

Nonconnected moduli spaces of nonnegative sectional curvature metrics on simply connected manifolds

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ABSTRACT

We show that in each dimension $4n + 3$, $n \geq 1$, there exist infinite sequences of closed smooth simply connected manifolds M of pairwise distinct homotopy type for which the moduli space of Riemannian metrics with nonnegative sectional curvature has infinitely many path components. Closed manifolds with these properties were known before only in dimension 7, and our result also holds for moduli spaces of Riemannian metrics with positive Ricci curvature. Moreover, in conjunction with work of Belegradek, Kwasiak and Schultz, we obtain that for each such M the moduli space of complete nonnegative sectional curvature metrics on the open simply connected manifold $M \times \mathbb{R}$ also has infinitely many path components.

1. Introduction

A central question in Riemannian geometry concerns the existence of complete metrics on smooth manifolds which satisfy certain prescribed curvature properties such as, for example, positivity of scalar or Ricci curvature, nonnegativity or negativity of sectional curvature, etc. On the other hand, once the respective existence question is settled, there is an equally important second one, namely: How ‘many’ metrics of the given type are there, and how ‘many’ different geometries of this kind does the manifold actually allow?

To answer these questions, one is naturally led to study the corresponding spaces of metrics that satisfy the curvature characteristics under investigation, as well as their respective moduli spaces, that is, the quotients of these spaces by the action of the diffeomorphism group given by pulling back metrics. When equipped with the topology of smooth convergence on compact subsets and the respective quotient topology, the topological properties and complexity of these objects constitute the appropriate means to measure the ‘number’ of different metrics and geometries, and we will adapt to this viewpoint.

There has been much activity and profound progress on these issues in the last decades, compare, for example, [3–5, 7–13, 15–20, 22–24, 26, 30–32, 34]. Since the very beginning, special importance has been given to the study of connectedness properties of spaces and moduli spaces of metrics with lower curvature bounds on closed as well as open manifolds. As these are also the main issue of the present note, let us now shortly review and comment on the most relevant developments in this respect, starting with closed manifolds:

Hitchin [22] showed first that there are closed manifolds for which the space of positive scalar curvature metrics is disconnected. Unfortunately, there is no explicit information on the actual number of path components available in Hitchin’s work. But for the standard spheres of dimension $4n + 3$, where $n \geq 1$, Carr [10] then proved that their respective spaces of positive scalar curvature metrics indeed have an *infinite* number of path components. Since $\pi_0(\text{Diff}(S^{4n+3}))$ is finite, the corresponding statement also holds for their moduli spaces [26, IV § 7].

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Results about the disconnectedness of moduli spaces of Riemannian metrics on a wide variety of examples are due to Kreck and Stolz [24]. They showed that for every closed simply connected spin manifold M of dimension $4n + 3$, where $n \geq 1$, with vanishing real Pontrjagin classes and $H^1(M; \mathbb{Z}/2) = 0$, the moduli space of metrics of positive scalar curvature, if not empty, has infinitely many path components. To achieve this, they introduced in this setting an invariant, the so-called s - (or also, nowadays Kreck–Stolz) invariant, whose absolute value is constant on path components of the moduli space.

Using the s -invariant together with work of Wang and Ziller about Einstein metrics on principal torus bundles over products of Kähler–Einstein manifolds [33], Kreck and Stolz went on to show that there are closed seven-manifolds for which the moduli space of metrics with positive Ricci curvature has infinitely many path components, and that in this dimension there are also closed manifolds which exhibit a disconnected moduli space of positive sectional curvature metrics.

Considering different metrics on the Kreck–Stolz examples, it was subsequently observed in [23] that the Kreck–Stolz result for positive Ricci curvature is actually also true for moduli spaces of nonnegative sectional curvature metrics. Note, however, that both these results only apply and are indeed confined to dimension 7.

David Wraith [34] was then the first to show that there is an infinite number of dimensions with closed manifolds for which the moduli space of Ricci positive metrics has infinitely many path components. Namely, he proved that for any homotopy sphere Σ^{4n+3} , $n \geq 1$, bounding a parallelizable manifold, the moduli space of metrics with positive Ricci curvature has an infinite number of path components distinguished by the above Kreck–Stolz invariant.

The main result of this note establishes a corresponding statement for moduli spaces of nonnegative sectional curvature metrics on infinite numbers of manifolds in an infinite range of dimensions.

THEOREM 1.1. *In each dimension $4n + 3$, $n \geq 1$, there exist infinite sequences of closed smooth simply connected manifolds of pairwise distinct homotopy type for which the moduli space of Riemannian metrics with nonnegative sectional curvature has infinitely many path components.*

Moreover, the manifolds figuring here also complement Wraith’s homotopy sphere results. Indeed, they do actually also show that there is an infinite number of dimensions in which there exist infinite sequences of closed smooth simply connected manifolds of pairwise distinct homotopy type for which the moduli space of metrics with positive Ricci curvature has an infinite number of path components. The same holds true if one restricts to metrics with positive Ricci and nonnegative sectional curvature.

The manifolds in Theorem 1.1 are total spaces of principal circle bundles over $\mathbb{C}P^{2n} \times \mathbb{C}P^1$ and agree in dimension 7 with the examples considered by Kreck and Stolz. Their topology has been studied by Wang and Ziller in their work on Einstein metrics on principal torus bundles mentioned above. These manifolds can also be viewed as quotients of the product of round spheres $S^{4n+1} \times S^3$ by free isometric circle actions. As in [23] we equip these manifolds with the submersion metric. We show that in any fixed dimension $4n + 3$ there exist among these Riemannian manifolds infinite sequences with the following properties: these manifolds belong to a fixed diffeomorphism type and their metrics have nonnegative sectional and positive Ricci curvature, but pairwise distinct absolute s -invariants. Since there are indeed infinite sequences of manifolds of pairwise distinct homotopy types with these properties, this will imply Theorem 1.1.

The study of moduli spaces of nonnegative sectional curvature metrics on *open* manifolds was initiated in the paper [23], with follow-ups in particular in the works [3–5].

Using [5, Proposition 2.8] one obtains by means of stabilizing the manifolds in Theorem 1.1 with \mathbb{R} , in each dimension $4n$, where $n \geq 2$, many new examples of simply connected open manifolds whose moduli spaces of nonnegative sectional curvature metrics have infinitely many path components. The following result generalizes [5, Corollary 1.11] to an infinite number of dimensions.

COROLLARY 1.2. *For each of the manifolds M in Theorem 1.1, the moduli space of complete Riemannian metrics with nonnegative sectional curvature on $M \times \mathbb{R}$ has infinitely many path components.*

The remaining parts of the present note are structured as follows. Section 2 contains the relevant preliminaries needed for the proofs of Theorem 1.1 and the corollary and the latter are presented in Section 3.

2. Preliminaries

2.1. Spaces and moduli spaces of metrics

Let M be an n -dimensional smooth manifold (without boundary), let $S^2 T^*M$ denote the second symmetric power of the cotangent bundle of M , and let $C^\infty(M, S^2 T^*M)$ be the real vector space of smooth symmetric $(0, 2)$ tensor fields on M . We always topologize $C^\infty(M, S^2 T^*M)$ and its subsets with the smooth topology of uniform convergence on compact subsets (compare, for example, [25, 28]). If M is compact, then $C^\infty(M, S^2 T^*M)$ is a Fréchet space.

The space $\mathcal{R}(M)$ of all (complete) Riemannian metrics on M is the subspace of $C^\infty(M, S^2 T^*M)$ consisting of all sections which are complete Riemannian metrics on M . Note that $\mathcal{R}(M)$ is a convex cone in $C^\infty(M, S^2 T^*M)$, that is, if $a, b > 0$ and $g_1, g_2 \in \mathcal{R}(M)$, then $ag_1 + bg_2 \in \mathcal{R}(M)$. In particular, $\mathcal{R}(M)$ is contractible and any open subset of $\mathcal{R}(M)$ is locally path-connected.

Let us now define moduli spaces of Riemannian metrics. If M is a finite-dimensional smooth manifold, let $\text{Diff}(M)$ be the group of self-diffeomorphisms of M (which is a Fréchet Lie group if M is compact). Then $\text{Diff}(M)$ acts on $\mathcal{R}(M)$ by pulling back metrics, that is, one has the action

$$\text{Diff}(M) \times \mathcal{R}(M) \rightarrow \mathcal{R}(M), \quad (g, \phi) \mapsto \phi^*(g).$$

The moduli space

$$\mathcal{M}(M) := \mathcal{R}(M)/\text{Diff}(M)$$

of (complete) Riemannian metrics on M is the quotient space of $\mathcal{R}(M)$ by the above action of the diffeomorphism group $\text{Diff}(M)$, equipped with the quotient topology.

Note that usually $\text{Diff}(M)$ will not act freely on $\mathcal{R}(M)$. Moreover, due to the fact that different Riemannian metrics may have isometry groups of different dimension, the moduli space $\mathcal{M}(M)$ will in general not have any kind of manifold structure. Note that $\mathcal{M}(M)$ is locally path-connected since it is the quotient of the locally path-connected space $\mathcal{R}(M)$. Thus there is no difference between connected and path-connected components of open subsets of $\mathcal{M}(M)$. Note also that a lower bound on the number of components in a moduli space is also a lower bound on the number of components for the respective space of metrics.

One can similarly form spaces and moduli spaces of metrics satisfying various curvature conditions, as these conditions are invariant under the action of $\text{Diff}(M)$. We will here employ the following notation (and always tacitly assume that our metrics are complete):

The space of all metrics with positive scalar curvature on M shall be denoted $\mathcal{R}_{scal>0}(M)$. The corresponding spaces of positive Ricci and nonnegative sectional curvature will be, respectively, denoted $\mathcal{R}_{Ric>0}(M)$ and $\mathcal{R}_{sec\geq 0}(M)$. The respective moduli spaces will be denoted as $\mathcal{M}_{scal>0}(M) := \mathcal{R}_{scal>0}(M)/\text{Diff}(M)$, $\mathcal{M}_{Ric>0}(M) := \mathcal{R}_{Ric>0}(M)/\text{Diff}(M)$, etc.

Following [24] we will consider in Section 3 closed smooth simply connected manifolds M for which the absolute s -invariant is well defined and constant on path-connected components of the moduli space $\mathcal{M}_{scal>0}(M)$. Note that in this situation the invariant is also constant on path components of the moduli space $\mathcal{M}_{Ric>0}(M)$. In other words, two metrics $g_0, g_1 \in \mathcal{R}_{Ric>0}(M)$ with $|s|(g_0) \neq |s|(g_1)$ belong to different path components of $\mathcal{M}_{Ric>0}(M)$.

If, in addition, g_0, g_1 are metrics of nonnegative sectional curvature then they also belong to different path components of the moduli space $\mathcal{M}_{sec\geq 0}(M)$. An elegant way to see this (compare also [5, Proposition 2.7]) uses the Ricci flow: Suppose that γ is a path in $\mathcal{M}_{sec\geq 0}(M)$ with end points represented by g_0 and g_1 . Consider the Ricci flow on $\mathcal{R}(M)$. Since the Ricci flow is invariant under diffeomorphisms it descends to a local flow on the moduli space. As shown by Böhm and Wilking [6], γ evolves under this flow instantly to a path in $\mathcal{M}_{Ric>0}(M)$. Concatenation of the evolved path and the trajectories of the end points of γ then yields a path in $\mathcal{M}_{Ric>0}(M)$ connecting the end points of γ , thereby contradicting $|s|(g_0) \neq |s|(g_1)$.

2.2. The Atiyah–Patodi–Singer index theorem

An essential ingredient for the definition of the Kreck–Stolz invariant is the index theorem of Atiyah, Patodi and Singer for manifolds with boundary. In this section we briefly recall the index theorem for the Dirac and the signature operator. For more details and the general discussion we refer to [1, 2].

Let W be a $4m$ -dimensional compact spin manifold with boundary $\partial W = M$ and let g_W be a Riemannian metric on W which is a product metric near M . Let g_M denote the induced metric on M . Then one can define the Dirac operator $D^+(W, g_W) : C^\infty(W, S^+) \rightarrow C^\infty(W, S^-)$, where S^\pm are the half-spinor bundles. This operator is Fredholm after imposing the APS-boundary conditions, that is, after restricting to spinors φ on W for which the restriction to M is in the kernel of the orthogonal projection P onto the space spanned by eigenfunctions for nonnegative eigenvalues [1, p. 55]. Its index will be denoted by $\text{ind } D^+(W, g_W)$.

In general, the projection P and the APS-boundary condition do not depend continuously on the metric due to potential zero eigenvalues of the Dirac operator on the boundary. Hence, the index $\text{ind } D^+(W, g_W) \in \mathbb{Z}$ may jump under variations of g_W . However, if $g_W(t)$ is a path of metrics as above such that the induced metrics $g_M(t)$ on M have *positive* scalar curvature for all t then, by Lichnerowicz’s argument, the APS-boundary condition does depend continuously on t and $\text{ind } D^+(W, g_W(t))$ is constant in t (see [2], p. 417).

Taking a slightly different point of view, suppose that (M, g_M) is a $(4m - 1)$ -dimensional spin manifold of positive scalar curvature which bounds a spin manifold W . Let g_W be any extension of g_M to W which is a product metric near the boundary. From the above it follows that $\text{ind } D^+(W, g_W)$ is independent of the chosen extension and only depends on the bordism W and the path-component of g_M in the space of metrics on M of positive scalar curvature $\mathcal{R}_{scal>0}(M)$. Moreover the index vanishes if g_W can be chosen to be of positive scalar curvature (see also [24, Remark 2.2]).

The index $\text{ind } D^+(W, g_W)$ can be computed with the index theorem of Atiyah, Patodi and Singer [1, Theorem 4.2]. If (M, g_M) has positive scalar curvature one obtains

$$\text{ind } D^+(W, g_W) = \int_W \hat{A}(p_1(W, g_W), \dots, p_m(W, g_W)) - \frac{\eta(D(M, g_M))}{2}, \quad (1)$$

where $p_i(W, g_W)$ are the Pontrjagin forms of (W, g_W) , \hat{A} is the multiplicative sequence for the \hat{A} -genus, $D(M, g_M)$ is the Dirac operator on M and $\eta(D(M, g_M))$ is its η -invariant.

Next we recall the signature theorem for manifolds with boundary. Let W be a $4m$ -dimensional compact oriented manifold with boundary $\partial W = M$ and let g_W be a Riemannian metric on W which is a product metric near M . Let g_M denote the induced metric on M . Then one can consider the signature operator on W which becomes a Fredholm operator after imposing the APS-boundary conditions. Applying their index theorem to this operator Atiyah, Patodi and Singer proved the following signature theorem [1, Theorem 4.14]:

$$\text{sign } W = \int_W \mathcal{L}(p_i(W, g_W)) - \eta(B(M, g_M)). \quad (2)$$

Here \mathcal{L} is the multiplicative sequence for the L -genus and $\eta(B(M, g_M))$ is the η -invariant of the signature operator on the boundary. Recall that the signature of W , $\text{sign } W$, is by definition the signature of the (nondegenerate) quadratic form defined by the cup product on the image of $H^*(W, M)$ in $H^*(W)$. In particular, the right-hand side of equation (2) is a purely topological invariant.

2.3. The Kreck–Stolz s -invariant

The index of the Dirac operator, the signature and the integrals involving multiplicative sequences considered in the previous subsection depend on the choice of the bordism. However, under favorable circumstances one can combine the data to define a nontrivial invariant of the boundary itself (cf. [14, 24]). In this section we give a brief introduction to the so-called s -invariant which was used by Kreck and Stolz in their study of moduli space of metrics of positive scalar curvature.

A starting point in [14, 24] is to consider a certain linear combination $\hat{\mathcal{A}} + a_m \cdot \mathcal{L}$ of the multiplicative sequences for the Dirac and signature operator. Recall that the part of degree at most $4m$ -part of a multiplicative sequence is a polynomial in the variables p_1, \dots, p_m , where p_1, \dots, p_m are variables which may be thought of as universal rational Pontrjagin classes. By choosing $a_m = 1/(2^{2m+1} \cdot (2^{2m-1} - 1))$ in the linear combination above one obtains in degrees at most $4m$, a polynomial $N_m(p_1, \dots, p_{m-1})$ not involving p_m (cf. also [21]).

Now suppose that W is a $4m$ -dimensional spin manifold with boundary M . In [14] Eells and Kuiper considered the situation where the real Pontrjagin classes $p_i(W)$, $i < m$, can be lifted uniquely to $H^*(W, M; \mathbb{R})$ and the natural homomorphism $H^1(W; \mathbb{Z}/2) \rightarrow H^1(M; \mathbb{Z}/2)$ is surjective. This is, for example, the case if M is simply connected with $b_{2m-1}(M) = 0$ and $b_{4i-1}(M) = 0$, where $0 < i < m$. In this situation Eells and Kuiper showed that the modulo 1 reduction of

$$\langle j^{-1}(N_m(p_1(W), \dots, p_{m-1}(W))), [W, M] \rangle - a_m \cdot \text{sign } (W) \quad (3)$$

(or half of it if m is odd) is independent of the choice of W and can be used to detect different smooth structures of M . Here $j : H^*(W, M; \mathbb{R}) \rightarrow H^*(W; \mathbb{R})$ is the natural homomorphism induced by inclusion and $\langle \cdot, [W, M] \rangle$ is the Kronecker product with the fundamental homology class.

In [24] Kreck and Stolz used the APS-index theorem to define an invariant, the so-called s -invariant, which refines the Eells–Kuiper invariant. Its relevance for the study of moduli spaces is summarized in Proposition 2.1.

Suppose, as in the last subsection, that g_W is a metric of W which is a product metric near the boundary and let g_M denote the induced metric on M . The integral $\int_W N_m(p_1(W, g_W), \dots, p_{m-1}(W, g_W))$ can be computed via the APS-index theorem. Suppose (M, g_M) has positive scalar curvature. Then, using equations (1) and (2), one obtains

$$\text{ind } D^+(W, g_W) + a_m \text{sign } W = \int_W N_m(p_i(W, g_W)) - \frac{\eta(D(M, g_M))}{2} - a_m \eta(B(M, g_M)).$$

Under favorable circumstances, for example, if all real Pontrjagin classes of M vanish, Kreck and Stolz show that the integral $\int_W N_m(p_1(W, g_W), \dots, p_{m-1}(W, g_W))$ is equal to

$$\int_M d^{-1}(N_m(p_i(M, g_M))) + \langle j^{-1}(N_m(p_i(W))), [W, M] \rangle,$$

where $d^{-1}(N_m(p_i(M, g_M)))$ is a term which only depends on (M, g_M) , that is, is independent of the choice of the bordism W and the metric g_W (see [24, p. 829]). Collecting the summands involving (M, g_M) one obtains the s -invariant of Kreck and Stolz

$$s(M, g_M) := -\frac{\eta(D(M, g_M))}{2} - a_m \eta(B(M, g_M)) + \int_M d^{-1}(N_m(p_i(M, g_M))). \quad (4)$$

Let $t(W) := -(\langle j^{-1}(N_m(p_i(W))), [W, M] \rangle - a_m \cdot \text{sign } W)$. Note that the rational number $t(W)$ only depends on the topology of W and reduces modulo 1 to the negative of the invariant considered by Eells and Kuiper (see equation (3)). Now, still assuming that (M, g_M) has positive scalar curvature equations (1) and (2) give

$$s(M, g_M) = \text{ind } D^+(W, g_W) + t(W). \quad (5)$$

Note that the right-hand side of (5) only depends on the bordism W and on the path component of g_M in the space $\mathcal{R}_{\text{scal}>0}(M)$ of metrics on M of positive scalar curvature, whereas the left-hand side only depends on (M, g_M) , that is, is independent of the chosen bordism. Hence, both sides depend only on the path component of g_M in $\mathcal{R}_{\text{scal}>0}(M)$. If (W, g_W) has positive scalar curvature as well then, by Lichnerowicz's argument, $\text{ind } D^+(W, g_W)$ vanishes and $s(M, g_M)$ provides a refinement of the Eells–Kuiper invariant (for all of this see [24, Section 2]).

If $H^1(M; \mathbb{Z}/2) = 0$, then the spin structure of M is uniquely determined up to isomorphism by the orientation. In this case the absolute s -invariant does not change under the action of the diffeomorphism group $\text{Diff}(M)$ on $\mathcal{R}_{\text{scal}>0}(M)$. In summary, one has

PROPOSITION 2.1 [24, Proposition 2.14]. *If M is a closed connected spin manifold of dimension $4m - 1$ with vanishing real Pontrjagin classes and $H^1(M; \mathbb{Z}/2) = 0$, then s induces a map*

$$|s| : \pi_0(\mathcal{R}_{\text{scal}>0}(M)/\text{Diff}(M)) \rightarrow \mathbb{Q}.$$

Kreck and Stolz also derive an explicit formula for $s(M, g_M)$ in the case where M is the total space of an S^1 -principal bundle. We will give this formula in the next section in a particular case.

2.4. Circle bundles over products of projective spaces

This subsection describes the Riemannian manifolds which will be used in the proof of Theorem 1.1. As in [24] we will consider S^1 -principal bundles over the product of two complex projective spaces. The total spaces of these bundles have been studied by Wang and Ziller [33] in their work about Einstein metrics on principal torus bundles, and we will employ some of their cohomological computations.

Let x (respectively, y) be the positive generators of the integral cohomology ring of $\mathbb{C}P^{2n}$ (respectively, $\mathbb{C}P^1$). Let $M_{k,l}$ be the total space of the S^1 -principal bundle P over $B := \mathbb{C}P^{2n} \times \mathbb{C}P^1$ with Euler class $c := lx + ky$, where k, l are coprime positive integers[†]. We summarize

[†]We follow here the notation of Kreck and Stolz. In the notation of Wang and Ziller [33] $M_{k,l}$ is denoted by $M_{l,k}^{2n,1} \cong M_{k,l}^{1,2n}$.

the relevant topological properties of the manifolds $M_{k,l}$ (see also [33, Proposition 2.1 and Proposition 2.3]).

PROPOSITION 2.2. (1) *The spaces $M_{k,l}$ are closed smooth simply connected $(4n+3)$ -dimensional manifolds.*

(2) *The integral cohomology ring of $M_{k,l}$ is given by*

$$H^*(M_{k,l}; \mathbb{Z}) \cong \mathbb{Z}[u, v] / ((lv)^2, v^{2n+1}, uv^2, u^2),$$

where $\deg u = 4n+1$ and $\deg v = 2$. In particular, $H^*(M_{k,l}; \mathbb{Z})$ does not depend on k .

(3) *The rational cohomology ring of $M_{k,l}$ is isomorphic to the one of $S^{4n+1} \times \mathbb{C}P^1$ and all rational Pontrjagin classes of $M_{k,l}$ vanish.*

(4) *The manifolds $M_{k,l}$ are all formal.*

(5) *For fixed l , the manifolds $M_{k,l}$ fall into finitely many diffeomorphism types.*

Proof. The first assertion follows from (k, l) coprime. The second one can be verified using a spectral sequence argument (cf. [33, Proposition 2.1]). The third statement is a direct consequence of the second. The fourth statement holds since $H^*(S^{4n+1} \times \mathbb{C}P^1; \mathbb{Q})$ is intrinsically formal, which follows by direct computation. The last statement is a consequence of results from Sullivan's surgery theory, compare [29, Theorem 13.1 and the proof of Theorem 12.5]. These imply that if a collection of simply connected closed smooth manifolds of dimension at least 5 all have isomorphic integral cohomology rings, the same rational Pontrjagin classes, and if their minimal model is a formal consequence of their rational cohomology ring, then there are only finitely many diffeomorphism types among them. \square

Let L denote the complex line bundle associated to the S^1 -principal bundle $P \rightarrow B$ and let $W = D(L)$ be the total space of the disk bundle with boundary $\partial W = S(L) = M_{k,l}$. Note that the tangent bundle of W is isomorphic to $\pi^*(TB \oplus L)$, where $\pi : W \rightarrow B$ is the projection. Hence, the total Stiefel–Whitney class of W is equal to

$$w(W) \equiv \pi^*((1+x)^{2n+1} \cdot (1+y)^2 \cdot (1+lx+ky)) \pmod{2}. \quad (6)$$

Since the restriction of $\pi^*(L)$ to $M_{k,l}$ is trivial, the tangent bundle of $M_{k,l}$ is stably isomorphic to $\pi^*(TB)$.

The classes x and y of $H^2(B; \mathbb{Z})$ pull back under $P \rightarrow B$ to $-kv$ and lv , respectively (cf. [33]). Hence,

$$w(M_{k,l}) = (1+lv)^2 \cdot (1-kv)^{2n+1}. \quad (7)$$

Restricting formulas (6) and (7) to w_2 and recalling that $M_{k,l}$ and W are simply connected, one obtains the following:

LEMMA 2.3. *$M_{k,l}$ is spin if and only if k is even. In this case l is odd because of $(k, l) = 1$, and, for a fixed orientation, $M_{k,l}$ as well as W admit a unique spin structure.*

From now on we will assume that k is even (and thus l is odd). We equip W with the orientation induced from the standard orientation of B and the complex structure of L . From the lemma above we see that $M_{k,l}$ and W admit unique spin structures and that W is a spin bordism for $M_{k,l}$.

Recall from equation (5) that the computation of the s -invariant of $M_{k,l}$ involves the topological term $t(W) := -(\langle j^{-1}(N_{n+1}(p_i(W))), [W, M] \rangle - a_{n+1} \cdot \text{sign } W)$. This term has already been computed by Kreck and Stolz. For the spin bordism $W = D(L)$ as above we thus obtain

LEMMA 2.4.

$$t(W) = - \left\langle \frac{1}{c} \cdot \left(\hat{\mathcal{A}}(TB) \cdot \frac{c/2}{\sinh c/2} + a_{n+1} \cdot \mathcal{L}(TB) \cdot \frac{c}{\tanh c} \right), [B] \right\rangle.$$

Proof. By [24, Lemma 4.2, part 2]

$$t(W) = - \left\langle \hat{\mathcal{A}}(TB) \cdot \frac{1}{2 \sinh c/2} + a_{n+1} \cdot \mathcal{L}(TB) \cdot \frac{1}{\tanh c}, [B] \right\rangle + a_{n+1} \cdot \text{sign}(B_c).$$

Here $\hat{\mathcal{A}}$ and \mathcal{L} are the polynomials in the Pontrjagin classes associated to the \hat{A} - and L -genus, $a_{n+1} := 1/(2^{2n+3} \cdot (2^{2n+1} - 1))$ and $\text{sign}(B_c)$ is the signature of the symmetric bilinear form

$$H^{2n}(B) \otimes H^{2n}(B) \rightarrow \mathbb{Q}, \quad (u, v) \mapsto \langle u \cdot v \cdot c, [B] \rangle. \quad (8)$$

It remains to show that the signature term $\text{sign}(B_c)$ vanishes. We note that the bilinear form (8) is represented with respect to the basis $(x^n, x^{n-1}y)$ by the matrix $\begin{pmatrix} k & l \\ i & 0 \end{pmatrix}$. Hence, $\text{sign}(B_c)$ vanishes for $l \neq 0$. \square

3. Proofs of Theorem 1.1 and Corollary 1.2

In this section we prove our main theorem and the corollary. Using the manifolds $M_{k,l}$ from above, we will employ the absolute Kreck–Stolz invariant $|s|$ to differentiate between the path components of the moduli spaces of nonnegative sectional curvature as well as positive Ricci curvature metrics.

We will first show that each $M_{k,l}$ admits a metric $g_M = g_{k,l}$ of nonnegative sectional curvature and positive Ricci curvature which is connected in $\mathcal{M}_{scal>0}(M)$ to a metric that extends to a metric g_W of positive scalar curvature on the associated disk bundle W such that g_W is a product metric near the boundary. This will imply that the s -invariant of $(M_{k,l}, g_M)$ is equal to the topological term $t(W)$ given in Lemma 2.4 (see equation (5)).

Next we consider, for fixed n , a certain infinite sequence of S^1 -principal bundles $P_m \rightarrow \mathbb{C}P^{2n} \times \mathbb{C}P^1$ with diffeomorphic total spaces, where each total space P_m can also be described as the quotient of $S^{4n+1} \times S^3$, equipped with the standard nonnegatively curved product metric, by the action of an isometric free S^1 -action. Note also that, in fact, there is a free and isometric $T^2 = S^1 \times S^1$ -action on $S^{4n+1} \times S^3$ which is given by the product of the circle Hopf actions on the respective sphere factors, and all S^1 -actions that yield the P_m are subactions of this fixed one.

The induced metric g_m on P_m has nonnegative sectional and positive Ricci curvature by the O’Neill formulas, and an upper sectional curvature bound which is independent of m . We show that the absolute s -invariants of the g_m are pairwise different (for $n = 1$ this was already shown by Kreck and Stolz) and, hence, belong to different connected components of the moduli space of positive scalar (and positive Ricci) curvature metrics. As explained above, this implies that the metrics g_m also belong to different connected components of the moduli space of nonnegative sectional curvature metrics (see Subsection 2.1).

Let us now discuss the metrics on $M_{k,l}$ and its associated disk bundle W .

PROPOSITION 3.1. *Each manifold $M := M_{k,l}$ admits a metric g_M of simultaneously nonnegative sectional curvature and positive Ricci curvature which is connected in $\mathcal{M}_{scal>0}(M)$ to a metric that extends to a metric g_W of positive scalar curvature on the associated disk bundle W such that g_W is a product metric near the boundary.*

Proof. Consider the circle Hopf action on odd-dimensional spheres $S^{2r+1} \subset \mathbb{C}^{r+1}$ given by multiplication of unit complex numbers from the right. This gives rise to an isometric and free action from the right of a two-dimensional torus T on any product of round unit spheres. Here we consider the T -action on $S^{4n+1} \times S^3$ with quotient $\mathbb{C}P^{2n} \times \mathbb{C}P^1$. We note that the T -orbits are products of geodesics and, hence, totally geodesic flat tori in $S^{4n+1} \times S^3$.

The manifold $M_{k,l}$ can be described as the quotient of $S^{4n+1} \times S^3$ by a certain circle subaction of T (the subaction depends on (k, l)). Let g_M be the submersion metric with respect to this action. Then $(M_{k,l}, g_M)$ has nonnegative sectional and positive Ricci curvature by the O'Neill formulas. The (noneffective) action of T on $M_{k,l}$ gives rise also to the S^1 -principal bundle $M_{k,l} \rightarrow M_{k,l}/T = \mathbb{C}P^{2n} \times \mathbb{C}P^1$. When we equip the base of this fibration with the submersion metric, it is not difficult to see that this Riemannian submersion has totally geodesic fibers. Now, after shrinking our metric along the fibers if necessary, following the arguments in [24, Section 4], we see that the metric can be extended to a metric g_W of positive scalar curvature on the associated disk bundle W such that g_W is a product metric near the boundary. \square

Thus, combining the proposition above with Lemma 2.4 and equation (5), it follows that $s(M_{k,l}, g_M)$ is equal to

$$-\left\langle \frac{1}{c} \cdot \left(\hat{A}(TB) \cdot \frac{c/2}{\sinh c/2} + a_{n+1} \cdot \mathcal{L}(TB) \cdot \frac{c}{\tanh c} \right), [B] \right\rangle.$$

Let us now compute the invariant $s(k, l) := s(M_{k,l}, g_M)$. Recall that $M_{k,l}$ is the total space of the S^1 -principal bundle over $B = \mathbb{C}P^{2n} \times \mathbb{C}P^1$ with Euler class $c := lx + ky$ where x and y are the positive generators of the integral cohomology rings of $\mathbb{C}P^{2n}$ and $\mathbb{C}P^1$, respectively, k, l are coprime positive integers, k is even and l is odd.

PROPOSITION 3.2. *The expression $s(k, l)/k := s(M_{k,l}^{4n+3}, g_M)/k$ is a Laurent polynomial in l of degree $2n$.*

Proof. Since the total Pontrjagin class of $B = \mathbb{C}P^{2n} \times \mathbb{C}P^1$ is given by $p(B) = (1 + x^2)^{2n+1}$, we see that $s(k, l)$ is equal to

$$-\left\langle \frac{1}{c} \left(\left(\frac{x/2}{\sinh x/2} \right)^{2n+1} \cdot \frac{c/2}{\sinh c/2} + a_{n+1} \left(\frac{x}{\tanh x} \right)^{2n+1} \cdot \frac{c}{\tanh c} \right), [B] \right\rangle. \quad (9)$$

Here

$$\frac{t/2}{\sinh t/2} = 1 + \sum_{j \geq 1} (-1)^j \frac{(2^{2j-1} - 1)}{(2j)! \cdot 2^{2j-1}} B_j \cdot t^{2j} =: 1 + \sum_{j \geq 1} \hat{a}_{2j} \cdot t^{2j}$$

and

$$\frac{t}{\tanh t} = 1 + \sum_{j \geq 1} (-1)^{j-1} \frac{2^{2j}}{(2j)!} B_j \cdot t^{2j} =: 1 + \sum_{j \geq 1} b_{2j} \cdot t^{2j}$$

are the characteristic power series associated to the \hat{A} - and L -genus (cf. [21], § 1.5, [27], Appendix B). The B_j are the Bernoulli numbers, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66, \dots$, and are all nonzero. In particular, $\hat{a}_{2j} = [(1 - 2^{2j-1})/2^{4j-1}] \cdot b_{2j}$.

Note that x^2 is equal to $c \cdot [(lx - ky)/l^2]$ and, hence, divisible by c . To evaluate the expression in (9) on the fundamental cycle $[B]$, one first divides out in the inner expression the terms of cohomological degree $4n + 4$ by c and then computes the coefficients of $x^{2n} \cdot y$. We claim that the result is k times a Laurent polynomial in l of degree $2n$. To see this we rewrite the formula

for $s(k, l)$ in the following form:

$$-s(k, l) = \left\langle \frac{1}{c} \left(\left(\frac{x/2}{\sinh x/2} \right)^{2n+1} + a_{n+1} \left(\frac{x}{\tanh x} \right)^{2n+1} \right), [B] \right\rangle \\ + \left\langle \left(\frac{x/2}{\sinh x/2} \right)^{2n+1} \cdot \left(\sum_{j \geq 1} \hat{a}_{2j} c^{2j-1} \right) + a_{n+1} \left(\frac{x}{\tanh x} \right)^{2n+1} \cdot \left(\sum_{j \geq 1} b_{2j} c^{2j-1} \right), [B] \right\rangle.$$

Using $x^2/c = (lx - ky)/l^2$ we see that the first summand is a rational multiple of $k \cdot l^{-2}$ and the second summand is of the form $k \cdot p(l)$, where $p(l)$ is a polynomial in l of degree at most $2n$. The term of degree $2n$ in $k \cdot p(l)$ is equal to

$$\langle \hat{a}_{2n+2} \cdot c^{2n+1} + a_{n+1} \cdot b_{2n+2} \cdot c^{2n+1}, [B] \rangle = (2n+1) \cdot k \cdot (\hat{a}_{2n+2} + a_{n+1} \cdot b_{2n+2}) \cdot l^{2n}.$$

Using $a_{n+1} := 1/(2^{2n+3} \cdot (2^{2n+1} - 1))$ and $\hat{a}_{2j} = (1 - 2^{2j-1})/(2^{4j-1}) \cdot b_{2j}$, it then follows that the coefficient of l^{2n} in $p(l)$ is, indeed, nonzero. \square

Proof of Theorem 1.1. We fix a positive odd integer l_0 for which the Laurent polynomial $p(l) := s(k, l)/k$ does not vanish. Note that there are infinitely many of such integers. Note also that $s(k, l_0) = k \cdot p(l_0)$ takes pairwise different absolute values for positive even integers k . It follows from Sullivan's surgery theory (see Proposition 2.2) that there exists an infinite sequence of such integers $(k_m)_m$ with diffeomorphic $M_{k_m, l}$. As before let $P_m := M_{k_m, l}$ be equipped with the submersion metric.

Recall that P_m has nonnegative sectional and positive Ricci curvature. Since the Riemannian manifolds P_m have pairwise different absolute s -invariant, they belong to pairwise different connected components of the moduli spaces $\mathcal{M}_{sec \geq 0}(M)$ and $\mathcal{M}_{Ric > 0}(M)$ (see Subsections 2.1 and 2.3).

From the classification results of Section 2 we conclude that there exists, in each relevant dimension, an infinite sequence of manifolds of pairwise distinct homotopy type which satisfy these properties. \square

Proof of Corollary 1.2. Consider any manifold M as in Theorem 1.1. From the results in Section 2.4 we know that there is an infinite sequence of nonnegatively curved Riemannian manifolds (M_k, g_k) that are all diffeomorphic to M but have pairwise distinct absolute s -invariants. For each k we fix a diffeomorphism $\varphi_k : M \rightarrow M_k$ and denote by \bar{g}_k the pull-back metric $\varphi_k^*(g_k)$ on M . We obtain an infinite sequence of nonnegatively curved metrics $(\bar{g}_k)_k$ on M which represent pairwise different path components of $\mathcal{M}_{sec \geq 0}(M)$.

Now crossing with (\mathbb{R}, dt^2) gives us nonnegatively curved complete metrics $\bar{g}_k \times dt^2$ on the simply connected manifold $M \times \mathbb{R}$ with souls (M, \bar{g}_k) belonging to different path components of $\mathcal{M}_{sec \geq 0}(M)$. By [5, Proposition 2.8] the moduli space of complete Riemannian metrics with nonnegative sectional curvature on $M \times \mathbb{R}$ is homeomorphic to the disjoint union of the moduli spaces of nonnegative sectional curvature metrics of all possible pairwise nondiffeomorphic souls of metrics in $\mathcal{M}_{sec \geq 0}(M \times \mathbb{R})$.

In particular, the Riemannian manifolds $(M, \bar{g}_k) \times (\mathbb{R}, dt^2)$ belong to pairwise different path components of the moduli space of complete Riemannian metrics with nonnegative sectional curvature on $M \times \mathbb{R}$. \square

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