Group actions on semimatroids

Emanuele Delucchi[a,*], Sonja Riedel[a,b]

[a] *Departement de Mathématiques, Université de Fribourg, chemin du musée 23, 1700 Fribourg, Switzerland*
[b] *Institute for Algebra, Geometry, Topology and Their Applications, Fachbereich Mathematik und Informatik, Universität Bremen, Bibliothekstraße 1, 28359 Bremen, Germany*

We initiate the study of group actions on (possibly infinite) semimatroids and geometric semilattices. To every such action is naturally associated an orbit-counting function, a two-variable "Tutte" polynomial and a poset which, in the representable case, coincides with the poset of connected components of intersections of the associated toric arrangement. In this structural framework we recover and strongly generalize many enumerative results about arithmetic matroids, arithmetic Tutte polynomials and toric arrangements by finding new combinatorial interpretations beyond the representable case. In particular, we thus find a class of natural examples of nonrepresentable arithmetic matroids. Moreover, we discuss actions that give rise to matroids over $\mathbb{Z}$ with natural combinatorial interpretations. As a stepping stone toward our results we also prove an extension of the cryptomorphism between semimatroids and geometric semilattices to the infinite case.

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* Corresponding author.
E-mail addresses: emanuele.delucchi@unifr.ch (E. Delucchi), sriedel@math.uni-bremen.de (S. Riedel).

0. Introduction

This paper is about group actions on combinatorial structures. There is an extensive literature on enumerative aspects of group actions, from Pólya's classical work [32] to, e.g., recent results on polynomial invariants of actions on graphs [7]. The chapter on group actions in Stanley's book [35] offers a survey of some of the results in this vein, together with a sizable literature list. Moreover, group actions on (finite) partially ordered sets have been studied from the point of view of representation theory [33], of homotopy theory [26], and of the poset's topology [3,36].

Here we consider group actions on (possibly infinite) semimatroids and geometric semilattices from a structural perspective. We develop an abstract setting that fits different contexts arising in the literature, allowing us to unify and generalize many recent results.

Motivation. Our original motivation came from the desire to better understand the different new combinatorial structures that have been introduced in the wake of recent work of De Concini–Procesi–Vergne [14,15] on toric arrangements and partition functions, and have soon gained independent research interest. Our motivating goals are

- to organize these different structures into a unifying theoretical framework and to develop new combinatorial interpretations also in the nonrepresentable case;
- to understand the geometric side of this theory, in particular in terms of a suitable abstract class of posets (an "arithmetic" analogue of geometric lattices).

To be more precise, let us consider a list $a_1, \dots, a_n \in \mathbb{Z}^d$ of integer vectors. Such a list gives rise to an *arithmetic matroid* (d'Adderio–Moci [9] and Brändén–Moci [5]) with an associated *arithmetic Tutte polynomial* [29], and a *matroid over the ring* $\mathbb{Z}$ (Fink–Moci [20]). Moreover, by interpreting the a_i as characters of the torus $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{C}^*) \simeq (\mathbb{C}^*)^d$ we obtain a *toric arrangement* in $(S^1)^d \subseteq (\mathbb{C}^*)^d$ defined by the kernels of the characters, with an associated *poset of connected components* of intersections of these hypersurfaces. In this case, the arithmetic Tutte polynomial computes the characteristic polynomial of the arrangement's poset and the Poincaré polynomial of the arrangement's complement, as well as the Ehrhart polynomial of the zonotope spanned by the a_i and the dimension of the associated Dahmen–Micchelli space [29]. Other contexts of application of arithmetic matroids include the theory of spanning trees of simplicial complexes [17] and interpretations in graph theory [10]. After a first version of this paper was submitted, we learned about current work of Aguiar and Chan [1] focusing on toric arrangements defined by graphs. Although they stay in the "representable" realm, their interesting work refines some statistics related to arithmetic matroids and fits well into our setup.

On an abstract level, arithmetic matroids offer a theory supporting some notable properties of the arithmetic Tutte polynomial, while matroids over rings are a very general and strongly algebraic theory with different applications for suitable choices of the "base ring".

However, outside the case of lists of integer vectors in abelian groups, arithmetic Tutte polynomials and arithmetic matroids have few combinatorial interpretations. For instance, the poset of connected components of intersections of a toric arrangement – which provides combinatorial interpretations for many an evaluation of arithmetic Tutte polynomials – has no counterpart in the case of nonrepresentable arithmetic matroids. Moreover, from a structural point of view it is striking (and unusual for matroidal objects) that there is no known cryptomorphism for arithmetic matroids, while for matroids over a ring a single one was recently presented [19].

In previous research – e.g. by Ehrenborg, Readdy and Slone [18] and Lawrence [25] on enumeration on the torus, and by Kamiya, Takemura and Terao [22,23] on characteristic quasipolynomials of affine arrangements – posets and "multiplicities" related to (but not satisfying the strict requirements of those arising with) arithmetic matroids were brought to light, calling for a systematic study of the abstract properties of "periodic" combinatorial structures.

Further motivation comes from recent progress in the study of complements of arrangements on products of elliptic curves [4] which, combinatorially and topologically, can be seen as quotients of "doubly periodic" subspace arrangements. In this context our work is an attempt at a unified combinatorial treatment of linear, toric and elliptic arrangements.

Results. We initiate the study of actions of groups by automorphisms on semimatroids (for short "G-semimatroids"). Helpful intuition comes, once again, from the case of integer vectors, where the associated toric arrangement is covered naturally by a periodic affine hyperplane arrangement: here semimatroids, introduced by Ardila [2] (independently by Kawahara [24]), enter the picture as abstract combinatorial descriptions of finite arrangements of affine hyperplanes. In particular, we obtain the following results (see also Table 1 for a quick overview).

- An equivalence (a.k.a. *cryptomorphism*) between G-semimatroids, which are defined in terms of certain set systems, and group actions on geometric semilattices (in the sense of Wachs and Walker [37]), based on a theorem extending Ardila's equivalence between semimatroids and geometric semilattices to the infinite case (Theorem E).
- Under appropriate conditions every G-semimatroid gives rise to an underlying finite (poly)matroid (Theorem A). Additional conditions can be imposed so that orbit enumeration determines an arithmetic matroid (often nonrepresentable). In fact, we see that the defining properties of arithmetic matroids arise in a natural "hierarchy" according to progressively stricter requirements on the action (Theorem B and Theorem C).
- In particular, we obtain the first natural class of examples of nonrepresentable arithmetic matroids.
- To every G-semimatroid is naturally associated a poset $\mathcal{P}$ obtained as a quotient of the geometric semilattice of the semimatroid acted upon. In particular, this gives a

natural abstract generalization of the poset of connected components of intersections of a toric arrangement.
- To every G-semimatroid is associated a two-variable polynomial which evaluates as the characteristic polynomial of $\mathcal{P}$ (Theorem F) and, under mild conditions on the action, satisfies a natural deletion–contraction recursion (Theorem G) and a generalization of Crapo's basis-activity decomposition (Theorem H). In particular, for every arithmetic matroid arising from group actions we have a new combinatorial interpretation of the coefficients of the arithmetic Tutte polynomial in terms of enumeration on $\mathcal{P}$ subsuming Brändén and Moci's interpretation [5, Theorem 6.3] in the representable case.
- Any G-semimatroid satisfying appropriate algebraic conditions gives rise to a matroid over $\mathbb{Z}$, and we discuss conditions under which the single modules have combinatorial interpretations (Theorem D).

Structure of the paper. First, in Section 1 we recall the definitions of semimatroids, arithmetic matroids and matroids over a ring. Then we devote Section 2 to explaining our guiding example, namely the "representable" case of a $\mathbb{Z}^d$-action by translations on an affine hyperplane arrangement. Then, Section 3 gives a panoramic run-through of the main definitions and results, in order to establish the "Leitfaden" of our work. Before delving into the technicalities of the proofs, in Section 4 we will discuss some specific examples (mostly arising from actions on arrangements of pseudolines) in order to illustrate and distinguish the different concepts we introduce. Then we will move towards proving the announced results. First, in Section 5 we prove the cryptomorphism between finitary semimatroids and finitary geometric semilattices. Section 6 is devoted to the construction of the underlying (poly)matroid and semimatroid of an action. Then, in Section 7 we will focus on *translative* actions (Definition 3.2), for which the orbit-counting function gives rise to a *pseudo-arithmetic semimatroid* over the action's underlying semimatroid. Subsequently, in Section 8, we will further (but mildly) restrict to *almost-arithmetic* actions, and recover "most of" the properties required in the definition of arithmetic matroids. In Section 9 we will then discuss the much more restrictive condition on the action which ensures that our orbit-count function fully satisfies the definition of an arithmetic matroid, and we will discuss combinatorial interpretations of some associated matroids over $\mathbb{Z}$. The closing Section 10 is devoted to the study of certain "Tutte" polynomials associated to G-semimatroids.

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1. The main characters

We start by introducing some basic definitions and terminology, sometimes modified with respect to the standard literature in order to better fit our setting. The reader may, in a first reading, skip the technical details; however, a quick look at the main examples we offer in this section might be illuminating and help the intuition later on.

1.1. *Finitary semimatroids*

We start by recalling the definition of a semimatroid, which we state without finiteness assumptions on the ground set. This relaxation substantially impacts the theory developed by Ardila [2], much of which rests on the fact that any finite semimatroid can be viewed as a certain substructure of a (finite) 'ambient' matroid. Here we list the definition and some immediate observations, while Section 5 will be devoted to proving the cryptomorphism with geometric semilattices. We note that equivalent structures were also introduced by Kawahara [24] under the name quasi-matroids, with a view on studying the associated Orlik–Solomon algebra.

The motivation for introducing these structures was, in both [2] and [24], the combinatorial study of affine hyperplane arrangements. In particular, keeping an eye on Example 1.6 below will help make the following definition plausible. For a pictorial representation of an instance of this definition that does not arise from hyperplane arrangements we point to Example 1.7, which we will also keep as a running example throughout the paper.

Definition 1.1 *(Compare [2, Definition 2.1]).* A *finitary semimatroid* is a triple $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk}_{\mathcal{C}})$ consisting of a (possibly infinite) set S, a non-empty finite dimensional simplicial complex $\mathcal{C}$ on S and a bounded function $\mathrm{rk}_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$ satisfying the following conditions.

(R1) If $X \in \mathcal{C}$, then $0 \leq \mathrm{rk}_{\mathcal{C}}(X) \leq |X|$.
(R2) If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $\mathrm{rk}_{\mathcal{C}}(X) \leq \mathrm{rk}_{\mathcal{C}}(Y)$.
(R3) If $X, Y \in \mathcal{C}$ and $X \cup Y \in \mathcal{C}$, then $\mathrm{rk}_{\mathcal{C}}(X) + \mathrm{rk}_{\mathcal{C}}(Y) \geq \mathrm{rk}_{\mathcal{C}}(X \cup Y) + \mathrm{rk}_{\mathcal{C}}(X \cap Y)$.
(CR1) If $X, Y \in \mathcal{C}$ and $\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$.
(CR2) If $X, Y \in \mathcal{C}$ and $\mathrm{rk}_{\mathcal{C}}(X) < \mathrm{rk}_{\mathcal{C}}(Y)$, then $X \cup y \in \mathcal{C}$ for some $y \in Y - X$.

If only (R1), (R2), (R3) are known to hold, we call $\mathcal{S}$ a *locally ranked triple*.

A *finite* semimatroid is a finitary semimatroid with a finite ground set. Finiteness of locally ranked triples is defined accordingly.

Here, and in the following, we often write rk instead of $\mathrm{rk}_{\mathcal{C}}$ and omit braces when representing singleton sets, thus writing $\mathrm{rk}(x)$ for $\mathrm{rk}(\{x\})$ and $X \cup x$ for $X \cup \{x\}$, when no confusion can occur.

We call S the *ground set*, $\mathcal{C}$ the *collection of central sets* and rk the *rank function* of the finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$, respectively. The *rank* of the semimatroid is the maximum value of rk on $\mathcal{C}$ and we will denote it by $\mathrm{rk}(\mathcal{S})$. A set $X \in \mathcal{C}$ is called *independent* if $|X| = \mathrm{rk}(X)$. A *basis* of $\mathcal{S}$ is an inclusion-maximal independent set.

Remark 1.2. We adopt the convention that every $x \in S$ is a vertex of $\mathcal{C}$, i.e., $\{x\} \in \mathcal{C}$ for all $x \in S$. Although this is not required in [2], it will not affect our considerations while simplifying the formalism. See also Remark 1.12.

Definition 1.3. A finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ is *simple* if $\mathrm{rk}(x) = 1$ for all $x \in S$ and $\mathrm{rk}(x, y) = 2$ for all $\{x, y\} \in \mathcal{C}$ with $x \neq y$.

A *loop* of a locally ranked triple $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ is any $s \in S$ with $\mathrm{rk}(s) = 0$. Two elements $s, t \in S$ that are not loops are called parallel if $\{s, t\} \in \underline{\mathcal{C}}$ and $\mathrm{rk}(\{s, t\}) = 1$. The triple $\mathcal{S}$ is called *simple* if it has no loops and no parallel elements. An *isthmus* of $\mathcal{S}$ is any $s \in S$ such that, for every $X \in \mathcal{C}$, $X \cup s \in \mathcal{C}$ and $\mathrm{rk}(X \cup s) = \mathrm{rk}(X) + 1$.

Remark 1.4. A *matroid* is, by definition, a finite semimatroid where every subset is central. Equivalently (and more classically), a matroid is given by a finite ground set S and a rank function $\mathrm{rk} : 2^S \to \mathbb{N}$ satisfying (R1), (R2), (R3). The *dual* to a matroid (S, rk) is (S, rk^*), where $\mathrm{rk}^*(X) := \mathrm{rk}(S \setminus X) - |X| - \mathrm{rk}(S)$ for all $X \subseteq S$.

Remark 1.5. A *polymatroid* is given by a finite ground set S and a rank function $\mathrm{rk} : 2^S \to \mathbb{N}$ satisfying (R2), (R3) and $\mathrm{rk}(\emptyset) = 0$. Polymatroids will appear furtively but naturally in our considerations; we refer e.g. to [38, §18.2] for background on these structures.

Example 1.6 *(The representable case, see Proposition 2.2 in [2]).* Given a positive integer d and a field $\mathbb{K}$, an *affine hyperplane* is an affine subspace of dimension $d-1$ in the vector space $\mathbb{K}^d$. An *arrangement of hyperplanes* in $\mathbb{K}^d$ is a collection $\mathscr{A}$ of affine hyperplanes in $\mathbb{K}^d$. The arrangement is called *locally finite* if every point in $\mathbb{K}^d$ has a neighborhood that intersects only finitely many hyperplanes of $\mathscr{A}$. A subset $X \subseteq \mathscr{A}$ is *central* if $\cap X \neq \emptyset$. Let $\mathcal{C}_{\mathscr{A}}$ denote the set of central subsets of $\mathscr{A}$ and define the rank function $\mathrm{rk}_{\mathscr{A}} : \mathcal{C}_{\mathscr{A}} \to \mathbb{N}$ as $\mathrm{rk}_{\mathscr{A}}(X) := d - \dim \cap X$.

Then, the triple $(\mathscr{A}, \mathcal{C}_{\mathscr{A}}, \mathrm{rk}_{\mathscr{A}})$ is a finitary semimatroid. It is simple if all elements of $\mathscr{A}$ are distinct, and it is a matroid if all elements of $\mathscr{A}$ are linear subspaces (i.e., they contain the origin of $\mathbb{K}^d$). $\triangle$

Example 1.7 *(Pseudoline arrangements).* There are cases of nonrepresentable semimatroids in which we can still take advantage of a pictorial illustration — one such instance is given by *arrangements of pseudolines* in the sense of Grünbaum [21], i.e., sets of homeomorphic images of $\mathbb{R}$ in $\mathbb{R}^2$ ("pseudolines") such that

(1) every point of $\mathbb{R}^2$ has a neighborhood intersecting only finitely many pseudolines,

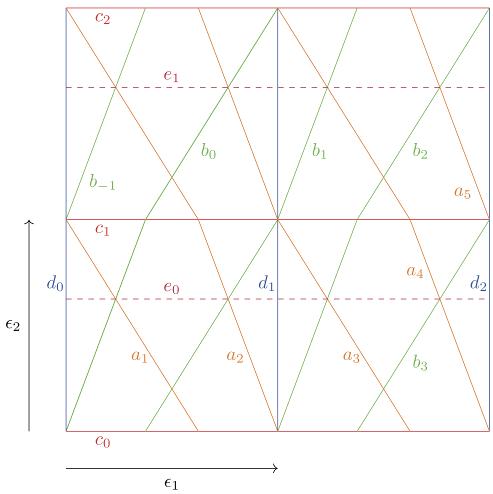

Fig. 1. A non-stretchable pseudoline arrangement (it should be thought of as repeating and tiling the plane).

(2) any two pseudolines intersect at most in one point (and if they intersect, they do so transversally).

Fig. 1 shows such an arrangement of pseudolines. The definitions of Example 1.6 can be carried over to this context. The triple $(S, \mathcal{C}, \mathrm{rk})$ associated to this pseudoline arrangement is given by

$$S = \{a_i \mid i \in \mathbb{Z}\} \cup \{b_i \mid i \in \mathbb{Z}\} \cup \{c_i \mid i \in \mathbb{Z}\} \cup \{d_i \mid i \in \mathbb{Z}\} \cup \{e_i \mid i \in \mathbb{Z}\},$$

$$\begin{aligned}\mathcal{C} = &\{\emptyset\} \cup \{a_i\}_i \cup \{b_i\}_i \cup \{c_i\}_i \cup \{d_i\}_i \cup \{e_i\}_i \cup \{a_i, b_j\}_{i,j} \cup \{a_i, c_j\}_{i,j} \\ &\cup \{a_i, d_j\}_{i,j} \cup \{a_i, e_j\}_{i,j} \cup \{b_i, c_j\}_{i,j} \cup \{b_i, d_j\}_{i,j} \cup \{b_i, e_j\}_{i,j} \cup \{c_i, d_j\}_{i,j} \\ &\cup \{d_i, e_j\}_{i,j} \cup \{a_{2i+k}, b_{2i-k}, c_k\}_{i,k} \cup \{a_{2i+k}, b_{2i-k}, d_k\}_{i,k} \cup \{a_k, b_{k-2i-1}, e_i\}_{i,k} \\ &\cup \{a_{2i+k}, c_k, d_i\}_{i,k} \cup \{b_{2i-k}, c_k, d_i\}_{i,k} \cup \{a_{2i+k}, b_{2i-k}, c_k, d_i\}_{i,k},\end{aligned}$$

$$\mathrm{rk}(X) = \mathrm{codim}(\cap X) \text{ for all } X \in \mathcal{C}$$

and one easily checks that this defines a finitary semimatroid.

For readability's sake, here and in all following examples we omit to specify that all indices run over $\mathbb{Z}$ and that the union is taken over sets of sets, and we use the shorthand notation $\{a_i, b_j\}_{i,j}$ for $\{\{a_i, b_j\} \mid i, j \in \mathbb{Z}\}$.

Notice that this triple cannot be obtained from an arrangement of straight lines: such an arrangement is called *non-stretchable*. △

We now state some basic facts and definitions about semimatroids for later reference. Except where otherwise specified, the proofs parallel those given in [2, Section 2].

Definition 1.8. Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a finitary semimatroid and let $X \in \mathcal{C}$. The *closure of X in $\mathcal{C}$* is

$$\mathrm{cl}(X) := \{x \in S \mid X \cup x \in \mathcal{C}, \mathrm{rk}(X \cup x) = \mathrm{rk}(X)\}.$$

A *flat* of a finitary semimatroid $\mathcal{S}$ is a set $X \in \mathcal{C}$ such that $\mathrm{cl}(X) = X$. The set of flats of $\mathcal{S}$ ordered by containment forms the *poset of flats of* $\mathcal{S}$, which we denote by $\mathcal{L}(\mathcal{S})$.

Remark 1.9. For all $X \in \mathcal{C}$ we have $\mathrm{cl}(X) = \max\{Y \supseteq X \mid X \in \mathcal{C}, \mathrm{rk}(X) = \mathrm{rk}(Y)\}$, i.e., the closure of X is the maximal central set containing X and having same rank as X. In particular, we have a monotone function $\mathrm{cl} : \mathcal{C} \to \mathcal{C}$.

Remark 1.10. A poset is the poset of flats of a matroid if and only if it is a geometric lattice (see Definition 5.1). In Section 5 we will prove a similar correspondence between finitary semimatroids and geometric semilattices (Theorem E).

We now introduce the notions of deletion and contraction for locally ranked triples. Example 1.13 below will illustrate the case of pseudoline arrangements.

Definition 1.11. Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a locally ranked triple. For every $T \subseteq S$ let $\mathcal{C}_{\setminus T} := \mathcal{C} \cap 2^{S \setminus T}$ and define the *deletion* of T from $\mathcal{S}$ as

$$\mathcal{S} \setminus T := (S \setminus T, \mathcal{C}_{\setminus T}, \mathrm{rk}),$$

where we slightly abuse notation and write rk for $\mathrm{rk}|_{\mathcal{C}_{\setminus T}}$. Moreover, we will denote by $\mathcal{S}[T] := \mathcal{S} \setminus (S \setminus T)$ the *restriction* to T.

Furthermore, for every central set $X \in \mathcal{C}$ let

$$\mathcal{C}_{/X} := \{Y \in \mathcal{C}_{\setminus X} \mid Y \cup X \in \mathcal{C}\}, \qquad S_{/X} := \{s \in S \mid \{s\} \in \mathcal{C}_{/X}\}$$

and define the *contraction* of X in $\mathcal{S}$ as

$$\mathcal{S}/X := (S_{/X}, \mathcal{C}_{/X}, \mathrm{rk}_{/X}),$$

where, for every $Y \in \mathcal{C}_{/X}$, $\mathrm{rk}_{/X}(Y) := \mathrm{rk}_{\mathcal{C}}(Y \cup X) - \mathrm{rk}_{\mathcal{C}}(X)$.

Remark 1.12. This definition applies in particular to the case where $\mathcal{S}$ is a semimatroid and, in this case, differs slightly from that given in [2]: since we assume every element of the ground set of a semimatroid to be contained in a central set, we need to further constrain the ground set of the contraction.

Example 1.13. Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be the semimatroid of Example 1.7 (see Fig. 1). If $T := \{e_i\}_{i \in \mathbb{Z}}$, then

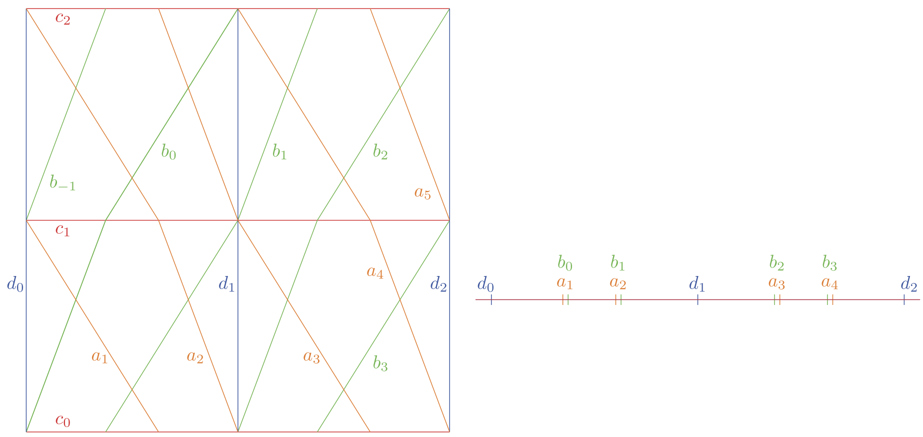

Fig. 2. Arrangements of pseudolines corresponding to the deletion $\mathcal{S} \setminus \{e_i\}_i$ (l.h.s.), and the contraction $\mathcal{S}/\{e_0\}$ (r.h.s.), where $\mathcal{S}$ is the semimatroid of Example 1.7. Again, we show only local pieces of these infinite arrangements, and the pictures must be thought of as being repeated in order to fill the plane (resp. the line).

$$\mathcal{C}_{\setminus T} = \mathcal{C} \setminus (\{e_i\}_i \cup \{a_i, e_j\}_{i,j} \cup \{b_i, e_j\}_{i,j} \cup \{d_i, e_j\}_{i,j} \cup \{a_k, b_{k-2i-1}, e_i\}_{i,k}),$$

and $\mathcal{S} \setminus T$ is the semimatroid associated to the arrangement on the left-hand side in Fig. 2.

The contraction of $\mathcal{S}$ to $e_0 \in S$ has ground set $S_{/\{e_0\}} = S \setminus (\{c_i\}_{i\in\mathbb{Z}} \cup \{e_i\}_{i\in\mathbb{Z}})$ and family of central sets $\mathcal{C}_{/\{e\}} = \{\emptyset\} \cup \{a_i\}_i \cup \{b_i\}_i \cup \{d_i\}_i \cup \{a_i, b_{i-1}\}_i$ with rank function $\mathrm{rk}_{/\{e_0\}}$ given by

$$\begin{aligned}
&\mathrm{rk}_{/\{e_0\}}(\emptyset) = \mathrm{rk}(\{e_0\}) - \mathrm{rk}(\{e_0\}) = 0;\\
&\mathrm{rk}_{/\{e_0\}}(\{a_i\}) = \mathrm{rk}(\{a_i, e_0\}) - \mathrm{rk}(\{e_0\}) = 1,\\
&\text{similarly } \mathrm{rk}_{/\{e_0\}}(\{b_i\}) = \mathrm{rk}_{/\{e_0\}}(\{d_i\}) = 1;\\
&\mathrm{rk}_{/\{e_0\}}(\{a_i, b_{i-1}\}) = \mathrm{rk}(\{a_i, b_{i-1}, e_0\}) - \mathrm{rk}(\{e_0\}) = 1;
\end{aligned}$$

where i ranges over the integers. This triple is represented by the arrangement of points depicted on the right-hand side in Fig. 2. $\triangle$

Proposition 1.14. *Let $\mathcal{S} = (S, \mathrm{rk}, \mathcal{C})$ be a finitary semimatroid. For every $T \subset S$, $\mathcal{S} \setminus T$ is a finitary semimatroid and, for every $X \in \mathcal{C}$, $\mathcal{S}/X$ is a finitary semimatroid.*

Proof. The proof of [2, Proposition 7.5 and 7.7] adapts straightforwardly. $\square$

Definition 1.15. To every finite locally ranked triple $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ we associate the following polynomial.

$$T_{\mathcal{S}}(x,y) := \sum_{X\in\mathcal{C}} (x-1)^{\mathrm{rk}(\mathcal{S})-\mathrm{rk}(X)}(y-1)^{|X|-\mathrm{rk}(X)}$$

Remark 1.16. If $\mathcal{S}$ is a finite semimatroid, this is exactly the *Tutte polynomial* of $\mathcal{S}$ introduced and studied by Ardila [2]. In particular, if $\mathcal{S}$ is a matroid, this is the associated Tutte polynomial.

A celebrated result about Tutte polynomials of matroids is the following "activities decomposition theorem" due to Crapo (for terminology we refer to [31]).

Proposition 1.17 *([8, Theorem 1]). Let $\mathcal{S}$ be a matroid with set of bases $\mathcal{B}$ and fix a total ordering $<$ on S. Then,*

$$T_{\mathcal{S}}(x,y) = \sum_{B\in\mathcal{B}} x^{|I(B)|}y^{|E(B)|},$$

where, for every $B\in\mathcal{B}$,

- $I(B)$ *is the set of* internally active elements *of B, i.e., the set of all $b\in B$ which are $<$-minimal in some codependent subset of $S\setminus(B\setminus b)$.*
- $E(B)$ *is the set of* externally active elements *of B, i.e., the set of all $e\in S\setminus B$ that are $<$-minimal in some dependent subset of $B\cup e$.*

Remark 1.18. Representable arithmetic Tutte polynomials satisfy an analogue to Crapo's theorem (see Remark 1.25). One of our results is the generalization of this theorem to all centered translative G-semimatroids (Theorem H).

1.2. Arithmetic (semi)matroids and their Tutte polynomials

We extend the definition of arithmetic matroids given in [5] and [9] to include the case where the underlying structure is a finite semimatroid.

Definition 1.19 *(Compare Section 2 of [5]).* Let $\mathcal{S}=(S,\mathcal{C},\mathrm{rk})$ be a locally ranked triple. A *molecule* of $\mathcal{S}$ is any triple (R,F,T) of disjoint sets with $R\cup F\cup T\in\mathcal{C}$ and such that, for every A with $R\subseteq A\subseteq R\cup F\cup T$,

$$\mathrm{rk}(A)=\mathrm{rk}(R)+|A\cap F|.$$

Remark 1.20. Once a total ordering of the ground set S is fixed, the notion of basis activities for matroids briefly recapped in Proposition 1.17 above allows us to associate to every basis B a molecule $(B\setminus I(B), I(B), E(B))$.

Definition 1.21 *(Extending Moci and Brändén [5]).* Let $\mathcal{S}=(S,\mathcal{C},\mathrm{rk})$ be a finite locally ranked triple and $m:\mathcal{C}\to\mathbb{R}$ any function. If (R,F,T) is a molecule, define

$$\rho(R, R \cup F \cup T) := (-1)^{|T|} \sum_{R \subseteq A \subseteq R \cup F \cup T} (-1)^{|R \cup F \cup T| - |A|} m(A).$$

We call the pair $(\mathcal{S}, m)$ *arithmetic* if the following axioms are satisfied:

(P) For every molecule (R, F, T),

$$\rho(R, R \cup F \cup T) \geq 0.$$

(A1) For all $A \subseteq S$ and $e \in S$ with $A \cup e \in \mathcal{C}$:
  (A.1.1) If $\mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A)$ then $m(A \cup \{e\})$ divides $m(A)$.
  (A.1.2) If $\mathrm{rk}(A \cup \{e\}) > \mathrm{rk}(A)$ then $m(A)$ divides $m(A \cup \{e\})$.

(A2) For every molecule (R, F, T)

$$m(R)m(R \cup F \cup T) = m(R \cup F)m(R \cup T).$$

Following [5] we use the expression *pseudo-arithmetic* to denote the case where m only satisfies (P). An *arithmetic matroid* is an arithmetic pair $(\mathcal{S}, m)$ where $\mathcal{S}$ is a matroid.

Remark 1.22. Following [9], the *dual* to an arithmetic matroid $(\mathcal{S}, m)$ is the pair $(\mathcal{S}^*, m^*)$, where $\mathcal{S}^*$ is the dual matroid to $\mathcal{S}$ and $m^*(A) := m(S \setminus A)$.

Example 1.23. To every set of integer vectors, say $a_1, \ldots, a_n \in \mathbb{Z}^d$, is associated a matroid on the ground set $[n] := \{1, \ldots, n\}$ with rank function

$$\mathrm{rk}(I) := \dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(a_i)_{i \in I}),$$

and a multiplicity function $m(I)$ defined for every $I \subseteq [n]$ as the greatest common divisor of the minors of the matrix with columns $(a_i)_{i \in I}$. These determine an arithmetic matroid [9]. We say that the vectors a_i *realize* this arithmetic matroid which we call then *representable*. $\triangle$

To every arithmetic pair $(\mathcal{S}, m)$ we associate an arithmetic Tutte polynomial as a straightforward extension of Moci's definition from [29].

Definition 1.24. Given an arithmetic pair $(\mathcal{S}, m)$, set

$$T_{(\mathcal{S},m)}(x, y) := \sum_{X \in \mathcal{C}} m(X)(x-1)^{\mathrm{rk}(\mathcal{S}) - \mathrm{rk}(X)}(y-1)^{|X| - \mathrm{rk}(X)}.$$

Remark 1.25. When $(\mathcal{S}, m)$ is an arithmetic matroid, the polynomial $T_{(\mathcal{S},m)}(x, y)$ enjoys a rich structure theory, investigated for instance in [5,9]. When this arithmetic matroid is representable, say by a set of vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$, the arithmetic Tutte polynomial specializes e.g. to the characteristic polynomial of the associated toric arrangement (see

Section 2) and to the Ehrhart polynomial of the zonotope obtained as the Minkowski sum of the a_i. Moreover, always in the representable case, Crapo's decomposition theorem (Proposition 1.17) has an analogue [5, Theorem 6.3] which gives a combinatorial interpretation of the coefficients of the polynomial in terms of counting integer points of zonotopes and intersections in the associated toric arrangement.

1.3. Matroids over rings

We give the general definition and some properties of matroids over rings. Further explanations and proofs of statements can be found in [20].

Definition 1.26 *(Fink and Moci [20]).* Let E be a finite set, R a commutative ring and $M : 2^E \to R$-mod any function associating an R-module to each subset of E. This defines a *matroid over* R if

(R) for any $A \subset E$, $e_1, e_2 \in E$, there is a pushout square

$$\begin{tikzcd} M(A) \arrow[r] \arrow[d] & M(A \cup \{e_1\}) \arrow[d] \\ M(A \cup \{e_2\}) \arrow[r] & M(A \cup \{e_1, e_2\}) \end{tikzcd}$$

such that all morphisms are surjections with cyclic kernel.

Remark 1.27 *([20, Section 6.1]).* Every matroid over the ring $R = \mathbb{Z}$ induces an arithmetic matroid on the ground set E with rank function satisfying $\mathrm{rk}(E) - \mathrm{rk}(A) = \mathrm{rank}_{\mathbb{Z}} M(A)$ and $m(A)$ equal to the cardinality of the torsion part of $M(A)$, for all $A \subseteq E$. We call (E, rk) the *underlying matroid* to $M_{\mathfrak{S}}$ and (E, rk, m) the *underlying arithmetic matroid* to $M_{\mathfrak{S}}$.

Remark 1.28 *(See Definition 2.2 in [20]).* A matroid M over a ring R is called *representable* if there is a finitely generated R-module N and a list $(x_e)_{e \in E}$ of elements of N such that for all $A \subseteq E$ we have that $M(A)$ is isomorphic to the quotient $N/(\sum_{e \in A} R x_e)$. Realizability is preserved under duality.

2. Geometric intuition: periodic arrangements

As an introductory example we describe the arithmetic matroid and the matroid over $\mathbb{Z}$ associated to periodic hyperplane arrangements, highlighting the structures we will encounter in the general theory later.

Let $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{C}$ and recall that an *affine hyperplane arrangement* is a locally finite set $\mathscr{A}$ of hyperplanes in $\mathbb{K}^d$. It is called *periodic* if it is (globally) invariant under the action of a group acting on $\mathbb{K}^d$ by translations.

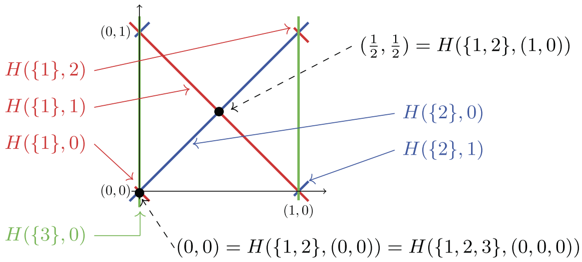

Fig. 3. A drawing of a "piece" (in fact, a neighborhood of a fundamental region) of the arrangement $\mathscr{A}$ of Example 2.1, with explicit labeling of some of the $H(X,k)$s.
Notice that $H(\{1,2,3\},(1,0,0)) = \emptyset$, and that $H(\emptyset,0) = \mathbb{R}^2$.

For simplicity, we will consider the standard action of $\mathbb{Z}^d$ on $\mathbb{K}^d$, with $k \in \mathbb{Z}^d$ acting as $t_k(x) = x + \sum_i k_i \varepsilon_i$, where $\varepsilon_1, \dots, \varepsilon_d$ is the standard basis of $\mathbb{K}^d$, and we will suppose the arrangement $\mathscr{A}$ being given by a finite list of integer vectors $a_1, \dots, a_n \in \mathbb{Z}^d$ (which we think of as the columns of a $d \times n$ matrix A) together with a corresponding list $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ of real numbers as follows.

For $X \subseteq [n]$ let $A[X]$ be the $d \times |X|$ matrix obtained by restricting A to the relevant columns. Moreover, given $k \in \mathbb{Z}^X$ we define the subspace

$$H(X,k) := \{x \in \mathbb{K}^d \mid \forall i \in X : a_i^T x = \alpha_i + k_i\}.$$

Then,

$$\mathscr{A} = \{H(\{i\},j) \mid i \in [n], j \in \mathbb{Z}\}.$$

Example 2.1. The periodic arrangement given by

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Is the set (see Fig. 3)

$$\begin{aligned} \mathscr{A} = \quad & \{H(\{1\},j) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = j\} \mid j \in \mathbb{Z}\} \\ \cup\, & \{H(\{2\},j) = \{x \in \mathbb{R}^2 \mid x_1 - x_2 = j\} \mid j \in \mathbb{Z}\} \\ \cup\, & \{H(\{3\},j) = \{x \in \mathbb{R}^2 \mid x_1 = j\} \mid j \in \mathbb{Z}\} \qquad \triangle \end{aligned}$$

The poset of intersections of $\mathscr{A}$ is the set

$$\mathcal{L}(\mathscr{A}) := \{\cap \mathscr{K} \mid \mathscr{K} \subseteq \mathscr{A}\} \setminus \{\emptyset\}$$

ordered by reverse inclusion (i.e., $x \leq y$ if $x \supseteq y$), see for instance [30]. This is a *geometric semilattice* in the sense of Wachs and Walker [37], see also Definition 5.2.

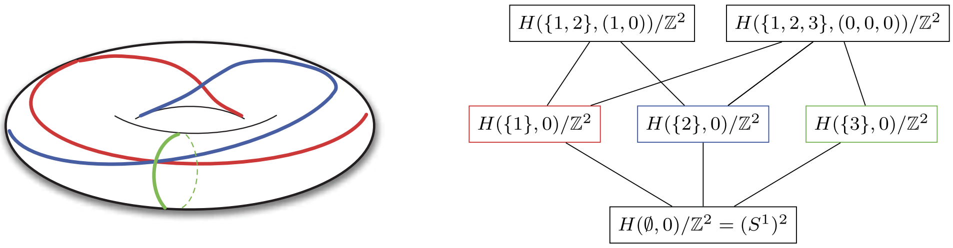

Fig. 4. Left-hand side: a drawing of the toric arrangement $\underline{\mathscr{A}}$ associated to the periodic line arrangement $\mathscr{A}$ of Example 2.1. Right-hand side: the poset of layers $\mathcal{C}(\underline{\mathscr{A}})$.

A closer look at the definition will reveal that $\mathcal{L}(\mathscr{A})$ is the poset of all nonempty $H(X,k)$, ordered by reverse inclusion.

Remark 2.2. The *toric arrangement* associated to $\mathscr{A}$ is the set

$$\underline{\mathscr{A}} := \{H/\mathbb{Z}^d \mid H \in \mathscr{A}/\mathbb{Z}^d\}$$

of quotients of orbits of the action on $\mathscr{A}$. (Notice that $\mathbb{Z}^d$ acts on the set $\mathscr{A}$ by permuting the hyperplanes and, for every $H_0 \in \mathscr{A}$, it acts on the space $H = \mathbb{Z}^d H_0$ by translations; in particular $H/\mathbb{Z}^d = H_0/\mathbb{Z}^d$ is a torus.)

The *poset of layers* of $\underline{\mathscr{A}}$ is the set $\mathcal{C}(\mathscr{A})$ of connected components of the intersections of elements of $\underline{\mathscr{A}}$, ordered by reverse inclusion. (See Fig. 4.). This poset is an important feature of toric arrangements: when $\mathbb{K} = \mathbb{C}$, we have an arrangement in the complex torus $\mathbb{C}^d/\mathbb{Z}^d$ (customarily given as a family of level sets of characters, see e.g. [11, §2.1]) and $\mathcal{C}(\underline{\mathscr{A}})$ encodes much of the homological data about the arrangement's complement (see e.g. [6,13]). When $\mathbb{K} = \mathbb{R}$, this is the poset considered in [18,25] pertaining to enumeration of the induced cell structure on the compact torus $\mathbb{R}^d/\mathbb{Z}^d \simeq (S^1)^d$.

Remark 2.3. We see that $\mathcal{C}(\underline{\mathscr{A}})$ is the quotient (in the sense of Definition 3.21) of the poset $\mathcal{L}(\mathscr{A})$ under the induced action of $\mathbb{Z}^d$ (where the element $\varepsilon_l \in \mathbb{Z}^d$ maps $H(\{i\},j)$ to $H(\{i\}, j + \langle \varepsilon_l \mid a_i \rangle))$.

For $X \subseteq [n]$ and $k \in \mathbb{Z}^X$ define

$$W(X) := \{k \in \mathbb{Z}^X \mid H(X,k) \neq \emptyset\}. \tag{1}$$

We call $\mathscr{A}$ *centered* if $\alpha_i = 0$ for all $i = 1, \dots, n$ and assume this for simplicity throughout this section. Notice that the toric arrangements considered in [29] can be obtained from actions on centered arrangements.

Remark 2.4. If $\mathscr{A}$ is centered, then $W(X) = (A[X]^T \mathbb{R}^d) \cap \mathbb{Z}^X$ for all $X \subseteq [n]$, thus $W(X)$ is a pure subgroup (hence a direct summand) of $\mathbb{Z}^X$.

Remark 2.5. Notice that $H(X,k)$ is the preimage of $\alpha + k$ with respect to the linear function $\mathbb{R}^d \to \mathbb{R}^X$, $x \mapsto A[X]^T x$, thus $H(X,k)$ is connected whenever nonempty.

Lemma 2.6. *If $\mathscr{A}$ is centered, the map*

$$\varphi_X : k \mapsto H(X,k) = \bigcap_{i \in X} H(\{i\}, k_i)$$

is a bijection between $W(X)$ and the connected components of $\bigcup_{k \in \mathbb{Z}^X} H(X,k)$.

Proof. The map φ_X is well-defined and surjective by definition of $W(X)$. It is injective by Remark 2.5, as $A[X]^T$-preimages of distinct elements are disjoint. □

Example 2.7 *(Continued from Example 2.1).*

$$\begin{aligned} W(\{1,2,3\}) &= \{k \in \mathbb{Z}^3 \mid A^T x = k \text{ for some } x \in \mathbb{R}^2\} \\ &= \{k \in \mathbb{Z}^3 \mid k_1 + k_2 = 2k_3\} \\ W(\{1,2\}) &= \{k \in \mathbb{Z}^2 \mid x_1 + x_2 = k_1,\, x_1 - x_2 = k_2 \text{ for some } x \in \mathbb{R}^2\} = \mathbb{Z}^2 \end{aligned} \qquad \triangle$$

Remark 2.8. We say that $\mathbb{Z}^d$ acts on $\mathbb{Z}^{\{i\}}$ by $\varepsilon_l(j) = j + \langle \varepsilon_l \mid a_i \rangle$ and, by coordinatewise extension, we obtain an action of $\mathbb{Z}^d$ on $\mathbb{Z}^X$ for all $X \subseteq [n]$. This induces an action of $\mathbb{Z}^d$ on $W(X)$ which is the action on $W(X)$ of its subgroup $A[X]^T\mathbb{Z}^d$ by addition and coincides with the "natural" action described in Remark 2.3.

Definition 2.9. For $X \subseteq [n]$ let $I(X) := A[X]^T\mathbb{Z}^d$ and consider

$$Z(X) := \mathbb{Z}^X / I(X).$$

Example 2.10 *(Continued from Example 2.7).* In the case $X = \{1,2,3\}$, we have $I(\{1,2,3\}) = A^T\mathbb{Z}^2 = \begin{pmatrix}1\\1\\1\end{pmatrix}\mathbb{Z} + \begin{pmatrix}1\\-1\\0\end{pmatrix}\mathbb{Z} = W(X)$. Since $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$ is a unimodular basis of $\mathbb{Z}^3$,

$$Z(\{1,2,3\}) = (\mathbb{Z}\varepsilon_1 \oplus W(X))/I(X) = \mathbb{Z}\varepsilon_1 \simeq \mathbb{Z}.$$

In the case $X = \{1,2\}$ we have $W(\{1,2\}) = \mathbb{Z}^2$ and $I(\{1,2\}) = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\mathbb{Z}^2$. Hence,

$$Z(\{1,2\}) = W(\{1,2\})/I(\{1,2\}) = \left\{\begin{pmatrix}0\\0\end{pmatrix} + I(\{1,2\}), \begin{pmatrix}1\\0\end{pmatrix} + I(\{1,2\})\right\} \simeq \mathbb{Z}/2\mathbb{Z}. \qquad \triangle$$

In general, we have the following description.

Lemma 2.11. *There is a direct sum decomposition of abelian groups*

$$Z(X) \simeq \mathbb{Z}^\eta \oplus W(X)/I(X),$$

where $\eta = |X| - \operatorname{rk} A[X]^T$*, the* nullity *of* X*, is the rank of* $Z(X)$ *as a* $\mathbb{Z}$*-module.*

Proof. The decomposition $\mathbb{Z}^X \simeq \mathbb{Z}^\eta \oplus W(X)$ exists by Remark 2.4, and $Z(X)$ decomposes as stated because $I(X) \subseteq W(X)$. For the claim on the rank, notice that both $W(X)$ and $I(X)$ are, by construction, free abelian groups of rank $\operatorname{rk} A[X]^T$, thus the quotient on the right hand side is pure torsion. □

Remark 2.12. Arithmetic matroids were introduced by d'Adderio and Moci in [9] in order to study, in the centered case, the combinatorial properties of the rank and multiplicity functions on the subsets of $[n]$, where every X has $\operatorname{rk}(X) := \operatorname{rk} A[X]$ and $m(X) := [\mathbb{Z}^d \cap A[X]\mathbb{R}^X : A[X]\mathbb{Z}^X]$. Since, by Remark 2.4 and Remark 2.8,

$$|W(X)/I(X)| = [W(X) : I(X)] = [\mathbb{Z}^X \cap A[X]^T\mathbb{R}^d : A[X]^T\mathbb{Z}^d],$$

classical work of McMullen [27] shows that $m(X) = |W(X)/I(X)|$. We thus recover in a geometric way the multiplicity function from [9].

Remark 2.13. The function φ_X of Lemma 2.6 induces a (natural) bijection between the elements of $W(X)/I(X)$ and the layers of $\left[\bigcup_{k\in\mathbb{Z}^X} H(X,k)\right]/\mathbb{Z}^d$ in the toric arrangement $\mathscr{A}$ (cf. Remark 2.2). This bijection exhibits the enumerative results proved in [9].

Example 2.14 *(Continued from Example 2.10).* Let us consider $X = \{1, 2\}$. We have seen that the family $\left[\bigcup_{k\in\mathbb{Z}^X} H(X,k)\right]/\mathbb{Z}^2$ equals

$$\{H(X,\, (0,0) + A[X]^T\mathbb{Z}^2),\ H(X,\, (1,0) + A[X]^T\mathbb{Z}^2)\}.$$

The map φ_X is then defined by

$$\varphi_X\left(\begin{pmatrix} i \\ j \end{pmatrix}\right) = H(X, (i,j)).$$

We have also previously seen that

$$W(X)/I(X) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix} + I(X), \begin{pmatrix} 1 \\ 0 \end{pmatrix} + I(X)\right\}.$$

Hence we can easily compute

$$\varphi_X\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + I(X)\right) = H(X, (0,0) + I(X))$$

$$\varphi_X\left(\begin{pmatrix}1\\0\end{pmatrix}+I(X)\right)=H(X,(1,0)+I(X))$$

which, since by definition $I(X)=A[X]^T\mathbb{Z}^2$, is a bijection as stated. $\triangle$

Remark 2.15. As proved in [29], the arithmetic Tutte polynomial associated to this arithmetic matroid evaluates to many interesting invariants — for instance to the characteristic polynomial of the poset $\mathcal{C}(\mathscr{A})$. Thus, it counts the number of chambers of the associated toric arrangement in $(S^1)^d$. Moreover, the quotient of the induced action on the complexification of $\mathscr{A}$ is an arrangement of subtori in $(\mathbb{C}^*)^d$, and the arithmetic Tutte polynomial specializes to the Poincaré polynomial of its complement.

For $Y\subseteq X\subseteq[n]$ we consider $\mathbb{Z}^{X\setminus Y}\subseteq\mathbb{Z}^X$ as an intersection of coordinate subspaces and let $\pi_{X,Y}$ denote the coordinate projection of $\mathbb{Z}^X$ onto $\mathbb{Z}^{X\setminus Y}$. Since $I(X\setminus Y)=I(X)\cap\mathbb{Z}^{X\setminus Y}$, the map $\pi_{X,Y}$ restricts to a surjection $I(X)\to I(X\setminus Y)$ and induces a map $\pi_{X,Y}:Z(X)\to Z(X\setminus Y)$ which, if $|Y|=1$, has cyclic kernel.

Lemma 2.16. *For $X\subseteq[n]$, $i,j\in X$, the diagram*

$$\begin{tikzcd} Z(X) \arrow[r, "\pi_{X,\,i}"] \arrow[d, "\pi_{X,\,j}"] & Z(X\setminus i) \arrow[d, "\pi_{X\setminus i,\,j}"] \\ Z(X\setminus j) \arrow[r, "\pi_{X\setminus j,\,i}"] & Z(X\setminus\{i,j\}) \end{tikzcd}$$

is a pushout square of epimorphisms with cyclic kernels.

Proof. This can be verified either directly or with Lemma 10.8, where the rows of the required diagram arise from short exact sequences of the type $0\to I(X)\to\mathbb{Z}^X\to Z(X)\to 0$, and the morphisms between the sequences are induced by the projections $\pi_{*,*}$. $\square$

Theorem 2.17. *The assignment $I\mapsto Z([n]\setminus I)$ defines a matroid over $\mathbb{Z}$ on the ground set $[n]$. The underlying arithmetic matroid is dual to that associated to the list $X:=\{a_1,\dots,a_n\}\subset\mathbb{Z}^d$ in Example 1.23, see [9].*

Proof. The previous lemma shows that this in fact defines a matroid over $\mathbb{Z}$. For the duality claim let us write rk for matrix rank, rk_X for the rank function of the arithmetic matroid associated to the list X, and rk_Z for the rank function of the underlying arithmetic matroid, respectively. Now, by Remark 1.27 and Lemma 2.11 we have

$$\mathrm{rk}_Z([n])-\mathrm{rk}_Z(I)=\mathrm{rank}_{\mathbb{Z}}\,Z(I\setminus[n])=|I^c|-\mathrm{rk}(A^T[I^c]),$$

where we write $I^c:=[n]\setminus I$. Moreover, $\mathrm{rk}_X(I)=\mathrm{rk}\,A[X]=\mathrm{rk}\,A^T[I]$ (see Example 1.23). Therefore we conclude

$$\mathrm{rk}_X(I^c) = |I^c| + \mathrm{rk}_Z(I) - \mathrm{rk}_Z([n]),$$

which is the very definition of rk_X being the rank function of the dual of the matroid defined by rk_Z (see [31, Proposition 2.1.9]).

Similarly, let us write m_X for the multiplicity function of the arithmetic matroid associated to the list X, and m_Z for the rank function of the underlying arithmetic matroid, respectively. Lemma 2.11 implies $m_Z(J) = |W(J^c)/I(J^c)|$ for all $J \subseteq [n]$. By Remark 2.12, we conclude $m_Z(I) = m_X(I^c)$, corresponding to the relationship between multiplicity functions of dual arithmetic matroids in [9]. $\square$

3. Overview: setup and main results

Throughout, we fix a finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ on the ground set S with set of central sets $\mathcal{C}$, rank function $\mathrm{rk} : \mathcal{C} \to \mathbb{N}$ and semilattice of flats $\mathcal{L}$.

Let G be a group acting on S. Given $x \in S$ write $g(x)$ (or simply gx) for its image under $g \in G$, and Gx for its orbit. Moreover, given $X \subseteq S$ let

$$\underline{X} := \{Gx \mid x \in X\} \subseteq S/G$$

denote the set of orbits met by X. Write

$$gX := \{g(x) \mid x \in X\},$$

to signify the induced action of G on the power set 2^S.

Remark 3.1. As a support for the intuition, the reader can think of the representable case described in Example 1.6, namely that of a periodic arrangement of hyperplanes. As a tangible instance, consider Example 2.1: there, the elements of the semimatroid are the hyperplanes $H(\{i\}, j)$, and the action of $\mathbb{Z}^2$ is by standard translation, i.e., such that $k \in \mathbb{Z}^2$ sends $H(\{i\}, j)$ to $H(\{i\}, j + \langle k \mid a_i \rangle)$ (compare Remark 2.3).

3.1. *Group actions on semimatroids*

We now discuss group actions on a set S that carries the structure of a semimatroid. In order to get a sense of the objects and notions introduced in the following definition the reader may already keep an eye on Example 3.7 and Fig. 7.

Definition 3.2 *(G-semimatroids).* An *action* of G on a semimatroid $\mathcal{S} := (S, \mathcal{C}, \mathrm{rk})$ is an action of G on the set S, whose induced action on 2^S preserves rank and centrality. A G-semimatroid

$$\mathfrak{S} = G \circlearrowright (S, \mathcal{C}, \mathrm{rk})$$

is a semimatroid together with a G-action. We define then

$$E_{\mathfrak{S}} := S/G; \qquad \mathcal{C}_{\mathfrak{S}} = \mathcal{C}/G; \qquad \underline{\mathcal{C}} := \{\underline{X} \mid X \in \mathcal{C}\};$$

where we take quotients of sets, i.e., $E_{\mathfrak{S}}$ and $\mathcal{C}_{\mathfrak{S}}$ are families of orbits. We call such an action

- *centered* if there is an $X \in \mathcal{C}$ with $\underline{X} = E_{\mathfrak{S}}$,
- *weakly translative* if, for all $g \in G$ and all $x \in S$, $\{x, g(x)\} \in \mathcal{C}$ implies $\mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(\{x\})$.
- *translative* if, for all $g \in G$ and all $x \in S$, $\{x, g(x)\} \in \mathcal{C}$ implies $g(x) = x$.

Moreover, for $A \subseteq E_{\mathfrak{S}}$ define

$$\underline{\mathrm{rk}}(A) := \max\{\mathrm{rk}_{\mathcal{C}}(X) \mid \underline{X} \subseteq A\}$$

and write $\mathrm{rk}(\mathfrak{S}) := \underline{\mathrm{rk}}(E_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S})$ for the rank of the G-semimatroid $\mathfrak{S}$.

Remark 3.3. We call a G-semimatroid $\mathfrak{S}$ *representable* if it arises from a periodic affine arrangement (see beginning of Section 2). In particular, $\mathcal{S}$ is representable in the sense of Example 1.6.

Remark 3.4. Every translative action is weakly translative. Moreover, every weakly translative action on a simple semimatroid is translative.

Remark 3.5. We will sometimes find it useful to consider the set system $\mathcal{C}_{\mathfrak{S}}$ as a poset, with the natural order defined by $GX \leq GY$ if $X \subseteq gY$ for some $g \in G$ (notice that this is well-defined: in fact, it is the poset-quotient of the poset of simplices of $\mathcal{C}$ ordered by inclusion).

Definition-assumption 3.6. The action is called *cofinite* if the set $\mathcal{C}_{\mathfrak{S}}$ is finite (in particular, $E_{\mathfrak{S}}$ is finite). We will assume this throughout without further mention.

Theorem A. *Every G-action on $\mathcal{S}$ gives rise to a polymatroid on the ground set $E_{\mathfrak{S}}$ with rank function $\underline{\mathrm{rk}}$ (see Remark 1.5). This polymatroid is a matroid if and only if the action is weakly translative: in this case the triple*

$$\mathcal{S}_{\mathfrak{S}} := (E_{\mathfrak{S}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$$

is locally ranked and satisfies (CR2). The triple $\mathcal{S}_{\mathfrak{S}}$ is a matroid if and only if $\mathfrak{S}$ is centered.

Proof. The first part of the claim is Proposition 6.4. The second part follows from Proposition 6.4 and Proposition 6.7. $\square$

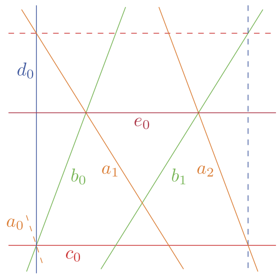

Fig. 5. A picture of the fundamental region of the $\mathbb{Z}^2$-semimatroid of Example 3.7, obtained from the natural action by translations on the pseudoline arrangement of Fig. 1.

Example 3.7. As an illustration consider the semimatroid $\mathcal{S}$ described in Example 1.7 (and Fig. 1) with an action of the group $\mathbb{Z}^2$ given by

$$\varepsilon_1(a_i) = a_{i+2},\ \varepsilon_1(b_i) = b_{i+2},\ \varepsilon_1(c_i) = c_i,\ \varepsilon_1(d_i) = d_{i+1},\ \varepsilon_1(e_i) = e_i$$
$$\varepsilon_2(a_i) = a_{i+1},\ \varepsilon_2(b_i) = b_{i-1},\ \varepsilon_2(c_i) = c_{i+1},\ \varepsilon_2(d_i) = d_i,\ \varepsilon_2(e_i) = e_{i+1}$$

where, as above, $\varepsilon_1, \varepsilon_2$ is the standard basis of $\mathbb{Z}^2$.

This action gives rise to a well-defined $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$, with

$$E_{\mathfrak{S}} = \{a, b, c, d, e\},\ \ \underline{\mathcal{C}} = 2^{\{a,b,c,d\}} \cup 2^{\{a,b,e\}} \cup 2^{\{e,d\}}$$

and rank function defined via $\underline{\mathrm{rk}}(\emptyset) = 0$ and, for $A \subseteq E_{\mathfrak{S}}$, $\underline{\mathrm{rk}}(A) = 1$ if $|A| = 1$, else $\underline{\mathrm{rk}}(A) = 2$. A sketch of the fundamental region of this action is given in Fig. 5, and the associated $\mathcal{C}_{\mathfrak{S}}$ is shown in Fig. 6.

In this case, $\mathcal{S}_{\mathfrak{S}}$ does not satisfy (CR1). For instance, with $X := \{a, b, c\}$ and $Y := \{a, b, e\}$, we have $X, Y \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(X \cap Y) = \underline{\mathrm{rk}}(\{a, b\}) = 2 = \underline{\mathrm{rk}}(X)$, but $X \cup Y = \{a, b, c, e\} \notin \underline{\mathcal{C}}$. $\triangle$

Remark 3.8. Notice that $\mathcal{S}_{\mathfrak{S}}$ not being a semimatroid is not a consequence of $\mathfrak{S}$ not being representable. In fact, Fig. 7 shows that the properties of being representable, centered and $\mathcal{S}_{\mathfrak{S}}$ being a semimatroid can appear in any combination not explicitly covered in Theorem A.

We want to study the "sets of orbits that give rise to central sets": the following definition makes this sentence precise, and Example 3.11 below illustrates it.

Definition 3.9. Let $\mathfrak{S}$ be a G-semimatroid. Given $A \subseteq E_{\mathfrak{S}}$ we define

$$\lceil A \rceil^{\mathcal{C}} := \{X \in \mathcal{C} \mid \underline{X} = A\} \subseteq \mathcal{C}.$$

For any given $A \subseteq E_{\mathfrak{S}}$, the set $\lceil A \rceil^{\mathcal{C}}$ carries a natural G-action, and we will be concerned with the study of its orbit set, i.e., the set

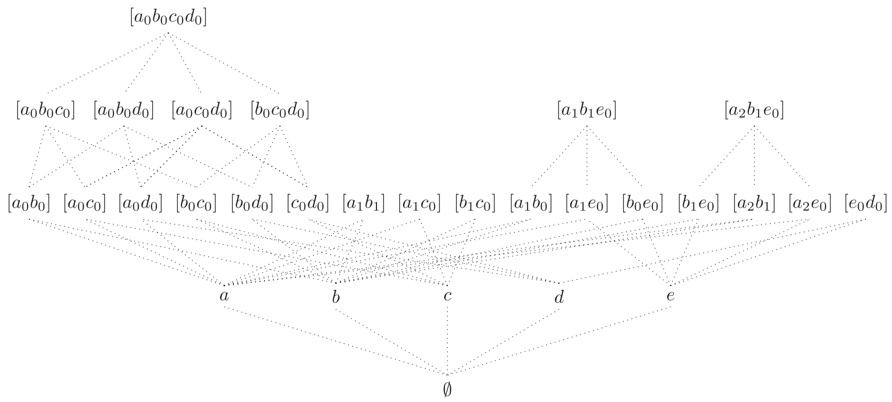

Fig. 6. The set system $\mathcal{C}_{\mathfrak{G}}$, with dotted lines representing the Hasse diagram of the associated poset. We use shorthand notation, where we write, e.g., $[a_0b_0c_0]$ for the orbit $\mathbb{Z}^2\{a_0, b_0, c_0\}$.

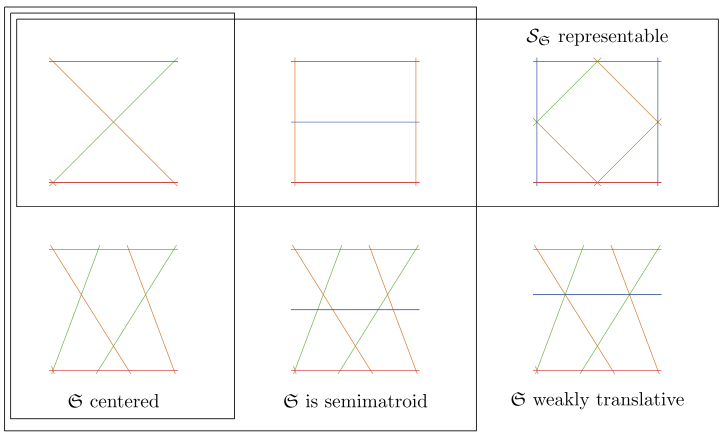

Fig. 7. This diagram depicts the fundamental regions of different cases of $\mathbb{Z}^2$-actions on arrangements of (PL-)pseudolines. The full arrangement can be recovered in each case by tiling the plane with copies of the respective picture, identifying the edges of adjacent squares, just as the arrangement of Fig. 1 is obtained from the fundamental region of Fig. 5.
These examples realize every combination of centered, representable and "$\mathcal{S}_{\mathfrak{G}}$ is semimatroid", within weakly translative actions (with the only constraint that for centered actions $\mathcal{S}_{\mathfrak{G}}$ always is a semimatroid – indeed in this case $\mathcal{S}_{\mathfrak{G}}$ is a matroid).

$$\lceil A \rceil^{\mathcal{C}}/G = \{\mathcal{T} \in \mathcal{C}_{\mathfrak{G}} \mid \lfloor \mathcal{T} \rfloor = A\}$$

where, for any orbit $\mathcal{T} = G\{t_1, \ldots, t_k\} \in \mathcal{C}_{\mathfrak{G}}$ we write

$$\lfloor \mathcal{T} \rfloor := \{Gt_1, \ldots, Gt_k\},$$

so that $\lfloor \cdot \rfloor$ defines a map $\mathcal{C}_{\mathfrak{G}} \to \underline{\mathcal{C}}$. For every $A \subseteq E_{\mathfrak{G}}$, let then

$$m_{\mathfrak{G}}(A) := |\lceil A \rceil^{\mathcal{C}}/G|.$$

$$
\underline{\mathcal{C}} = \left\{ \begin{array}{ccccccccc}
 & & & & \{a,b,c,d\}^{(1)} & & & & \\
 & \{a,b,c\}^{(1)} & & \{a,b,d\}^{(1)} & \{a,c,d\}^{(1)} & \{b,c,d\}^{(1)} & & \{a,b,e\}^{(2)} & \\
\{a,b\}^{(4)} & \{a,c\}^{(2)} & \{b,c\}^{(2)} & \{a,d\}^{(1)} & \{c,d\}^{(1)} & \{b,d\}^{(1)} & \{a,e\}^{(2)} & \{b,e\}^{(2)} & \{e,d\}^{(1)} \\
a^{(1)} & & b^{(1)} & & c^{(1)} & & d^{(1)} & & e^{(1)} \\
 & & & & \emptyset^{(1)} & & & &
\end{array} \right\}
$$

Fig. 8. The set $\underline{\mathcal{C}}$ for Example 3.7, with the multiplicity $m_{\mathfrak{S}}(A)$ written as a superscript of every set $A \in \underline{\mathcal{C}}$.

Remark 3.10. We illustrate the relationships between the previous definitions by fitting them into a diagram.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "/G"] & \mathcal{C}_{\mathfrak{S}} \arrow[r, "\lfloor \cdot \rfloor"] & \underline{\mathcal{C}} \subseteq 2^{E_{\mathfrak{S}}} \\
\lceil A \rceil^{\mathcal{C}} \arrow[u, phantom, "\cup\mid"] \arrow[r, mapsto, dashed, "\text{preimage of}"] & \lceil A \rceil^{\mathcal{C}}/G \arrow[u, phantom, "\cup\mid"] \arrow[r, mapsto, dashed, "\text{preimage of}"] & A \arrow[u, phantom, "\cup\mid"] \\
X \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & GX \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \underline{X} \arrow[u, phantom, "\Cup"]
\end{tikzcd}
$$

The number $m_{\mathfrak{S}}(A)$ is nonzero if and only if $A \in \underline{\mathcal{C}}$. We will often tacitly consider the restriction of $m_{\mathfrak{S}}$ to its support, which in the cofinite case defines a multiplicity function $m_{\mathfrak{S}} : \underline{\mathcal{C}} \to \mathbb{N}_{>0}$.

Example 3.11. In our running example (the $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$ of Example 3.7), we consider for instance the set $\{a, b\} \in \underline{\mathcal{C}}$. Then,

$$\lceil \{a, b\} \rceil^{\mathcal{C}} = \{\{a_i, b_j\} \mid i, j \in \mathbb{Z}\}$$

and so

$$\lceil \{a, b\} \rceil^{\mathcal{C}}/\mathbb{Z}^2 = \left\{\mathbb{Z}^2\{a_0, b_0\}, \mathbb{Z}^2\{a_1, b_0\}, \mathbb{Z}^2\{a_1, b_1\}, \mathbb{Z}^2\{a_2, b_1\}\right\},$$

thus $m_{\mathfrak{S}}(\{a, b\}) = 4$. Repeating this procedure for all elements of $\underline{\mathcal{C}}$ we obtain the multiplicities written as "exponents" next to the corresponding sets in Fig. 8. △

Definition 3.12. We call the action of G

- *normal* if, for all $x \in S$, $\mathrm{stab}(x)$ is a normal subgroup of G,
- *almost arithmetic* if it is translative and normal.

Remark 3.13. The two above-defined conditions are independent from each other and from the previous definitions. Indeed: the action of the symmetric group on its associated braid arrangement (see e.g. [30, Example 1.9]) is neither normal nor translative; the permutation action of the symmetric group on n distinct points in $\mathbb{R}$ is translative but not normal; the nontrivial action of $\mathbb{Z}_2$ on the uniform matroid of rank 1 on two elements is normal but not translative; every representable G-semimatroid is translative.

Theorem B. *If $\mathfrak{S}$ is a G-semimatroid associated to an almost-arithmetic action, then the pair $(\mathcal{S}_{\mathfrak{S}}, m_{\mathfrak{S}})$ is pseudo-arithmetic (see Definition 1.21). If $\mathcal{S}_{\mathfrak{S}}$ is a semimatroid, $m_{\mathfrak{S}}$ defines a pseudo-arithmetic semimatroid whose arithmetic Tutte polynomial, which we will call $T_{\mathfrak{S}}(x, y)$ (cf. Definition 3.28), satisfies an analogue of Crapo's decomposition formula (Theorem H) generalizing the combinatorial interpretation of [5, Theorem 6.3].*

Proof. This is proved as Proposition 8.6 and Theorem H. □

Remark 3.14. If, in addition to satisfying the conditions of Theorem B, $\mathfrak{S}$ is also centered, then $\mathcal{S}_{\mathfrak{S}}$ is a matroid and $m_{\mathfrak{S}}$ defines a pseudo-arithmetic matroid on $E_{\mathfrak{S}}$ in the sense of [5]. Notice that this way we can produce a natural class of nonrepresentable arithmetic matroids, e.g., by the action associated to non-stretchable pseudoarrangements (see Fig. 5).

Definition 3.15. If the action of G is translative, for every $X \in \mathcal{C}$ we have that $\operatorname{stab}(X) = \cap_{x \in X} \operatorname{stab}(x)$. If, moreover, the action is normal, it follows that, for every $X \in \mathcal{C}$, $\operatorname{stab}(X)$ is a normal subgroup of G. We can then define the group

$$\Gamma(X) := G/\operatorname{stab}(X)$$

and, for $g \in G$, write $[g]_X := g + \operatorname{stab}(X) \in \Gamma(X)$. For any $X \subseteq S$ consider then the group

$$\Gamma^X := \prod_{x \in X} \Gamma(x)$$

and the natural map

$$h'_X : G \to \Gamma^X, \quad h'_X(g) = ([g]_x)_{x \in X}.$$

Given $\gamma \in \Gamma^X$, let

$$\gamma.X := \{\gamma_x(x) \mid x \in X\}$$

and, for all $X \in \mathcal{C}$, define

$$W(X) := \{\gamma \in \Gamma^X \mid \gamma.X \in \mathcal{C}\}.$$

Since $X \in \mathcal{C}$ implies $\operatorname{im}(h'_X) \subseteq W(X)$, we can restrict h'_X to $W(X)$ as follows.

$$h_X : G \to W(X), \quad h_X(g) := h'_X(g).$$

Remark 3.16. In order to help the intuition, notice that this definition of $W(X)$ coincides, in the representable case, with that given in Equation (1).

Example 3.17. In our running example (from Example 1.7 and 3.7) we can illustrate the construction of $W(X)$ by taking, e.g., $X = \{a_0, b_0, c_0\} \in \mathcal{C}$. We have

$$\mathrm{stab}(a_0) = \mathbb{Z}\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right), \quad \mathrm{stab}(b_0) = \mathbb{Z}\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right), \quad \mathrm{stab}(c_0) = \mathbb{Z}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),$$

hence

$$\begin{aligned}
\Gamma(a_0) &= \mathbb{Z}^2/\,\mathrm{stab}(a_0) = \{\left(\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right) + \mathrm{stab}(a_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z} \\
\Gamma(b_0) &= \mathbb{Z}^2/\,\mathrm{stab}(b_0) = \{\left(\begin{smallmatrix} 0 \\ -k \end{smallmatrix}\right) + \mathrm{stab}(b_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z} \\
\Gamma(c_0) &= \mathbb{Z}^2/\,\mathrm{stab}(c_0) = \{\left(\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right) + \mathrm{stab}(c_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}
\end{aligned}$$

where we take the isomorphism with $\mathbb{Z}$ to send $k \in \mathbb{Z}$ to the element listed in the braces.

Then, $\Gamma^X = \Gamma(a_0) \times \Gamma(b_0) \times \Gamma(c_0) \simeq \mathbb{Z}^3$ and for $\gamma \in \Gamma^X$, say $\gamma = (i, j, l) \in \mathbb{Z}^3$, our choice of the isomorphisms with $\mathbb{Z}$ above implies that

$$\gamma.\{a_0, b_0, c_0\} = \{a_i, b_j, c_l\}$$

and thus we see that $\gamma.\{a_0, b_0, c_0\} \in \mathcal{C}$ if and only if $i - l = j + l$ is an even number (compare Example 1.7). Therefore

$$W(X) = \{(2h + l, 2h - l, l) \mid h, l \in \mathbb{Z}\}$$

is clearly seen to be a subgroup of Γ^X. We leave it to the reader to check that this applies to every X, thus the $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$ is arithmetic (though not centered, neither representable, and $\mathcal{S}_{\mathfrak{S}}$ is not a semimatroid). $\triangle$

Definition 3.18. An almost-arithmetic action is called *arithmetic* if $W(X)$ is a subgroup of Γ^X for all $X \in \mathcal{C}$.

Theorem C. *If $\mathfrak{S}$ is an arithmetic G-semimatroid, then the pair $(\mathcal{S}_{\mathfrak{S}}, m_{\mathfrak{S}})$ is arithmetic. If, moreover, $\mathfrak{S}$ is centered, then $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is an arithmetic matroid.*

Proof. This is a combination of Proposition 8.6 and Lemma 9.6. $\square$

3.2. Matroids over $\mathbb{Z}$

Under appropriate circumstances, the objects defined in Notation 3.15 give rise to a matroid over $\mathbb{Z}$ defined on the ground set $E_{\mathfrak{S}}$. In fact, for every arithmetic G-semimatroid, the groups $\Gamma(X)$ and Γ^X from Definition 3.15 do not depend on the choice of X in $\lceil \underline{X} \rceil^{\mathcal{C}}$

(Lemma 9.1). Moreover, if we assume that all groups $\Gamma(x)$ are cyclic, then every group Γ^X is abelian, and in particular all notions introduced in Notation 3.15 above do not depend on the choice of X inside $\lceil \underline{X} \rceil^{\mathcal{C}}$ (Lemma 9.3). So given $A \in \underline{\mathcal{C}}$ it makes sense to write $\Gamma(A)$, Γ^A, $W(A)$ etc., see Section 9 for a more thorough discussion.

Definition 3.19. Let $\mathfrak{S}$ denote an arithmetic and centered G-semimatroid such that, for all $a \in E_{\mathfrak{S}}$, the group $\Gamma(a)$ is cyclic. Given $A \subseteq E_{\mathfrak{S}}$ we will write $A^c := E_{\mathfrak{S}} \setminus A$ and define

$$M_{\mathfrak{S}}(A) := \Gamma^{A^c}/h'_{A^c}(G).$$

Theorem D. *Let $\mathfrak{S}$ denote an arithmetic and centered G-semimatroid such that, for all $a \in E_{\mathfrak{S}}$, the group $\Gamma(a)$ is cyclic. Then the abelian groups $M_{\mathfrak{S}}(A)$, where A runs over all subsets of $E_{\mathfrak{S}}$, define a representable matroid over $\mathbb{Z}$. Moreover, if the groups $\Gamma(a)$ are infinite cyclic, the underlying matroid of $M_{\mathfrak{S}}$ is the dual to $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$. If, additionally, $W(A)$ is a pure subgroup of Γ^A we have an isomorphism*

$$M_{\mathfrak{S}}(A) \simeq \mathbb{Z}^{|A^c| - \underline{\mathrm{rk}}(A^c)} \oplus W(A^c)/h_{A^c}(G)$$

and the underlying arithmetic matroid is dual to $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$.

Proof. This statement combines those of Proposition 9.13, Corollary 9.15, Corollary 9.16, Proposition 9.17 and Corollary 9.18. $\square$

Remark 3.20. In general, a toric arrangement in $(\mathbb{C}^*)^d$ is given as a family of level sets of characters of $(\mathbb{C}^*)^d$ (see e.g. [11, §2.1]). By lifting the toric arrangement to the universal covering space of the torus one recovers a periodic affine hyperplane arrangement $\mathscr{A}$. If $\mathfrak{S}$ is the $\mathbb{Z}^d$-semimatroid associated to this action as in Section 2, then $M_{\mathfrak{S}}$ is dual to the matroid over $\mathbb{Z}$ associated to the characters defining the toric arrangement (see Theorem 2.17).

3.3. Group actions on finitary geometric semilattices

The main tool allowing us to establish a poset-theoretic formulation of the theory of G-semimatroids is the following cryptomorphism result between finitary semimatroids and finitary geometric semilattices. Its proof is the object of Section 5.

Theorem E. *A poset $\mathcal{L}$ is a finitary geometric semilattice if and only if it is isomorphic to the poset of flats of a finitary semimatroid. Furthermore, each finitary geometric semilattice is the poset of flats of an unique simple[1] finitary semimatroid (up to isomorphism).*

[1] See Definition 1.3.

We now discuss some basics about group actions on finitary geometric semilattices.

Definition 3.21. An action of G on a geometric semilattice $\mathcal{L}$ is given by a group homomorphism of G in the group of poset automorphisms of $\mathcal{L}$. We define

$$\mathcal{P}_{\mathfrak{S}} := \mathcal{L}/G,$$

the set of orbits of elements of $\mathcal{L}$ partially ordered such that $GX \leq GY$ if there is g with $X \leq gY$ (where as usual we identify a group element in G with the automorphism to which it corresponds).

Remark 3.22. The fact that automorphisms of $\mathcal{L}$ preserve rank implies that the above binary relation on $\mathcal{P}_{\mathfrak{S}}$ is indeed a partial order. For another appearance of this definition of a "quotient poset" see, e.g., [36].

Example 3.23 *(Toric arrangements).* If $\mathfrak{S}$ arises from a periodic arrangement of hyperplanes as in Section 2, then $\mathcal{P}_{\mathfrak{S}}$ is the poset of layers of the associated toric arrangement (cf. Remark 2.2 and Remark 2.3). △

Example 3.24 *(Toric pseudoarrangements).* If $\mathfrak{S}$ is the $\mathbb{Z}^2$-semimatroid associated to a periodic arrangement of pseudolines (see, e.g., Example 1.7) then $\mathcal{P}_{\mathfrak{S}}$ is the poset of layers of the associated pseudoarrangement on the torus.

The higher-dimensional analogue of this construction needs a (combinatorial) notion of a "periodic affine arrangement of pseudoplanes" whose intersection poset is a geometric semilattice. A forthcoming paper [16] will provide such a notion by defining finitary affine oriented matroids and studying their topological representation. If $\mathfrak{S}$ arises from an appropriate $\mathbb{Z}^d$-action on a rank d finitary affine oriented matroid, then $\mathcal{P}_{\mathfrak{S}}$ is the poset of layers of the associated pseudoarrangement on the torus. △

Remark 3.25. It is clear that every action on a semimatroid induces an action on its semilattice of flats, and every action on a geometric semilattice induces an action on the associated simple semimatroid. It is an exercise to reformulate the requirements of the different kinds of actions in terms of the poset – where, however, the distinction between weakly translative and translative does not show. In our proofs we will mostly use the semimatroid language, in order to treat the most general case, and will call an action on a geometric semilattice *cofinite, weakly translative, translative, normal, arithmetic*, etc., if the corresponding G-semimatroid is.

Example 3.26. The poset $\mathcal{P}_{\mathfrak{S}}$ for the $\mathbb{Z}^2$-semimatroid of Example 3.7 can be read off the picture of the fundamental region in Fig. 5, and gives the poset depicted in Fig. 9. △

The poset $\mathcal{P}_{\mathfrak{S}}$ can also be obtained through a "closure operator" on $\mathcal{C}_{\mathfrak{S}}$.

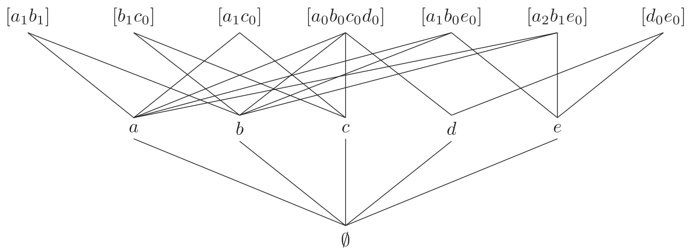

Fig. 9. The poset $\mathcal{P}_{\mathfrak{S}}$ for the (nonrepresentable) $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$ of our running Example 3.7, where we use the same conventions as in Fig. 6.

Definition 3.27. Given a G-semimatroid $\mathfrak{S} : G \circlearrowright (S, \mathcal{C}, \mathrm{rk})$, define the function

$$\kappa_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \to \mathcal{P}_{\mathfrak{S}}, \quad GX \mapsto G\,\mathrm{cl}(X)$$

where cl denotes the closure operator associated to $(S, \mathcal{C}, \mathrm{rk})$ (see Remark 1.9).

The function $\kappa_{\mathfrak{S}}$ is independent from the choice of representatives (since the action is rank-preserving) and thus defines a "closure operator" $\kappa_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}_{\mathfrak{S}}$ whose closed sets are exactly the elements of $\mathcal{P}_{\mathfrak{S}}$.

Think of $\mathcal{C}_{\mathfrak{S}}$ as a poset with the natural order given by $GX \leq GY$ if there is $g \in G$ with $gX \subseteq Y$, and let $\mathcal{C}$ and $\underline{\mathcal{C}}$ be ordered by inclusion. Then, for every weakly translative $\mathfrak{S}$-semimatroid we have the following commutative diagram of order-preserving functions.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "/G"] \arrow[d, "\mathrm{cl}"] & \mathcal{C}_{\mathfrak{S}} \arrow[r, "\lfloor\cdot\rfloor"] \arrow[d, "\kappa_{\mathfrak{S}}"] & \underline{\mathcal{C}} \arrow[r, "\subseteq"] & 2^{E_{\mathfrak{S}}} \arrow[d, "\underline{\mathrm{cl}}"] \\
\mathcal{L} \arrow[r, "/G"] & \mathcal{P}_{\mathfrak{S}} \arrow[rr, "\underline{\mathrm{cl}}\lfloor\cdot\rfloor"] & & \mathcal{L}_0
\end{tikzcd}$$

3.4. Tutte polynomials of group actions

Definition 3.28. To every G-semimatroid $\mathfrak{S}$ we associate the polynomial

$$T_{\mathfrak{S}}(x, y) := \sum_{A \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A)(x-1)^{\underline{\mathrm{rk}}(E_{\mathfrak{S}}) - \underline{\mathrm{rk}}(A)}(y-1)^{|A| - \underline{\mathrm{rk}}(A)}.$$

This definition is natural in its own right, as can be seen in Section 10.1 and Section 10.2. If the action is centered (so in particular $\mathcal{S}_{\mathfrak{S}}$ is a matroid), we recover Definition 1.24 and in particular, in the representable, resp. arithmetic case, Moci's arithmetic Tutte polynomial [29].

Our first result is valid in the full generality of weakly translative actions, and concerns the characteristic polynomial of the poset $\mathcal{P}_{\mathfrak{S}}$: we point, e.g., to [34] for background on characteristic polynomials of posets, and to our Section 10.1 for the precise definition.

Theorem F. *Let $\mathfrak{S}$ be any weakly translative and loopless G-semimatroid, and let $\chi_{\mathfrak{S}}(t)$ denote the characteristic polynomial of the poset $\mathcal{P}_{\mathfrak{S}}$. Then,*

$$\chi_{\mathfrak{S}}(t) = (-1)^r T_{\mathfrak{S}}(1-t, 0).$$

Proof. The proof is given at the end of Section 10.1 □

Example 3.29. For our running example we have (e.g., from Fig. 8)

$$\begin{aligned} T_{\mathfrak{S}}(x,y) &= (x-1)^2 + 5(x-1) + 16 + 6(y-1) + (y-1)^2 \\ &= x^2 + y^2 + 3x + 4y + 7 \end{aligned}$$

and, from Fig. 9,

$$\chi_{\mathfrak{S}}(t) = t^2 - 5t + 11.$$

An elementary computation now verifies Theorem F in this case. △

The polynomials $T_{\mathfrak{S}}(x,y)$ associated to translative actions satisfy a deletion–contraction recursion. Deletion and contraction for G-semimatroids correspond, in the representable case, to removing a set of orbits of hyperplanes, respectively considering the periodic arrangement induced on any (nonempty) intersection of hyperplanes.

Definition 3.30. For every $T \subseteq E_{\mathfrak{S}}$, G acts on $\mathcal{S} \setminus \cup T$. We denote the associated G-semimatroid by $\mathfrak{S} \setminus T$ and call this the *deletion* of T. We follow established matroid terminology and denote by $\mathfrak{S}[T] := \mathfrak{S} \setminus (S \setminus \cup T)$ the *restriction* to T.

Remark 3.31. A comparison with Definition 1.11 shows that $\mathcal{S}_{\mathfrak{S}[T]} = \mathcal{S}_{\mathfrak{S}}[T]$ and that, for every $A \subseteq T$, $m_{\mathfrak{S}[T]}(A) = m_{\mathfrak{S}}(A)$.

Definition 3.32. Recall $\mathcal{C}_{\mathfrak{S}} := \mathcal{C}/G$. For all $\mathcal{T} \in \mathcal{C}_{\mathfrak{S}}$ define the *contraction* of $\mathfrak{S}$ to $\mathcal{T}$ by choosing a representative $T \in \mathcal{T}$ and considering the action of $\mathrm{stab}(T)$ on the contraction $\mathcal{S}/T$. This defines the $\mathrm{stab}(T)$-semimatroid $\mathfrak{S}/\mathcal{T}$.

Remark 3.33. Clearly $\mathfrak{S}/\mathcal{T}$ does not depend on the choice of the representative $T \in \mathcal{T}$. Moreover, for all $e \in E_{\mathfrak{S}}$ we will abuse notation and write $\mathfrak{S}/e$ as a shorthand for $\mathfrak{S}/\{e\}$.

Remark 3.34. By Proposition 10.7, weak translativity, translativity, normality and arithmeticity of actions are preserved under taking contractions and restrictions.

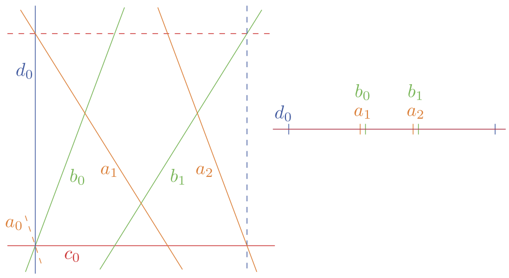

Fig. 10. Illustration for Example 3.35.

Theorem G. *Let $\mathfrak{S}$ be a translative G-semimatroid and let $e \in E_{\mathfrak{S}}$. Then*

(1) *if e is neither a loop nor an isthmus*[2] *of $\mathcal{S}_{\mathfrak{S}}$,*

$$T_{\mathfrak{S}}(x,y) = T_{\mathfrak{S}/e}(x,y) + T_{\mathfrak{S}\setminus e}(x,y);$$

(2) *if e is an isthmus, $T_{\mathfrak{S}}(x,y) = (x-1)T_{\mathfrak{S}\setminus e}(x,y) + T_{\mathfrak{S}/e}(x,y)$;*
(3) *if e is a loop, $T_{\mathfrak{S}}(x,y) = T_{\mathfrak{S}\setminus e}(x,y) + (y-1)T_{\mathfrak{S}/e}(x,y)$.*

Proof. The proof is given at the end of Section 10.4. □

Example 3.35. If $\mathfrak{S}$ is the $\mathbb{Z}^2$-semimatroid of our running example, then $\mathfrak{S} \setminus e$ is given by the induced $\mathbb{Z}^2$-action on the semimatroid $\mathcal{S} \setminus \{e_i\}_{i\in\mathbb{Z}}$ associated to the periodic arrangement of Fig. 2.(a). Moreover, $\mathfrak{S}/e$ is the $\mathbb{Z}$-semimatroid given by the action of $\operatorname{stab}(e_0) = \mathbb{Z}\binom{1}{0} \simeq \mathbb{Z}$ on the finitary semimatroid associated to the periodic arrangement of Fig. 2.(b). A picture of the fundamental regions of these two actions is given in Fig. 10, from which we can compute

$$\begin{aligned} T_{\mathfrak{S}\setminus e}(x,y) &= (x-1)^2 + 4(x-1) + 11 + 4(y-1) + (y-1)^2 \\ &= x^2 + y^2 + 2x + 2y + 5 \\ T_{\mathfrak{S}/e}(x,y) &= (x-1) + 5 + 2(y-1) = x + 2y + 2 \end{aligned}$$

and easily verify that the sum of these polynomials equals $T_{\mathfrak{S}}(x,y) = x^2+y^2+3x+4y+7$ (Example 3.29). △

4. Some examples

Example 4.1 *(Reflection groups).* Let G be a finite or affine complex reflection group acting on the intersection poset of its reflection arrangement. This setting has been considered extensively, especially in the finite case (see e.g. the treatment of Orlik and Terao [30]). These actions are not translative, and thus fall at the margins of our present

[2] See Definition 1.3.

Table 1
A tabular overview of our setup and our results.

G-semimatroid $\mathfrak{G}$	Loc. ranked triple $\mathcal{S}_{\mathfrak{G}}$	Multiplicity $m_{\mathfrak{G}}$	Poset $\mathcal{P}_{\mathfrak{G}}$	Polynomial $T_{\mathfrak{G}}(x, y)$	Modules $M_{\mathfrak{G}}$
Weakly translative	well-defined (Theorem A)		$\chi_{\mathcal{P}_{\mathfrak{G}}}(t) = (-1)^r T_{\mathfrak{G}}(1-t, 0)$ (Theorem F)		
Translative	Pseudo-arithmetic (Proposition 7.28)			Deletion–contraction recursion (Theorem G)	
Translative and normal	Almost-arithmetic (P, A.1.2, A2) (Theorem B)				
...and $\mathcal{S}_{\mathfrak{G}}$ a semimatroid				Activity decomposition (Theorem H)	
Arithmetic	Arithmetic (Theorem C)				
Centered	Matroid				
Representable and centered	Arithmetic matroid dual to that of [5]		Poset of layers of toric arrangement	Arithmetic Tutte polynomial	Representable matroid over $\mathbb{Z}$

treatment. Still, we would like to mention them as a motivation for further investigation of non-translative actions — e.g., the case where $(E, \underline{\mathrm{rk}})$ is a polymatroid. △

Example 4.2 *(Toric arrangements).* The natural setting in order to develop a combinatorial framework for toric arrangements is that of the group $\mathbb{Z}^d$ acting by translations on an affine hyperplane arrangement on $\mathbb{C}^d$ (see Section 2). Such actions will often fail to be centered. Therefore we will try to state our results as much as possible without centrality assumptions, adding them only when needed in order to establish a link to the arithmetic and algebraic matroidal structures appeared in the literature. △

The next examples will refer to Fig. 10 and Fig. 11. These are to be interpreted as the depiction of a fundamental region for an action of $\mathbb{Z}^2$ by unit translations in orthogonal directions (vertical and horizontal) on an arrangement of pseudolines in $\mathbb{R}^2$ (see Example 1.7) which, then, can be recovered by "tiling" the plane by translates of the depicted squares. Notice that the intersection poset of any arrangement of pseudolines is trivially a geometric semilattice, and thus defines a simple semimatroid. We will call a, b, c, d the orbits of the a_i, b_i, c_i, d_i, respectively. (Thus, $a = Ga_0 = \{a_i\}_{i\in\mathbb{Z}}$, etc.)

Example 4.3. The $\mathbb{Z}^2$-semimatroid described in Fig. 11 is clearly almost-arithmetic, but it cannot be arithmetic because the multiplicity $m_{\mathfrak{G}}(\{c, b, a\}) = 3$ does not divide $m_{\mathfrak{G}}(\{c, a\}) = 4$, violating (A.1.1). △

Example 4.4. One readily verifies that the $\mathbb{Z}^2$-semimatroid described at the left-hand side of Fig. 10 is arithmetic. However, $M_{\mathfrak{G}}$ is not a matroid over $\mathbb{Z}$. Indeed, the requirement of Definition 1.26 fails for the square

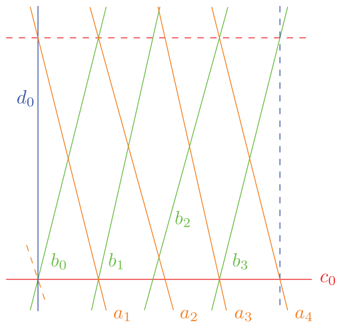

Fig. 11. Figure for Example 4.3.

$$\begin{tikzcd}
M_{\mathfrak{S}}(\{b\}) \cong \mathbb{Z} \arrow[r, "?"] \arrow[d, "?"'] & M_{\mathfrak{S}}(\{b, c\}) \cong \mathbb{Z}_2 \arrow[d] \\
M_{\mathfrak{S}}(\{a, b\}) \cong \mathbb{Z}_4 \arrow[r] & M_{\mathfrak{S}}(\{a, b, c\}) \cong \{0\}
\end{tikzcd}$$

where the condition that the maps be surjections with cyclic kernel determines everything up to leaving two possibilities for the left-hand side vertical map: neither of these gives the required pushout. △

Remark 4.5. Examples where $M_{\mathfrak{S}}$ is a nonrepresentable matroid over $\mathbb{Z}$ can easily be generated in a trivial way, e.g. by considering trivial group actions on nonrepresentable matroids. We do not know whether there is a periodic pseudoarrangement for which $M_{\mathfrak{S}}$ is a nonrepresentable matroid over $\mathbb{Z}$.

Example 4.6 *(The representable case).* The arrangement on the top left of Fig. 7 is a periodic affine arrangement in the sense of Section 2: thus, the associated $M_{\mathfrak{S}}$ is a representable matroid over $\mathbb{Z}$. △

Example 4.7 *(Crystallographic root systems).* An important family of representable examples is that of the periodic hyperplane arrangements arising as the reflection arrangements of the affine Coxeter groups associated to crystallographic root systems, where the weight lattice acts by translation. In this setting, some enumerative results in terms of Dynkin diagrams were obtained by Moci [28]. △

5. Finitary geometric semilattices

In this section we study posets associated to finitary semimatroids. This leads us to consider geometric semilattices in the sense of Wachs and Walker [37]. Our goal is to prove a finitary version of the equivalence between simple semimatroids and geometric semilattices given in [2].

We start by recalling some basic terminology about partially ordered sets. The reader already familiar with poset theory may skip to Definition 5.2. We refer to Stanley's book [34] for a comprehensive introduction to this topic.

A partially ordered set (for short *poset*) is a set $\mathcal{P}$ endowed with a partial order relation, i.e., a transitive, antisymmetric and reflexive binary relation which we denote by $\leq$. As is customary, we write $p < q$ if $p \leq q$ and $p \neq q$. Given $x \in \mathcal{P}$ we write $\mathcal{P}_{\leq x} := \{p \in \mathcal{P} \mid p \leq x\}$ for the set of elements below x, and define $\mathcal{P}_{\geq x}$ analogously. We say that the poset $\mathcal{P}$ is *bounded below* (resp. bounded above) if it possesses a unique minimal (resp. maximal) element, that is an element $\hat{0} \in \mathcal{P}$ (resp. $\hat{1} \in \mathcal{P}$) with $\mathcal{P}_{\geq \hat{0}} = \mathcal{P}$ (resp. $\mathcal{P}_{\leq \hat{1}} = \mathcal{P}$). If $\mathcal{P}$ is bounded below and bounded above, we call it simply *bounded*.

The *join* of a subset $X \subseteq \mathcal{P}$, written $\vee X$, if it exists, is defined by

$$\mathcal{P}_{\geq \vee X} = \{p \in \mathcal{P} \mid p \geq x \text{ for all } x \in X\}.$$

Analogously the *meet* $\wedge X$, if it exists, is defined by

$$\mathcal{P}_{\geq \wedge X} = \{p \in \mathcal{P} \mid p \leq x \text{ for all } x \in X\}.$$

If $X = \{x, y\}$, we write $x \vee y := \vee X$ and $x \wedge y := \wedge X$.

If the meet of any two elements exists, then so does the meet of every finite set of elements, and $\mathcal{P}$ is called *meet-semilattice*. *Join-semilattices* are defined accordingly. If $\mathcal{P}$ is both a meet- and a join-semilattice, then it is called a *lattice*.

A *chain* in $\mathcal{P}$ is any totally ordered subset, i.e., any $\omega = \{p_1, \ldots, p_k\} \subseteq \mathcal{P}$ such that $p_0 < p_1 < \cdots < p_k$. The *length* of such a chain is $\ell(\omega) = |\omega| - 1$. In this paper **we assume throughout that all posets are chain-finite**, i.e., all chains have finite length. The (closed) *interval* between $p, q \in \mathcal{P}$ is the set

$$[p, q] := \{x \in \mathcal{P} \mid p \leq x \leq q\}.$$

We say that q *covers* p if $[p, q] = \{p, q\}$. The *atoms* of a bounded below poset $\mathcal{P}$ are the elements that cover $\hat{0}$. A bounded below poset $\mathcal{P}$ is called *atomic* if every element is a join of atoms, i.e., if for every $p \in \mathcal{P}$ there is a set A of atoms of $\mathcal{P}$ such that $p = \vee A$.

A poset $\mathcal{P}$ is called *ranked* if there is a function $\mathrm{rk} : \mathcal{P} \to \mathbb{N}$ such that $\mathrm{rk}(q) = \mathrm{rk}(p) + 1$ whenever q covers p. If $\mathcal{P}$ is bounded below we assume $\mathrm{rk}(\hat{0}) = 0$, and the condition above is equivalent to the fact that, for every $x \in \mathcal{P}$, all maximal chains of $\mathcal{P}_{\leq x}$ have the same (finite) length. In general, a poset $\mathcal{P}$ is called *graded* if all maximal chains have the same (finite) length. A ranked lattice $\mathcal{P}$ is called *semimodular* if, for all $x, y \in \mathcal{P}$, $\mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y) = \mathrm{rk}(x) + \mathrm{rk}(y)$.

Definition 5.1. A *geometric lattice* is a finite, atomic and semimodular lattice.

A set A of atoms of a ranked, bounded below poset, is called *independent* if the join $\vee A$ exists and satisfies $\mathrm{rk}(\vee A) = |A|$.

A *morphism of posets* is an order preserving map, i.e., a morphism between posets $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \preccurlyeq)$ is a function $f : \mathcal{P} \to \mathcal{Q}$ such that $f(p_1 \leq p_2)$ implies $f(p_1) \preccurlyeq f(p_2)$, for all $p_1, p_2 \in \mathcal{P}$. An *isomorphism* of posets is a bijective morphism of posets with order-preserving inverse.

Definition 5.2 *(See Theorem 2.1 in [37])*. A (chain-finite) ranked meet-semilattice $\mathcal{L}$ is called a *finitary geometric semilattice* if it satisfies the following conditions.

(G3) *There is* $N \in \mathbb{N}$ *such that* every (maximal) interval in $\mathcal{L}$ is a (finite) geometric lattice *with at most* N *atoms.*

(G4) For every independent set A of atoms of $\mathcal{L}$ and every $x \in \mathcal{L}$ such that $\mathrm{rk}_{\mathcal{L}}(x) < \mathrm{rk}_{\mathcal{L}}(\vee A)$, there is $a \in A$ with $a \not\leq x$ and such that $x \vee a$ exists.

Remark 5.3. The definition given in [37] of a finite geometric semilattice is that of a finite ranked meet-semilattice which satisfies:

(G1) Every element is a join of atoms.

(G2) The collection of independent sets of atoms is the set of independent sets of a matroid.

In the finite case, Wachs and Walker prove that this is equivalent to Definition 5.2, which we choose to take as our definition because of its more immediate generalization to the infinite case. We keep, for consistency, the labeling of the conditions as in [37].

In passing to the infinite case we have added the part of (G3) that is written in italic. If the poset is finite, then this addition is redundant, and it does not appear in [37]. Notice that Theorem E remains valid if the italic part of (G3) and the requirement finite-dimensionality of $\mathcal{C}$ in Definition 1.1 are simultaneously dropped (these only play a role in the parts of the proof marked by (†)).

Remark 5.4. In view of the proof of Theorem E and for later reference we note that finitary semimatroids satisfy the following properties (i.e., a "local" version of (R2) and a stronger version of (CR1) and (CR2)).

(R2') If $X \cup x \in \mathcal{C}$ then $\mathrm{rk}(X \cup x) - \mathrm{rk}(X)$ equals 0 or 1.

(CR1') If $X, Y \in \mathcal{C}$ and $\mathrm{rk}(X) = \mathrm{rk}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$ and $\mathrm{rk}(X \cup Y) = \mathrm{rk}(Y)$.

(CR2') If $X, Y \in \mathcal{C}$ and $\mathrm{rk}(X) < \mathrm{rk}(Y)$, then $X \cup y \in \mathcal{C}$ and $\mathrm{rk}(X \cup y) = \mathrm{rk}(X) + 1$ for some $y \in Y - X$.

The proof is analogous to that in the finite case given in [2, Section 2].

Proof of Theorem E. Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk}_\mathcal{C})$ be a finitary semimatroid. Recall from Definition 1.8 the closure operator cl and the poset of flats $\mathcal{L}(\mathcal{S})$ of $\mathcal{S}$. We begin by showing that $\mathcal{L}(\mathcal{S})$ is a geometric semilattice.

- $\mathcal{L}(\mathcal{S})$ *is a chain-finite ranked meet semilattice.* Given flats X, Y of $\mathcal{S}$, the subset $X \cap Y$ is also central and its closure $\mathrm{cl}(X \cap Y) \in \mathcal{L}(\mathcal{S})$ is a lower bound of X and Y in $\mathcal{L}(\mathcal{S})$ by Remark 1.9. Now suppose $A \in \mathcal{L}(\mathcal{S})$ is a lower bound of X, Y in $\mathcal{L}(\mathcal{S})$, thus $A \subseteq X, Y$. In particular, this means $A \subseteq X \cap Y \subseteq \mathrm{cl}(X \cap Y)$. Therefore, the set $\mathrm{cl}(X \cap Y)$ is the meet of X and Y in $\mathcal{L}(\mathcal{S})$. Now, (CR2') implies that $\mathcal{L}(\mathcal{S})$ is ranked with rank function $\mathrm{rk}_\mathcal{L} := \mathrm{rk}_\mathcal{C}$. In particular, an infinite chain in $\mathcal{L}(\mathcal{S})$ would violate boundedness of the rank function $\mathrm{rk}_\mathcal{C}$.
- *Condition (G3).* If X is a maximal flat of $\mathcal{S}$, then in particular $\mathrm{rk}_\mathcal{C}$ is defined for every subset of X and satisfies axioms (R1–R3). Thus $\mathrm{rk}_\mathcal{C}$ defines a matroid M on X whose closure operator coincides with cl (since X is closed, cl restricts to a function $2^X \to 2^X$), and thus the lattice of flats of M is isomorphic to the interval $[\hat{0}, X]$ in $\mathcal{L}(\mathcal{S})$, proving that this interval is indeed a geometric lattice.
  (†) For the bound on the number of atoms of intervals, notice that a top simplex X of $\mathcal{C}$ is a maximal flat of $\mathcal{S}$, hence its cardinality is at least the number of atoms in $\mathcal{L}(\mathcal{S})_{\leq X}$. Thus, if d is the (finite) dimension of the simplicial complex $\mathcal{C}$, the poset $\mathcal{L}(\mathcal{S})$ satisfies (G3) with $N = d + 1$.
- *Condition (G4).* Now let A be an independent set of atoms in $\mathcal{L}(\mathcal{S})$ and X a flat of $\mathcal{S}$ such that $\mathrm{rk}_\mathcal{C}(X) < \mathrm{rk}_\mathcal{C}(\vee A) = \mathrm{rk}_\mathcal{C}(\mathrm{cl}(\cup A)) = \mathrm{rk}_\mathcal{C}(\cup A)$. By (CR2), there is an element $a \in \cup A \backslash X$ such that $X \cup a \in \mathcal{C}$. In particular, $\mathrm{cl}(\{a\})$ is an atom from A such that $\mathrm{cl}(\{a\}) \nleq X$ in $\mathcal{L}(\mathcal{S})$. Furthermore, by Remark 1.9 the set $X \cup \mathrm{cl}(\{a\})$ is a subset of $\mathrm{cl}(X \cup a)$ – and hence central as well. So the join $X \vee \mathrm{cl}(\{a\}) = \mathrm{cl}(X \cup \mathrm{cl}(\{a\}))$ exists and (G4) is satisfied.
  This concludes the proof that $\mathcal{L}(\mathcal{S})$ is a finitary geometric semilattice.

Conversely, let $\mathcal{L}$ be a finitary geometric semilattice. Let $S_\mathcal{L}$ denote the set of atoms of $\mathcal{L}$ and set

$$\mathcal{C}_\mathcal{L} = \{X \subseteq S_\mathcal{L} \mid \vee X \in \mathcal{L}\}.$$

Moreover, we define the function

$$\mathrm{rk}_{\mathcal{C}_\mathcal{L}} : \mathcal{C}_\mathcal{L} \to \mathbb{N}, \quad X \mapsto \mathrm{rk}_\mathcal{L}(\vee X).$$

Now suppose $Y \subseteq X \in \mathcal{C}_\mathcal{L}$. Then $\vee X$ is an upper bound for Y and thus the join $\vee Y$ exists (since $\mathcal{L}$ is a meet-semilattice). Hence, the collection $\mathcal{C}_\mathcal{L}$ is an abstract simplicial complex. (†) Since $|X| \leq |S_\mathcal{L} \cap \mathcal{L}_{\leq \vee X}|$ for all $X \in \mathcal{C}_\mathcal{L}$, the cardinality of any simplex is bounded by N; thus $\mathcal{C}$ is finite-dimensional.
We will now show that $\mathcal{S}_\mathcal{L} := (S_\mathcal{L}, \mathcal{C}_\mathcal{L}, \mathrm{rk}_{\mathcal{C}_\mathcal{L}})$ is a finitary semimatroid with semilattice of flats $\mathcal{L}(\mathcal{S}_\mathcal{L})$ isomorphic to $\mathcal{L}$.

- *Axioms (R1)–(R3).* For every $X \in \mathcal{C}_{\mathcal{L}}$, the join $\vee X$ exists and the interval $[\hat{0}, \vee X]$ is a geometric lattice by (G3). Thus (e.g., by Remark 1.10) it defines a matroid M_X with ground set the atoms in $[\hat{0}, \vee X]$, whose rank function is a restriction of $\mathrm{rk}_{\mathcal{L}}$ (hence of $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}$). Thus (R1) holds for X because it holds in M_X. Moreover, (R2) holds for every $X \subseteq Y \in \mathcal{C}_{\mathcal{L}}$ because it holds in M_Y, and (R3) holds for every X, Y with $X \cup Y \in \mathcal{C}_{\mathcal{L}}$ because it does in $M_{X\cup Y}$.
- *Axiom (CR1).* Take $X, Y \in \mathcal{C}_{\mathcal{L}}$ with $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X \cap Y)$, i.e., $\mathrm{rk}_{\mathcal{L}}(\vee X) = \mathrm{rk}_{\mathcal{L}}(\vee(X \cap Y))$. Since $\mathcal{L}$ is a ranked poset, the former rank equality and the evident relation $\vee(X \cap Y) \leq \vee X$ imply $\vee(X \cap Y) = \vee X$. So

  $$\vee X = \vee(X \cap Y) \leq \vee Y,$$

  that is to say every upper bound of Y is also an upper bound of X. Hence $\vee(X \cup Y) = \vee Y$ and $X \cup Y \in \mathcal{C}$, and (CR1) is satisfied.
- *Axiom (CR2).* Let X, Y be in $\mathcal{C}_{\mathcal{L}}$ and such that $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X) < \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(Y)$. Choose an independent set $A \subseteq Y$ with $\vee A = \vee Y$. Property (CR2) for X and Y now follows applying (G4) to X and A.
- *There is a poset isomorphism $\mathcal{L} \simeq \mathcal{L}(\mathcal{S}_{\mathcal{L}})$.* Let $\varphi : \mathcal{L} \to \mathcal{L}(\mathcal{S}_{\mathcal{L}})$ be defined by

  $$\varphi(x) := \{a \in S_{\mathcal{L}} \mid a \leq x\}. \tag{2}$$

  For φ to be well-defined, we must check that, for all $x \in \mathcal{L}$, $\varphi(x)$ is a flat of $\mathcal{S}_{\mathcal{L}}$. Let $x \in \mathcal{L}$. First, by (G3) we have $\vee\varphi(x) = x$ and thus $\varphi(x) \in \mathcal{C}_{\mathcal{L}}$. Now suppose b is an element of S such that $\varphi(x) \cup \{b\} \in \mathcal{C}_{\mathcal{L}}$ and $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(\varphi(x) \cup b) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(\varphi(x))$. This means that $\mathrm{rk}_{\mathcal{L}}(\vee(\varphi(x) \cup b)) = \mathrm{rk}_{\mathcal{L}}(\vee\varphi(x))$, and since clearly $\vee(\varphi(x) \cup b) \leq \vee\varphi(x) = x$, from the fact that $\mathcal{L}$ is ranked we conclude $\vee(\varphi(x) \cup b) \leq \vee\varphi(x) = x$. In particular, $b \leq x$, so $b \in \varphi(x)$. This proves that the set $\varphi(x)$ is closed, hence a flat of $\mathcal{S}_{\mathcal{L}}$.
  The function φ is injective because $\mathcal{L}$ is atomic. To check surjectivity, let Y be a flat of $\mathcal{S}_{\mathcal{L}}$. We have to find some $x \in \mathcal{L}$ with $\varphi(x) = Y$, and indeed $x = \vee Y$ will do.
  Moreover, comparing the definition of φ in Equation (2) one readily checks the following equivalences

  $$\varphi(x) \leq \varphi(y) \Leftrightarrow \varphi(x) \subseteq \varphi(y) \Leftrightarrow x \leq y$$

  Thus, both φ and its inverse are order-preserving, and φ is the required isomorphism.

The semimatroid $\mathcal{S}_{\mathcal{L}} = (S_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}}, \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}})$ is simple by construction. We are left with showing that for every simple semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ with a poset-isomorphism

$$\psi : \mathcal{L}(\mathcal{S}) \xrightarrow{\cong} \mathcal{L}$$

we can construct an isomorphism between $\mathcal{S}$ and $\mathcal{S}_{\mathcal{L}}$.

Since $\mathcal{S}$ is simple, for every $x \in S$ the set $\{x\}$ is closed. Thus, ψ induces a natural bijection

$$\psi_S : S \to S_{\mathcal{L}}, \qquad \{\psi_S(x)\} = \psi(\{x\}).$$

To see that ψ_S induces a well-defined function $\mathcal{C} \to \mathcal{C}_{\mathcal{L}}$, consider any $X = \{x_1, \ldots, x_k\} \in \mathcal{C}$. Then, using the definition of ψ_S and the fact that ψ is an isomorphism,

$$\bigvee_{i=1}^{k} \{\psi_S(x_i)\} = \bigvee_{i=1}^{k} \psi(\{x_i\}) = \psi(\bigvee_{i=1}^{k} \{x_i\}),$$

hence the right-hand side exists in $\mathcal{L}$, and thus $\psi_S(X) \in \mathcal{C}_{\mathcal{L}}$.

An analogous argument using ψ_S^{-1} (together with the fact that ψ_S is monotone by definition) shows that in fact ψ_S induces an *isomorphism* of simplicial complexes $\mathcal{C} \cong \mathcal{C}_{\mathcal{L}}$.

It remains to show that ψ_S preserves ranks of central sets. For this consider any $X = \{x_1, \ldots, x_k\} \in \mathcal{C}$ and compute

$$\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{L}(\mathcal{S})}(\bigvee_i \{x_i\}) = \mathrm{rk}_{\mathcal{L}}(\bigvee_i \psi(\{x_i\})) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(\psi_S(X)). \quad \square$$

6. The underlying matroid of a group action

This section is devoted to the proof of Theorem A. Let $\mathfrak{S}$ be a G-semimatroid associated to an action of G on a semimatroid $(S, \mathcal{C}, \mathrm{rk})$. Recall from Section 3 the set $E_{\mathfrak{S}} := S/G$ of orbits of elements, the family $\underline{\mathcal{C}} = \{\underline{X} \subseteq E_{\mathfrak{S}} \mid X \in \mathcal{C}\}$, and that we only consider actions for which $E_{\mathfrak{S}}$ is finite.

For every $A \subseteq E_{\mathfrak{S}}$ define

$$J(A) := \{X \in \mathcal{C} \mid \underline{X} \subseteq A\}$$

and write $J_{\max}(A)$ for the set of inclusion-maximal elements of $J(A)$.

Lemma 6.1. *For every $X, Y \in J_{\max}(A)$, $\mathrm{rk}(X) = \mathrm{rk}(Y)$.*

Proof. By way of contradiction assume $\mathrm{rk}(X) > \mathrm{rk}(Y)$. Then with (CR2) we can find $x \in X \setminus Y$ with $Y \cup x \in \mathcal{C}$ and $\underline{Y \cup x} \subseteq A$, contradicting maximality of Y. $\square$

Definition 6.2. For any $A \subseteq E_{\mathfrak{S}}$ choose $X \in J_{\max}(A)$ and let $\underline{\mathrm{rk}}(A) := \mathrm{rk}(X)$, in agreement with Definition 3.2. Lemma 6.1 shows that this is well-defined and independent on the choice of X.

Remark 6.3. For all $A \subseteq E_{\mathfrak{S}}$ we have

$$\underline{\mathrm{rk}}(A) = \max\{\underline{\mathrm{rk}}(A') \mid A' \subseteq A, A' \in \underline{\mathcal{C}}\},$$

because $A' \subseteq A$ implies $J(A') \subseteq J(A)$.

Proposition 6.4. *The pair $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$ always satisfies (R2) and (R3), and thus defines a polymatroid on $E_{\mathfrak{S}}$. Moreover, $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$ satisfies (R1) if and only if the action is weakly translative.*

Proof.
• $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$ *is a polymatroid.* Property (R2) is trivial, we check (R3). Consider $A, A' \subseteq E_{\mathfrak{S}}$, and choose $B_0 \in J_{\max}(A \cap A')$. By Lemma 6.1,

$$\mathrm{rk}(B_0) = \underline{\mathrm{rk}}(A \cap A'). \tag{*}$$

In particular, $B_0 \in J(A)$ and thus we can find $B_1 \in J(A)$ such that

$$B_0 \cup B_1 \in J_{\max}(A) \tag{*}$$

and a maximal $B_2 \in J(A')$ such that $B_0 \cup B_1 \cup B_2$ is in $J(A' \cup A)$. Then,

$$B_0 \cup B_1 \cup B_2 \in J_{\max}(A' \cup A), \tag{*}$$

because otherwise we could complete it with some $B_2' \in J(A)$ in order to get an element of $J_{\max}(A \cup A')$ – but then, $B_0 \cup B_1 \cup B_2' \supseteq B_0 \cup B_1 \in J_{\max}(A)$, thus $B_2' = \emptyset$ by the choice of B_1. Using the identities (*) and axiom (R3) for $(S, \mathcal{C}, \mathrm{rk})$ we obtain

$$\begin{aligned}\underline{\mathrm{rk}}(A \cap A') + \underline{\mathrm{rk}}(A \cup A') - \underline{\mathrm{rk}}(A) &= \mathrm{rk}(B_0) + \mathrm{rk}(B_0 \cup B_1 \cup B_2) - \mathrm{rk}(B_0 \cup B_1) \\ &\leq \mathrm{rk}(B_0 \cup B_2) \leq \underline{\mathrm{rk}}(A'),\end{aligned}$$

where the last inequality follows from $\underline{B_0 \cup B_2} \subseteq A'$. This proves that $\underline{\mathrm{rk}}$ satisfies (R3).
• *Weakly translative implies (R1)*

Suppose that the action is weakly translative. For (R1) we need to show that $0 \leq \underline{\mathrm{rk}}(A) \leq |A|$ for every $A \subseteq E_{\mathfrak{S}}$. The left hand side inequality is trivial. Consider $A \subseteq E_{\mathfrak{S}}$ and choose $X \in J_{\max}(A)$.

Claim. *For every $x \in X$ with $g(x) \in X$ we have* $\mathrm{rk}(X) = \mathrm{rk}(X \setminus g(x))$.

Proof of claim. Using (R3) in $(S, \mathcal{C}, \mathrm{rk})$ on the sets $X \setminus g(x)$ and $\{x, g(x)\}$, we obtain

$$\mathrm{rk}(X) + \mathrm{rk}(x) \leq \mathrm{rk}(X \setminus g(x)) + \mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(X \setminus g(x)) + \mathrm{rk}(x)$$

where in the last equality we used weak translativity of the action. Thus we get $\mathrm{rk}(X) \leq \mathrm{rk}(X \setminus g(x))$ and, the other inequality being trivial from (R2), we have the claimed equality. $\square$

Choose a (central) system X' of representatives of the orbits in $\underline{X}$. We obtain the claimed inequality by computing

$$\underline{\mathrm{rk}}(A) = \mathrm{rk}(X) = \mathrm{rk}(X') \leq |X'| = |\underline{X}| \leq |A| \tag{3}$$

where the second equality holds because of the claim above.

• *(R1) implies weakly translative.* By contraposition. If the action is not weakly translative, choose $x \in S$ and $g \in G$ violating the weak translativity condition, and consider $A := \{Gx\}$. First notice that x cannot be a loop, since if $\mathrm{rk}(x) = 0$ then $\mathrm{rk}(g(x)) = 0$ and $\mathrm{rk}(\{x, g(x)\})$ must equal 0 (otherwise it would contain an independent set of rank 1, implying that x is not a loop), thus $\mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(x)$ and x would not violate the weak translativity condition. Hence it must be $\mathrm{rk}(x) = 1$, and we have $\underline{\mathrm{rk}}(A) \geq \mathrm{rk}(\{x, g(x)\}) > \mathrm{rk}(x) = 1 = |\{A\}|$, contradicting (R1). $\square$

Corollary 6.5. *If the action is weakly translative, for all $X \in \mathcal{C}$ we have $\mathrm{rk}(X) = \underline{\mathrm{rk}}(\underline{X})$.*

Proof. This is a consequence of Equation (3) in the previous proof, and of the discussion preceding it. $\square$

Remark 6.6. The matroid $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ is, in some sense an 'artificial' construct, although in some cases useful. For instance, when $(S, \mathcal{C}, \mathrm{rk})$ is the semimatroid of a periodic arrangement of hyperplanes in real space associated to a toric arrangement $\mathscr{A}$, then $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ is the matroid of the arrangement $\mathscr{A}_0$ which plays a key role in [6,11,12].

Proposition 6.7. *Let $\mathfrak{G}$ be weakly translative. Then $\mathcal{S}_{\mathfrak{G}} := (E_{\mathfrak{G}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$ is a locally ranked triple satisfying (CR2).*

Proof. Proposition 6.4 implies that (R1), (R2), (R3) hold.

For (CR2), let $A, B \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) < \underline{\mathrm{rk}}(B)$ and choose $X \in \lceil A \rceil^{\mathcal{C}}$ and $Y \in \lceil B \rceil^{\mathcal{C}}$. Then, by Corollary 6.5, $\mathrm{rk}(X) < \mathrm{rk}(Y)$. Using (CR2') in $\mathcal{S}$ (cf. Remark 5.4) we find $y \in Y \setminus X$ with $X \cup y \in \mathcal{C}$ and $\mathrm{rk}(X \cup y) > \mathrm{rk}(X)$. Set $b := \{Gy\}$. Then, $A \cup b = \underline{X \cup y} \in \underline{\mathcal{C}}$ and $b \in B \setminus A$ (otherwise $b \in A$, thus – using Corollary 6.5 – $\mathrm{rk}(X \cup y) = \underline{\mathrm{rk}}(A \cup b) = \underline{\mathrm{rk}}(A) = \mathrm{rk}(X)$, a contradiction). $\square$

7. Translative actions

We now proceed towards establishing Theorem B. The main idea in this section is to associate a diagram of finite sets and injective maps to every molecule of the quotient triple $\mathcal{S}_{\mathfrak{G}}$ (see Example 7.24 below). In the representable case, this structure specializes to the inclusion pattern of integer points in semiopen parallelepipeds as well as to that of layers of the associated toric arrangement. In general, these diagrams will allow us in later sections to extend to the general (nonrepresentable, non-arithmetic) case some combinatorial decompositions given in [9] for representable arithmetic matroids, most notably Theorem H.

Recall the definitions in Section 3 and, in particular, that $\mathfrak{S}$ denotes a G-semimatroid corresponding to the action of a group G on a semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$. In this section we suppose this action always to be cofinite and translative. In particular, we can consider the associated locally ranked triple $\mathcal{S}_{\mathfrak{S}} = (E_{\mathfrak{S}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$ with multiplicity function $m_{\mathfrak{S}}$.

7.1. Maps between sets of "central orbits"

Definition 7.1. Given $A \in \underline{\mathcal{C}}$ and $a_0 \in A$ define

$$w_{A,a_0} : \lceil A \rceil^{\mathcal{C}} \to \lceil A \setminus a_0 \rceil^{\mathcal{C}}, \quad X \mapsto X \setminus a_0, \tag{4}$$

and notice that, since it is G-equivariant, it induces a function

$$\underline{w}_{A,a_0} : \lceil A \rceil^{\mathcal{C}}/G \to \lceil A \setminus a_0 \rceil^{\mathcal{C}}/G. \tag{5}$$

Remark 7.2. When w_{A,a_0} is injective then $\underline{w}_{A,a_0}$ also is. This can be seen in many ways – for instance, by noting that any injective map of G-sets is a split monomorphism (e.g., see [40]), and the splitting G-map induces a splitting of $\underline{w}_{A,a_0}$.

Lemma 7.3. *Let $\mathfrak{S}$ be translative.*

(a) If $x_0 \in X \in \mathcal{C}$ with $\mathrm{rk}(X) = \mathrm{rk}(X \setminus x_0) + 1$, then $Y \cup g(x_0) \in \mathcal{C}$ for all $g \in G$ and all $Y \in \mathcal{C}$ with $\underline{Y} = \underline{X \setminus x_0}$.

(b) If $a_0 \in A \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) = \underline{\mathrm{rk}}(A \setminus a_0) + 1$, then w_{A,a_0} is surjective and, for any choice of $x_0 \in a_0$, a right inverse of w_{A,a_0} is given by

$$\widehat{w}_{A,a_0} : \lceil A \setminus a_0 \rceil^{\mathcal{C}} \to \lceil A \rceil^{\mathcal{C}}, \; Y \mapsto Y \cup x_0. \tag{6}$$

Moreover, $\underline{w}_{A,a_0}$ is surjective. In particular, $m_{\mathfrak{S}}(A) \geq m_{\mathfrak{S}}(A \setminus a_0)$.

(c) If $a_0 \in A \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) = \underline{\mathrm{rk}}(A \setminus a_0)$, then w_{A,a_0} is injective and thus $m_{\mathfrak{S}}(A) \leq m_{\mathfrak{S}}(A \setminus a_0)$.

Proof.

(a) Let X, x_0 be as in the claim. For all $g \in G$ consider the central set $g(X)$ of rank $\mathrm{rk}(g(X)) = \mathrm{rk}(X) > \mathrm{rk}(X \setminus x_0)$. By (CR2) there is some $y \in g(X) \setminus (X \setminus x_0)$ with $y \cup (X \setminus x_0) \in \mathcal{C}$ and $\mathrm{rk}(y \cup (X \setminus x_0)) = \mathrm{rk}(X)$. This y must be $g(x_0)$ because every other element $y' \in g(X) \setminus (X \cup g(x_0))$ is of the form $y' = g(x')$ ($\notin X$) for some $x' \in X$, thus $y' \cup (X \setminus x_0) \in \mathcal{C}$ would imply $\{x', g(x')\} \in \mathcal{C}$ which, since by construction $x' \neq g(x')$, is forbidden by the fact that the action is translative. Thus $(X \setminus x_0) \cup g(x_0) \in \mathcal{C}$ for all $g \in G$. Now consider any Y with $\underline{Y} = \underline{X \setminus x_0}$ and notice that with Lemma 6.1 we have the first equality in the following expression

$$\mathrm{rk}(Y) = \mathrm{rk}(X \setminus x_0) < \mathrm{rk}(X) = \mathrm{rk}((X \setminus x_0) \cup g(x_0))$$

(where the inequality holds by assumption and the last equality derives from the choice of $y = g(x_0)$ above). Thus by (CR2) there must be $x \in (X \setminus x_0) \cup g(x_0)$ with $Y \cup x \in \mathcal{C}$ and $\mathrm{rk}(Y \cup x) = \mathrm{rk}(Y) + 1$. Since Y consists of translates of elements of X, as above the fact that the action is translative forces $x = g(x_0)$.

Towards (b) and (c), choose any $X \in \lceil A \rceil^{\mathcal{C}}$ and let $x_0 \in X$ be a representative of a_0. By the definition of $\underline{\mathrm{rk}}$ (Definition 6.2) and since translativity allows us to apply Corollary 6.5, we conclude that $\mathrm{rk}(X \setminus x_0) = \mathrm{rk}(X)$ if and only if $\underline{\mathrm{rk}}(A \setminus a_0) = \underline{\mathrm{rk}}(A)$.

(b) Suppose $\underline{\mathrm{rk}}(A \setminus a_0) = \underline{\mathrm{rk}}(A) - 1$. Part (a) ensures that the function $\widehat{w}_{A,a_0}$ is well-defined. Clearly, it is injective and $w_{A,a_0} \circ \widehat{w}_{A,a_0} = \mathrm{id}$. In particular, w_{A,a_0} is surjective. Moreover, if we fix a representative $Y^{\mathcal{O}}$ of every element $\mathcal{O} \in \lceil A \setminus a_0 \rceil^{\mathcal{C}}/G$ we see that the assignment

$$\lceil A \setminus a_0 \rceil^{\mathcal{C}}/G \to \lceil A \rceil^{\mathcal{C}}/G, \quad \mathcal{O} \mapsto G\widehat{w}_{A,a_0}(Y^{\mathcal{O}}) \tag{7}$$

defines a (noncanonical) section of $\underline{w}_{A,a_0}$. This proves surjectivity of $\underline{w}_{A,a_0}$, which implies the stated inequality.

(c) Suppose now $\underline{\mathrm{rk}}(A \setminus a_0) = \underline{\mathrm{rk}}(A)$ and consider $X_1, X_2 \in \lceil A \rceil^{\mathcal{C}}$. Since the action is translative the sets $X_1 \cap a_0$ and $X_2 \cap a_0$ each consist of a single element, say $x_{0,1}$ and $x_{0,2}$ respectively. If moreover w_{A,a_0} maps both X_1, X_2 to the same $Y = X_1 \setminus a_0 = X_2 \setminus a_0$, then $Y \cup x_{0,1}$ and $Y \cup x_{0,2}$ are both central and of the same rank, equal to the rank of Y. By (CR1) then $Y \cup \{x_{0,1}, x_{0,2}\} \in \mathcal{C}$, thus $\{x_{0,1}, x_{0,2}\} \in \mathcal{C}$ and since the action is translative we must have $x_{0,1} = x_{0,2}$, hence $X_1 = X_2$. This proves that w_{A,a_0} is injective and, with Remark 7.2, the stated inequality. □

Remark 7.4. More generally, for every $A \in \underline{\mathcal{C}}$ and every $A' \subseteq A$ we can consider

$$w_{A,A'} : \lceil A \rceil^{\mathcal{C}} \to \lceil A \setminus A' \rceil^{\mathcal{C}}, \quad X \mapsto X \setminus \cup A'$$

and the associated map $\underline{w}_{A,A'} : \frac{\lceil A \rceil^{\mathcal{C}}}{G} \to \frac{\lceil A \setminus A' \rceil^{\mathcal{C}}}{G}$.

Notice that, given any enumeration $a'_1, \ldots, a'_l$ of A', we have

$$w_{A,A'} = w_{A,a'_1} \circ \cdots \circ w_{A,a'_l}, \quad \underline{w}_{A,A'} = \underline{w}_{A,a'_1} \circ \cdots \circ \underline{w}_{A,a'_l}.$$

Corollary 7.5. *For every molecule* (R, F, T) *of* $\mathcal{S}_{\mathfrak{S}}$,

(a) $w_{R\cup F\cup T,T}$ *and* $\underline{w}_{R\cup F\cup T,T}$ *are injective,*
(b) $w_{R\cup F\cup T,F}$ *and* $\underline{w}_{R\cup F\cup T,F}$ *are surjective.*

7.2. Labeling orbits

The purpose of this section is to provide the groundwork for proving that the objects that will be introduced in Section 7.3 are well-defined. Our main task will be to specify canonical representatives for orbits supported on a given molecule, in order for Equation (6) to induce a well-defined function between sets of orbits. The reader wishing to acquire a general view of our setup without delving into technicalities may skip this section with no harm.

Again, we consider throughout a G-semimatroid $\mathfrak{S}$ defined by an action on $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$, and we assume translativity.

Assumption-Notation 7.6. For this section we fix a molecule $\mathfrak{m} := (R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$ and a linear extension $\prec$ of the partial order defined by inclusion on 2^F, the set of subsets of F.[3] (In particular, $I \subseteq I' \subseteq F$ implies $I \preceq I'$.)

Definition 7.7. We choose representatives $X_R^{(1)}, \dots, X_R^{(k_R)}$ of the orbits in $\lceil R\rceil^{\mathcal{C}}/G$ and extend $\prec$ to a total order on the index set $\{(i, I) \mid i = 1, \dots, k_R,\ I \in 2^F\}$ via

$$(i, I) \prec (i', I') \Leftrightarrow \begin{cases} i < i', \\ \text{or } i = i' \text{ and } I \prec I'. \end{cases} \tag{8}$$

Moreover, choose and fix an element $x_f \in f$ for every $f \in F$. Then, for all $F' \subseteq F$ define $X_{F'} = \{x_f \mid f \in F'\}$.

We now can recursively define the blocks of an ordered partition of $\lceil R \cup F\rceil^{\mathcal{C}}/G$ as follows.

Definition 7.8. Set $\mathscr{Y}^{(1,\emptyset)} := \{G(X_R^{(1)} \cup X_F)\}$, and for each $(i, I) \succ (1, \emptyset)$ let

$$\mathscr{Y}^{(i,I)} := \left\{ \mathcal{O} \in \frac{\lceil R \cup F\rceil^{\mathcal{C}}}{G} \;\middle|\; \begin{array}{ll} \text{(i)} & \mathcal{O} \notin \bigcup_{(j,J)\prec(i,I)} \mathscr{Y}^{(j,J)} \\ \text{(ii)} & X_R^{(i)} \cup X_{F\setminus I} \subseteq Y \text{ for some } Y \in \mathcal{O} \end{array} \right\}.$$

Choose an enumeration

$$\mathscr{Y}^{(i,I)} = \{\mathcal{O}_1, \dots, \mathcal{O}_{h_{(i,I)}}\}$$

[3] E.g., represent the elements of 2^F as ordered zero-one-tuples and take the lexicographic order.

thereby defining the numbers $h_{(i,I)}$ (and setting $h_{(i,I)} = 0$ if $\mathscr{Y}^{(i,I)} = \emptyset$).

Remark 7.9. The sets $\mathscr{Y}^{(i,I)}$ do partition $\lceil R \cup F \rceil^{\mathcal{C}}/G$. First, (i) ensures that they have trivial intersections. Moreover, for every $\mathcal{O} \in \lceil R \cup F \rceil^{\mathcal{C}}/G$ consider the unique i with $X_R^{(i)} \subseteq Y$ for some $Y \in \mathcal{O}$. Now let I be $\prec$-minimal such that the expression in part (ii) holds, and we have $\mathcal{O} \in \mathscr{Y}^{(i,I)}$.

Remark 7.10. Let $\mathcal{O} \in \mathscr{Y}^{(i,I)}$. If $X_{F \setminus J} \subseteq Y$ for some $Y \in \mathcal{O}$ then $J \succeq I$. In particular, $J \subsetneq I$ implies $X_{F \setminus F_J} \nsubseteq Y$ for all $Y \in \mathcal{O}$.

Now we are ready to define representatives for orbits in $\lceil R \cup F \rceil^{\mathcal{C}}/G$.

Definition 7.11. Define the set

$$\mathscr{Z}_{R,F} := \{(i, I, j) \mid i = 1, \dots, k_R;\ I \in 2^F;\ j = 1, \dots, h_{(i,I)}\}$$

and consider on it the total ordering $\lhd$ given by

$$(i, I, j) \lhd (i', I', j') \Leftrightarrow \begin{cases} (i, I) \prec (i', I') \text{ or} \\ (i, I) = (i', I') \text{ and } j < j'. \end{cases}$$

For every $(i, I, j) \in \mathscr{Z}_{R,F}$ consider the corresponding orbit $\mathcal{O}_j \in \mathscr{Y}^{(i,I)}$ and choose a representative $Y_{R \cup F}^{(i,I,j)}$ of $\mathcal{O}_j$ with

$$X_R^{(i)} \cup X_{F \setminus I} \subseteq Y_{R \cup F}^{(i,I,j)} \in \mathcal{O}_j \tag{9}$$

(such a representative exists by requirement (2) of Definition 7.8).

Lemma 7.12. *We have* $Y_{R \cup F}^{(i,I,j)} \cap X_F = X_{F \setminus I}$.

Proof. Let J be such that $Y_{R \cup F}^{(i,I,j)} \cap X_F = X_{F \setminus J}$. Then $J \subseteq I$ by Equation (9). Moreover, if $J \subsetneq I$ then $J \prec I$, a contradiction to Remark 7.10. Hence $I = J$ as desired. □

For each $F' \subseteq F$ we now fix representatives of the orbits in $\lceil R \cup F' \rceil^{\mathcal{C}}/G$.

Definition 7.13. Given $F' \subseteq F$, for every $\mathcal{O} \in \lceil R \cup F' \rceil^{\mathcal{C}}/G$ let

$$z(\mathcal{O}) := \min_{\lhd}\{z \in \mathscr{Z}_{R,F} \mid \mathcal{O} \leq GY_{R \cup F}^{z} \text{ in } \mathcal{C}_{\mathfrak{G}}\}$$

and let $Y_{R \cup F'}^{\mathcal{O}} \in \mathcal{O}$ be the (unique) representative with

$$Y_{R \cup F'}^{\mathcal{O}} \subseteq Y_{R \cup F}^{z(\mathcal{O})}.$$

With these choices, let

$$\begin{aligned} \widehat{\underline{w}}_{R\cup F,F\setminus F'} : \lceil R\cup F'\rceil^{\mathcal{C}}/G &\to \lceil R\cup F\rceil^{\mathcal{C}}/G \\ \mathcal{O} &\mapsto G(Y^{\mathcal{O}}_{R\cup F'}\cup X_{F\setminus F'}). \end{aligned} \tag{10}$$

Lemma 7.14. *Let $F''\subseteq F'\subseteq F$. Then*

(a) for every $\mathcal{O}\in\lceil R\cup F'\rceil^{\mathcal{C}}/G$

$$Y^{z(\mathcal{O})}_{R\cup F} = Y^{\mathcal{O}}_{R\cup F'}\cup X_{F\setminus F'};$$

(b) for every $\mathcal{O}\in\lceil R\cup F''\rceil^{\mathcal{C}}/G$

$$Y^{G(Y^{\mathcal{O}}_{R\cup F''}\cup X_{F'\setminus F''})}_{R\cup F'} = Y^{\mathcal{O}}_{R\cup F''}\cup X_{F'\setminus F''}.$$

(c) Furthermore,

$$\widehat{\underline{w}}_{R\cup F,F\setminus F'}\circ\widehat{\underline{w}}_{R\cup F',F'\setminus F''} = \widehat{\underline{w}}_{R\cup F,F\setminus F''}.$$

Proof. In this proof, given any $\mathcal{O}\in\lceil R\cup F'\rceil^{\mathcal{C}}/G$ let us for brevity call $\mathscr{X}(\mathcal{O})$ the set over which the minimum is taken in Definition 7.13 in order to define $z(\mathcal{O})$.

(a) It is enough to show that $X_{F\setminus F'}\subseteq Y^{z(\mathcal{O})}_{R\cup F}$. In order to prove this, we consider

$$Y' := (Y^{z(\mathcal{O})}_{R\cup F}\setminus X')\cup X_{F\setminus F'}$$

where $X'\in\lceil F\setminus F'\rceil^{\mathcal{C}}$ is defined by $X'\subseteq Y^{z(\mathcal{O})}_{R\cup F}$ (notice that $|X'|=|X_F|$ since $Y^{z(\mathcal{O})}_{R\cup F}\in\lceil R\cup F\rceil^{\mathcal{C}}$ and the action is translative). The set Y' is central by Lemma 7.3.(a), because $\mathrm{rk}(Y^{z(\mathcal{O})}_{R\cup F'}) = \mathrm{rk}(Y^{z(\mathcal{O})}_{R\cup F}\setminus X') + |X'|$. Moreover, $GY'\geq\mathcal{O}$ in $\mathcal{C}_{\mathfrak{G}}$ since $Y^{\mathcal{O}}_{R\cup F}\subseteq Y'$.

If $X_{F\setminus F'}\subseteq Y^{z(\mathcal{O})}_{R\cup F}$, then $Y'=Y^{z(\mathcal{O})}_{R\cup F}$ and we are done. We will prove that if this is not the case, then $z(\mathcal{O})\neq\min\mathscr{X}(\mathcal{O})$, reaching a contradiction. Suppose then $X_{F\setminus F'}\not\subseteq Y^{z(\mathcal{O})}_{R\cup F}$, and write $z(\mathcal{O})=(i,I,j)$. By Lemma 7.12, we have $I=\{f\mid x_f\notin Y^{z(\mathcal{O})}_{R\cup F}\}$. Hence, setting

$$I_{Y'} := \{f\mid x_f\notin Y'\}$$

we have that $I_{Y'} = I\cap F'\subseteq I$, where the last containment is strict (otherwise $Y'=Y^{z(\mathcal{O})}_{R\cup F}$, hence $X_{F\setminus F'}\subseteq Y^{z(\mathcal{O})}_{R\cup F}$, contrary to our assumption). By definition, $I_{Y'}\subsetneq I$ implies $I_{Y'}\precneq I$. Moreover, for $z'=(i,I',j')$ defined by $GY'=\mathcal{O}_{j'}\in\mathscr{Y}^{(i,I')}$ we have in fact by Remark 7.10 that $I'\preceq I_{Y'}$. Therefore, $I'\preceq I_{Y'}\precneq I$. This implies that $z'=(i,I',j')\trianglelefteq(i,I,j)=z(\mathcal{O})$ and $z'\neq z(\mathcal{O})$. Thus, $GY^{z'}_{R\cup F}\in\mathscr{X}(\mathcal{O})$ but z' strictly precedes $z(\mathcal{O})$, and we reach the announced contradiction.

(b) Let $\mathcal{O}$ be as in the claim, and set $\mathcal{U} := G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\setminus F''})$. Then $\mathcal{O} \leq \mathcal{U}$ in $\mathcal{C}_{\mathfrak{S}}$, thus $\mathscr{Z}(\mathcal{O}) \supseteq \mathscr{Z}(\mathcal{U})$ and therefore $z(\mathcal{O}) \trianglelefteq z(\mathcal{U})$. Now, since $Y^{z(\mathcal{O})}_{R\cup F} = Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\setminus F''}$ by part (a), we see that $\mathcal{U} \leq GY^{z(\mathcal{O})}_{R\cup F}$ in $\mathcal{C}_{\mathfrak{S}}$, thus $z(\mathcal{U}) \trianglelefteq z(\mathcal{O})$. In summary, $z(\mathcal{U}) = z(\mathcal{O})$ and, as a subset of $Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\setminus F''}$, we see that $Y^{\mathcal{U}}_{R\cup F'} = Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\setminus F''}$ as claimed.

(c) For every $\mathcal{O} \in \lceil R \cup F'' \rceil^{\mathcal{C}}/G$ we compute

$$\begin{aligned} &\underline{\widehat{w}}_{R\cup F, F\setminus F'} \circ \underline{\widehat{w}}_{R\cup F', F'\setminus F''}(\mathcal{O}) = \underline{\widehat{w}}_{R\cup F, F\setminus F'}(G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\setminus F''})) \\ &= G(Y^{G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\setminus F''})}_{R\cup F'} \cup X_{F\setminus F'}) = G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\setminus F''}) = \underline{\widehat{w}}_{R\cup F, F\setminus F''}(\mathcal{O}), \end{aligned}$$

where in the third equality we used (b) and all other equalities hold by definition. □

Corollary 7.15. *For every $F' \subseteq F$, the function $\widehat{w}_{R\cup F, F\setminus F'}$ is injective.*

Proof. Let $f_1, \ldots, f_m$ be an enumeration of the elements of $F \setminus F'$ and for every $j = 1, \ldots, m$ set $F_j := F' \cup \{f_1, \ldots, f_j\}$. Then by Lemma 7.14.(c)

$$\widehat{w}_{R\cup F, F\setminus F'} = \widehat{w}_{R\cup F, f_m} \circ \widehat{w}_{R\cup F_{m-1}, f_{m-1}} \cdots \circ \widehat{w}_{R\cup F_2, f_1}$$

and each of the functions on the right-hand side is injective because it is an instance of the function described in Equation (7). The latter is used as a left-inverse to prove the surjectivity claim of Lemma 7.3.(b) and, as such, is injective. □

Definition 7.16. Given $F' \subseteq F$, $T' \subseteq T$, as a representative of the orbit $\mathcal{O} \in \lceil R \cup F' \cup T' \rceil^{\mathcal{C}}/G$ we choose

$$Y^{\mathcal{O}}_{R\cup F'\cup T'} := w^{-1}_{R\cup F'\cup T', T'}(Y^{\underline{w}_{R\cup F'\cup T', T'}(\mathcal{O})}_{R\cup F'}), \tag{11}$$

and we let $Y^{\mathcal{O}}_{T'} := Y^{\mathcal{O}}_{R\cup F'\cup T'} \setminus Y^{\underline{w}_{R\cup F'\cup T', T'}(\mathcal{O})}_{R\cup F'}$.

Remark 7.17. In order to prove that $Y^{\mathcal{O}}_{R\cup F'\cup T'}$ is well defined, we have to show that the right-hand side of Equation (11) is not empty; uniqueness will then follow from injectivity of $w_{R\cup F'\cup T', T'}$ (see Corollary 7.5). To see this, it is enough to notice that the function $w_{R\cup F'\cup T', T'}$ is onto when restricted to $\underline{w}_{R\cup F'\cup T', T'}(\mathcal{O})$: in fact, the latter is by definition $w_{R\cup F'\cup T', T'}(\mathcal{O})$, hence part of the image of $w_{R\cup F'\cup T', T'}$.

Example 7.18. We go back to our running example (Example 1.7), for which we depict in Fig. 12 a piece of the associated periodic arrangement, and consider there the molecule $(\emptyset, F, \emptyset)$, where $F = \{f_a, f_b\}$ is the set of orbits of the diagonal lines (drawn orange and green in the web version).

Choose representatives $x_a = a_0$ for the "northwest to southeast" (orange) lines, $x_b = b_0$ for the "southwest to northeast" (green) lines and denote their $(0, k)$-translate by a_k (resp. b_k).

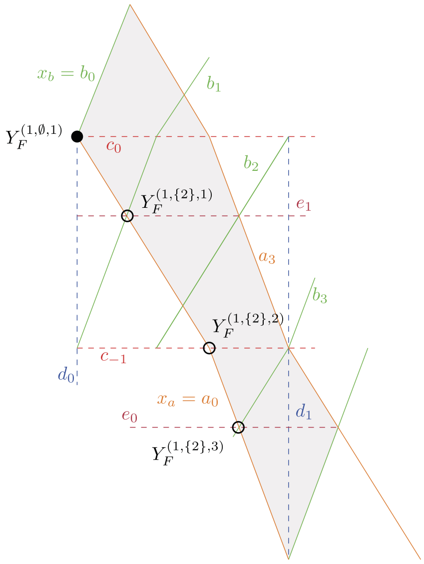

Fig. 12. An illustration for Example 7.18.

By Definitions 7.8 and 7.11, we get the following partition of $\lceil F \rceil^{\mathcal{C}}/G$,

$$\mathscr{Y}^{(1,\emptyset)} = \{\mathcal{O}_0\}, \ \mathscr{Y}^{(1,\{2\})} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}, \ \mathscr{Y}^{(1,\{1\})} = \mathscr{Y}^{(1,\{1,2\})} = \emptyset,$$

with representatives

$$Y_F^{(1,\emptyset,1)} = \{a_0, b_0\}, Y_F^{(1,\{2\},1)} = \{a_0, b_{k_1}\}, Y_F^{(1,\{2\},2)} = \{a_0, b_{k_2}\}, Y_F^{(1,\{2\},3)} = \{a_0, b_{k_3}\},$$

where $k_1 \neq 0 \mod 4$; $k_2 \neq 0, k_1 \mod 4$; and $k_3 \neq 0, k_1, k_2 \mod 4$. Without loss of generality, one could assume $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, and we get the situation depicted in Fig. 12. Moreover, by Definition 7.13 we get $Y_a^{\mathcal{O}_a} = a_0$ (where $\lceil f_a \rceil^{\mathcal{C}}/G = \{\mathcal{O}_a\}$), $Y_b^{\mathcal{O}_b} = b_0$ (where $\lceil f_b \rceil^{\mathcal{C}}/G = \{\mathcal{O}_b\}$), and $Y_\emptyset^\emptyset = \emptyset$ where $\lceil \emptyset \rceil^{\mathcal{C}}/G = \{\emptyset\}$.

Thus,

$$\underline{\widehat{w}}_{F,F}(\emptyset) = \underline{\widehat{w}}_{F,f_b}(\underline{\widehat{w}}_{f_a,f_a}(\emptyset)) = \underline{\widehat{w}}_{F,f_a}(\underline{\widehat{w}}_{f_b,f_b}(\emptyset)) = G(a_0 b_0) = \mathcal{O}_0.$$

Notice that an accurate choice of representatives is of the essence. For example, choosing $Y_a^{\mathcal{O}_a} = a_0$ and $Y_b^{\mathcal{O}_b} = b_1$ as representatives of $\mathcal{O}_a$, resp. $\mathcal{O}_b$,

$$\operatorname{im} \underline{\widehat{w}}_{F,f_b} = G(a_0 x_b) = G(a_0 b_0) \neq G(a_0 b_1) = G(x_a b_1) = \operatorname{im} \underline{\widehat{w}}_{F,f_a}. \qquad \triangle$$

Lemma 7.19. *For every $F' \subseteq F$ and $T' \subseteq T$,*

$$\widehat{\underline{w}}_{R\cup F, F\setminus F'} \circ \underline{w}_{R\cup F'\cup T', T'} = \underline{w}_{R\cup F\cup T', T'} \circ \widehat{\underline{w}}_{R\cup F\cup T', F\setminus F'}.$$

Proof. We check equality on every $\mathcal{O} \in \lceil R \cup F' \cup T' \rceil^{\mathcal{C}}$. On the right-hand side, using the definitions (Definition 7.1 and Definition 7.13), we find

$$\begin{aligned} &\underline{w}_{R\cup F\cup T', T'}(\widehat{\underline{w}}_{R\cup F\cup T', F\setminus F'}(\mathcal{O})) \\ &= \underline{w}_{R\cup F\cup T', T'}(G(Y^{\mathcal{O}}_{R\cup F'\cup T'} \cup X_{F\setminus F'})) = G((Y^{\mathcal{O}}_{R\cup F'\cup T'} \setminus \cup T') \cup X_{F\setminus F'}) \end{aligned}$$

while on the left-hand side we compute

$$\begin{aligned} &\widehat{\underline{w}}_{R\cup F, F\setminus F'}(\underline{w}_{R\cup F'\cup T', T'}(\mathcal{O})) \\ &= G(Y^{\underline{w}_{R\cup F'\cup T', T'}(\mathcal{O})}_{R\cup F'} \cup X_{F\setminus F'}) = G((Y^{\mathcal{O}}_{R\cup F'\cup T'} \setminus \cup T') \cup X_{F\setminus F'}) \end{aligned}$$

where the last equality uses Definition 7.16. □

7.3. Orbit count for molecules

Definition 7.20. Given a molecule (R, F, T) of a ranked triple, define the following (boolean) poset

$$\begin{gathered} P[R, F, T] := \{(F', T') \mid F' \subseteq F,\ T' \subseteq T\} \text{ with order} \\ (F', T') \leq (F'', T'') \Leftrightarrow F' \subseteq F'',\ T' \supseteq T''. \end{gathered}$$

Thus, the maximal element is $(F, \emptyset)$ and the minimal element $(\emptyset, T)$.

Definition 7.21. Let $\mathfrak{S}$ be a translative G-semimatroid and $\mathfrak{m} := (R, F, T)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. By composing the above functions we obtain, for every $(F', T') \in P[R, F, T]$, a function

$$f^{\mathfrak{m}}_{(F',T')} := \widehat{\underline{w}}_{R\cup F, F'} \circ \underline{w}_{R\cup F'\cup T', T'}. \tag{12}$$

Remark 7.22. Explicitly (cf. proof of Lemma 7.19),

$$\begin{aligned} f^{\mathfrak{m}}_{(F',T')} : \lceil R \cup F' \cup T' \rceil^{\mathcal{C}}/G &\to \lceil R \cup F \rceil^{\mathcal{C}}/G, \\ \mathcal{O} &\mapsto G((Y^{\mathcal{O}}_{R\cup F'\cup T'} \setminus \cup T') \cup X_{F\setminus F'}). \end{aligned}$$

Remark 7.23. The functions $f^{\mathfrak{m}}_{(F',T')}$ are injective by Corollary 7.5 and Corollary 7.15. In particular, with $A := R \cup F' \cup T'$,

$$m_{\mathfrak{S}}(A) = \left| \frac{\lceil R \cup F' \cup T' \rceil^{\mathcal{C}}}{G} \right| = |\operatorname{im} f^{\mathfrak{m}}_{(F',T')}|.$$

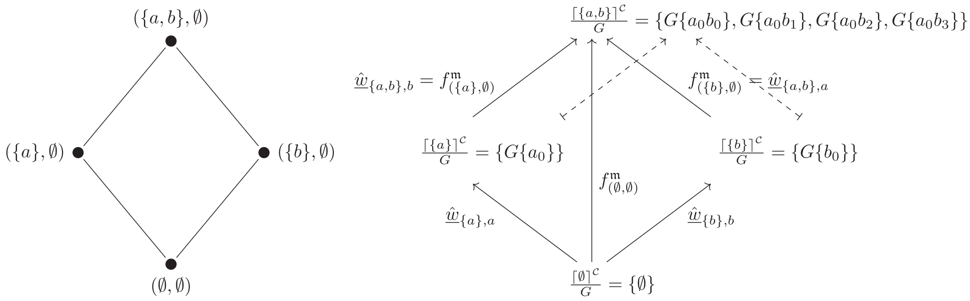

Fig. 13. The Hasse diagram of the poset $P[\emptyset, \{a,b\}, \emptyset]$ in the context of Example 1.7 and, on the right-hand side, the associated diagram of sets.

Example 7.24. In the context of our running example, Example 1.7, we have that $\mathfrak{m} := (\emptyset, \{a,b\}, \emptyset)$ is a molecule of $\mathcal{S}_{\mathfrak{S}}$. Fig. 13 depicts the associated poset and maps. $\triangle$

Lemma 7.25. *Let $\mathfrak{S}$ be translative and consider a molecule $\mathfrak{m} := (R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$.*

(a) For $(F', T'), (F'', T'') \in P[R, F, T]$ we have

$$\begin{aligned}\operatorname{im}(f^{\mathfrak{m}}_{(F',T')\wedge(F'',T'')}) &= \operatorname{im}(f^{\mathfrak{m}}_{(F'\cap F'',T'\cup T'')})\\ &= \operatorname{im} f^{\mathfrak{m}}_{(F',T')} \cap \operatorname{im} f^{\mathfrak{m}}_{(F'',T'')}.\end{aligned}$$

(b) In particular,

$$\operatorname{im} f^{\mathfrak{m}}_{(F',T')} \subseteq \operatorname{im} f^{\mathfrak{m}}_{(F'',T'')} \ \textit{if}\ (F',T') \le (F'',T'').$$

(c) The function

$$m_{\mathfrak{S}} : P[R,F,T] \to \mathbb{N}, \quad (F',T') \mapsto m_{\mathfrak{S}}(R \cup F' \cup T')$$

is (weakly) increasing.

Proof. Part (b) is an immediate consequence of (a) and by Remark 7.23 it implies (c). Thus it is enough to prove part (a), where the first equality is the definition of greatest lower bound in $P[R,F,T]$. We turn then to the second equality and consider the following diagram, where we write $F^* := F' \cap F''$ and $T^* := T' \cup T''$.

$$\begin{tikzcd}
\frac{\lceil R\cup F^*\cup T^*\rceil^{\mathcal{C}}}{G} \arrow[rr, "\underline{\hat{w}}_{R\cup F', F'\setminus F''}\circ \underline{w}_{R\cup F^*\cup T^*, T''\setminus T'}"] \arrow[d, "\underline{\hat{w}}_{R\cup F'', F''\setminus F'} \circ \underline{w}_{R\cup F^*\cup T^*, T'\setminus T''}"'] \arrow[drr, "f^{\mathfrak{m}}_{(F'\cap F'', T'\cup T'')}"] & & \frac{\lceil R\cup F'\cup T'\rceil^{\mathcal{C}}}{G} \arrow[d, "f^{\mathfrak{m}}_{(F',T')}"] \\
\frac{\lceil R\cup F''\cup T''\rceil^{\mathcal{C}}}{G} \arrow[rr, "f^{\mathfrak{m}}_{(F'',T'')}"] & & \frac{\lceil R\cup F\rceil^{\mathcal{C}}}{G}
\end{tikzcd}$$

This diagram is commutative by Lemma 7.19 and Remark 7.4, and we conclude that $\operatorname{im}(f^{\mathfrak{m}}_{(F'\cap F'',T'\cup T'')}) \subseteq \operatorname{im} f^{\mathfrak{m}}_{(F',T')} \cap \operatorname{im} f^{\mathfrak{m}}_{(F'',T'')}$. For the reverse inclusion let $\mathcal{O} \in \operatorname{im} f^{\mathfrak{m}}_{(F',T')} \cap \operatorname{im} f^{\mathfrak{m}}_{(F'',T'')}$ and choose $\mathcal{O}' \in \frac{\lceil R\cup F'\cup T'\rceil^{\mathcal{C}}}{G}$, $\mathcal{O}'' \in \frac{\lceil R\cup F''\cup T''\rceil^{\mathcal{C}}}{G}$ such that $\mathcal{O} = f^{\mathfrak{m}}_{(F',T')}(\mathcal{O}') = f^{\mathfrak{m}}_{(F'',T'')}(\mathcal{O}'')$. In particular,

$$Y^{\mathcal{O}} = Y^{\mathcal{O}'} \setminus Y^{\mathcal{O}'}_{T'} \cup X_{F\setminus F'} = Y^{\mathcal{O}''} \setminus Y^{\mathcal{O}''}_{T''} \cup X_{F\setminus F''}.$$

Thus, the set

$$\widehat{Y} := (Y^{\mathcal{O}'} \setminus Y^{\mathcal{O}'}_{T'}) \cap (Y^{\mathcal{O}''} \setminus Y^{\mathcal{O}''}_{T''})$$

is a subset of $Y^{\mathcal{O}}$, hence it is a central set and generates an orbit

$$\widehat{\mathcal{O}} := G\widehat{Y} \in \frac{\lceil R \cup (F' \cap F'')\rceil^{\mathcal{C}}}{G}.$$

Since $Y^{\mathcal{O}} = \widehat{Y} \cup X_{F\setminus(F'\cap F'')}$, Lemma 7.14.(a) implies $z(\mathcal{O}) = z(\widehat{\mathcal{O}'})$, hence $\widehat{Y} = Y^{\widehat{\mathcal{O}}}$ by definition of preferred representatives.

Now notice that $Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}'}_{T'}$ is central because it is a subset of $Y^{\mathcal{O}'}$. Similarly, also $Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}''}_{T''} \subseteq Y^{\mathcal{O}''}$ is central. Moreover, $\operatorname{rk}(Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}'}_{T''}) = \operatorname{rk}(Y^{\widehat{\mathcal{O}}}) = \operatorname{rk}(Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}'}_{T'})$ and thus, by (CR1) (see Definition 1.1), $Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}'}_{T'} \cup Y^{\mathcal{O}''}_{T''}$ is central, and we can compute

$$f^{\mathfrak{m}}_{(F'\cap F''),(T'\cup T'')}(G(Y^{\widehat{\mathcal{O}}} \cup Y^{\mathcal{O}'}_{T'} \cup Y^{\mathcal{O}''}_{T''})) = G(Y^{\widehat{\mathcal{O}}} \cup X_{F\setminus(F'\cap F'')}) = GY^{\mathcal{O}} = \mathcal{O}$$

proving $\mathcal{O} \in \operatorname{im} f^{\mathfrak{m}}_{(F'\cap F''),(T'\cup T'')}$, as was to be shown. □

Definition 7.26. Let $\mathfrak{m} := (R, F, T)$ be a molecule of $\mathcal{S}_{\mathfrak{G}}$. For every $(F', T') \in P[R, F, T]$ define the sets

$$Z^{\mathfrak{m}}(F', T') := \operatorname{im} f^{\mathfrak{m}}_{(F',T')}, \quad \overline{Z}^{\mathfrak{m}}(F', T') := Z^{\mathfrak{m}}(F', T') \setminus \bigcup_{(F'',T'')<(F',T')} Z^{\mathfrak{m}}(F'', T''),$$

and let

$$n_{\mathfrak{m}}(F', T') := |\overline{Z}^{\mathfrak{m}}(F', T')|.$$

The following equality holds then by Lemma 7.25.(a) and Remark 7.23.

$$m_{\mathfrak{G}}(R \cup T' \cup F') = |\operatorname{im} f^{\mathfrak{m}}_{(F',T')}| = \sum_{p\leq(F',T')} n_{\mathfrak{m}}(p) \tag{13}$$

Lemma 7.27. *If $\mathfrak{G}$ is translative, then for every molecule $\mathfrak{m} := (R, F, T)$ in $\mathcal{S}_{\mathfrak{G}}$ we have*

$$\rho(R, R \cup F \cup T) = n_{\mathfrak{m}}(F, \emptyset).$$

Proof. Let (R, F, T) be a molecule in $\mathcal{S}_{\mathfrak{G}}$ and in this proof let us write P for $P[R, F, T]$. We start by rewriting the expression in Definition 1.21 as a sum over elements of P.

$$\begin{aligned}\rho(R, R \cup F \cup T) &:= (-1)^{|T|} \sum_{R \subseteq A \subseteq R \cup F \cup T} (-1)^{|R \cup F \cup T| - |A|} m_{\mathfrak{G}}(A) \\ &= \sum_{F' \subseteq F} \sum_{T' \subseteq T} (-1)^{|F \setminus F'| + |T'|} m_{\mathfrak{G}}(R \cup F' \cup T')\end{aligned}$$

The poset P has rank function $\operatorname{rk}(F', T') = |F'| + |T \setminus T'|$, and by Möbius inversion (where we call μ_P the Möbius function of P) we can write explicitly the value of $n_{\mathfrak{m}}(F, \emptyset)$ from Equation (13).

$$\begin{aligned}n_{\mathfrak{m}}(F, \emptyset) &= \sum_{(F', T') \in P} \mu_P((F', T'), (F, \emptyset)) m_{\mathfrak{G}}(R \cup F' \cup T') \\ &= \sum_{(F', T') \in P} (-1)^{|F| + |T| - |F'| - |T \setminus T'|} m_{\mathfrak{G}}(R \cup F' \cup T') \\ &= \sum_{F' \subseteq F} \sum_{T' \subseteq T} (-1)^{|F \setminus F'| + |T'|} m_{\mathfrak{G}}(R \cup F' \cup T') \\ &= \rho(R, R \cup F \cup T) \quad \square\end{aligned}$$

Since the function $n_{\mathfrak{m}}$ is – by definition – never negative, as an easy corollary we obtain the following.

Proposition 7.28. *If $\mathfrak{G}$ is translative, then the pair $(\mathcal{S}_{\mathfrak{G}}, m_{\mathfrak{G}})$ satisfies property (P) of Definition 1.21 (and is thus called "pseudo-arithmetic").*

Definition 7.29. Fix $A \subseteq E_{\mathfrak{G}}$. Recall the poset $\mathcal{P}_{\mathfrak{G}}$ from Definition 3.21, the function $\kappa_{\mathfrak{G}}$ from Definition 3.27, and define

$$\eta_A : \mathcal{C}_{\mathfrak{G}} \to \mathbb{N}, \quad \eta_A(\mathcal{O}) := |\{a \in A \mid a \leq_{\mathcal{P}_{\mathfrak{G}}} \kappa_{\mathfrak{G}}(\mathcal{O})\}|.$$

Proposition 7.30. *Let $(R, \emptyset, T)$ be a molecule. Then,*

$$\sum_{L \subseteq T} \rho(R \cup L, R \cup T) x^{|L|} = \sum_{\mathcal{O} \in \lceil R \rceil^{\mathcal{C}} / G} x^{\eta_T(\mathcal{O})}.$$

Remark 7.31. Notice that, in terms of the poset $\mathcal{P}_{\mathfrak{G}}$,

$$\eta_T(\mathcal{O}) = |\{t \in T \mid \kappa_{\mathfrak{G}}(t) \leq_{\mathcal{P}_{\mathfrak{G}}} \kappa_{\mathfrak{G}}(\mathcal{O})\}|.$$

Thus, in the representable case we recover the number defined in [5, Section 6].

Proof of Proposition 7.30. Consider a molecule $\mathfrak{m} := (R, \emptyset, T)$. First notice that, for every $L \subseteq T$, $\mathfrak{m}_L := (R \cup L, \emptyset, T \setminus L)$ is also a molecule, and that by Equation (12) we have immediately

$$f^{\mathfrak{m}}_{(\emptyset, L\cup L')} = f^{\mathfrak{m}}_{(\emptyset, L)} \circ f^{\mathfrak{m}_L}_{(\emptyset, L')}$$

for every $L' \subseteq T \setminus L$. Therefore, with Lemma 7.25.(b) and Lemma 7.27 we can write the following

$$\begin{aligned}\rho(R\cup L, R\cup T) &= \left| \lceil R\cup L\rceil^{\mathcal{C}}/G \Big\backslash \bigcup_{t\in T\setminus L} \operatorname{im} f^{\mathfrak{m}_L}_{(\emptyset,\{t\})} \right| \\ &= \left| \operatorname{im} f^{\mathfrak{m}}_{(\emptyset,L)} \Big\backslash \bigcup_{t\in T\setminus L} \operatorname{im} f^{\mathfrak{m}}_{(\emptyset,L\cup\{t\})} \right| = |\overline{Z}^{\mathfrak{m}}(\emptyset, L)|\end{aligned}$$

where the second equality follows from injectivity of the functions $f^{\mathfrak{m}}$ and $f^{\mathfrak{m}_L}$.

Claim. *For all $\mathcal{O} \in \lceil R\rceil^{\mathcal{C}}/G$, if $\mathcal{O} \in \overline{Z}^{\mathfrak{m}}(\emptyset, L)$ then*

$$\{t \in T \mid t \leq_{\mathcal{C}_{\mathfrak{G}}} \kappa_{\mathfrak{G}}(\mathcal{O})\} = L.$$

In particular, we have that $\eta_T(\mathcal{O}) = |L|$.

Proof. Let $\mathcal{O} \in \overline{Z}^{\mathfrak{m}}(\emptyset, L)$. Then for every $t \in T$ we have $\mathcal{O} \in \operatorname{im} f^{\mathfrak{m}}_{(\emptyset,t)}$ if and only if there is a representative X_R of $\mathcal{O}$ and some $x_t \in t$ such that $X_R \cup x_t \in \mathcal{C}$. Since we know that $\underline{\mathrm{rk}}(R\cup t) = \underline{\mathrm{rk}}(R)$, the latter is equivalent to saying that $x_t \in \mathrm{cl}_{\mathcal{C}}(X_R)$, i.e., $t \leq \kappa_{\mathfrak{G}}(\mathcal{O})$ in $\mathcal{C}_{\mathfrak{G}}$. Now, by Lemma 7.25.(a) we have

$$\operatorname{im} f^{\mathfrak{m}}_{(\emptyset,L)} = \bigcap_{t\in L} \operatorname{im} f^{\mathfrak{m}}_{(\emptyset,t)}$$

and thus we see that $t \leq \kappa_{\mathfrak{G}}(\mathcal{O})$ if and only if $t \in L$. □

We can now return to the statement to be verified and write

$$\begin{aligned}\sum_{L\subseteq T} \rho(R\cup L, R\cup T)x^{|L|} &= \sum_{L\subseteq T} |\overline{Z}^{\mathfrak{m}}(\emptyset, L)| x^{|L|} = \sum_{L\subseteq T} \sum_{\mathcal{O}\in\overline{Z}^{\mathfrak{m}}(\emptyset,L)} x^{|L|} \\ &= \sum_{\mathcal{O}\in\lceil R\rceil^{\mathcal{C}}/G} x^{\eta_T(\mathcal{O})}\end{aligned}$$

where the last equality uses the Claim we just proved. □

8. Almost-arithmetic actions

We now turn to what we call "almost-arithmetic" actions (see Definition 3.12). The name is reminiscent of the fact that one additional condition on top of translativity (i.e., normality) already ensures that the multiplicity function satisfies "almost all" of the requirements for arithmetic matroids (see Definition 1.21): this is the gist of the main result of this section (Proposition 8.6).

We keep the notation $\mathfrak{S}$ to signify a G-semimatroid arising from an action on a semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$.

Lemma 8.1. *Let $\mathfrak{S}$ be almost-arithmetic and let $X \in \mathcal{C}$. Then*

(a) for all $X' \in \lceil \underline{X} \rceil^{\mathcal{C}}$ we have $\mathrm{stab}(X) = \mathrm{stab}(X')$,
(b) if $x_0 \in X$ and $\mathrm{rk}(X \setminus x_0) = \mathrm{rk}(X)$, *then* $\mathrm{stab}(X) = \mathrm{stab}(X \setminus x_0)$.

Proof. Item (a) is an immediate consequence of normality. In the claim of item (b), the inclusion $\mathrm{stab}(X) \subseteq \mathrm{stab}(X \setminus x_0)$ is evident. For the reverse inclusion, consider $g \in \mathrm{stab}(X \setminus x_0)$. Then $X \setminus x_0 \subseteq gX \cap X$, which justifies the first inequality in

$$\mathrm{rk}(X \setminus x_0) \leq \mathrm{rk}(gX \cap X) \leq \mathrm{rk}(X), \tag{$*$}$$

where the second inequality holds by (R2). Since by assumption $\mathrm{rk}(X) = \mathrm{rk}(X \setminus x_0)$, equality must hold throughout $(*)$ above, proving that $\mathrm{rk}(X) = \mathrm{rk}(gX \cap X)$. By (CR1), the latter implies $X \cup g(X) \in \mathcal{C}$ and, in particular, $\{x_0, g(x_0)\} \in \mathcal{C}$. Translativity of the action then ensures $g \in \mathrm{stab}(x_0)$ and thus $g \in \mathrm{stab}(X)$. □

Definition 8.2. Given $X_1, \ldots, X_k \in \mathcal{C}$ define

$$\theta_{X_1,\ldots,X_k} : G \to \prod_{i=1}^{k} \Gamma(X_i), \quad g \mapsto ([g]_{X_1}, \ldots, [g]_{X_k}).$$

Remark 8.3. By Lemma 8.1.(a), in an almost-arithmetic G-semimatroid this map does not depend on the choice of the X_i in $\lceil \underline{X_i} \rceil^{\mathcal{C}}$ for $i = 1, \ldots, k$.

Lemma 8.4. *Given an almost-arithmetic G-semimatroid $\mathfrak{S}$, consider $A \subseteq E_{\mathfrak{S}}$ and $a_1, \ldots, a_k \in E_{\mathfrak{S}}$ with $\underline{\mathrm{rk}}(A \cup \{a_1, \ldots, a_k\}) = \underline{\mathrm{rk}}(A) + k$. For every choice of $X \in \lceil A \rceil^{\mathcal{C}}$ and of $x_i \in a_i$, $i = 1, \ldots, k$,*

$$\frac{m_{\mathfrak{S}}(A \cup \{a_1, \ldots, a_k\})}{m_{\mathfrak{S}}(A)} = [\Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i) : \theta_{X,x_1,\ldots,x_k}(G)]$$

Proof. Let A and $a_1, \ldots, a_k$ be as in the statement and, in this proof, let us write $A' := A \cup \{a_1, \ldots, a_k\}$. Since the action is translative, with Lemma 7.3.(a) we obtain the following equality of sets.

$$\lceil A' \rceil^{\mathcal{C}} = \lceil A \rceil^{\mathcal{C}} \times \prod_{i=1}^{k} \lceil a_i \rceil^{\mathcal{C}}$$

The projection

$$p_A : \lceil A' \rceil^{\mathcal{C}} \to \lceil A \rceil^{\mathcal{C}}, \quad Y \mapsto Y \setminus \bigcup_{i=1}^{k} a_i$$

maps each of the $m_{\mathfrak{S}}(A')$ orbits of the action of G on $\lceil A' \rceil^{\mathcal{C}}$ to one of the $m_{\mathfrak{S}}(A)$ orbits of the action on $\lceil A \rceil^{\mathcal{C}}$. Thus, it is enough to prove that the number of $\lceil A' \rceil^{\mathcal{C}}$-orbits mapped to a fixed $\lceil A \rceil^{\mathcal{C}}$-orbit equals the right-hand side of the equation in the claim.

To this end, choose $X \in \lceil A \rceil^{\mathcal{C}}$ and consider the set of orbits in $\lceil A' \rceil^{\mathcal{C}}$ which project to GX, i.e., the orbits of the action of G on

$$p_A^{-1}(GX) = \{(g(X), x_1, \ldots, x_k) \mid g \in G, \forall i = 1, \ldots, k : x_i \in \lceil a_i \rceil^{\mathcal{C}}\}.$$

Notice that for every $a \in E_{\mathfrak{S}}$ and every $x \in a$ we have an equality $a = Gx = \lceil a \rceil^{\mathcal{C}}$ and a natural bijection of this set with $\Gamma(x)$. In fact, any choice of $x_i \in \lceil a_i \rceil^{\mathcal{C}}$ for $i = 1, \ldots, k$ and $X \in \lceil A \rceil^{\mathcal{C}}$ fixes a bijection $p_A^{-1}(GX) \to \Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i)$, and under this bijection the action of G on the right-hand side is given by composition with elements of the subgroup $\theta_{X,x_1,\ldots,x_k}(G)$ defined above. Therefore we have a bijection

$$p_A^{-1}(GX)/G \to (\Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i))/\theta_{X,x_1,\ldots,x_k}(G).$$

By Lemma 8.1.(a) and Remark 8.3, the group on the right hand side does not depend on the choice of $X \in \lceil A \rceil^{\mathcal{C}}$ and $x_i \in a_i$. This concludes the proof. $\square$

Lemma 8.5. *The multiplicity function associated to an almost-arithmetic G-semimatroid $\mathfrak{S}$ satisfies*

$$m_{\mathfrak{S}}(R) m_{\mathfrak{S}}(R \cup F \cup T) = m_{\mathfrak{S}}(R \cup T) m_{\mathfrak{S}}(R \cup F)$$

for every molecule (R, F, T) of $\mathcal{S}_{\mathfrak{S}}$.

Proof. We choose $X_{R\cup T} \in \lceil R \cup T \rceil^{\mathcal{C}}$ and let $X_R := X_{R\cup T} \setminus \cup T$, so that $X_R \in \lceil R \rceil^{\mathcal{C}}$. Moreover, write $F = \{f_1, \ldots, f_k\}$ and choose $x_i \in f_i$ for all $i = 1, \ldots, k$. From Lemma 8.4 we obtain the following equalities.

$$\frac{m(R \cup F)}{m(R)} = \left[\Gamma(X_R) \times \prod_{i=1}^{k} \Gamma(x_i) : \theta_{X_R, x_1, \ldots, x_k}(G)\right]$$

$$\frac{m(R \cup T \cup F)}{m(R \cup T)} = \left[\Gamma(X_{R \cup T}) \times \prod_{i=1}^{k} \Gamma(x_i) : \theta_{X_{R \cup T}, x_1, \ldots, x_k}(G)\right]$$

Since $\underline{\mathrm{rk}}(R \cup T) = \underline{\mathrm{rk}}(R)$, by Lemma 8.1.(b) we have $\mathrm{stab}(X_R) = \mathrm{stab}(X_{R \cup T})$, so (e.g., by inspection of Definition 3.15) the two right-hand sides are equal. $\square$

Proposition 8.6. *If $\mathfrak{S}$ is an almost-arithmetic action on a semimatroid, then $m_{\mathfrak{S}}$ satisfies properties (P), (A.1.2) and (A2) with respect to $\mathcal{S}_{\mathfrak{S}}$.*

Proof. This follows combining Lemma 7.28, Lemma 8.4 and Lemma 8.5. $\square$

We close the section on almost-arithmetic actions with a proposition about molecules of the form $(R, F, \emptyset)$, as a counterpart to Proposition 7.30 above.

Definition 8.7. Let $\mathfrak{S}$ be an almost-arithmetic G-semimatroid. Given a molecule $\mathfrak{m} := (R, F, \emptyset)$ of $\mathcal{S}_{\mathfrak{S}}$, choose an orbit $\mathcal{O} \in \lceil R \rceil^{\mathcal{C}}/G$ and fix a representative $X_R \in \mathcal{O}$. For every $F' \subseteq F$ let $\mathcal{X}(F') \subseteq \lceil R \cup F' \rceil^{\mathcal{C}}/G$ denote the subset consisting of orbits of the form GY with $X_R \subseteq gY$ for some $g \in G$, i.e.,

$$\mathcal{X}(F') := (\lceil R \cup F' \rceil^{\mathcal{C}}/G)_{\geq \mathcal{O}} \subseteq \mathcal{C}_{\mathfrak{S}}.$$

Fix a numbering of the elements of F and, recalling Definition 7.26, let

$$\widetilde{Z}_F^{\mathfrak{m}}(F') := \overline{Z}^{\mathfrak{m}}(F', \emptyset) \cap \mathcal{X}(F).$$

The sets $\{\widetilde{Z}_F^{\mathfrak{m}}(F')\}_{F' \subseteq F}$ partition $\mathcal{X}(F)$. Thus, for every $\mathcal{O} \in \mathcal{X}(F)$ we can consider the unique $F' \subseteq F$ for which $\mathcal{O} \in \widetilde{Z}_F^{\mathfrak{m}}(F')$ and define the number

$$\iota(\mathcal{O}) := |F| - |F'|.$$

Lemma 8.8. *Let $\mathfrak{S}$ be an almost-arithmetic G-semimatroid and let $\mathfrak{m} := (R, F, \emptyset)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. Then for all $F' \subseteq F$ we have*

$$|\widetilde{Z}_F^{\mathfrak{m}}(F')| = \frac{\rho(R, R \cup F')}{m_{\mathfrak{S}}(R)}.$$

In particular, this cardinality does not depend on the choice of the representative X_R and of the numbering of the elements of F.

Proof. By construction, $|Z^{\mathfrak{m}}(F', \emptyset) \cap \mathcal{X}(F)| = \sum_{(F'', \emptyset) \leq (F', \emptyset)} |\widetilde{Z}_F^{\mathfrak{m}}(F'')|$. Hence (following the notation of [34], to which we refer for basics about Möbius transforms), $|\widetilde{Z}_F^{\mathfrak{m}}(F')| = (\mu\Psi)(F', \emptyset)$, i.e., the evaluation at $(F', \emptyset)$ of the Möbius transform of the function

$$\Psi : P[R, F', \emptyset] \to \mathbb{Z}, \quad (F'', \emptyset) \mapsto |Z^{\mathfrak{m}}(F'', \emptyset) \cap \mathcal{X}(F)| = m_{\mathfrak{S}}(R \cup F'')/m_{\mathfrak{S}}(R)$$

(where the equality holds by Lemma 8.4). By the same computation as in the proof of Lemma 7.27, the Möbius transform $(\mu\Psi)$ then satisfies

$$(\mu\Psi)(F', \emptyset) = \rho(R, R \cup F')/m_{\mathfrak{S}}(R)$$

whence the claim. □

Proposition 8.9. *Let $\mathfrak{S}$ be almost-arithmetic and let $\mathfrak{m} := (R, F, \emptyset)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. Then, with the notations of Definition 8.7,*

$$\sum_{F' \subseteq F} \frac{\rho(R, R \cup F')}{m_{\mathfrak{S}}(R)} x^{|F \setminus F'|} = \sum_{\mathcal{O} \in \mathcal{X}(F)} x^{\iota(\mathcal{O})}.$$

Proof. The proof reduces to the following direct computation, where the first equality is Lemma 8.8 and the second holds by the discussion after Definition 8.7.

$$\begin{aligned}\sum_{F' \subseteq F} \frac{\rho(R, R \cup F')}{m_{\mathfrak{S}}(R)} x^{|F \setminus F'|} &= \sum_{F' \subseteq F} |\widetilde{Z}^{\mathfrak{m}}_F(F')| x^{|F \setminus F'|} \\ &= \sum_{F' \subseteq F} \sum_{\mathcal{O} \in \widetilde{Z}^{\mathfrak{m}}_F(F')} x^{|F \setminus F'|} = \sum_{\mathcal{O} \in \mathcal{X}(F)} x^{\iota(\mathcal{O})} \quad \square\end{aligned}$$

9. Arithmetic actions

In this section we assume that the actions under consideration are arithmetic. A glance back at Definition 3.18 will remind the reader that this assumption is much more restrictive (and more algebraic in nature) than almost-arithmetic.

Lemma 9.1. *Let $\mathfrak{S}$ be an arithmetic G-semimatroid and consider $A \subseteq E_{\mathfrak{S}}$. Then, for any two $X, Y \in \lceil A \rceil^{\mathcal{C}}$,*

$$\textit{(i)}\ \Gamma(X) = \Gamma(Y), \quad \textit{(ii)}\ \Gamma^X = \Gamma^Y.$$

Proof. Fix two sets $X, Y \in \lceil A \rceil^{\mathcal{C}}$ as in the claim. By Lemma 8.1.(a), $\mathrm{stab}(X) = \mathrm{stab}(Y)$, hence (i) follows immediately. Moreover, since every arithmetic action is translative, X and Y contain exactly one element x_a resp. y_a of every orbit in A: in fact, $X = \{x_a \mid a \in A\}$, $Y = \{y_a \mid a \in A\}$. In order to prove (ii), we recall Definition 3.15 and compute

$$\Gamma^X \overset{\text{def}}{=\!=} \prod_{a \in A} \Gamma(x_a) = \prod_{a \in A} \Gamma(y_a) \overset{\text{def}}{=\!=} \Gamma^Y,$$

where the equality in the middle is part (i) applied to $X = \{x_a\}$, $Y = \{y_b\}$. □

In particular, for arithmetic actions we can simplify Definition 3.15 as follows.

Definition 9.2. Given $A \in \underline{\mathcal{C}}$, choose $X \in \lceil A \rceil^{\mathcal{C}}$ and write

$$\Gamma^A := \Gamma^X, \quad \Gamma(A) := \Gamma(X).$$

By Lemma 9.1, these are well-defined and independent from the choice of X.

Lemma 9.3. *Let $\mathfrak{S}$ be an arithmetic G-semimatroid, consider $A \in \underline{\mathcal{C}}$ and pick any two $X, Y \in \lceil A \rceil^{\mathcal{C}}$. Then,*

(i) $W(X)$ and $W(Y)$ are conjugated subgroups of Γ^A.
(ii) $m_{\mathfrak{S}}(A) = [W(X) : h_X(G)]$

Proof.

(i) For every $a \in A$ choose $g_a \in G$ with $x_a = g_a(y_a)$; with this, define the A-tuple $\gamma_{YX} := ([g_a])_{a \in A} \in \Gamma^A$. We have immediately

$$(*)\ X = \gamma_{YX}.Y, \text{ hence } (**)\ \gamma_{YX} \in W(Y).$$

Claim. *$W(X) = \gamma_{YX} W(Y) \gamma_{YX}^{-1}$, thus $W(X)$ and $W(Y)$ are conjugate in Γ^A.*
Proof. By symmetry, it is enough to show $\gamma_{YX} W(Y) \gamma_{YX}^{-1} \subseteq W(X)$. Let, thus, $\gamma \in W(Y)$. For arithmetic actions multiplication is well defined in the group $W(Y)$, thus $(**)$ implies $\gamma_{YX}\gamma \in W(Y)$. With this,

$$(\gamma_{YX}\gamma{\gamma_{YX}}^{-1}).X \overset{(*)}{=} (\gamma_{YX}\gamma).Y \in \mathcal{C}$$

and therefore $(\gamma_{YX}\gamma{\gamma_{YX}}^{-1}) \in W(X)$. $\square$

(ii) The choice of X fixes a function

$$b_X : \lceil A \rceil^{\mathcal{C}} \to W(X), \quad \{g_x x \mid x \in X\} \mapsto ([g_x]_x)_{x \in X} \tag{14}$$

which is bijective by definition of $W(X)$. Moreover, for every $g \in G$ and $Y \in \lceil A \rceil^{\mathcal{C}}$,

$$b_X(gY) = h_X(g) b_X(Y). \tag{15}$$

Thus b_X induces a bijection of sets $\lceil A \rceil^{\mathcal{C}}/G \to W(X)/h_X(G)$ mapping an orbit GY to the coset $h_X(G)b_X(Y)$. We now compute

$$m_{\mathfrak{S}}(A) = |\lceil A \rceil^{\mathcal{C}}/G| = |W(X)/h_X(G)| = [W(X) : h_X(G)]. \quad \square$$

Definition 9.4. Let $\mathfrak{S}$ be an arithmetic G-semimatroid, and consider $A \in \underline{\mathcal{C}}$. Choose $X \in \lceil A \rceil^{\mathcal{C}}$ and $x_0 \in X$. The projection $\Gamma^X \to \Gamma^{X \setminus x_0}$ induces a group homomorphism

$$w_{X,x_0} : W(X) \to W(X \setminus x_0), \quad ([g_x]_x)_{x \in X} \mapsto ([g_x]_x)_{x \in X \setminus x_0}.$$

Remark 9.5. Let $\mathfrak{S}$ be arithmetic. Consider $A \in \underline{\mathcal{C}}$ and $a_0 \in A$, choose $X \in \lceil A \rceil^{\mathcal{C}}$, and let $x_0 \in a_0 \cap X$. The following diagram is commutative

$$\begin{tikzcd} \lceil A \rceil^{\mathcal{C}} \arrow[r, "b_X"] \arrow[d, "w_{A,a_0}"'] & W(X) \arrow[d, "w_{X,x_0}"'] & G \arrow[l, "h_X"'] \arrow[d, equal] \\ \lceil A \setminus a_0 \rceil^{\mathcal{C}} \arrow[r, "b_{X \setminus x_0}"] & W(X \setminus x_0) & G \arrow[l, "h_{X \setminus x_0}"'] \end{tikzcd} \tag{16}$$

where the maps b_* are defined in Equation (14).

9.1. Arithmetic matroids

Theorem C follows from the next lemma, which proves that arithmetic actions induce the last of the defining properties of arithmetic matroids which was not fulfilled by almost-arithmetic actions (Example 4.3 shows that this difference is nontrivial).

Lemma 9.6. *Let $\mathfrak{S}$ be a G-semimatroid associated to an arithmetic action. Then $m_{\mathfrak{S}}$ satisfies property (A.1.1) of Definition 1.21.*

Proof. Consider $A \in \underline{\mathcal{C}}$ and $a_0 \in A$ such that $\underline{\mathrm{rk}}(A \setminus a_0) = \underline{\mathrm{rk}}(A)$. Choose $X \in \lceil A \rceil^{\mathcal{C}}$ and $x_0 \in a_0 \cap X$. Using Lemma 9.3.(ii) we have $m_{\mathfrak{S}}(A \setminus a_0) = [W(X \setminus x_0) : h_{X \setminus x_0}(G)]$.

By Lemma 7.3, the condition on the ranks implies that w_{A,a_0} is injective. Commutativity of the left-hand side square in Diagram (16) implies that $w_{X,x0}$ is injective. Therefore (using again Lemma 9.3) we can write

$$m_{\mathfrak{S}}(A) = [W(X) : h_X(G)] = [\mathrm{im}(w_{X,x_0}) : h_{X \setminus x_0}(G)].$$

Now the claim follows from multiplicativity of the index in the chain of subgroups $h_{X \setminus x_0}(g) \subseteq \mathrm{im}(w_{X,x_0}) \subseteq W(X)$, which allows us to write

$$m_{\mathfrak{S}}(A \setminus a_0) = [W(X \setminus x_0) : \mathrm{im}(w_{X,x_0})] m_{\mathfrak{S}}(A)$$

proving in particular that $m_{\mathfrak{S}}(A)$ divides $m_{\mathfrak{S}}(A \setminus a_0)$, as claimed. □

9.2. Matroids over rings

We now outline a link to the theory of matroids over rings. We will give a direct combinatorial interpretation of some matroids over $\mathbb{Z}$ arising from group actions on semimatroids (and, in particular, from toric arrangements).

With this in mind, from now we will let $\mathfrak{S}$ denote an arithmetic G-semimatroid and consider the following condition.

(Cyc) For every $e \in E_{\mathfrak{S}}$, $\Gamma^{\{e\}}$ is a cyclic group.

Remark 9.7. An immediate consequence of (Cyc) is that, for every $A \subset E_{\mathfrak{S}}$, the group Γ^A is abelian. In particular, Lemma 9.3.(i) implies $W(X) = W(Y)$ and $h_X = h_Y$ for all $X, Y \in \lceil A \rceil^{\mathcal{C}}$.

Definition 9.8. Let $\mathfrak{S}$ denote an arithmetic G-semimatroid. Then, for every $A \subseteq E_{\mathfrak{S}}$ and every $a_0 \in A$ we have the following canonical group homomorphisms.

(i) $g_{A,a_0} : \Gamma(A) \to \Gamma(A \setminus a_0)$, induced by the inclusion $\operatorname{stab}(A) \subseteq \operatorname{stab}(A \setminus a_0)$.
(ii) $\pi_{A,a_0} : \Gamma^A \to \Gamma^{A \setminus a_0}$, the canonical projection along the a_0-coordinate.

When (Cyc) holds and if $A \in \underline{\mathcal{C}}$, with Remark 9.7 we can set $W(A) := W(X)$ and $h_A := h_X$ (see Definition 3.15), where X is any element $X \in \lceil A \rceil^{\mathcal{C}}$. We then have more canonical homomorphisms.

(iii) $w_{A,a_0} : W(A) \to W(A \setminus a_0)$, induced by π_{A,a_0} and equal to the map of Definition 9.4 (see Remark 9.9.(a) below).
(iv) $j_A : \Gamma(A) \to \Gamma^A$ and $j'_A : \Gamma(A) \to W(A)$, induced respectively by h'_A and h_A (see Remark 9.9.(b) below).

Remark 9.9.

(a) The maps w_{A,a_0} defined in (iii) above should be regarded as the natural "enriched" version of their namesakes from Definition 7.1. In fact, as maps of sets, the two correspond via the natural bijections $b_A : \lceil A \rceil^{\mathcal{C}} \to W(A)$ (cf. Equation (14)). More precisely the following diagram (of sets) commutes.

$$\begin{tikzcd}
\lceil A \rceil^{\mathcal{C}} \arrow[r, "w_{A,a_0}", "\text{Definition 7.1}"'] \arrow[d, "b_A"] & \lceil A \setminus a_0 \rceil^{\mathcal{C}} \arrow[d, "b_{A\setminus a_0}"] \\
W(A) \arrow[r, "w_{A,a_0}", "\text{Definition 9.8.(iii)}"'] & W(A \setminus a_0)
\end{tikzcd}$$

(b) The homomorphisms j_A and j'_A of Definition 9.8.(iv) are well defined and injective. In fact, since $\ker h_A = \ker h'_A = \operatorname{stab}(A)$, both h_A and h'_A factor uniquely by injective maps through the quotient $q : G \to \Gamma(A)$. We summarize with the following diagram.

$$\begin{tikzcd}
G \arrow[r, "h_A"] \arrow[d, "q"'] \arrow[dr, "h'_A" near start] & W(A) \arrow[d, hook, "\iota"] \\
\Gamma(A) \arrow[r, tail, "j_A"'] \arrow[ur, tail, "j'_A"' near start] & \Gamma^A
\end{tikzcd}$$

Definition 9.10. Given an arithmetic G-semimatroid $\mathfrak{S}$ satisfying (Cyc) define, for every $A \subseteq E_{\mathfrak{S}}$ such that $A^c \in \underline{\mathcal{C}}$, an abelian group

$$M_{\mathfrak{S}}(A) := \Gamma^{A^c} / \operatorname{im}(h'_{A^c}).$$

Moreover, for every $a_0 \in E_{\mathfrak{S}}$ let $\mu_{A,a_0} : M_{\mathfrak{S}}(A) \to M_{\mathfrak{S}}(A \cup a_0)$ be the unique group homomorphism that makes the following diagram of short exact sequences commute.

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(A^c) \arrow[r, "j_{A^c}"] \arrow[d, "g_{A^c,a_0}"] & \Gamma^{A^c} \arrow[r] \arrow[d, "\pi_{A^c,a_0}"] & M_{\mathfrak{S}}(A) \arrow[r] \arrow[d, dashed, "\mu_{A,a_0}"] & 0 \\
0 \arrow[r] & \Gamma(A^c \setminus a_0) \arrow[r, "j_{A^c\setminus a_0}"] & \Gamma^{A^c \setminus a_0} \arrow[r] & M_{\mathfrak{S}}(A \cup a_0) \arrow[r] & 0
\end{tikzcd} \tag{17}$$

Lemma 9.11. *Let $\mathfrak{S}$ be arithmetic, suppose that (Cyc) holds, and recall Definition 9.8. Then, for every $A \subseteq E_{\mathfrak{S}}$ and every $a_0 \in E_{\mathfrak{S}}$,*

(i) g_{A,a_0} is surjective with cyclic kernel;
(ii) π_{A,a_0} is surjective with cyclic kernel;
(iii) μ_{A,a_0} is surjective with cyclic kernel.

Proof. Part (ii) is clear from (Cyc). Surjectivity of g_{A,a_0} is also evident from the definition. With these preliminary remarks we can complete the diagram in Definition 9.10 with the kernels and cokernels of the vertical maps, obtaining the diagram in Fig. 14. We first check that the bottom row (dashed) is exact and thus we obtain $\operatorname{coker}(\mu_{A,a_0}) = 0$. Then, the nine lemma implies that the top row is exact: since we know that $\ker(\pi_{A^c,a_0})$ is cyclic, we can thus deduce cyclicity of $\ker(g_{A^c,a_0})$ and $\ker(\mu_{A,a_0})$. This concludes the proof of (i) and (iii). □

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(g_{A^c,a_0}) \arrow[r] \arrow[d] & \ker(\pi_{A^c,a_0}) \arrow[r] \arrow[d] & \ker(\mu_{A,a_0}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(A^c) \arrow[r] \arrow[d, "g_{A^c,a_0}"] & \Gamma^{A^c} \arrow[r] \arrow[d, "\pi_{A^c,a_0}"] & M_{\mathfrak{S}}(A) \arrow[r] \arrow[d, "\mu_{A,a_0}"] & 0 \\
0 \arrow[r] & \Gamma(A^c \setminus a_0) \arrow[r] \arrow[d] & \Gamma^{A^c \setminus a_0} \arrow[r] \arrow[d] & M_{\mathfrak{S}}(A \cup a_0) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r, dashed] & 0 \arrow[r, dashed] & \operatorname{coker}(\mu_{A,a_0}) \arrow[r, dashed] & 0
\end{tikzcd}$$

Fig. 14. Diagram for the proof of Lemma 9.11.

Lemma 9.12. *Let $\mathfrak{S}$ be arithmetic and suppose that (Cyc) holds. Then, for every $A \subseteq E_{\mathfrak{S}}$ such that $A^c \in \underline{\mathcal{C}}$ and every $a_0, b_0 \in E_{\mathfrak{S}}$, the following is a pushout square.*

$$\begin{tikzcd}
M_{\mathfrak{S}}(A) \arrow[r, "\mu_{A,a_0}"] \arrow[d, "\mu_{A,b_0}"] & M_{\mathfrak{S}}(A \cup \{a_0\}) \arrow[d, "\mu_{A\cup\{a_0\},b_0}"] \\
M_{\mathfrak{S}}(A \cup \{b_0\}) \arrow[r, "\mu_{A\cup\{b_0\},a_0}"] & M_{\mathfrak{S}}(A \cup \{a_0, b_0\})
\end{tikzcd}$$

Proof. The morphism of short exact sequences defining the maps $\mu_{*,*}$ described in Diagram (17) can be fit together to a square of short exact sequences as follows, where for simplicity we write $A_1 := A \cup \{a_0\}$, $A_2 := A \cup \{b_0\}$, $A_3 := A \cup \{a_0, b_0\}$, so that the right-hand side square is indeed the square appearing in the claim.

$$\begin{tikzcd}
 & 0 \arrow[r, dashed] & \Gamma(A^c) \arrow[r, dashed] \arrow[dl, dashed] \arrow[ddr, dashed] & \Gamma^{A^c} \arrow[r] \arrow[dl] \arrow[ddr] & M_{\mathfrak{S}}(A) \arrow[r] \arrow[dl] \arrow[ddr] & 0 & \\
0 \arrow[r, dashed] & \Gamma({A_1}^c) \arrow[r, dashed] \arrow[ddr, dashed] & \Gamma^{{A_1}^c} \arrow[r] \arrow[ddr] & M_{\mathfrak{S}}(A_1) \arrow[r] \arrow[ddr] & 0 & & \\
 & & 0 \arrow[r, dashed] & \Gamma({A_2}^c) \arrow[r] \arrow[dl] & \Gamma^{{A_2}^c} \arrow[r] \arrow[dl] & M_{\mathfrak{S}}(A_2) \arrow[r] \arrow[dl] & 0 \\
 & 0 \arrow[r, dashed] & \Gamma({A_3}^c) \arrow[r] & \Gamma^{{A_3}^c} \arrow[r] & M_{\mathfrak{S}}(A_3) \arrow[r] & 0 &
\end{tikzcd}$$

By part (i) and (ii) of Lemma 9.11, by exactness of the rows and with Definition 9.10, the part of the diagram drawn with solid arrows satisfies the assumptions of Lemma 10.8, which allows us to conclude that the right-hand side square is a pushout square, as was to be shown. □

Proposition 9.13. *Let $\mathfrak{S}$ be a centered, arithmetic G-semimatroid satisfying (Cyc). Then $M_{\mathfrak{S}}$ is a representable matroid over $\mathbb{Z}$.*

Proof. This follows combining Lemma 9.11.(iii) and Lemma 9.12. □

Lemma 9.14. *Let $\mathfrak{S}$ be an arithmetic G-semimatroid such that all groups Γ^a are infinite cyclic. Then for all $A \in \underline{\mathcal{C}}$ the rank of $W(A)$ as a $\mathbb{Z}$-module is*

$$\operatorname{rank}_{\mathbb{Z}}(W(A)) = \underline{\mathrm{rk}}(A)$$

Proof. Let $F \subseteq A$ be a maximal independent set in A, i.e., one with $|F| = \underline{\mathrm{rk}}(A)$. In particular, such an F satisfies $|F| = \underline{\mathrm{rk}}(F)$ and thus, by Lemma 7.3 and Definition 3.15

$$W(F) = \Gamma^F \simeq \mathbb{Z}^{|F|}. \tag{18}$$

Moreover, since $\underline{\mathrm{rk}}(F) = \underline{\mathrm{rk}}(A)$, by Lemma 7.3 and Remark 9.9.(a), the group homomorphism $w_{A,A\setminus F} : W(A) \to W(F)$ is injective and, by the additivity theorem for ranks, we have

$$\mathrm{rank}_{\mathbb{Z}}(W(F)) = \mathrm{rank}_{\mathbb{Z}}(W(A)) + \mathrm{rank}_{\mathbb{Z}}\left(\frac{W(F)}{w_{A,A\setminus F}(W(A))}\right) \tag{19}$$

Claim. $\mathrm{rank}_{\mathbb{Z}}\left(\frac{W(F)}{w_{A,A\setminus F}(W(A))}\right) = 0.$

Proof. We have $\Gamma(F) = \Gamma(A)$ by Lemma 8.1.(i), and the subgroup $j'_F(\Gamma(F)) \subseteq W(F)$ is contained in $w_{A,A\setminus F}(W(A))$. We thus obtain an isomorphism

$$\frac{W(F)}{w_{A,A\setminus F}(W(A))} \simeq \frac{W(F)/j'_F(\Gamma(F))}{w_{A,A\setminus F}(W(A))/j'_F(\Gamma(F))}.$$

The cardinality of $W(F)/j'_F(\Gamma(F))$ equals $m_{\mathfrak{S}}(F)$ and is, in particular, finite. Thus both groups above are finite and have rank zero as $\mathbb{Z}$-modules. □

With the claim we can conclude by the following computation (where we use Equation (18), Equation (19) and the definition of F).

$$\mathrm{rank}_{\mathbb{Z}}(W(A)) = \mathrm{rank}_{\mathbb{Z}}(W(F)) = |F| = \underline{\mathrm{rk}}(A) \quad \square$$

Corollary 9.15. *Let $\mathfrak{S}$ be a centered arithmetic G-semimatroid such that all groups Γ^a are infinite cyclic. Then for every $A \in \underline{\mathcal{C}}$ the rank of $M_{\mathfrak{S}}(A^c)$ as a $\mathbb{Z}$-module is*

$$\mathrm{rank}_{\mathbb{Z}}(M_{\mathfrak{S}}(A^c)) = |A| - \underline{\mathrm{rk}}(A)$$

Proof. First, notice that Remark 9.9.(b) implies exactness of the sequence

$$0 \longrightarrow \Gamma(A) \xrightarrow{j'_A} W(A) \longrightarrow W(A)/j'_A(\Gamma(A)) \longrightarrow 0$$

Since the group $W(A)/j'_A(\Gamma(A))$ has finite cardinality (equal to $m_{\mathfrak{S}}(A)$), the additivity theorem for ranks of abelian groups implies

$$\mathrm{rank}_{\mathbb{Z}}(W(A)) = \mathrm{rank}_{\mathbb{Z}}(j'_A(\Gamma(A))).$$

In particular, using the definitions, Lemma 9.14 and Remark 9.9 we conclude

$$\mathrm{rank}_{\mathbb{Z}}(M_{\mathfrak{S}}(A^c)) = \mathrm{rank}_{\mathbb{Z}}(\Gamma^A/j_A(\Gamma(A)))$$
$$= \mathrm{rank}_{\mathbb{Z}}(\Gamma^A) - \mathrm{rank}_{\mathbb{Z}}(j_A(\Gamma(A))) = |A| - \underline{\mathrm{rk}}(A). \quad \square$$

Corollary 9.16. *Let $\mathfrak{S}$ be a centered arithmetic G-semimatroid such that all groups Γ^a are infinite cyclic. Then the underlying matroid of $M_{\mathfrak{S}}$ is the dual to $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$.*

Proof. By Remark 1.27 and Corollary 9.15 the rank function rk of the underlying matroid satisfies

$$\mathrm{rk}(E_{\mathfrak{S}}) - \mathrm{rk}(A) = \mathrm{rank}_{\mathbb{Z}}(M_{\mathfrak{S}}(A)) = |A^c| - \underline{\mathrm{rk}}(A^c)$$

For all $A \subseteq E_{\mathfrak{S}}$. Thus $\underline{\mathrm{rk}}(A^c) = \mathrm{rk}(A) - |A^c| - \underline{\mathrm{rk}}(E_{\mathfrak{S}})$, proving that $(E_{\mathfrak{S}}, \underline{\mathrm{rk}})$ and $(E_{\mathfrak{S}}, \mathrm{rk})$ are dual (see, e.g., [31, Proposition 2.1.9]). $\square$

We end by describing a situation where the torsion elements of the modules $M_{\mathfrak{S}}$ can be interpreted combinatorially.

Proposition 9.17. *Let $\mathfrak{S}$ be a centered arithmetic G-semimatroid such that all groups Γ^a are infinite cyclic and consider $A \subseteq E_{\mathfrak{S}}$. If $W(A)$ is a pure subgroup of Γ^A, then*

$$M_{\mathfrak{S}}(A) \simeq \mathbb{Z}^{|A^c| - \underline{\mathrm{rk}}(A)} \oplus W(A)/h_A(G)$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \ker(\varphi) \arrow[d] & \\
0 \arrow[r] & \Gamma(A) \arrow[r, "j_A"] \arrow[d, "j'_A"] & \Gamma^A \arrow[r] \arrow[d, "="] & M_{\mathfrak{S}}(A) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & W(A) \arrow[r, hook] \arrow[d] & \Gamma^A \arrow[r, "\epsilon"] \arrow[d] & L(A) \arrow[r] \arrow[d] & 0 \\
 & C(A) \arrow[r] & 0 \arrow[r] & \mathrm{coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we have an isomorphism $\ker(\varphi) \simeq W(A)/j'_A(\Gamma(A))$. Moreover, exactness of the second row at $L(A)$ implies that the last row is exact at $\mathrm{coker}(\varphi)$, and the latter is thus trivial. Summarizing, we have the following exact sequence.

$$0 \longrightarrow W(A)/j'_A(\Gamma(A)) \longrightarrow M_{\mathfrak{S}}(A) \longrightarrow L(A) \longrightarrow 0$$

The purity assumption on $W(A)$ means that $L(A)$ is a free abelian group and implies that this sequence splits. Remark 9.9.(b) then shows $j'_A(\Gamma(A)) = h_A(G)$, proving the claimed isomorphism. $\square$

Corollary 9.18. *Recall Remark 1.27. With the assumptions of Proposition 9.17, the underlying arithmetic matroid of $M_{\mathfrak{S}}$ is the dual to $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$.*

Proof. After Corollary 9.16 we only have to show that $m_{\mathfrak{S}}(A)$ equals the cardinality of the torsion part of $M_{\mathfrak{S}}(E_{\mathfrak{S}} \setminus A)$, which is a direct consequence of Lemma 9.3.(ii). $\square$

Remark 9.19. The map b_X of Equation (14) induces a bijection between $\lceil A \rceil^{\mathcal{C}}/G$ and the group $C(A)$ defined in the diagram for the proof of Proposition 9.17. We regard the group structure thus induced from $C(A)$ as additional data that can be extracted from $\mathfrak{G}$. Recent results in the topology of toric arrangements [6, Example 7.3.2] show that this additional data has an algebraic-topological significance.

10. Tutte polynomials of group actions

In this section we study the Tutte polynomial associated to a group action on a semimatroid and, as an application, we extend to the generality of group actions on semimatroids (in particular, beyond the representable case) two important combinatorial interpretations of Tutte polynomials of toric arrangements.

Recall our standard setup, e.g., from Section 3. We let $\mathfrak{G}$ denote the action of a group G on a finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$. Write $\mathcal{L} = \mathcal{L}(\mathcal{S})$ for the geometric semilattice of flats of $\mathcal{S}$, and let $\mathcal{P}_{\mathfrak{G}}$ denote the quotient poset of $\mathcal{L}$ (see Definition 3.21). Moreover, recall the set $\mathcal{C}_{\mathfrak{G}}$ of orbits of the action on $\mathcal{C}$ and the "underlying" locally ranked triple $\mathcal{S}_{\mathfrak{G}} = (E_{\mathfrak{G}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$

We will make use of standard terminology about posets (see Section 5 for a review).

10.1. The characteristic polynomial of $\mathcal{P}_{\mathfrak{G}}$

Remark 10.1. Since G acts on $\mathcal{L}$ by rank-preserving maps, the poset $\mathcal{P}_{\mathfrak{G}}$ is ranked. With slight abuse of notation we will call $\underline{\mathrm{rk}}$ the rank function on $\mathcal{P}_{\mathfrak{G}}$, which satisfies

$$\underline{\mathrm{rk}}(p) = \underline{\mathrm{rk}}(x_p) \qquad \text{if } p = Gx_p.$$

We can thus define the *characteristic polynomial of* $\mathcal{P}_{\mathfrak{G}}$ (e.g., following [34, §3.10]) as

$$\chi_{\mathfrak{G}}(t) := \sum_{p \in \mathcal{P}_{\mathfrak{G}}} \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p) t^{r - \underline{\mathrm{rk}}(p)},$$

where r is the rank of $\mathcal{S}_{\mathfrak{G}}$ and $\mu_{\mathfrak{G}}$ is the Möbius function of $\mathcal{P}_{\mathfrak{G}}$ (notice that $\mathcal{P}_{\mathfrak{G}}$ has a unique minimal element corresponding to the empty subset of $E_{\mathfrak{G}}$).

Lemma 10.2. *Let $\mathfrak{G}$ be weakly translative. Then, for every $x \in \mathcal{L}$, the intervals $[\hat{0}, Gx]$ in $\mathcal{P}_{\mathfrak{G}}$ and $[\hat{0}, x]$ in $\mathcal{L}$ are poset-isomorphic. In particular, intervals in $\mathcal{P}_{\mathfrak{G}}$ are geometric lattices.*

Proof. Choose $x_p \in \mathcal{L}$, set $p := Gx_p \in \mathcal{P}_{\mathfrak{G}}$ and consider any $q \in [\hat{0}, p]$. Since $q \leq_{\mathcal{P}_{\mathfrak{G}}} p$, by definition there is $x_q \in q$ with $x_q \leq_{\mathcal{L}} x_p$.

Every other $x'_q \in q$ with $x'_q \leq_{\mathcal{L}} x_p$ has the form $x'_q = gx_q$ for some $g \in G$. Then, for every atom x_a of $\mathcal{L}$ with $x_a \leq_{\mathcal{L}} x_q \leq_{\mathcal{L}} x_p$, $gx_a \leq_{\mathcal{L}} x_p$. In particular, for every $s \in x_a$, $\{s, gs\} \in \mathcal{C}$ and by weak translativity $\mathrm{rk}\{s, gs\} = 1$. Thus $gx_a \subseteq \mathrm{cl}_{\mathcal{C}}\, x_a = x_a$ and, by symmetry, $x_a = gx_a$. This is true for all atoms $x_a \leq_{\mathcal{L}} x_p$ and hence, because the interval $[\hat{0}, x_p]$ is atomic, we have $x_q = x'_q$.

Therefore the mapping

$$[\hat{0}, p]_{\mathcal{P}_{\mathfrak{G}}} \to [\hat{0}, x_p]_{\mathcal{L}}, \; q \mapsto x_q$$

is well-defined and order preserving. So is clearly its inverse

$$[\hat{0}, x_p]_{\mathcal{L}} \to [\hat{0}, p]_{\mathcal{P}_{\mathfrak{G}}}, \; x \mapsto Gx$$

and thus the two intervals are poset-isomorphic. □

Proof of Theorem F. Let us first consider some $p \in \mathcal{P}_{\mathfrak{G}}$ with $p > \hat{0}$. By Hall's theorem [34, Proposition 3.8.5] the number $\mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p)$ is the reduced Euler characteristics of the "open interval" $[\hat{0}, p] \setminus \{\hat{0}, p\}$.

By Lemma 10.2, the interval $[\hat{0}, p]$ is a geometric lattice with set of atoms $A(p)$, and thus it induces a matroid structure on the set $\cup A(p) \subseteq E_{\mathfrak{G}}$ (with rank function $\underline{\mathrm{rk}}$). Let cl_p denote the associated closure operator.

Following [39], the reduced Euler characteristics of $[\hat{0}, p]$ can be computed by means of the *atomic complex*: this is the simplicial complex on the vertex set $A(p)$ and with set of simplices $\Delta_p = \{B \subseteq A(p) \mid \vee B < p\}$. We obtain

$$\mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p) = \sum_{A \in \Delta_p} (-1)^{|A|-1} = \sum_{A \in D_p} (-1)^{|A|},$$

where $D_p := \{A \subseteq A(p) \mid \vee A = p\}$ and the second equality is derived from the boolean identity $\sum_{A \subseteq A(p)} (-1)^{|A|} = 0$. Moreover, setting

$$\widetilde{D}_p := \{\widetilde{A} \subseteq E_{\mathfrak{G}} \mid \mathrm{cl}_p(\widetilde{A}) = p\}$$

and using the fact that $\mathcal{S}$ loopless implies $\mathcal{S}_{\mathfrak{G}}$ loopless, we can compute

$$\begin{aligned} \sum_{\widetilde{A} \in \widetilde{D}_p} (-1)^{|\widetilde{A}|} &= \sum_{A \in D_p} \sum_{\substack{\widetilde{A} = \coprod_{a \in A} X_a \\ \mathrm{cl}_p(X_a) = a}} (-1)^{|\widetilde{A}|} \\ &= \sum_{A \in D_p} \prod_{a \in A} \left[\sum_{\emptyset \neq X_a \subseteq a} (-1)^{|X_a|} \right] = \sum_{A \in D_p} (-1)^{|A|} = \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p). \end{aligned}$$

Notice that the equality $\sum_{\widetilde{A} \in \widetilde{D}_p} (-1)^{|\widetilde{A}|} = \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p)$, which we just proved for $p > \hat{0}$, holds trivially for $p = \hat{0}$. Moreover, $\widetilde{A} \in \widetilde{D}_p$ implies in particular $\underline{\mathrm{rk}}(\widetilde{A}) = \underline{\mathrm{rk}}(p)$. We can rewrite

$$\chi_{\mathfrak{S}}(t) = \sum_{p\in\mathcal{P}_{\mathfrak{S}}} \mu_{\mathcal{P}_{\mathfrak{S}}}(\hat{0},p)t^{r-\underline{\mathrm{rk}}(p)} = \sum_{p\in\mathcal{P}_{\mathfrak{S}}} \sum_{\widetilde{A}\in\widetilde{D}_p} (-1)^{|\widetilde{A}|}t^{r-\underline{\mathrm{rk}}(p)}$$
$$= \sum_{\widetilde{A}\in\underline{\mathcal{C}}} (-1)^{|\widetilde{A}|} \sum_{p\in P_{\widetilde{A}}} t^{r-\underline{\mathrm{rk}}(\widetilde{A})}$$

where for every $\widetilde{A}\in\underline{\mathcal{C}}$ we let

$$P_{\widetilde{A}} := \{p\in\mathcal{P}_{\mathfrak{S}} \mid \widetilde{A}\in\widetilde{D}_p\} = \lceil\widetilde{A}\rceil^{\mathcal{C}}/G,$$

which is a set with exactly $m_{\mathfrak{S}}(\widetilde{A})$ elements (see Definition 3.9). Thus,

$$\chi_{\mathfrak{S}}(t) = \sum_{A\in\underline{\mathcal{C}}} (-1)^{|A|} m_{\mathfrak{S}}(A)t^{r-\underline{\mathrm{rk}}(A)}$$
$$= (-1)^r \sum_{A\in\underline{\mathcal{C}}} m_{\mathfrak{S}}(A)(-1)^{|A|-\mathrm{rk}(A)}(-t)^{r-\underline{\mathrm{rk}}(A)}$$
$$= (-1)^r T_{\mathfrak{S}}(1-t,0)$$

where, as above, r denotes the rank of $\mathcal{S}_{\mathfrak{S}}$. □

10.2. The corank-nullity polynomial of $\mathcal{C}_{\mathfrak{S}}$

The corank-nullity polynomial of the poset $\mathcal{C}_{\mathfrak{S}}$ is

$$s(\mathcal{C}_{\mathfrak{S}};u,v) = \sum_{GX\in\mathcal{C}_{\mathfrak{S}}} u^{(r-\mathrm{rk}(X))}v^{(|X|-\mathrm{rk}(X))}.$$

Proposition 10.3. *If $\mathfrak{S}$ is translative,*

$$T_{\mathfrak{S}}(x,y) = s(\mathcal{C}_{\mathfrak{S}};x-1,y-1).$$

Proof. When $\mathfrak{S}$ is translative, for every $X\in\mathcal{C}$ we have $|X| = |\underline{X}|$. Moreover, by Corollary 6.5, $\mathrm{rk}(X) = \underline{\mathrm{rk}}(\underline{X})$. Then,

$$s(\mathcal{C}_{\mathfrak{S}};u,v) = \sum_{GX\in\mathcal{C}_{\mathfrak{S}}} u^{(r-\mathrm{rk}(X))}v^{(|X|-\mathrm{rk}(X))} = \sum_{A\in\underline{\mathcal{C}}} \sum_{\substack{GX\in\mathcal{C}_{\mathfrak{S}}\\ \underline{X}=A}} u^{(r-\underline{\mathrm{rk}}(A))}v^{(|A|-\underline{\mathrm{rk}}(A))}$$

and the claim follows by setting $u = x-1$ and $v = y-1$. □

10.3. Activities

We now turn to a generalization and new combinatorial interpretation of the basis-activity decomposition of arithmetic Tutte polynomials as defined in [5].

Remark 10.4. Since we will not need details here, but only the statement of the next lemma, we refer to Ardila [2] for the definition of internal and external activity of bases of a finite semimatroid.

Lemma 10.5 *(Proposition 9.11 of [2]). Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ is a finite semimatroid with set of bases $\mathcal{B}$ and let a total ordering of S be fixed. For every basis $B \in \mathcal{B}$ let $E(B)$, resp. $I(B)$, denote the set of externally, resp. internally active elements with respect to B and write $R_B := B \setminus I(B)$. Then, $(R_B, I(B), E(B))$ is a molecule, and*

$$\mathcal{C} = \biguplus_{B \in \mathcal{B}} [R_B, B \cup E(B)]$$

We use this decomposition, which generalizes that for matroids proved in [8], in order to rewrite the sum in Definition 3.28 as a sum over all bases.

Theorem H. *Let $\mathfrak{S}$ be an almost-arithmetic G-semimatroid such that $\mathcal{S}_{\mathfrak{S}}$ is a semimatroid. Let $\mathcal{B}_{\mathfrak{S}}$ denote the set of bases of $\mathcal{S}_{\mathfrak{S}}$ and fix a total ordering of $E_{\mathfrak{S}}$. For $B \in \mathcal{B}_{\mathfrak{S}}$ let $E(B)$, resp. $I(B)$ denote the set of externally, resp. internally active elements with respect to B, and write $R_B := B \setminus I(B)$. Then*

$$T_{\mathfrak{S}}(x, y) = \sum_{B \in \mathcal{B}_{\mathfrak{S}}} \left(\sum_{p \in \mathcal{Z}(B)} x^{\iota(p)} \right) \left(\sum_{c \in \lceil R_B \rceil^{\mathcal{C}}/G} y^{\eta_{E(B)}(c)} \right)$$

where

$\eta_{E(B)}(c)$ *is the number of $e \in E(B)$ with $e \leq \kappa_{\mathfrak{S}}(c)$ in $\mathcal{C}_{\mathfrak{S}}$ (Definition 7.29),*
$\mathcal{Z}(B)$ *denotes the set $\mathcal{X}(I(B))$ associated to the molecule $(R_B, I(B), \emptyset)$ in Definition 8.7 and, accordingly,*
$\iota(p)$ *is the number defined in Definition 8.7.*

In particular, the theorem holds when $\mathfrak{S}$ is centered, in which case it extends [9, Theorem 6.3] to the nonrepresentable (and non-arithmetic) case.

Proof. First, using Lemma 10.5 we rewrite

$$T_{\mathfrak{S}}(x, y) = \sum_{B \in \mathcal{B}} \sum_{A \in \mu(B)} m_{\mathfrak{S}}(A)(x-1)^{\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) - \underline{\mathrm{rk}}(A)}(y-1)^{|A| - \underline{\mathrm{rk}}(A)}$$

and then, using [5, Lemma 4.3] (whose proof only uses axiom (A2)) we obtain

$$T_{\mathfrak{S}}(x, y) =$$
$$\sum_{B \in \mathcal{B}} \left(\sum_{F \subseteq I(B)} \frac{\rho(R_B, R_B \cup (I(B) \setminus F))}{m(R_B)} x^{|F|} \right) \left(\sum_{T \subseteq E(B)} \rho(R_B \cup T, R_B \cup E(B)) y^{|T|} \right).$$

Here, in every summand the right-hand side factor is ready to be treated with Proposition 7.30 applied to the molecule $(R_B, \emptyset, E(B))$, while the left-hand side factor equals the claimed polynomial by Proposition 8.9 applied to the molecule $(R_B, I(B), \emptyset)$. □

10.4. Deletion–contraction recursion

We have seen (Section 3) that the matroid operations of contraction and deletion extend in a natural way to the context of G-semimatroids. In this section we study these operations, showing that the Tutte polynomial of a translative action decomposes as a weighted sum of the polynomial of any single-element contraction and that of the corresponding deletion.

Recall the definitions and notations from Section 1.1 and Section 3. In the following, given a locally ranked triple $\mathcal{S}$ we will write $\mathcal{C}(\mathcal{S})$ for its associated simplicial complex (the triple's "second component").

Lemma 10.6. *Let $\mathfrak{S} : G \circlearrowleft (S, \mathcal{C}, \mathrm{rk})$ be a weakly translative G-semimatroid, and fix $e \in E_{\mathfrak{S}}$. Then,*

(1) there is a surjection $\phi : \mathcal{C}(\mathcal{S}_{\mathfrak{S}/e}) \to \mathcal{C}(\mathcal{S}_{\mathfrak{S}}/e)$ with $\mathrm{rk}_{\mathfrak{S}}(\phi(A) \cup e) - \mathrm{rk}_{\mathfrak{S}}(e) = \mathrm{rk}_{\mathfrak{S}/e}(A)$ which, if the action is translative, also satisfies $|\phi(A)| = |A|$;
(2) $\mathcal{P}_{\mathfrak{S}/e} = (\mathcal{P}_{\mathfrak{S}})_{\geq e}$.

Moreover,

(3) $m_{\mathfrak{S}}(A \cup e) = \sum_{A' \in \phi^{-1}(A)} m_{\mathfrak{S}/e}(A').$

Proof. Let us choose a fixed representative $x_e \in e$. In order to prove (1), we start by recalling that, by definition,

$$\mathcal{C}_{(\mathfrak{S}/e)} = (\mathcal{C}_{/x_e})/\operatorname{stab}(x_e).$$

From now, throughout this proof, we write $H := \operatorname{stab}(x_e)$. Recall also the natural order on $\mathcal{C}_{\mathfrak{S}}$ (Remark 3.5) and define

$$\widetilde{\phi} : \mathcal{C}_{\mathfrak{S}/e} \to (\mathcal{C}_{\mathfrak{S}})_{\geq e}, \quad H\{x_1, \ldots, x_k\} \mapsto G\{x_1, \ldots, x_k, x_e\}.$$

The function $\widetilde{\phi}$ is a bijection, because the assignment

$$G\{x_1, \ldots, x_k, gx_e\} \mapsto H\{g^{-1}x_1, \ldots, g^{-1}x_k\}$$

determines a well-defined inverse to $\widetilde{\phi}$.

In order to prove (2) we notice that $\widetilde{\phi}$ commutes with the relevant closure operators, i.e.,

$$\widetilde{\phi} \circ \kappa_{\mathfrak{S}/e} = \kappa_{\mathfrak{S}} \circ \widetilde{\phi}.$$

Bijectivity of $\widetilde{\phi}$ implies then that $\mathcal{P}_{\mathfrak{S}/e} = \kappa_{\mathfrak{S}}((\mathcal{C}_{\mathfrak{S}})_{\geq e})$, and the latter is easily seen to equal $(\mathcal{P}_{\mathfrak{S}})_{\geq e}$. Thus, (2) holds.

Consider now the map

$$\phi : \underline{\mathcal{C}_{/x_e}} \to \underline{\mathcal{C}}_{/e}; \{Hx_1, \ldots, Hx_k\} \mapsto \{Gx_1, \ldots, Gx_k\}$$

and the following diagram (recall Remark 3.9)

$$\begin{tikzcd}
\mathcal{C}_{\mathfrak{S}/e} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\lfloor \cdot \rfloor"'] & (\mathcal{C}_{\mathfrak{S}})_{\geq e} \arrow[d, "\lfloor \cdot \rfloor \setminus \{e\}"] \\
\underline{\mathcal{C}_{/x_e}} \arrow[r, "\phi"] & \underline{\mathcal{C}}_{/e}
\end{tikzcd}$$

where commutativity is evident once we evaluate all maps on a specific argument as follows.

$$\begin{tikzcd}
H\{x_1, \ldots, x_k\} \arrow[r, mapsto] \arrow[d, mapsto] & G\{x_1, \ldots, x_k, x_e\} \arrow[d, mapsto] \\
\{Hx_1, \ldots, Hx_k\} \arrow[r, mapsto] & \{Gx_1, \ldots, Gx_k\}
\end{tikzcd}$$

Now, for every $A \in \underline{\mathcal{C}}_{/e}$ the map $\widetilde{\phi}$ gives a bijection between the $\lfloor \cdot \rfloor \setminus \{e\}$-preimage of A and the $\lfloor \cdot \rfloor$-preimage of $\phi^{-1}(A)$, which proves (3). Claim (1) follows by inspecting the definition of the rank and, for the claim about cardinality, by noticing that if $Hx_1 \neq Hx_2$ and $gx_1 = x_2$ for some $g \in G$, then $\{x_1, gx_1\} \in \mathcal{C}$ and by translativity $x_1 = gx_1 = x_2$, a contradiction. $\square$

Proposition 10.7. *Let $\mathfrak{S}$ denote a G-semimatroid and fix $e \in E_{\mathfrak{S}}$. If $\mathfrak{S}$ is weakly translative – resp. translative, normal, arithmetic –, then so are $\mathfrak{S}/e$ and $\mathfrak{S}\setminus e$ as well. Moreover, if $\mathfrak{S}$ is weakly translative and cofinite, then $\mathfrak{S}/e$ and $\mathfrak{S} \setminus e$ are also cofinite.*

Proof. The treatment of $\mathfrak{S} \setminus e$ is trivial: indeed, the same group acts on a smaller set of elements with the same constraints. We will thus examine the case $\mathfrak{S}/e$. Choose $x_e \in e$ and let $H := \mathrm{stab}(x_e)$.

– $\mathfrak{S}$ *weakly translative.* To check weak translativity for $\mathfrak{S}/e$ consider some $y \in S_{/x_e}$ and suppose $\{y, hy\} \in \mathcal{C}_{/x_e}$ for some $h \in H$. This means by definition that $\{y, hy, x_e\} \in \mathcal{C}$, thus $\{y, hy\} \in \mathcal{C}$ and, by weak translativity of $\mathfrak{S}$, we have $\mathrm{rk}_{\mathcal{C}}(\{y, hy\}) = \mathrm{rk}_{\mathcal{C}}(\{y\})$. Now by (R3) we know

$$\mathrm{rk}_{\mathcal{C}}(\{y\}) + \mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) \leq \mathrm{rk}_{\mathcal{C}}(\{y, x_e\}) + \mathrm{rk}_{\mathcal{C}}(\{y, hy\}).$$

By subtracting $\mathrm{rk}_{\mathcal{C}}(\{y\})$ from both sides we obtain the inequality $\mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) \leq \mathrm{rk}_{\mathcal{C}}(\{y, x_e\})$ and, by (R2), $\mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) = \mathrm{rk}_{\mathcal{C}}(\{y, x_e\})$. We are now left with computing

$$\begin{aligned} \mathrm{rk}_{\mathcal{C}/x_e}(\{y, hy\}) &\overset{\text{def.}}{=} \mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) - \mathrm{rk}_{\mathcal{C}}(\{x_e\}) \\ &= \mathrm{rk}_{\mathcal{C}}(\{y, x_e\}) - \mathrm{rk}_{\mathcal{C}}(\{x_e\}) \overset{\text{def.}}{=} \mathrm{rk}_{\mathcal{C}/x_e}(\{y\}) \end{aligned}$$

– $\mathfrak{S}$ *translative.* As above, consider some $y \in S_{/x_e}$ and suppose $\{y, hy\} \in \mathcal{C}_{/x_e}$ for some $h \in H$. This means that $\{y, hy, x_e\} \in \mathcal{C}$, thus $\{y, hy\} \in \mathcal{C}$ and, by translativity of $\mathfrak{S}$, $hy = y$ as required.
– $\mathfrak{S}$ *normal.* Let $X \in \mathcal{C}_{/x_e}$ then $\mathrm{stab}_H(X) = \mathrm{stab}_G(X) \cap H$ is normal in G because it is the intersection of two normal subgroups. *A fortiori* it is normal in H.
– $\mathfrak{S}$ *arithmetic.* Let $X = \{x_1, \ldots, x_k\} \in \mathcal{C}_{/x_e}$. For all i there is a natural group homomorphism

$$\omega_i : \Gamma_{/e}(x_i) = H/\,\mathrm{stab}_H(x_i) \hookrightarrow G/\,\mathrm{stab}_G(x_i) = \Gamma(x_i)$$

and these induce a natural group homomorphism

$$\omega : \Gamma^X_{/e} \to \Gamma^{X \cup x_e}, \quad (\gamma_1, \ldots, \gamma_k) \mapsto (\mathrm{id}, \omega_1(\gamma_1), \ldots, \omega_k(\gamma_k)).$$

Now consider $\gamma, \gamma' \in W_{/e}(X)$. Then clearly $\omega(\gamma), \omega(\gamma') \in W(X \cup x_e)$ and, by arithmeticity of $\mathfrak{S}$,

$$\omega(\gamma)\omega(\gamma') = (\mathrm{id}, \omega_1(\gamma_1)\omega_1(\gamma_1'), \ldots) = (\mathrm{id}, \omega_1(\gamma_1\gamma_1'), \ldots) \in W(X \cup x_e).$$

Now, this means that $\omega(\gamma\gamma').(X \cup x_e) = \gamma\gamma'.X \cup \{x_e\} \in \mathcal{C}$, hence $\gamma\gamma'.X \in \mathcal{C}_{/x_e}$ thus by definition $\gamma\gamma' \in W_{/e}(X)$.
– $\mathfrak{S}$ *(weakly translative and) cofinite.* Cofiniteness of $\mathfrak{S} \setminus e$ is trivial, and that of $\mathfrak{S}/e$ is a consequence of Lemma 10.6.(3). $\square$

We can now state and prove the desired recursion for Tutte polynomials of translative G-semimatroids, generalizing the corresponding result of [5] for the arithmetic and centered case.

Proof of Theorem G. In this proof for greater clarity we will write $\mathrm{rk}_{\mathfrak{S}}$, resp. $\mathrm{rk}_{\mathfrak{S}/e}$ for the rank functions of $\mathcal{S}_{\mathfrak{S}}$, resp. $\mathcal{S}_{\mathfrak{S}/e}$ (in particular, $\mathrm{rk}_{\mathfrak{S}}$ corresponds to what we called $\underline{\mathrm{rk}}$ previously).

We follow [2, Proposition 8.2], where the analogous results for semimatroids are proved, and start by rewriting the definition.

$$
\begin{aligned}
T_{\mathfrak{S}}(x,y) :&= \sum_{A \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A)}(y-1)^{|A|-\mathrm{rk}_{\mathfrak{S}}(A)} \\
&= \underbrace{\sum_{A \in \underline{\mathcal{C}},\ e \notin A}}_{\text{i.e., } A \in \underline{\mathcal{C}}_{\setminus e} = \mathcal{C}(\mathcal{S}_{\mathfrak{S}\setminus e})} m_{\mathfrak{S}}(A)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A)}(y-1)^{|A|-\mathrm{rk}_{\mathfrak{S}}(A)} \\
&\quad + \sum_{A \cup e \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A \cup e)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A \cup e)}(y-1)^{|A \cup e|-\mathrm{rk}_{\mathfrak{S}}(A \cup e)}
\end{aligned}
$$

The second summand can be rewritten as follows by Lemma 10.6.

$$
\underbrace{\sum_{A \in \underline{\mathcal{C}}_{/e}} \sum_{A' \in \phi^{-1}(A)}}_{A' \in \mathcal{C}(\mathcal{S}_{\mathfrak{S}/e})} m_{\mathfrak{S}/e}(A')(x-1)^{r(\mathcal{S}_{\mathfrak{S}/e})-\mathrm{rk}_{\mathfrak{S}/e}(A')}(y-1)^{|A'|+1-\mathrm{rk}_{\mathfrak{S}/e}(A')-\mathrm{rk}_{\mathfrak{S}}(e)}
$$

If e is neither a loop nor an isthmus, by Remark 3.31 and Lemma 10.6 we have $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e})$ and $\mathrm{rk}_{\mathfrak{S}}(e) = 1$, thus the two summands are exactly $T_{\mathfrak{S}\setminus e}(x,y)$ and $T_{\mathfrak{S}/e}(x,y)$, respectively. If e is an isthmus, $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e})-1$ (and $\mathrm{rk}_{\mathfrak{S}}(e) = 1$) and thus we have $T_{\mathfrak{S}}(x,y) = (x-1)T_{\mathfrak{S}\setminus e}(x,y) + T_{\mathfrak{S}/e}(x,y)$. Finally, when e is a loop we have $\mathrm{rk}_{\mathfrak{S}}(e) = 0$ (but still $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e})$) and we easily get the claimed identity. □

Appendix A. An algebraic lemma

We give the proof of the following auxiliary lemma for completeness' sake and in order not to clutter the exposition in the main text.

Lemma 10.8. *Consider the following commutative diagram of abelian groups with exact rows and where the arrows $\twoheadrightarrow$ denote epimorphisms.*

$$
\begin{tikzcd}
 & & & B_0 \arrow[rr] \arrow[dl] \arrow[dd] & & C_0 \arrow[dl] \arrow[dd] \\
 & & B_1 \arrow[rr, two heads] \arrow[dd, two heads] & & C_1 \arrow[dd] & \\
 & A_2 \arrow[rr] \arrow[dl] & & B_2 \arrow[rr, two heads] \arrow[dl] & & C_2 \arrow[dl] \\
A_3 \arrow[rr] & & B_3 \arrow[rr, two heads] & & C_3 &
\end{tikzcd}
$$

If the square of the B_i is a pushout square, then so is the square of the C_i.

Proof. We name the arrows in the diagram as below and we verify the pushout property by considering a co-cone of the diagram spanned by C_0, C_1, C_2, which consists of a group H and two arrows h_1, h_2 such that $h_1 \circ \hat{c}_1 = h_2 \circ \hat{c}_2$. One verifies that the group H with the morphisms $\hat{h}_1 := h_1 \circ \epsilon_1$, $\hat{h}_2 := h_2 \circ \epsilon_2$ defines a co-cone on the diagram spanned by B_0, B_1, B_2. Since by assumption the B_i span a pushout square, there is a unique arrow φ with

$$\varphi \circ b_1 = \hat{h}_1 = h_1 \circ \epsilon_1, \quad \varphi \circ b_2 = \hat{h}_2 = h_2 \circ \epsilon_2$$

$$\begin{tikzcd}
 & & & B_0 \arrow[rrr] \arrow[dl] \arrow[ddr] & & & C_0 \arrow[dl, "\hat{c}_1"'] \arrow[ddr, "\hat{c}_2"] & \\
 & & B_1 \arrow[rrr, two heads, "\epsilon_1"] \arrow[ddr, "b_1"'] & & & C_1 \arrow[ddr, "c_1"] \arrow[dddr, dashed, "h_1"'] & & \\
 & A_2 a \arrow[rrr, "j_2"] \arrow[dl, two heads] & & & B_2 \arrow[rrr, two heads, "\epsilon_2"] \arrow[dl, "b_2"] & & & C_2 \arrow[dl, "c_2"] \arrow[ddl, dashed, "h_2"] \\
A_3 \arrow[rrr, "j_3"] \arrow[drrrrrr, dotted, bend right, "0"'] & & & B_3 \arrow[rrr, two heads, "\epsilon_3"] \arrow[drrr, dotted, "\exists!\varphi"'] & & & C_3 & \\
 & & & & & & H &
\end{tikzcd}$$

Notice that

$$\varphi \circ j_3 \circ a = \varphi \circ b_2 \circ j_2 = \hat{h}_2 \circ j_2 = h_2 \circ \underbrace{\epsilon_2 \circ j_2}_{=0} = 0 = 0 \circ a$$

and, since a is an epimorphism, by right cancellation we obtain

$$\varphi \circ j_3 = 0.$$

Exactness of the bottom row, by the universal property of cokernels, shows that there exist a unique g with $g \circ \epsilon_3 = \varphi$.

Claim. *For every $g' : C_3 \to H$ and every $i = 1, 2$,*

$$g' \circ c_i = h_i \text{ is equivalent to } g' \circ \epsilon_3 = \varphi.$$

Proof. By right cancellativity of epimorphisms, $g' \circ c_i = h_i$ is equivalent to

$$g' \circ c_i \circ \epsilon_i = h_i \circ \epsilon_i.$$

By commutativity of the diagram, the left-hand side of this equation equals $g' \circ \epsilon_3 \circ b_i$. By the definition of φ, the right-hand side equals $\varphi \circ b_i$. Again, by right-cancellativity of the epimorphism b_i we obtain the claimed equivalence. □

Using the claim we see immediately that our g satisfies $g \circ c_1 = h_1$ and $g \circ c_2 = h_2$. Moreover, for every g' with the same commutativity properties the claim implies that $g' \circ \epsilon_3 = \varphi$, and by the uniqueness in the definition of g we must have $g' = g$. This concludes the proof that the square of the C_i is a pushout. □

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