Permutation invariant properties of primitive cubic quadruples

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Dedicated to the 60th birthday of Sibylle von Burg-Baldini on July 29, 2016, whose ZH car number is 1729

Abstract Based on a specific quadratic Hopf map between the Euclidean spaces of dimension four and three that is associated with Euler's complete rational parameterization of the four cubes equation, we study the permutation invariant properties of the primitive integer cubic quadruples that solve this equation. Fixing the coordinate with maximum height and taking it positive, our main result describes the six positive primitive triples that leave it invariant under the inverted cubic map to this Hopf map and permute the remaining integer coordinates. The obtained invariant primitive triples are ordered in the so-called integer triple ordering, so that the minimum triple with respect to this ordering determines each primitive cubic quadruple uniquely. Implications for the counting and enumeration of all primitive cubic quadruples are mentioned. A list of all primitive cubic quadruples with positive maximum height below 100 and their minimum invariant triples is given. The relationship with the famous Taxicab and Cabtaxi numbers is also explained.

Keywords Cubic quadruple · Sum of two cubes · Taxicab number · Cabtaxi number · Euler rational parameterization · Quadratic Hopf map · Integer triple ordering

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1 Introduction

Unlike sums of squares, the representation of numbers by sums of cubes has been identified as a much more difficult problem (e.g., [10, Chap. D], [14, Chap. 2], [5, Sect. 6.4.6]). A special case is the four cubes Diophantine equation

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$$p^3 + q^3 + r^3 + s^3 = 0 \tag{1.1}$$

in non-zero integers p, q, r, s. Partial and complete solutions in integers and rational numbers of this equation have been discussed by many authors (e.g., [7, pp. 550–561], [6], [17, Part 6]). It is also known that Ramanujan had been quite interested in cubic quadruples (see [1, p. 201]).

Euler's complete rational solution of (1.1) is based on the parameterization (e.g., [12, Theorem 9.1, pp. 290–291], [9, 18])

$$\begin{aligned} &p = \rho F(a, b, c), \quad q = \rho F(-a, b, -c), \quad r = \rho F(-a, -b, c), \\ &s = \rho F(a, -b, -c), \end{aligned} \tag{1.2}$$

where ρ is a rational number and

$$F(a, b, c) = 9a^3 + 3a^2(3b + c) + 3a(b - c)^2 + (3b^2 + c^2)(b + c). \tag{1.3}$$

An elementary calculation shows that this result is equivalent to Theorem 49 in [8, pp. 58–59]. According to the Theorem of Berndt and Bhargava [3, pp. 646–647] , a family of solutions equivalent to Euler's general solution is found in the third notebook of [19, p. 387].

Given a (non-trivial) quadruple (p, q, r, s) solving (1.1) (non-trivial means $(p + q)(p + r)(p + s) \neq 0$), the quadratic map

$$\begin{aligned} a &= rs - pq, \\ b &= (p + s)^2 - (q + r)^2 + 3(qr - ps), \\ c &= (p + q + r + s)^2 - 3(p + r)(q + s), \end{aligned} \tag{1.4}$$

yields a triple (a, b, c) satisfying (1.2). Allowing rational numbers, the maps (1.2) and (1.4) describe the fact that all rational solutions (p, q, r, s) to (1.1) up to non-zero rational factors are in bijective correspondence with all triples (a, b, c) up to non-zero rational factors (Theorem 1 in [9]). Restricted to integer triples (a, b, c), the following result holds ([9, Corollary 2]). Every integral primitive solution of (1.1) can be written uniquely as

$$(p, q, r, s) = (P, Q, R, S)/D, \tag{1.5}$$

with $P = F(a, b, c)$, $Q = F(-a, b, -c)$, $R = F(-a, -b, c)$, $S = F(a, -b, -c)$, and the greatest common divisor $D = \gcd(P, Q, R, S)$ is assumed to take integral non-negative values. The latter has been completely described by Gar-el and Vaserstein [9, Theorem 3], and Pine [18, Theorem 3.1] (see Sect. 2, Theorem 2.1).

As shown in Sect. 2, the described approach to primitive integer cubic quadruples is an application of the arithmetic theory of quadratic maps initiated by Ono [15, Sect. 7]. Although the representation (1.5) is uniquely determined, there exist different primitive triples which map to the same cubic quadruple (up to permutations). Section 3 constructs, for fixed positive integer p (with maximum height), the six positive primitive triples $(a, b, c) \in Z_{>0}^3$, called *invariant primitive triples*, that leave p invariant and permute the remaining integer coordinates q, r, s of a primitive cubic quadruple. The invariant primitive triples are ordered in the so-called *integer triple ordering*, denoted by $\leq_T$, and each primitive cubic quadruple is uniquely determined by its *minimum triple* with respect to this ordering. The presented method completes and simplifies considerably the results of Chapter 4 in [18]. In particular, a simple and explicit definition of his triple (A, B, C) is obtained, which additionally turns out to be primitive. As a new important property, we show that primitive triples (a, b, c) with $3 \mid c$ must not be considered when counting the number of distinct primitive cubic quadruples generated by the map (1.5). Indeed, in virtue of the integer triple ordering relation $(c/3, a, b) \leq_T (a, b, c)$ and Theorem 3.1, these triples are not minimum triples. This divisibility property has not been taken into account in [18] and should invalidate parts of his counting asymptotic method. To illustrate, we list in Sect. 4 all primitive cubic quadruples with maximum height below 100. We make a distinction between equal and unequal sums of two cubes, the two different forms that solutions of the four cubes equation can take. They are also known under the name Taxicab and Cabtaxi numbers. We conclude with a brief analysis of two different complete integer parameterizations of equal or/and unequal sums of two cubes, one by Euler (see [7, pp. 552–554]) and the other by Choudhry [6]. In contrast to the above uniquely defined minimum invariant triples, the generated cubic quadruples are not uniquely defined in Euler's and Choudhry's parameterizations, as shown through examples.

2 Euler's rational parameterization and the associated quadratic Hopf map

The approach by Gar-el and Vaserstein [9,18] to the four cubes equations (1.1) through (1.5) is based on Euler's complete rational parameterization as introduced in Sect. 1, Eqs. (1.2)–(1.3), and the inverse quadratic map (1.4). In modern algebraic nomenclature, this approach is an application of the arithmetic of quadratic maps initiated by Ono [15]. The quadratic map (1.4) can be viewed as a (quadratic) Hopf map in the sense of Ono [15, Sect. 5.2]. Indeed, the *Hopf map* $\varphi : Z^4 \to Z^3$ associated with cubic quadruples maps a (non-trivial) quadruple $\pi = (p, q, r, s) \in Z^4$ to a triple $\varphi(\pi) = (\varphi_1, \varphi_2, \varphi_3)$ such that (non-trivial means $(p+q)(p+r)(p+s) \neq 0$)

$$\begin{aligned} \varphi_1 &= rs - pq, \\ \varphi_2 &= (p+s)^2 - (q+r)^2 + 3(qr - ps), \\ \varphi_3 &= (p+q+r+s)^2 - 3(p+r)(q+s). \end{aligned} \tag{2.1}$$

The *inverted cubic map* $\psi\colon Z^3 \to Z^4$ sends a triple $(a, b, c) \in Z^3$ to a quadruple $\psi(a, b, c) = (P, Q, R, S)$, which solves $P^3 + Q^3 + R^3 + S^3 = 0$, such that

$$\begin{aligned} &P = F(a, b, c), \quad Q = F(-a, b, -c), \quad R = F(-a, -b, c), \quad S = F(a, -b, -c), \\ &F(a, b, c) = 9a^3 + 3a^2(3b + c) + 3a(b - c)^2 + (3b^2 + c^2)(b + c). \end{aligned} \tag{2.2}$$

Allowing rational numbers, the quadratic Hopf map (2.1) and its inverse (2.2) describe the fact that all rational solutions $(p, q, r, s) \in Q^4$ to (1.1) up to non-zero rational factors are in bijective correspondence with all triples $(a, b, c) \in Q^3$ up to non-zero rational factors ([9, Theorem 1]). Restricted to integers, the following result holds (Corollary 2 in [9]). Every integral primitive solution to (1.1) can be written uniquely as

$$\pi = (p, q, r, s) = \psi(a, b, c)/D = (P, Q, R, S)/D, \quad D = \gcd(P, Q, R, S), \tag{2.3}$$

where the greatest common divisor D is assumed to take integral non-negative values. The latter is completely described by the following result ([9, Theorem 3], [18, Theorem 3.1]).

Theorem 2.1 (Greatest common divisor of the inverted coordinates) *If* $\gcd(a, b, c) = 1$*, then one has* $D = 2^{e(2)}3^{e(3)}\alpha\beta\gamma$ *with* $\alpha = \gcd(a, c^2+3b^2)$, $\beta = \gcd(b, c^2+3a^2)$, $\gamma = \gcd(c, b^2+3a^2)$*. Further, if 2 exactly divides abc, then* $e(2) = 2$*, and if 4 divides one of a, b or c and the other two are odd, then* $e(2) = 1$*; otherwise* $e(2) = 0$*. If 3 divides c but does not divide ab, then* $e(3) = 1$*; otherwise* $e(3) = 0$*. Moreover, if a prime p divides the product* $\gcd(\alpha, \beta) \cdot \gcd(\alpha, \gamma) \cdot \gcd(\beta, \gamma)$*, then* $p = 2, 3$.

Using the explicit description of the inverse map $\psi(a, b, c) = D \cdot (p, q, r, s)$ to the Hopf map (2.1), it is possible to study the properties of the four cubes equation and presumably count and enumerate them exhaustively, a still open problem, even asymptotically (e.g., [18]). Although the representation (2.3) is uniquely determined, there exist different triples, which map to the same solution (up to permutations). In the next section, for fixed positive integer p (with maximum height), we construct the six positive primitive triples $(a, b, c) \in Z^3_{>0}$ that leave p invariant and permute the remaining integer coordinates q, r, s of a primitive cubic quadruple. We like to mention that the present study has been inspired by Hürlimann [13], who studied primitive cuboids using permutation invariant properties of the first so-called classical quadratic Hopf map between the Euclidean spaces of dimension 4 and 3 that has been introduced by Hopf [11]. Since all of the Hopf maps considered by Ono involve only quadratic forms, the present paper based on the Hopf map $\varphi\colon Z^4 \to Z^3$ and its inverse cubic map $\psi\colon Z^3 \to Z^4$ is an example that goes beyond the previous work.

3 Permutation invariant primitive triple generation of primitive cubic quadruples

In the following, only non-trivial solutions of the four cubes equation $p^3 + q^3 + r^3 + s^3 = 0$ are considered. For this, by Proposition 2.1 and Lemma 2.2 in [18], it suffices to consider (strictly) positive integer triples $(a, b, c) \in Z^3_{>0}$, which map to the primitive cubic quadruples $(p, q, r, s) \in Z^4$ using the inverse cubic map $\psi(a, b, c) = D\cdot(p, q, r, s)$ to the Hopf map (2.1). Given a triple $(a, b, c) \in Z^3_{>0}$, its *maximum height* is denoted by $H_{\max}(a, b, c) = \max(a, b, c)$, its minimum height by $H_{\min}(a, b, c) = \min(a, b, c)$, and its *sum height* by $H_{sum}(a, b, c) = a + b + c$. The following partial ordering between integer triples is crucial in our investigation.

Definition 3.1 A triple $(a, b, c) \in Z^3_{>0}$ precedes another triple $(\alpha, \beta, \gamma) \in Z^3_{>0}$, which is not a permutation of $(a, b, c) \in Z^3_{>0}$, in the *integer triple ordering*, written as $(a, b, c) \leq_T (\alpha, \beta, \gamma)$, if $H_{\max}(a, b, c) \leq H_{\max}(\alpha, \beta, \gamma)$, $H_{sum}(a, b, c) \leq$

$H_{sum}(\alpha,\beta,\gamma)$, and $H_{\min}(a,b,c) < H_{\min}(\alpha,\beta,\gamma)$ in case $H_{\max}(a,b,c) = H_{\max}(\alpha,\beta,\gamma)$, $H_{sum}(a,b,c) = H_{sum}(\alpha,\beta,\gamma)$.

Furthermore, without loss of generality, it suffices to consider primitive cubic quadruples $(p,q,r,s) \in Z^4$ satisfying the four cubes equation $p^3+q^3+r^3+s^3=0$ such that $p>0$ is the maximum height of the associated positive quadruple, i.e., $p = \max(|p|,|q|,|r|,|s|)$. Such a cubic quadruple generates 6 permutation representations of the same solution denoted by

$$\pi_1=(p,q,r,s),\quad \pi_2=(p,r,s,q),\quad \pi_3=(p,s,q,r),$$
$$\pi_4=(p,r,q,s),\quad \pi_5=(p,q,s,r),\quad \pi_6=(p,s,r,q).$$

The following main result characterizes the invariant primitive triples that generate the permutation invariant solutions $\pi_i,\ i=1,\ldots,6$, of the four cubes equation.

Theorem 3.1 (Primitive cubic quadruples and their invariant primitive triples) *Suppose the positive primitive triple (a,b,c) generates the primitive quadruple $\pi_1 = (p,q,r,s)$ with $p=\max(|p|,|q|,|r|,|s|)$. Suppose it is the minimum triple in the sense that it precedes in the integer triple ordering the five primitive triples that generate the remaining permutations $\pi_i,\ i=2,\ldots,6$. The four cases below characterize via the quadratic Hopf map the invariant primitive triples $\varphi(\pi_i)=(\varphi_{i,1},\varphi_{i,2},\varphi_{i,3}) = d_i(a_i.b_i,c_i),\ d_i=\gcd(\varphi_{i,1},\varphi_{i,2},\varphi_{i,3}),\ i=1,\ldots,6$, that generate the π_i's via the inverse cubic map $\psi(a_i.b_i,c_i)=D\cdot\pi_i,\ i=1,\ldots,6$, as explained in Sect. 2. The primitive triples displayed in the tables below are increasingly ordered according to the integer triple ordering.*

Case 1 $a+b+c\equiv 0 \pmod 2,\ 3\nmid c$
Case 1.1 $3\nmid b \quad\Leftrightarrow\quad p+q\equiv -(r+s)\equiv 1,2 \pmod 3$
There exists a positive primitive integer triple (A,B,C) defined by

$$A=\frac{qs-pr}{3b},\quad B=\frac{rs-pq}{3a},\quad C=\frac{qr-ps}{c}, \tag{3.1}$$

and one has the following table of invariant primitive triples:

π_i	d_i	a_i	b_i	c_i
π_1	$3B$	a	b	c
π_2	$3A$	b	c	$3a$
π_3	C	c	$3a$	$3b$
π_4	$3b$	A	B	C
π_5	$3a$	B	C	$3A$
π_6	c	C	$3A$	$3B$

Case 1.2 $3\mid b \quad\Leftrightarrow\quad p+q\equiv (r+s)\equiv 0 \pmod 3$
There exists a positive primitive integer triple (A,B,C) defined by

$$A=\frac{qs-pr}{3b},\quad B=\frac{rs-pq}{3a},\quad C=\frac{qr-ps}{3c}, \tag{3.2}$$

and one has the following table of invariant primitive triples:

π_i	d_i	a_i	b_i	c_i
π_1	$3B$	a	b	c
π_2	$3A$	b	c	$3a$
π_3	$3C$	c	$3a$	$3b$
π_6	$3c$	C	A	B
π_4	$3b$	A	B	$3C$
π_5	$3a$	B	$3C$	$3A$

Case 2 $a+b+c \equiv 1 \pmod 2$, $3 \nmid c$
Case 2.1 $3 \nmid b \quad \Leftrightarrow \quad p+q \equiv -(r+s) \equiv 1, 2 \pmod 3$
There exists a positive primitive integer triple (A, B, C) defined by

$$A = \frac{qs - pr}{12b}, \quad B = \frac{rs - pq}{12a}, \quad C = \frac{qr - ps}{4c}, \tag{3.3}$$

and one has the following table of invariant primitive triples:

π_i	d_i	a_i	b_i	c_i
π_1	$12B$	a	b	c
π_2	$12A$	b	c	$3a$
π_3	$4C$	c	$3a$	$3b$
π_4	$12b$	A	B	C
π_5	$12a$	B	C	$3A$
π_6	$4c$	C	$3A$	$3B$

Case 2.2 $3 \mid b \quad \Leftrightarrow \quad p+q \equiv (r+s) \equiv 0 \pmod 3$
There exists a positive primitive integer triple (A, B, C) defined by

$$A = \frac{qs - pr}{12b}, \quad B = \frac{rs - pq}{12a}, \quad C = \frac{qr - ps}{12c}, \tag{3.4}$$

and one has the following table of invariant primitive triples:

π_i	d_i	a_i	b_i	c_i
π_1	$12B$	a	b	c
π_2	$12A$	b	c	$3a$
π_3	$12C$	c	$3a$	$3b$
π_6	$12c$	C	A	B
π_4	$12b$	A	B	$3C$
π_5	$12a$	B	$3C$	$3A$

Proof As a first step, we show that the displayed triples generate the correct permutations of cubic quadruples. Clearly, by assumption, the triple (a, b, c) generates π_1. The coordinates of the inverse cubic map $\psi(a, b, c)$ are given by

$$\begin{aligned} p \cdot D(a,b,c) &= F(a,b,c), \quad q \cdot D(a,b,c) = F(-a,b,-c),\\ r \cdot D(a,b,c) &= F(-a,-b,c), \quad s \cdot D(a,b,c) = F(a,-b,-c), \end{aligned} \tag{3.5}$$

with (see Theorem 2.1)

$$D(a,b,c) = 2^{e_2}\alpha\beta\gamma,\ \alpha = (a, c^2+3b^2),\ \beta = (b, c^2+3a^2),\ \gamma = (c, b^2+3a^2). \tag{3.6}$$

On the other hand, in the notation of Sect. 2, one has the identity

$$\begin{aligned} F(a,b,c) &= 9a^3 + 3a^2(3b+c) + 3a(b-c)^2 + (3b^2+c^2)(b+c)\\ &= c^3 + c^2(3a+b) + 3c(a-b)^2 + 3(3a^2+b^2)(a+b), \end{aligned} \tag{3.7}$$

which implies the relationships

$$\begin{aligned} &F(b,c,3a) = 3\cdot F(a,b,c), \quad F(c,3a,3b) = 9\cdot F(a,b,c),\\ &F(-b,c,-3a) = 3\cdot F(-a,-b,c), \quad F(-c,3a,-3b) = 9\cdot F(a,-b,-c),\\ &F(-b,-c,3a) = 3\cdot F(a,-b,-c), \quad F(-c,-3a,3b) = 9\cdot F(-a,b,-c),\\ &F(b,-c,-3a) = 3\cdot F(-a,b,-c), \quad F(c,-3a,-3b) = 9\cdot F(-a,-b,c). \end{aligned} \tag{3.8}$$

Now, consider Case 1.1. Since $3 \nmid bc$, one sees that $D(b,c,3a) = 2^{e_2}3\alpha'\beta'\gamma'$, with

$$\begin{aligned} &\alpha' = (b, 9a^2+3c^2) = (b, 3(c^2+3a^2)) = \beta,\\ &\beta' = (c, 9a^2+3b^2) = (c, 3(b^2+3a^2)) = \gamma, \quad \gamma' = (3a, c^2+3b^2) = \alpha. \end{aligned}$$

Hence $D(b,c,3a) = 3\cdot D(a,b,c)$. Similarly, one has $D(c,3a,3b) = 9\cdot D(a,b,c)$. Using this, as well as (3.5) and (3.8), one sees that $\psi(b,c,3a) = \pi_2$ and $\psi(c,3a,3b) = \pi_3$. Further, by definition of the quadratic Hopf map φ, the following identities hold true:

$$d_1a_1 = d_5a_5 = rs - pq \tag{I1}$$
$$d_2a_2 = d_4a_4 = qs - pr \tag{I2}$$
$$d_3a_3 = d_6a_6 = qr - ps \tag{I3}$$
$$d_1b_1 = d_4b_4 = (p+s)^2 - (q+r)^2 + 3(qr-ps) \tag{I4}$$
$$d_2b_2 = d_6b_6 = (p+q)^2 - (r+s)^2 + 3(rs-pq) \tag{I5}$$
$$d_3b_3 = d_5b_5 = (p+r)^2 - (q+s)^2 + 3(qs-pr) \tag{I6}$$
$$d_1c_1 = d_6c_6 = (p+q+r+s)^2 - 3(p+r)(q+s) \tag{I7}$$
$$d_2c_2 = d_5c_5 = (p+q+r+s)^2 - 3(p+s)(q+r) \tag{I8}$$
$$d_3c_3 = d_4c_4 = (p+q+r+s)^2 - 3(p+q)(r+s) \tag{I9}$$

With this and Definition (3.1), one sees that
$\varphi(\pi_4) = d_4(a_4.b_4, c_4) = (d_2a_2, d_1b_1, d_3c_3) = 3b\cdot(A,B,C)$, and hence $\psi(A,B,C) = \pi_4$.

The other identifications $\psi(B, C, 3A) = \pi_5$ and $\psi(C, 3A, 3B) = \pi_6$ follow similarly. The same derivation for Cases 1.2, 2.1, and 2.2 works by taking into account the modified definitions of (A, B, C) in (3.2)–(3.4). Next, we derive the different representations of $d_i,\ i = 1, 2, 3$. For this, Lemmas 3.1 and 3.2 will be used. Let us begin with Case 1. With (3.9), the identity (I7) implies that $d_1 c \equiv 0 \pmod 3$, and hence $d_1 = 3m$ for some integer m. However, by (I1), one has $d_1 a = 3ma = rs - pq$, and hence $m = B$ by (3.1)–(3.2) and $d_1 = 3B$. Similarly, with (3.9), the identity (I5) implies that $d_2 c \equiv (p+q+r+s)(p+q-r-s) \equiv 0 \pmod 3$, and hence $d_2 = 3m$ for some integer m. To be compatible with (I2), one must have $d_2 b = 3mb = qs - pr$, and hence $m = A$ by (3.1)–(3.2) and $d_2 = 3A$. With (3.9), the identity (I9) is equivalent to $d_3 b \equiv -(p+q)(r+s) \pmod 3$. Now, Case 1.1 occurs exactly when $3 \nmid d_3 b$, in particular $3 \nmid b$. Comparing (I3) with (3.1), one has $d_3 c = qr - ps = Cc$, that is $3 \nmid d_3 = C$. In Case 1.2, we use that $p + s \equiv q + r \equiv 0 \pmod 3$, a fact which follows from identity (I8) using that $d_2 c_2 = 9aA$ and (3.9). Now, the identity (I4) implies that $3bB \equiv 3(qr - ps) \equiv 3d_3 c \pmod 9$, or $bB \equiv d_3 c \pmod 3$. Since $3 \nmid c$ and $3 \mid b$, it follows that $3 \mid d_3$, and to be compatible with (3.2) one must have $d_3 = 3C$. The divisibility properties by 3 of the d_i's in Case 2 are identical and derived in the same manner. To obtain the divisibility properties by 4 in Case 2, one applies Lemma 3.2. In the situation $a+b+c \equiv 1 \pmod 2$, one knows by Lemma 3.5 in [18] that $2 \nmid D$. Let $2^{e(a)} \parallel a$, $2^{e(b)} \parallel b$, $2^{e(c)} \parallel c$ be the exact powers of 2 that divide $a,\ b,\ c$. Then (3.10) in Lemma 3.2 implies the following divisibility properties:

$$4 \cdot 2^{e(a)+e(c)} \mid rs - pq = d_1 a \quad \Rightarrow \quad 4 \cdot 2^{e(c)} \mid d_1 \quad \Rightarrow \quad 4 \mid d_1,$$
$$4 \cdot 2^{e(a)+e(b)} \mid qs - pr = d_2 b \quad \Rightarrow \quad 4 \cdot 2^{e(a)} \mid d_2 \quad \Rightarrow \quad 4 \mid d_2,$$
$$4 \cdot 2^{e(b)+e(c)} \mid qr - ps = d_3 c \quad \Rightarrow \quad 4 \cdot 2^{e(b)} \mid d_3 \quad \Rightarrow \quad 4 \mid d_3.$$

The representations of the d_i's in Case 2 follow. It remains to show that the triple (A, B, C) is primitive. This follows from Lemma 3.3. In Case 1.1, one has $\kappa = 0$ and with (3.1) one has $\Delta = \gcd(3aB, 3bA, cC) = 1$, and hence $\gcd(A, B, C) = 1$ because $\gcd(a, b, c) = 1$. In Case 1.2, one has $\kappa = 1$ and with (3.1) one has $\Delta = \gcd(3aB, 3bA, 3cC) = 3$, and hence $\gcd(A, B, C) = 1$ because $\gcd(a, b, c) = 1$. In Case 2.1, one has $\kappa = 0$, and inserting (3.3) into the expression for Δ in Lemma 3.3, one sees that $\gcd(aB, bA, cC) = 2^\delta$. If (a, b, c) are all odd, then $\delta = 0$ and $\gcd(A, B, C) = 1$ follows from $\gcd(a, b, c) = 1$. If two of a, b, c are even, then $\delta \geq 1$. Write $a = 2^{e(a)}a'$, $b = 2^{e(b)}b'$, $c = 2^{e(c)}c'$, with $2 \nmid a'b'c'$ and $\gcd(a', b', c') = 1$, so that $\gcd(2^{e(a)}a'B, 2^{e(b)}b'A, 2^{e(c)}c'C) = 2^\delta$. If now a is odd, that is $e(a) = 0$, then $\delta = \min(e(b), e(c)$ and $2^\delta \mid B$, from which it follows that necessarily $\gcd(A, B, C) = 1$. The same conclusion holds if b or c is odd. Similarly, in Case 2.1 one has $\kappa = 1$, and inserting (3.4) into the expression for Δ in Lemma 3.3, one sees again that $\gcd(aB, bA, cC) = 2^\delta$. Proceeding as in Case 2.1, one concludes that $\gcd(A, B, C) = 1$. The result is shown. □

Lemma 3.1 *A non-trivial primitive cubic quadruple (p, q, r, s) satisfies the congruence*

$$p + q + r + s \equiv 0 \pmod 3 \tag{3.9}$$

Proof The identities (I7)–(I9) imply that $(p + q + r + s)^2 \equiv 0 \pmod 3$, and (3.9) follows. □

Lemma 3.2 *The coordinates* (3.5) *of the inverse cubic map* $\psi(a,b,c)$ *of a triple* $(a,b,c) \in Z^3_{>0}$ *satisfy the following identities (with* $D = D(a,b,c)$*):*

$$\begin{aligned} D^2(rs-pq) &= 12ac \cdot U, \quad U = (3a^2+3b^2+c^2)^2 + 12a^2b^2, \\ D^2(qs-pr) &= 12ab \cdot V, \quad V = 3(3a^2+b^2+c^2)^2 + 4b^2c^2, \\ D^2(qr-ps) &= 4bc \cdot W, \quad W = (9a^2+3b^2+c^2)^2 + 12a^2c^2. \end{aligned} \tag{3.10}$$

Proof This follows through calculation using (3.5) and (3.7). □

Lemma 3.3 *Assume that* $\gcd(p,q,r,s) = 1$ *and set* $\Delta = \gcd[(rs-pq),(qs-pr),(qr-ps)]$, $\kappa = 0$ *if* $3 \nmid b$ *and* $\kappa = 1$ *if* $3 \mid b$. *Further, let* $2^{e(a)} \parallel a$, $2^{e(b)} \parallel b$, $2^{e(c)} \parallel c$ *be the exact powers of 2 that divide* $a,\ b,\ c$, *and set* $\delta = \max\{\min(e(a),e(b)), \min(e(a),e(c)), \min(e(b),e(c)\}$. *Then, one has* $\Delta = 3^\kappa$ *if* $a+b+c \equiv 0 \pmod 2$ *and* $\Delta = 3^\kappa \cdot 4 \cdot 2^\delta$ *if* $a+b+c \equiv 1 \pmod 2$.

Proof This follows using Lemma 3.2, Theorem 2.1, and some additional facts as follows. From Eqs. (3.10), one obtains the identity

$$D^2\Delta = \gcd(12acU, 12abV, 4bcW). \tag{3.11}$$

where $\psi(a,b,c) = D\cdot(p,q,r,s)$ and $D = D(a,b,c) = 2^{e(2)}3^{e(3)}\alpha\beta\gamma$ is determined by Theorem 2.1. Further, the following facts are needed:

(F1) $a+b+c \equiv 0 \pmod 2 \quad \Leftrightarrow \quad 8 \parallel D$,
$a+b+c \equiv 1 \pmod 2 \quad \Leftrightarrow \quad 2 \nmid D$
(F2) $3 \nmid c \quad \Leftrightarrow \quad 3 \nmid D$
(F3) If $\xi \neq 2, 3$ is a prime, then $\xi^e \parallel D \quad \Leftrightarrow \quad \xi^e \parallel \text{one of } \alpha, \beta, \gamma$

All three properties are found in [18]: (F1) is Lemma 3.5, (F2) is Lemma 3.6, and (F3) is Proposition 3.4. From (F2), it follows that $e(3) = 0$ and $3 \nmid D$ because $3 \nmid c$. The values of Δ are determined separately for the four cases of Theorem 3.1.

Case 1.1 $\quad a+b+c \equiv 0 \pmod 2$, $3 \nmid b$

(I) $3 \nmid W$ and $3 \nmid bc$ imply that $3 \nmid D^2\Delta$, in particular $3 \nmid \Delta$
(II) Since $a+b+c \equiv 0 \pmod 2$, two of $a,\ b,\ c$ are odd, and one is even. Let us distinguish between three sub-cases.
(IIa) Let a be even, and $b,\ c$ be odd. One can write

$$V = (b^2+3c^2)(3b^2+c^2) + 9a^2(3a^2+2b^2+2c^2). \tag{3.12}$$

Since $2^4 \parallel (b^2+3c^2)(3b^2+c^2)$, one sees that $2^4 \parallel U,\ V,\ W$, and hence $2^6 \parallel D^2\Delta$, and $2 \nmid \Delta$ because $2^3 \parallel D$ by property (F2).
(IIb) Let b be even, and a, c be odd. One has

$$W = (c^2+3a^2)(c^2+27a^2) + 3b^2(18a^2+3b^2+2c^2). \tag{3.13}$$

Since $2^4 \parallel (c^2+3a^2)(c^2+27a^2)$, one sees that $2^4 \parallel U,\ V,\ W$, and hence $2^6 \parallel D^2\Delta$, and $2 \nmid \Delta$ because $2^3 \parallel D$.

(IIc) Let c be even, and $a,\ b$ be odd. One has

$$U = 3(b^2 + 3a^2)(3b^2 + a^2) + c^2(6a^2 + 6b^2 + c^2). \tag{3.14}$$

Since $2^4 \parallel (b^2 + 3a^2)(3b^2 + a^2)$, one sees that $2^4 \parallel U,\ V,\ W$, and hence $2^6 \parallel D^2\Delta$, and $2 \nmid \Delta$ because $2^3 \parallel D$.

(III) Let $\xi \neq 2, 3$ be a prime that satisfies $\xi^e \parallel D$. Using (F3), one distinguishes between three sub-cases.

(IIIa) Let $\xi^e \parallel \alpha = \gcd(a, c^2 + 3b^2)$. Then, one has $\xi^{3e} \parallel U$, $\xi^{2e} \parallel W$. From (3.12), one gets $V \equiv (b^2 + 3c^2)(3b^2 + c^2) \pmod{\xi^{2e}}$, and hence $\xi^e \parallel V$. It follows that $\xi^{2e} \parallel D^2\Delta$, and hence $\xi \nmid \Delta$ because $\xi^e \parallel D$.

(IIIb) Let $\xi^e \parallel \beta = \gcd(b, c^2 + 3a^2)$. Then, one has $\xi^{2e} \parallel U$, $\xi^{3e} \parallel V$. From (3.13), one gets $W \equiv (c^2 + 3a^2)(c^2 + 27a^2) \pmod{\xi^{2e}}$, and hence $\xi^e \parallel W$. It follows that $\xi^{2e} \parallel D^2\Delta$, and hence $\xi \nmid \Delta$.

(IIIc) Let $\xi^e \parallel \gamma = \gcd(c, b^2 + 3a^2)$. Then, one has $\xi^{2e} \parallel V$, $\xi^{3e} \parallel W$. From (3.14), one gets $U \equiv 3(b^2+3a^2)(3b^2+a^2) \pmod{\xi^{2e}}$, and hence $\xi^e \parallel U$. It follows that $\xi^{2e} \parallel D^2\Delta$, hence $\xi \nmid \Delta$.

(IV) If a prime $\xi \neq 2, 3$ divides the right-hand side of (3.11), then $\xi \nmid \Delta$. Through the application of Lemma 3.4, one sees that either $\xi \mid D$ or $\xi \mid \bar{D}$. In the first case, one has $\xi \nmid \Delta$ in virtue of (III). In the second case, repeat the arguments in (III) for the triple (A, B, C), as defined in Theorem 3.1, to see that $\xi \nmid \Delta$.

Taken together, the steps (I)–(IV) show that $\Delta = 1$ in Case 1.1. Applying the same method to the other cases, one obtains the following values:

Case 1.2 $\Delta = 3$

Case 2.1 $\Delta = 4 \cdot 2^\delta$

Case 2.2 $\Delta = 3 \cdot 4 \cdot 2^\delta$

The required details for this are left to the interested reader. □

Lemma 3.4 *Suppose that $\psi(a, b, c) = D \cdot \pi_1$ and $\psi(A, B, C) = \bar{D} \cdot \pi_4$. If a prime $\xi \neq 2, 3$ divides $D^2\Delta = D^2 \gcd(aB, bA, cC)$, then either $\xi \mid D$ or $\xi \mid \bar{D}$.*

Proof Rewrite the identities (3.10), respectively (3.12)–(3.14), in the form

$$\begin{aligned} U &= (c^2 + 3a^2)^2 + 3b^2(10a^2 + 3b^2 + 2c^2), \\ V &= 3(b^2 + 3a^2)^2 + c^2(18a^2 + 10b^2 + 3c^2), \\ W &= (3b^2 + c^2)^2 + 3a^2(27a^2 + 18b^2 + 10c^2). \end{aligned} \tag{3.15}$$

If $\xi \mid D$, nothing must be shown. Therefore, let $\xi \nmid D$. Two cases are distinguished:

(I) ξ |abc

(Ia) $\xi \mid a$ but $\xi \nmid bc$.
With (3.11), one sees that $\xi \mid W$. But $W \equiv (3b^2 + c^2)^2 \pmod{\xi^2}$ by (3.15), which implies that $\xi \mid c^2 + 3b^2$, and hence $\xi \mid \alpha = \gcd(a, c^2 + 3b^2)$ and $\xi \mid D$.

(Ib) $\xi \mid b$ but $\xi \nmid ac$
With (3.11), one sees that $\xi \mid U$. But $U \equiv (c^2 + 3a^2)^2 \pmod{\xi^2}$ by (3.15), which implies that $\xi \mid c^2 + 3a^2$, and hence $\xi \mid \beta = \gcd(b, c^2 + 3a^2)$ and $\xi \mid D$.

(Ic) $\xi \mid c$ but $\xi \nmid ab$

With (3.11), one sees that $\xi \mid V$. But $V \equiv 3(b^2 + 3a^2)^2 \pmod{\xi^2}$ by (3.15), which implies that $\xi \mid b^2 + 3a^2$, and hence $\xi \mid \gamma = \gcd(c, b^2 + 3a^2)$ and $\xi \mid D$.

(II) $\xi \nmid abc$

Since $\xi \mid \gcd(aB, bA, cC)$, one must have $\xi \mid \gcd(A, B, C)$, and hence $\xi \mid ABC$. By construction, the triple (A, B, C) generates the primitive quadruple $\pi_4 = (p, r, q, s)$ such that $\psi(A, B, C) = \bar{D} \cdot (p, r, q, s)$ with $\bar{D} = \bar{D}(A, B, C) = 2^{e(2)} 3^{e(3)} \bar{\alpha}\bar{\beta}\bar{\gamma}$ as determined by Theorem 2.1. Applying Lemma 3.2, one obtains similarly to (3.11) the identity

$$\bar{D}^2 \Delta = \gcd(12AC\bar{U}, 12AB\bar{V}, 4BC\bar{W}),$$

with

$$\begin{aligned} \bar{D}^2(qs - pr) &= 12AC \cdot \bar{U}, & \bar{U} &= (3A^2 + 3B^2 + C^2)^2 + 12A^2B^2, \\ \bar{D}^2(rs - pq) &= 12AB \cdot \bar{V}, & \bar{V} &= 3(3A^2 + B^2 + C^2)^2 + 4B^2C^2, \\ \bar{D}^2(qr - ps) &= 4BC \cdot \bar{W}, & \bar{W} &= (9A^2 + 3B^2 + C^2)^2 + 12A^2C^2. \end{aligned}$$

Repeating the arguments under (I), one concludes that $\xi \mid \bar{D}$. □

4 The equal and unequal sums of two cubes with maximum height below 100

As many other elementary number theoretical questions, the birth of the Diophantine four cubes equation is due to Fermat who asked to divide numbers composed of two cubes into two other cubes. Two months after posing this new problem, Frenicle in 1657 (see [7, p. 552]) gave the solution

$$1729 = 12^3 + 1^3 = 10^3 + 9^3. \tag{4.1}$$

In the twentieth century, this number became famous due to Ramanujan's remark about Hardy taking a London taxi numbered 1729: "It is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways". One finds (4.1) in the second notebook of Ramanujan [19, p. 225] (see also [2, pp. 199–200]). Nowadays, this number is called Taxicab(2), where Taxicab(n) is the smallest number expressible in n ways as a sum of two cubes. In generalization to this, one can allow differences of cubes and define Cabtaxi(n) as the smallest number expressible in n ways as a sum or difference of two cubes. For example, Cabtaxi(2) is given by

$$91 = 6^3 - 5^3 = 4^3 + 3^3. \tag{4.2}$$

Numbers with the property (4.1) (respectively (4.2)) can also be called *equal (unequal) sums of two cubes*. These are the two different forms that can arise as solutions of the four cubes equation. Many mathematicians have studied these numbers including Silverman [20], Ono [16], Boyer [4], and the references therein (see also Sloane's

Table 1 Equal sums of two cubes and their minimum invariant triples

Nr	p	q	r	s	d	a	b	c
1	12	−10	1	−9	111	1	2	7
2	16	−9	−15	2	57	2	1	1
3	27	10	−19	−24	186	1	7	2
4	34	−33	−15	2	273	4	1	7
5	34	−33	9	−16	489	2	1	13
6	39	−26	17	−36	402	1	7	26
7	40	12	−33	−31	543	1	4	1
8	51	−38	−43	12	474	3	1	2
9	53	−29	8	−50	1137	1	6	13
10	55	−54	−24	17	183	14	1	19
11	58	9	−57	−22	183	4	7	1
12	60	−22	−59	3	381	3	2	1
13	67	−51	30	−58	1677	1	4	19
14	69	−56	−61	42	186	7	1	2
15	76	−69	−48	5	1668	3	1	4
16	76	−73	−38	17	258	19	3	26
17	80	15	−54	−71	2634	1	5	2
18	82	51	−75	−64	309	2	13	1
19	89	−86	2	−41	1893	4	3	19
20	93	−92	11	−30	2742	3	1	14
21	94	−63	−84	23	210	19	7	10
22	96	50	−59	−93	687	1	26	7
23	97	−33	−96	20	183	7	4	1
24	97	−66	−90	47	543	4	1	1
25	98	−63	24	−89	4038	1	5	14
26	98	−92	35	−59	993	7	6	61
27	99	−92	29	−60	7368	1	1	8

OEIS). Tables 1 and 2 illustrate the main result. Besides the equal and unequal sums of two cubes with maximum height $p = \max(|p|, |q|, |r|, |s|) < 100$, we list their uniquely defined minimum invariant triples.

It might interest the reader to conclude with a brief analysis of other known so-called complete integer parameterizations of equal or/and unequal sums of two cubes. Let us begin with Euler, who in a series of papers (see [7, pp. 552–554]) has derived the following general representation of unequal sums of two cubes in the form $p^3 = q^3 + r^3 + s^3$. For arbitrary integers f, g, h, k, one has $D \cdot (p, q, r, s) = (P, Q, R, S)$, $D = \gcd(P, Q, R, S)$, with

$$\begin{aligned} &P = (k+h)t + (3k-h)u, \quad Q = (k-h)t + (3k-h)u, \\ &R = (f+g)t + (3g-f)u, \quad S = (f-g)t + (3g+f)u, \\ &t = 3 \cdot [k(h^2+3k^2) - g(f^2+3g^2)], \quad u = f(f^2+3g^2) - h(h^2+3k^2). \end{aligned} \tag{4.3}$$

Table 2 Unequal sums of two cubes and their minimum invariant triples

Nr	p	$(-q)$	$(-r)$	$(-s)$	d	a	b	c
1	6	5	4	3	42	1	1	2
2	9	1	8	6	57	1	2	1
3	19	18	3	10	372	1	1	4
4	20	17	7	14	219	2	3	7
5	25	4	17	22	474	1	3	2
6	28	18	21	19	129	7	10	13
7	29	15	27	11	732	1	1	1
8	41	33	6	32	1545	1	2	5
9	41	2	40	17	381	2	3	1
10	44	23	16	41	1668	1	3	4
11	46	27	30	37	336	7	13	16
12	46	3	36	37	210	7	19	10
13	53	29	34	44	1011	3	6	7
14	54	53	19	12	1545	2	1	5
15	58	42	15	49	453	7	16	31
16	67	51	54	22	921	5	4	7
17	69	38	61	36	438	11	13	14
18	70	57	54	7	336	13	7	16
19	71	23	70	14	201	13	12	7
20	72	34	65	39	453	11	14	13
21	75	38	43	66	1896	3	7	8
22	76	72	33	31	1299	5	4	13
23	81	74	25	48	654	11	13	38
24	82	60	69	19	201	31	22	37
25	84	28	53	75	2109	3	8	7
26	85	64	50	61	4245	2	3	5
27	87	55	26	78	2271	3	8	13
28	87	79	38	48	669	13	14	37
29	87	79	20	54	723	11	14	43
30	88	21	84	43	1092	5	7	4
31	88	31	86	25	1626	3	3	2
32	89	86	40	17	1389	6	3	13
33	90	38	25	87	5595	1	4	5
34	90	58	59	69	489	19	32	43
35	93	32	85	54	582	13	19	14
36	96	53	19	90	3399	2	7	11
37	97	45	69	79	9816	1	2	2

Another complete integer parameterization of unequal sums of two cubes is due to [6, Theorem 3]. One has $D \cdot (p, q, r, s) = (P, Q, R, S)$, $D = \gcd(P, Q, R, S)$ with

$$
\begin{aligned}
&P = b \cdot (a^3 + b^3 + c^3), \quad Q = P - 3ab(a^2 - ac + c^2), \\
&R = (2a - c)b^3 - (a^4 - 2a^3c + 3a^2c^2 - 2ac^3 + c^4), \quad S = R - 3ab^3,
\end{aligned} \tag{4.4}
$$

where a, b, c are positive integers satisfying the inequalities $a > c, \quad b^3 > a^3 + (a - c)^3$.

In contrast to the uniquely defined minimum invariant triples, the generated cubic quadruples are not uniquely defined in Euler's and Choudhry's parameterizations. Two examples illustrate this fact. The choices $(f, g, h, k) = (0, 2, 1, 1), (1, 1, 3, 1)$ in (4.3) generate permutations of the solution $(9, 6, 1, 8)$. The choices $(a, b, c) = (2, 3, 1), \ (3, 7, 2)$ in (4.4) generate permutations of the solution $(6, 3, 4, 5)$. Similarly to what has been achieved, one can ask for permutation invariant properties of these integer parameterizations that circumvent this difficulty.

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