ by one element and have the same cardinality, is again a cut set.

Lemma 5.5. *Let G be a graph such that J_G is unmixed. Let $S, T \in \mathcal{M}(G)$ with $|S| = |T|$ and $|S \setminus T| = 1$.*

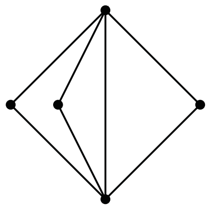
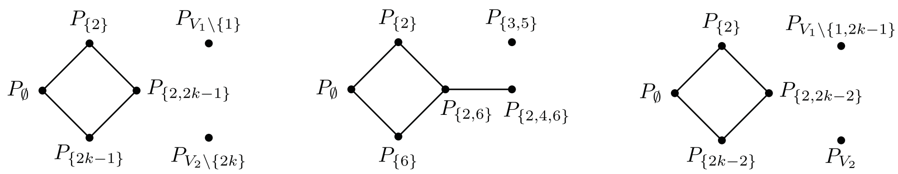


Fig. 13. A graph G without cut vertices, with J_G unmixed and $\mathcal{D}(G)$ disconnected.



(a) The dual graph of $J_{M_{k,k}}$. (b) The dual graph of $J_{M_{3,4}}$. (c) The dual graph of $J_{M_{k-1,k}}$, $k \geq 5$.

Fig. 14. The dual graphs of $J_{M_{h,k}}$.

(i) *If* $\{P_S, P_T\} \in \mathcal{D}(G)$, *then* $S \cap T \in \mathcal{M}(G)$.
(ii) *If* $S \cup T \in \mathcal{M}(G)$ *and* G *is bipartite, then* $S \cap T \in \mathcal{M}(G)$.

Proof. Let $S = (T \setminus \{t\}) \cup \{s\}$ and let $G_1, \ldots, G_r$ be the connected components of $G_{\overline{S\cap T}}$. Suppose first that $s \in G_i$ and $t \in G_j$ with $i \neq j$. Let $z \in S \cap T$ such that $c_G((S \cap T) \setminus \{z\}) = c_G(S \cap T)$. Since $z \in S$ and $S \in \mathcal{M}(G)$, z joins at least two components of $G_{\overline{S}}$. Then in $G_{\overline{S}}$ it is only adjacent to some components of $(G_i)_{\overline{\{s\}}}$. This implies that it does not join any components in $G_{\overline{T}}$, a contradiction, since $T \in \mathcal{M}(G)$.

Assume now that $s, t \in G_1$ and suppose first that $r = |S \cap T| + 1$. We claim that $S \cap T \in \mathcal{M}(G)$. In this case, $G_{\overline{S}}$ has $r + 1$ connected components, say $H_1, H_2, G_2, \ldots, G_r$. Consider the set

$$Z = \{z \in S \cap T : \text{ adding } z \text{ to } G_{\overline{S}} \text{ it connects only } H_1 \text{ and } H_2\}.$$

We show that $X = (S \cap T) \setminus Z \in \mathcal{M}(G)$. For every $x \in X$, we know that $c_G(S \setminus \{x\}) < c_G(S)$. In particular, adding x to $G_{\overline{S}}$, it joins some connected components and at least one of them is G_i with $i \geq 2$. Hence, $c_G(X \setminus \{x\}) < c_G(X)$. Moreover, $c_G(X) = |S \cap T| - |Z| + 1$, by the unmixedness of J_G. On the other hand, by definition of Z and since $S \in \mathcal{M}(G)$, it follows that $c_G(X) = r = |S| = |S \cap T| + 1$. Thus, $Z = \emptyset$ and $S \cap T = X \in \mathcal{M}(G)$.

Suppose now that $H_1, \ldots, H_i, G_2, \ldots, G_r$ are the connected components of $G_{\overline{S}}$, with $i \geq 3$, and that $t \in H_1$. In the same way let $K_1, \ldots, K_i, G_2, \ldots, G_r$ be the connected components of $G_{\overline{T}}$ and let $s \in K_1$. We show that this case cannot occur.

Following the same argument of the proof of Theorem 5.2(c), we conclude that the connected components of $G_{\overline{S\cup T}}$ are $H_2, \ldots, H_i, K_2, \ldots, K_i, G_2, \ldots, G_r$ and the connected components of $H_1 \cap K_1$, if it is not empty.

(i) If $H_1 \cap K_1 \neq \emptyset$, it follows that $V(H_1) = V(K_2 \cup \cdots \cup K_i \cup (H_1 \cap K_1) \cup \{t\})$ and $V(K_1) = V(H_2 \cup \cdots \cup H_i \cup (H_1 \cap K_1) \cup \{s\})$. In this case, t is a cut vertex of $G_{\overline{S}}$, hence $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$ by Theorem 5.2(c), a contradiction.

Let now $H_1 \cap K_1 = \emptyset$, then $V(H_1) = V(K_2 \cup \cdots \cup K_i \cup \{t\})$ and $V(K_1) = V(H_2 \cup \cdots \cup H_i \cup \{s\})$. Since $i \geq 3$, t is a cut vertex of H_1, hence $\{P_S, P_T\}$ is not an edge of $\mathcal{D}(G)$ by Theorem 5.2(c), a contradiction.

(ii) In this case, since both S and $S \cup T$ are cut sets of G and $i \geq 3$, we have that $i = 3$ and $H_1 \cap K_1 = \emptyset$. Therefore, the connected components of $G_{\overline{S\cup T}}$ are $H_2, H_3, K_2, K_3, G_2, \ldots, G_r$.

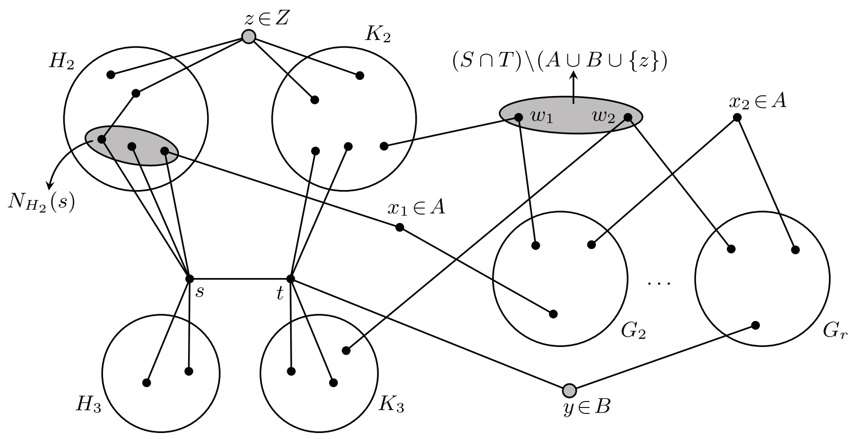


Fig. 15. The set W in grey.

We know that s is adjacent to $x \in H_1$ and that $s \in K_1$. Hence, s is not adjacent to any vertices of K_2 or K_3. Thus, $x = t$, since $V(H_1) = V(K_2 \cup K_3 \cup \{t\})$. This means that $\{s, t\} \in E(G)$. Let

$$Z = \{z \in S \cap T : \text{ adding } z \text{ to } G_{\overline{S \cup T}} \text{ it connects only some } H_i \text{ with some } K_j\}.$$

Notice that, there are no vertices in $S \cap T$ that only connects H_2 to H_3 or K_2 to K_3 in $G_{\overline{S\cup T}}$. In fact, if $z \in S \cap T$ only connects H_2 to H_3 in $G_{\overline{S\cup T}}$, then $c_G(T \setminus \{z\}) = c_G(T)$, a contradiction, since $T \in \mathcal{M}(G)$. The same holds for K_2 and K_3.

As above, since $S \cup T \in \mathcal{M}(G)$, it follows that $(S \cap T) \setminus Z \in \mathcal{M}(G)$ and, by the unmixedness of J_G, $|Z| = 1$, say $Z = \{z\}$. Without loss of generality, we may assume that z connects at least H_2 and K_2.

Since s and t are adjacent in G, one of them is in the same bipartition set of z. Without loss of generality, assume that this vertex is t, thus $N_{H_2}(s) \cap N_{H_2}(z) = \emptyset$. Let

$$A = \{x \in S \cap T : \text{ if } \{x, v\} \in E(G) \text{ for some } v \in G_1, \text{ then } v \in N_{H_2}(s)\}.$$

Notice that, A contains also all vertices of $S \cap T$ that connect only some G_j's in $G_{\overline{S\cap T}}$, with $j \geq 2$. We claim that

$$W = ((S \cap T) \setminus A) \cup N_{H_2}(s) \in \mathcal{M}(G).$$

In Fig. 15 the set W is coloured in grey and the circles represent the connected components of $G_{\overline{S\cup T}}$, where only some vertices are drawn.

Notice that $z \in W$. Let $w \in W$. Adding $w = z$ to $G_{\overline{W}}$, we connect a vertex of $H_2 \setminus N_{H_2}(s)$ with K_2 whereas, adding $w \in N_{H_2}(s)$ to $G_{\overline{W}}$, we connect s to $H_2 \setminus N_{H_2}(s)$. Moreover, if $w \in (S \cap T) \setminus (A \cup \{z\})$, we know that, in $G_{\overline{S\cap T}}$, w connects G_i for some $i \geq 2$ to a vertex v of $G_1 \setminus N_{H_2}(s)$. By construction, in $G_{\overline{W}}$ the connected components containing v and G_i are different and w still connects them. This proves that $W \in \mathcal{M}(G)$.

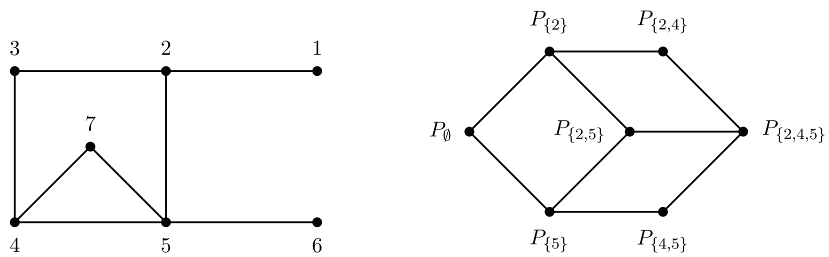
Since J_G is unmixed, we have that $c_G(W) = |W| + 1$ and a connected component of $G_{\overline{W}}$ is the subgraph induced on $H_3 \cup K_2 \cup K_3 \cup \{s, t\}$. Thus, removing t from $G_{\overline{W}}$, this component splits in three components, $H_3 \cup \{s\}, K_2, K_3$. Therefore, if $W \cup \{t\}$ is a cut set of G, we get $c_G(W \cup \{t\}) = c_G(W) + 2 = |W| + 3$, which contradicts the unmixedness of J_G.

Hence, we may assume that $W \cup \{t\} \notin \mathcal{M}(G)$. Thus there exists $y \in N_G(t)$ that joins t with only one connected component of $G_{\overline{W}}$ (i.e., $c_G((W \cup \{t\}) \setminus \{y\}) = c_G(W \cup \{t\})$). In this case, we define

$$B = \{y \in S \cap T : \{y, t\} \in E(G) \text{ and } N_G(y) \setminus \{t\} \text{ is contained in one connected component of } G_{\overline{W}}\},$$

where $|B| \geq 1$, since $W \cup \{t\} \notin \mathcal{M}(G)$. We claim that

$$W' = (W \setminus B) \cup \{t\} \in \mathcal{M}(G).$$



(a) The graph G. (b) The dual graph of G.

Fig. 16. A non-bipartite graph G with J_G Cohen-Macaulay, where the intersection of two cut sets is not always a cut set.

Notice that $z \in W'$. The proof is similar to the case of W. We only notice that, adding t to $G_{\overline{W'}}$, we connect at least K_2, K_3 and the connected component containing s. Moreover, each element in B does not connect different connected components of $G_{\overline{W}}$ and any two elements of B are not adjacent (since they are adjacent to t and G is bipartite). Thus, $|W'| < |W \cup \{t\}|$ and

$$c_G(W') = c_G(W \cup \{t\}) = c_G(W) + 2 = |W| + 3 = |W \cup \{t\}| + 2 > |W'| + 1,$$

which contradicts the unmixedness of J_G. $\square$

Remark 5.6. It could be true that, if G is bipartite and J_G is unmixed, then $S \cap T \in \mathcal{M}(G)$ for every $S, T \in \mathcal{M}(G)$. Both assumptions are needed: in fact, if G is the graph in Fig. 16, one can check with `Macaulay2` [7] that J_G is Cohen–Macaulay and thus $\mathcal{D}(G)$ is connected. Nevertheless, $\{2, 4\}, \{4, 5\} \in \mathcal{M}(G)$ and $\{2, 4\} \cap \{4, 5\} = \{4\} \notin \mathcal{M}(G)$.

On the other hand, if G is the cycle of length 6 with consecutive labelled vertices, then J_G is not unmixed, $\{1, 3\}, \{1, 5\} \in \mathcal{M}(G)$ and $\{1, 3\} \cap \{1, 5\} = \{1\} \notin \mathcal{M}(G)$.

The next result is important for Theorem 6.1, since at the same time provides the equivalence (b) $\Leftrightarrow$ (d) and has important consequences for the proof of (b) $\Rightarrow$ (c).

Theorem 5.7. *Let G be a bipartite graph. If $\mathcal{D}(G)$ is connected, then for every non-empty $S \in \mathcal{M}(G)$, there exists $s \in S$ such that $S \setminus \{s\} \in \mathcal{M}(G)$.*

Proof. By contradiction, let $T \in \mathcal{M}(G)$ such that $T \setminus \{t\} \notin \mathcal{M}(G)$ for every $t \in T$. Notice that $|T| \geq 2$, otherwise $T \setminus \{t\} = \emptyset \in \mathcal{M}(G)$.

Let $W \in \mathcal{M}(G)$, $W \neq T$, such that there exists a path $\mathcal{P} : P_T = P_{S_0}, P_{S_1}, \ldots, P_{S_k}, P_{S_{k+1}} = P_W$ in $\mathcal{D}(G)$. Assume $\mathcal{P}$ is a shortest path from P_T to P_W.

Claim: For $1 \leq i \leq k+1$, $|S_i| > |S_{i-1}|$. In particular, $|W| > |T|$.

We proceed by induction on $k \geq 0$.

Let $k = 0$. First notice that $|W| \geq |T|$, otherwise by Theorem 5.2(a), $W = T \setminus \{t\} \in \mathcal{M}(G)$ for some $t \in T$, a contradiction. If $|W| = |T|$, since $\{P_T, P_W\}$ is an edge of $\mathcal{D}(G)$, by Theorem 5.2(a), we have that $W = (T \setminus \{t\}) \cup \{w\}$, for some $t \in T$ and $w \notin T$. By Lemma 5.5, we have that $W \cap T = T \setminus \{t\} \in \mathcal{M}(G)$, a contradiction. Then $|W| > |T|$.

Let $k \geq 1$. By induction, $|S_i| > |S_{i-1}|$, for every $1 \leq i \leq k$. In particular, by Theorem 5.2(b), $S_i = T \cup \{s_1, \ldots, s_i\}$ for $i = 1, \ldots, k$ and $s_j \notin T$, for $j = 1, \ldots, k$. Set $S = S_k$.

If $|W| < |S|$, then $|W| = |S| - 1$ by Theorem 5.2(a). Hence $W = S \setminus \{s\}$ for some $s \in S$.

First suppose that $s \in T$. Thus, $W = (T \setminus \{s\}) \cup \{s_1, \ldots, s_k\}$. Since $|W| = |S_{k-1}|$, $|W \setminus S_{k-1}| = 1$ and $W \cup S_{k-1} = S \in \mathcal{M}(G)$, by Lemma 5.5(ii) it follows that $S_{k-1} \cap W = (T \setminus \{s\}) \cup \{s_1, \ldots, s_{k-1}\} \in \mathcal{M}(G)$. For every $i = 1, \ldots, k-2$, let $T_i = S_{i+1} \cap \cdots \cap S_{k-1} \cap W$. By induction on $i \leq k-2$, assume that $T_i \in \mathcal{M}(G)$, then $T_{i-1} = S_i \cap T_i = (T \setminus \{s\}) \cup \{s_1, \ldots, s_i\} \in \mathcal{M}(G)$ by Lemma 5.5 (ii), since $|S_i| = |T_i|$, $|T_i \setminus S_i| = 1$ and $S_i \cup T_i = S_{i+1} \in \mathcal{M}(G)$. In particular, $T_0 = S_1 \cap \cdots \cap S_{k-1} \cap W = (T \setminus \{s\}) \cup \{s_1\} \in \mathcal{M}(G)$,

$|T_0| = |T|$, $|T_0 \setminus T| = 1$ and $T_0 \cup T = S_1 \in \mathcal{M}(G)$. Again, by Lemma 5.5 (ii), $T \cap T_0 = T \setminus \{s\} \in \mathcal{M}(G)$, a contradiction.

Now assume that $s \in S \setminus T$, where $s = s_j$ for some $j \in \{1, \ldots, k\}$. Since $|W| = |S_{k-1}|$ and $|W \setminus S_{k-1}| = 1$, by Lemma 5.5(ii), $S_{k-1} \cap W = S_{k-1} \setminus \{s_j\} \in \mathcal{M}(G)$. For every $i = j, \ldots, k-2$, let $T_i = S_{i+1} \cap \cdots \cap S_{k-1} \cap W$. By induction on $i \le k-2$, assume that $T_i \in \mathcal{M}(G)$, then $T_{i-1} = S_i \cap T_i = S_i \setminus \{s_j\} \in \mathcal{M}(G)$ by Lemma 5.5(ii), since $|S_i| = |T_i|$, $|T_i \setminus S_i| = 1$ and $S_i \cup T_i = S_{i+1} \in \mathcal{M}(G)$. In particular, $T_{j-1} = S_j \cap \cdots \cap S_{k-1} \cap W = S_j \setminus \{s_j\} = S_{j-1} \in \mathcal{M}(G)$. Therefore,

$$\mathcal{P}' : P_{S_0} = P_T, P_{S_1}, \ldots, P_{S_{j-1}} = P_{T_{j-1}}, P_{T_j}, \ldots, P_{T_{k-2}}, P_W$$

is a path from P_T to P_W, shorter than $\mathcal{P}$, a contradiction.

If $|W| = |S|$, then $W = (S \setminus \{x\}) \cup \{y\}$ for some x, y. If $x \in T$, then $W = (T \setminus \{x\}) \cup \{s_1, \ldots, s_k, y\}$. By Lemma 5.5(i), $W \cap S = (T \setminus \{x\}) \cup \{s_1, \ldots, s_k\} \in \mathcal{M}(G)$. We may proceed in a similar way to the case $|W| < |S|$, setting $T_i = S_{i+1} \cap \cdots \cap S_k \cap W$ for $i = 1, \ldots, k-1$.

Now assume $x \in S \setminus T$, where $x = s_j$ for some $j \in \{1, \ldots, k\}$. Since $|W| = |S|$, by Lemma 5.5(i), $S \cap W = S \setminus \{x\} \in \mathcal{M}(G)$. Again, we may proceed as in the case $|W| < |S|$, setting $T_i = S_{i+1} \cap \cdots \cap S_k \cap W$ for $i = j, \ldots, k-1$.

In both cases we find a contradiction. In conclusion, we proved that, if there exists a path from P_T to P_W in $\mathcal{D}(G)$, then $|W| > |T| \ge 2$. Thus, there is no path from P_T to $P_\emptyset$ in $\mathcal{D}(G)$, hence $\mathcal{D}(G)$ is disconnected. $\square$

Using the following result, we may reduce to consider bipartite graphs G with exactly two cut vertices and $\mathcal{D}(G)$ connected.

Theorem 5.8. *Let G be a bipartite graph with at least three cut vertices and such that J_G is unmixed.*

(a) *There exist G_1 and G_2 such that $G = G_1 * G_2$ or $G = G_1 \circ G_2$.*
(b) *If $\mathcal{D}(G)$ is connected, then $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ are connected.*

Proof. (a) By Proposition 2.3, G has exactly two leaves. Let v be a cut vertex that is not a neighbour of a leaf and let H_1 and H_2 be the connected components of $G_{\overline{\{v\}}}$. If v is a leaf of both $G_{V(H_1)\cup\{v\}}$ and $G_{V(H_2)\cup\{v\}}$, then $G = G_{V(H_1)\cup\{v\}} * G_{V(H_2)\cup\{v\}}$.

Assume that v is not a leaf of $G_{V(H_1)\cup\{v\}}$ and of $G_{V(H_2)\cup\{v\}}$. Then, given two new vertices w_1 and w_2, for $i = 1, 2$ we set G_i to be the graph $(G_{V(H_i)\cup\{v\}}) \cup \{v, w_i\}$. It follows that $G = G_1 \circ G_2$.

Now assume by contradiction that v is a leaf of $G_{V(H_2)\cup\{v\}}$, but not of $G_{V(H_1)\cup\{v\}}$, and let w be the only neighbour of v in $G_{V(H_2)\cup\{v\}}$. Hence, w is a cut vertex of G and we may assume that it is not a leaf of $G_{V(H_2)}$, otherwise $G = G_{V(H_1)\cup\{v,w\}} * G_{V(H_2)}$.

The graphs $G_{V(H_1)\cup\{v\}}$ and $G_{V(H_2)}$ are bipartite with bipartitions $V_1 \sqcup V_2$ and $W_1 \sqcup W_2$, respectively. Without loss of generality, assume that $v \in V_1$ and $w \in W_1$ and let $S = V_1 \setminus \{\ell : \ell \text{ is a leaf of } G\}$. This is a cut set of G: indeed in $G_{\overline{S}}$ all vertices of V_2 are either isolated or connected with only one leaf of G, hence every element of S connects at least one vertex of V_2 with some other connected component. Therefore, since J_G is unmixed, $G_{\overline{S}}$ has $|S| + 1$ connected components, $G_{V(H_2)}$ is one of them and the vertices of $G_{V(H_1)}$ not in S form the remaining $|S|$ connected components. In the same way, the set $T = W_1 \setminus \{\ell : \ell \text{ is a leaf of } G\} \in \mathcal{M}(G)$ and $G_{\overline{T}}$ consists of the connected component $G_{V(H_1)\cup\{v\}}$ and of $|T|$ connected components on the vertices of $G_{V(H_2)}$ that are not in T. Notice that $S \cup T$ is a cut set of G: in fact, adding either v or w to $G_{\overline{S\cup T}}$, we join at least two connected components, since v is not a leaf of $G_{V(H_1)\cup\{v\}}$ and w is not a leaf of $G_{V(H_2)}$. Then $G_{\overline{S\cup T}}$ has $|S|$ connected components on the vertices of $G_{V(H_1)\cup\{v\}}$ and $|T|$ on the vertices of $G_{V(H_2)}$. Hence, $c_G(S \cup T) = |S| + |T|$, a contradiction.

(b) We prove the statement for G_1, the argument for G_2 is the same. Let P_S be the primary components of J_{G_1}, $S_0 \in \mathcal{M}(G_1)$ and $k = |S_0|$. Thus, $S_0 \in \mathcal{M}(G)$ by Theorems 4.2 and 4.5. Moreover, by Theorem 5.7, there exists $s_1 \in S_0$ such that $S_1 = S_0 \setminus \{s_1\} \in \mathcal{M}(G)$. Applying repeatedly Theorem 5.7, we find a finite sequence of cut sets $S_2 = S_0 \setminus \{s_1, s_2\}, S_3 = S_0 \setminus \{s_1, s_2, s_3\}, \ldots, S_k = S_0 \setminus S_0 = \emptyset \in \mathcal{M}(G)$. Notice that $S_i \in \mathcal{M}(G_1)$ for $i = 1, \ldots, k$ and, by Theorem 5.2, $\{P_{S_i}, P_{S_{i+1}}\}$ is an edge of $\mathcal{M}(G_1)$ for $i = 1, \ldots, k-1$. Hence,

$$\mathcal{P} : P_{S_0}, P_{S_1}, P_{S_2}, \ldots, P_k = P_\emptyset,$$

is a path from P_S to $P_\emptyset$ in $\mathcal{D}(G_1)$. Therefore, $\mathcal{D}(G_1)$ is connected. $\square$

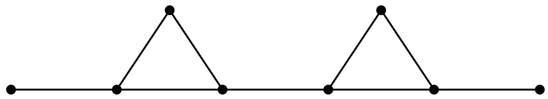


Fig. 17. A non-bipartite graph that cannot be split using $*$ and $\circ$.

Remark 5.9. If the graph G is not bipartite, Theorem 5.8(a) does not hold. For instance, the ideal J_G of the graph in Fig. 17 is unmixed, indeed Cohen–Macaulay, and G has four cut vertices, but it is not possible to split it using the operations $*$ and $\circ$.

The remaining part of the section is useful to prove that a bipartite graph G with exactly two cut vertices and $\mathcal{D}(G)$ connected is of the form F_m.

Corollary 5.10. *Let G be a bipartite graph such that $\mathcal{D}(G)$ is connected. Then every non-empty cut set $S \in \mathcal{M}(G)$ contains a cut vertex.*

Proof. Let $S \in \mathcal{M}(G)$ and $k = |S|$. We may assume $k \geq 2$. By Theorem 5.7, there exists $s \in S$ such that $T = S \setminus \{s\} \in \mathcal{M}(G)$. By induction, T contains a cut vertex and the claim follows. $\square$

Remark 5.11. All assumptions in Theorem 5.8 and Corollary 5.10 are needed. In fact, both claims do not hold if we only assume G bipartite but $\mathcal{D}(G)$ is not connected. For instance, let $G = M_{3,4}$. Then $\{3, 5\}$ is a cut set that does not contain any cut vertex (see Example 2.2).

On the other hand, both results do not hold if $\mathcal{D}(G)$ is connected but G is not bipartite. For example, if G is the graph in Fig. 12(a), then $\{3, 4\} \in \mathcal{M}(G)$, but 3 and 4 are not cut vertices of G.

Corollary 5.12. *Let G be a bipartite graph with bipartition $V_1 \sqcup V_2$ and with exactly two cut vertices v_1 and v_2. If $\mathcal{D}(G)$ is connected, then $\{v_1, v_2\} \in E(G)$. In particular $|V_1| = |V_2|$.*

Proof. Let f_i be the leaf adjacent to v_i for $i = 1, 2$. Assume that $\{v_1, v_2\} \notin E(G)$. Then $S_i = N_G(v_i) \setminus \{f_i\}$ is a cut set of G for $i = 1, 2$. Moreover, S_1 and S_2 do not contain cut vertices. By Corollary 5.10 it follows that $\mathcal{D}(G)$ is disconnected, a contradiction. The last part of the claim follows from Remark 2.4. $\square$

Lemma 5.13. *Let G be a bipartite graph with bipartition $V_1 \sqcup V_2$, $|V_1| = |V_2|$ and with exactly two cut vertices. If $\mathcal{D}(G)$ is connected, then there exists a vertex of G with degree 2.*

Proof. Suppose by contradiction that all the vertices of G, except the two leaves, have degree greater than 2. Let f be the only leaf of G in V_1 and consider $T = V_1 \setminus \{f\}$. Clearly $G_{\overline{T}}$ is the disjoint union of $|V_2| - 1$ isolated vertices and the edge $\{v_2, f\}$, where $v_2 \in V_2$ is a cut vertex. Therefore, T is a cut set and we claim that it is an isolated vertex in $\mathcal{D}(G)$.

Notice that T is not contained in any other cut set. Moreover, suppose that S is a cut set of G such that $S \subset T$ and $T \setminus S = \{v\}$. Since $S \subset V_1$, it follows that $\deg_{G_{\overline{S}}}(v) > 2$. Then $c_G(S) = c_G(T \setminus \{v\}) \leq c_G(T) - 2 = |V_1| - 2$, since $G_{\overline{T}}$ consists of isolated vertices and one edge. This contradicts the unmixedness of J_G.

Finally, let T' be a cut set such that $T \setminus T' = \{v\}$ and $T' \setminus T = \{v'\}$. If we set $S = T \setminus \{v\} = T' \setminus \{v'\}$, it follows that v' has to be a cut vertex of $G_{\overline{S}}$. As consequence, $v' = v_2$ is the cut vertex in V_2, and $\{v, v'\} \in E(G)$. On the other hand, as before, $G_{\overline{S}}$ has at most $|V_2| - 2$ connected components, then $c_G(T') = c_G(S) + 1 \leq |V_2| - 1$. This contradicts the unmixedness of J_G, because $|T'| = |V_2| - 1$. Therefore, Theorem 5.2 implies that T is an isolated vertex in $\mathcal{D}(G)$ against our assumption. $\square$

Proposition 5.14. *Let H be a bipartite graph with bipartition $V_1 \sqcup V_2$ and $|V_1| = |V_2|$. Let v and f be two new vertices and let G be the bipartite graph with $V(G) = V(H) \cup \{v, f\}$ and $E(G) = E(H) \cup \{\{v, x\} : x \in V_1 \cup \{f\}\}$. If $\mathcal{D}(G)$ is connected, then $\mathcal{D}(H)$ is connected.*

Proof. Let f_2 be the leaf of G in V_2 and w its only neighbour, which is a cut vertex. Lemma 2.6(b) and 5.13 imply that there is a vertex with degree 2 in G. Thus, by Proposition 2.8,

$$\mathcal{M}(G) = \{\emptyset, V_1\} \cup \{S \cup \{v\} : S \in \mathcal{M}(H)\} \cup \{T \subset V_1 : T \in \mathcal{M}(H)\}.$$

Let us denote by P_S the primary components of J_G and by Q_S those of J_H. Using Theorem 5.2, we can give a complete description of the edges of $\mathcal{D}(G)$:

(i) $\{P_\emptyset, P_T\} \in E(\mathcal{D}(G))$ if and only if either $T = \{v\}$ or $T = \{w\}$,
(ii) $\{P_{V_1}, P_T\} \in E(\mathcal{D}(G))$ if and only if either $T = V_1 \setminus \{f_1\}$ or $T = (V_1 \setminus \{f_1\}) \cup \{v\}$;
(iii) if $S_1, S_2 \in \mathcal{M}(H)$, then $\{P_{S_1\cup\{v\}}, P_{S_2\cup\{v\}}\} \in E(\mathcal{D}(G))$ if and only if $\{Q_{S_1}, Q_{S_2}\} \in E(\mathcal{D}(H))$;
(iv) if $T_1, T_2 \in \mathcal{M}(G)$ are strictly contained in V_1, then we have $\{P_{T_1}, P_{T_2}\} \in E(\mathcal{D}(G))$ if and only if $\{Q_{T_1}, Q_{T_2}\} \in E(\mathcal{D}(H))$;
(v) if $S, T \in \mathcal{M}(H)$ and $T \subsetneq V_1$, then $\{P_{S\cup\{v\}}, P_T\} \in E(\mathcal{D}(G))$ if and only if $S = T$.

If $S \in \mathcal{M}(H)$, it is enough to prove that Q_S is in the same connected component as $Q_\emptyset$ in $\mathcal{D}(H)$. By (iii), this is equivalent to prove that in $\mathcal{D}(G)$ there exists a path $P_{\{v\}} = P_{U_1}, P_{U_2}, \ldots, P_{U_r} = P_{S\cup\{v\}}$ such that U_i contains v for all i. Since $\mathcal{D}(G)$ is connected, we know that there exists a path $\mathcal{P}$ from $P_{\{v\}}$ to $P_{S\cup\{v\}}$. We first note that, if $\mathcal{P}$ contains $P_\emptyset$ or P_{V_1}, we may avoid them: in fact, by (i) and (ii), they only have two neighbours; for P_{V_1} they are adjacent by (v), whereas we may replace $P_\emptyset$ with $P_{\{v,w\}}$ by (iii) and (v). Let i be the smallest index for which U_i does not contain v. This means that $U_i \subsetneq V_1$ and $U_{i-1} = U_i \cup \{v\}$ by (v). Moreover, U_{i+1} does not contain v, otherwise it would be equal to U_{i-1} (again by (v)). Therefore, $U_{i+1} \subsetneq V_1$ and $\{Q_{U_i}, Q_{U_{i+1}}\} \in E(\mathcal{D}(H))$ by (iv). Thus, replacing U_i with $U_{i+1} \cup \{v\}$ in $\mathcal{P}$, we get a new path from $P_{\{v\}}$ to $P_{S\cup\{v\}}$, by (iii) and (iv). Repeating the same argument finitely many times, we eventually find a path from $P_{\{v\}}$ to $P_{S\cup\{v\}}$ that involves only cut sets containing v. Thus $\mathcal{D}(H)$ is connected by (iii). □

6. The main theorem

In this section we prove the main theorem of the paper and give some applications.

Theorem 6.1. *Let G be a connected bipartite graph. The following properties are equivalent:*

(a) *J_G is Cohen–Macaulay;*
(b) *the dual graph $\mathcal{D}(G)$ is connected;*
(c) *$G = A_1 * A_2 * \cdots * A_k$, where $A_i = F_m$ or $A_i = F_{m_1} \circ \cdots \circ F_{m_r}$, for some $m \geq 1$ and $m_j \geq 3$;*
(d) *J_G is unmixed and for every non-empty $S \in \mathcal{M}(G)$, there exists $s \in S$ such that $S \setminus \{s\} \in \mathcal{M}(G)$.*

Proof. The implication (a) ⇒ (b) follows by Hartshorne's Connectedness Theorem [8, Proposition 1.1, Corollary 2.4, Remark 2.4.1].

(b) ⇒ (c): We may assume that G has more than two vertices. Recall that, since $\mathcal{D}(G)$ is connected, then J_G is unmixed. By Proposition 2.3, G has exactly two leaves, hence at least two cut vertices v_1, v_2, which are their neighbours. We proceed by induction on the number $h \geq 2$ of cut vertices of G.

Let $h = 2$. We claim that $G = F_m$, for some $m \geq 2$. Let $V(G) = V_1 \sqcup V_2$ be the bipartition of the vertex set of G. By Corollary 5.12, we have that $\{v_1, v_2\} \in E(G)$ and $|V_1| = |V_2|$, with $v_i \in V_i$ for $i = 1, 2$. We proceed by induction on $m = |V_1| = |V_2|$. If $m = 2$, then $G = F_2$. Let $m > 2$ and consider the graph H obtained removing v_2 and the leaf adjacent to it. Lemma 2.6(b) implies that v has degree m and H has exactly two cut vertices, whereas by Proposition 5.14, $\mathcal{D}(H)$ is connected. Hence, by induction, it follows that $H = F_{m-1}$ and $G = F_m$ by construction.

Assume now $h > 2$. Let v be a cut vertex of G such that $v \neq v_1, v_2$. By Theorem 5.8, there exist two graphs G_1 and G_2 such that $G = G_1 * G_2$ or $G = G_1 \circ G_2$ and $\mathcal{D}(G_1)$, $\mathcal{D}(G_2)$ are connected. If $G = G_1 * G_2$, by induction they are of the form $A_1 * A_2 * \cdots * A_k$, for some $k \geq 1$, where $A_i = F_m$, with $m \geq 1$, or $A_i = F_{m_1} \circ \cdots \circ F_{m_r}$, with $m_j \geq 3$ for $j = 1, \ldots, r$.

On the other hand, if $G = G_1 \circ G_2$, it follows that $G_1 = A_1 * A_2 * \cdots * A_s$ and $G_2 = B_1 * B_2 * \cdots * B_t$, where each A_i and B_i are equal to F_m, for some $m \geq 1$, or to $F_{m_1} \circ \cdots \circ F_{m_r}$, with $m_j \geq 3$ for $j = 1, \ldots, r$. By Theorem 4.5, it follows that if $A_s = F_m$ or $B_1 = F_m$, then $m \geq 3$.

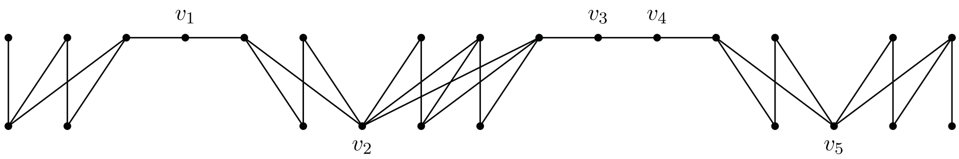


Fig. 18. The graph $G = F_3 * F_3 \circ F_4 * F_1 * F_3 \circ F_3$.

(c) $\Rightarrow$ (a): Let G be a graph as in (c). We proceed by induction on $k \geq 1$.

If $k = 1$, then $G = F_m$ for some $m \geq 1$, or $G = F_{m_1} \circ \cdots \circ F_{m_r}$, with $m_j \geq 3$ for $j = 1, \ldots, r$. In the first case the claim follows from Proposition 3.3, in the latter from Theorem 4.9.

Let $k > 1$ and consider the graphs $G_1 = A_1 * A_2 * \cdots * A_{k-1}$ and $G_2 = A_k$. By induction, J_{G_1} is Cohen–Macaulay and, by the previous argument, also J_{G_2} is Cohen–Macaulay. Then, the claim follows from Theorem 4.2.

(b) $\Leftrightarrow$ (d): The first implication follows from Theorem 5.7. Conversely, let $S \in \mathcal{M}(G)$, $S \neq \emptyset$, and P_S be the primary components of J_G. It suffices to show that there exists a path from $P_\emptyset$ to P_S. If $|S| = 1$, the claim follows by Theorem 5.2(b). If $|S| > 1$, by assumption, there exists $s \in S$ such that $S \setminus \{s\} \in \mathcal{M}(G)$ and, by induction, there exists a path from $P_\emptyset$ to $P_{S\setminus\{s\}}$. Thus, Theorem 5.2(b) implies that $\{P_{S\setminus\{s\}}, P_S\}$ is an edge of $\mathcal{D}(G)$. $\square$

Since A/I Cohen–Macaulay implies that A/I is S_2 and in this case $\mathcal{D}(I)$ is connected [8, Corollary 2.4], for a bipartite graph G, R/J_G is Cohen–Macaulay if and only if R/J_G is S_2 by Theorem 6.1.

Theorem 6.1 can be restated in the following way. Let G be a connected bipartite graph. If it has exactly two cut vertices, then J_G is Cohen–Macaulay if and only if $G = F_m$ for some $m \geq 1$. If it has more than two cut vertices, then J_G is Cohen–Macaulay if and only if there exist two bipartite graphs G_1, G_2 such that J_{G_1}, J_{G_2} are Cohen–Macaulay and $G = G_1 * G_2$ or $G = G_1 \circ G_2$.

Fig. 18 shows a graph G obtained by a sequence of operations $*$ and $\circ$ on a finite set of graphs of the form F_m. More precisely, $G = F_3 * F_3 \circ F_4 * F_1 * F_3 \circ F_3$ and v_i denotes the only common vertex between two consecutive blocks. By Theorem 6.1, J_G is Cohen–Macaulay.

It is interesting to notice that Theorem 6.1 gives, at the same time, a classification of other known classes of Cohen–Macaulay binomial ideals associated with graphs. We recall that, given a graph G, the *Lovász–Saks–Schrijver ideal* L_G (see [11]), the *permanental edge ideal* Π_G (see [11, Section 3]) and the *parity binomial edge ideal* $\mathcal{I}_G$ (see [12]) are defined respectively as

$$L_G = (x_i x_j + y_i y_j : \{i, j\} \in E(G)),$$
$$\Pi_G = (x_i y_j + x_j y_i : \{i, j\} \in E(G)),$$
$$\mathcal{I}_G = (x_i x_j - y_i y_j : \{i, j\} \in E(G)).$$

Corollary 6.2. *Let G be a bipartite connected graph. Then Theorem 6.1 holds for L_G, Π_G and $\mathcal{I}_G$.*

Proof. Let G be a bipartite graph with bipartition $V(G) = V_1 \sqcup V_2$. Then the binomial edge ideal J_G can be identified respectively with L_G, Π_G and $\mathcal{I}_G$ by means of the isomorphisms induced by:

$$(x_i, y_i) \overset{L_G}{\longmapsto} \begin{cases} (x_i, y_i) & \text{if } i \in V_1 \\ (y_i, -x_i) & \text{if } i \in V_2, \end{cases} \qquad (x_i, y_i) \overset{\Pi_G}{\longmapsto} \begin{cases} (x_i, y_i) & \text{if } i \in V_1 \\ (-x_i, y_i) & \text{if } i \in V_2, \end{cases}$$
$$(x_i, y_i) \overset{\mathcal{I}_G}{\longmapsto} \begin{cases} (x_i, y_i) & \text{if } i \in V_1 \\ (y_i, x_i) & \text{if } i \in V_2. \end{cases}$$

Notice that the first transformation is more general than the one described in [11, Remark 1.5].

Thus, for bipartite graphs, these four classes of binomial ideals are essentially the same and Theorem 6.1 classifies which of these ideals are Cohen–Macaulay. $\square$

As a final application, using condition (d) in Theorem 6.1 we show that [2, Conjecture 1.6] holds for Cohen–Macaulay binomial edge ideals of bipartite graphs. Recall that the *diameter*, diam(G), of a graph

G is the maximal distance between two of its vertices. A homogeneous ideal I in $A = K[x_1, \dots, x_n]$ is called *Hirsch* if $\operatorname{diam}(\mathcal{D}(I)) \leq \operatorname{ht}(I)$. In [2], the authors conjecture that every Cohen–Macaulay homogeneous ideal generated in degree two is Hirsch.

Corollary 6.3. *Let G be a bipartite connected graph such that J_G is Cohen–Macaulay. Then J_G is Hirsch.*

Proof. Let $S \in \mathcal{M}(G)$ be a cut set of G and let $n = |V(G)|$. We may assume $n \geq 3$, otherwise $\mathcal{D}(J_G)$ is a single vertex. Since J_G is unmixed, $G_{\overline{S}}$ has exactly $|S| + 1$ connected components and we claim that $|S| \leq \lceil \frac{n}{2} \rceil - 1$. In fact, if $|S| \geq \lceil \frac{n}{2} \rceil$, we would have

$$|V(G)| \geq |S| + |S| + 1 \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1 \geq \frac{n}{2} + \frac{n}{2} + 1 = n + 1,$$

a contradiction. Consider now another cut set T of G. By Theorem 6.1(d), it follows that there is a path connecting P_S and P_T, containing $P_\emptyset$ and with length $|S| + |T| \leq 2(\lceil \frac{n}{2} \rceil - 1) \leq n - 1$. Thus, $\operatorname{diam}(\mathcal{D}(J_G)) \leq n - 1 = \operatorname{ht}(J_G)$. $\square$

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