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GROUPS OF PL-HOMEOMORPHISMS ADMITTING NONTRIVIAL INVARIANT CHARACTERS

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We show that several classes of groups G of PL-homeomorphisms of the real line admit nontrivial homomorphisms $\chi : G \to \mathbb{R}$ that are fixed by every automorphism of G. The classes enjoying the stated property include the generalizations of Thompson's group F studied by K. S. Brown (1992), M. Stein (1992), S. Cleary (1995), and Bieri and Strebel (2016), but also the class of groups investigated by Bieri, Neumann, and Strebel (Theorem 8.1 in *Invent. Math.* 90 (1987), 451–477). It follows that every automorphism of a group in one of these classes has infinitely many associated twisted conjugacy classes.

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1. Introduction

This paper stems from two articles [Bleak et al. 2008; Gonçalves and Kochloukova 2010] about twisted conjugacy classes of Thompson's group F. In order to describe the aim of the cited papers, we recall some terminology. Let G be a group and α

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an automorphism of G. Then α gives rise to an action $\mu_\alpha : G \times |G| \to |G|$ of G on its underlying set $|G|$, defined by

$$\mu_\alpha(g, x) = g \cdot x \cdot \alpha(g)^{-1}. \tag{1-1}$$

The orbits of this action are called *twisted conjugacy classes*, or *Reidemeister* classes, of α. The twisted conjugacy classes of the identity automorphism, for instance, are nothing but the conjugacy classes.

Two questions now arise, firstly, whether a given automorphism α has infinitely many orbits and, secondly, whether every automorphism of G has infinitely many orbits. As the latter property will be central to this paper, we recall the definition of property R_∞:

Definition 1.1. A group G is said to have *property* R_∞ if the action μ_α has infinitely many orbits for every automorphism $\alpha : G \xrightarrow{\sim} G$.

The problem of determining whether a given group, or a class of groups, satisfies property R_∞ has attracted the attention of several researchers. The problem is rendered particularly interesting by the fact there does not exist a uniform method of solution. Indeed, a variety of techniques and ad hoc arguments from several branches of mathematics have been used to tackle the problem, notably combinatorial group theory in [Gonçalves and Wong 2009], geometric group theory in [Levitt and Lustig 2000], C^*-algebras in [Fel'shtin and Troitsky 2012], and algebraic geometry in [Mubeena and Sankaran 2014b].

Bleak, Fel'shtyn, and Gonçalves [Bleak et al. 2008] show that Thompson's group F enjoys property R_∞, while Gonçalves and Kochloukova [2010] establish the same property for Thompson's group F, but also for many other groups G having the peculiarity that the complement of their BNS-invariant $\Sigma^1(G)$ is made up of finitely many rank 1 points. In this paper, we generalize both approaches and prove in this way that many classes of groups of PL-homeomorphisms have property R_∞.

1A. ***A useful fact.*** The papers by Bleak et al. and by Gonçalves and Kochloukova both exploit the following observation: let α be an automorphism of a group G, let $\psi : G \to B$ be a homomorphism into an *abelian* group, and assume ψ is fixed by α. Then ψ is constant on twisted conjugacy classes of α; indeed, if the elements x and y lie in the same twisted conjugacy class there exists $z \in G$ so that

$$y = z \cdot x \cdot \alpha(z)^{-1};$$

the computation

$$\psi(y) = \psi(z \cdot x \cdot \alpha(z)^{-1}) = \psi(x) \cdot \psi(z) \cdot ((\psi \circ \alpha)(z))^{-1} = \psi(x)$$

proves the claim. A group G therefore *has property* R_∞ if it admits a homomorphism onto an infinite, abelian group that is fixed by every automorphism of G.[1] For $B \subset \mathbb{R}^\times_{>0}$, the classes of groups G admitting such a nontrivial homomorphism include various generalizations of Thompson's group F [Brown 1987a; Stein 1992; Cleary 1995; 2000; Bieri and Strebel 2016].

1B. ***Approach used by Bleak, Fel'shtyn, and Gonçalves.*** The authors [2008] establish that Thompson's group F has property R_∞ by using the mentioned fact. To find the homomorphism ψ, they use a representation of F by piecewise linear homeomorphisms of the real line: F is isomorphic to the group of all piecewise linear homeomorphisms f with supports in the unit interval $I = [0, 1]$, slopes a power of 2, and *break points*, i.e., points where the left and right derivatives differ, in the group $\mathbb{Z}[1/2]$ of dyadic rationals; see, e.g., [Cannon et al. 1996, p. 216, §1]. This representation affords them with two homomorphisms σ_ℓ and σ_r, given by the right derivative in the *left* end point 0 and the left derivative in *right* end point 1 of I, respectively. In formulae,

$$\sigma_\ell(f) = \lim_{t \searrow 0} f'(t) \quad \text{and} \quad \sigma_r(f) = \lim_{t \nearrow 1} f'(t). \tag{1-2}$$

The images of σ_ℓ and σ_r are both equal to gp(2), the (multiplicative) cyclic group generated by the natural number 2. Theorem 3.3, the main result of [Bleak et al. 2008], can be rephrased by saying that the homomorphism

$$\psi : F \to \mathrm{gp}(2), \quad f \mapsto \sigma_\ell(f) \cdot \sigma_r(f)$$

is fixed by every automorphism α of F. Its proof uses the very detailed information about Aut F established by M. Brin [1996].

1C. ***A generalization.*** The stated description of Thompson's group F invites one to introduce generalized groups of type F in the following manner.

Definition 1.2. Let $\mathrm{PL}_o(\mathbb{R})$ denote the group of all increasing PL-homeomorphisms of the real line with only finitely many break points. Fix a closed interval $I \subseteq \mathbb{R}$, a subgroup P of the multiplicative group of positive reals $\mathbb{R}^\times_{>0}$ and a subgroup A of the additive group $\mathbb{R}_{\mathrm{add}}$ of the field $\mathbb{R}$ that is stable under multiplication by P. Define $G(I; A, P)$ to be the subset of $\mathrm{PL}_o(\mathbb{R})$ made up of all PL-homeomorphisms g that satisfy the following conditions:

(a) the support $\mathrm{supp}\, g = \{t \in \mathbb{R} \mid g(t) \neq t\}$ of g is contained in I,

(b) the slopes of the finitely many line segments forming the graph of g lie in P,

(c) the break points of g lie in A, and

(d) g maps A onto A.

[1] One can find generalizations of, and also many variations on, the stated observation; see, e.g., [Gonçalves and Wong 2003, Formula (2.2)] or [Fel'shtyn and Troitsky 2015, Claim 2 in Theorem 4.4].

Remarks 1.3. (a) The subset $G(I; A, P)$ is closed under composition[2] and inversion. The set $G(I; A, P)$ equipped with these operations is a group; by abuse of notation, it will also be denoted by $G(I; A, P)$.

(b) We shall always require that neither P nor A be reduced to the neutral element. These requirements imply that A contains arbitrary small positive elements and thus A is a dense subgroup of $\mathbb{R}$. As concerns the interval I we shall restrict attention to three types: compact intervals with endpoints 0 and $b \in A_{>0}$, the half-line $[0, \infty[$ and the line $\mathbb{R}$; we refer the reader to [Bieri and Strebel 2016, Sections 2.4 and 16.4] for a discussion of the groups associated to other intervals.

(c) The idea of introducing and studying the groups $G(I; A, P)$ goes back to the papers [Brin and Squier 1985; Bieri and Strebel 1985].

1C1. *The homomorphisms σ_ℓ, σ_r, and ψ.* The definitions of σ_ℓ and σ_r, given in (1-2), admit straightforward extensions to the groups $G(I; A, P)$; note, however, that in case of the half-line $[0, \infty[$, the number $\sigma_r(f)$ will denote the slope of f near $+\infty$, and similarly for $I = \mathbb{R}$ and σ_ℓ, σ_r. The homomorphisms σ_ℓ and σ_r allow one then to introduce an analogue of $\psi : F \to \mathrm{gp}(2)$, namely,

$$\psi : G = G(I; A, P) \to P, \quad g \mapsto \sigma_\ell(g) \cdot \sigma_r(g). \tag{1-3}$$

There remains the question whether this homomorphism ψ is fixed by every automorphism of G. In the case of Thompson's group F the question has been answered in the affirmative by exploiting the detailed information about $\mathrm{Aut}\, F$ obtained by Brin [1996]. Such a detailed description is not to be expected for every group of the form $G(I; A, P)$; indeed, the results in [Brin and Guzmán 1998] show that the structure of the automorphism group gets considerably more involved if one passes from the group $G([0, 1]; \mathbb{Z}[1/2], \mathrm{gp}(2))$, the group isomorphic to F, to the groups $G([0, 1]; \mathbb{Z}[1/n], \mathrm{gp}(n))$ with n an integer greater than 2.

1C2. *The first main results.* It turns out that one does not need very detailed information about $\mathrm{Aut}\, G(I; A, P)$ in order to construct a nontrivial homomorphism $\psi : G(I; A, P) \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of the group $G(I; A, P)$; it suffices to go back to the findings in the memoir [Bieri and Strebel 1985] and to supplement them by some auxiliary results based upon them.[3] One outcome is the following theorem.

Theorem 1.4. *Assume the interval I, the group of slopes P and the $\mathbb{Z}[P]$-module A are as in* Definition 1.2 *and in* Remark 1.3(b). *Then there exists an epimorphism*

[2] In this article we use *left* actions and the composition of functions familiar to analysts; thus $g_2 \circ g_1$ denotes the function $t \mapsto g_2(g_1(t))$ and ${}^{g_1}g_2$ the homeomorphism $g_1 \circ g_2 \circ g_1^{-1}$.

[3] The memoir [Bieri and Strebel 1985] has recently been published as [Bieri and Strebel 2016].

$\psi : G(I; A, P) \twoheadrightarrow P$ *that is fixed by every automorphism of* G. *Furthermore, the group* $G(I; A, P)$ *therefore has property* R_∞.

Remark 1.5. Let I, A, and P be as before and let $B = B(I; A, P)$ be the subgroup of $G(I; A, P)$ made up of all elements g that are the identity near the endpoints. Then B is a characteristic subgroup of $G(I; A, P)$ and variations of Theorem 1.4 hold for many subgroups G of $G(I; A, P)$ with $B \subset G$. For further details, see Theorems 3.8, 4.4, and 5.5.

1D. ***Route taken by Gonçalves and Kochloukova.*** For details, see [Gonçalves and Kochloukova 2010]. The proof of Theorem 1.4 does not exploit information about $\operatorname{Aut} G(I; A, P)$ that is as precise as that going into the proof of the main result of [Bleak et al. 2008]. It uses, however, nontrivial features of the automorphisms of $G(I; A, P)$. Gonçalves and Kochloukova [loc. cit.] put forward the novel idea of replacing detailed information about $\operatorname{Aut} G$ by information about the form of the BNS-invariant of the group G; they carry out this program for the generalized Thompson group $F_{n,0}$ with $n \geq 2$, a group isomorphic to $G([0, 1]; \mathbb{Z}[1/n], \operatorname{gp}(n))$, and for many other groups, as well.

In a nutshell, their idea is this. Suppose G is a finitely generated group for which the complement of $\Sigma^1(G)$ is *finite*.[4] Then every automorphism of G permutes the finitely many rays in $\Sigma^1(G)^c$. This suggests that it might be possible to construct a new ray $\mathbb{R}_{>0} \cdot \chi_0$ that is fixed by $\operatorname{Aut} G$. If one succeeds in doing so, then $\mathbb{R} \cdot \chi_0$ will be a 1-dimensional subrepresentation of the finite dimensional real vector space $\operatorname{Hom}(G, \mathbb{R})$, acted on by $\operatorname{Aut} G$ via

$$(\alpha, \chi) \mapsto \chi \circ \alpha^{-1}.$$

A priori, this invariant line need not be fixed pointwise.

Gonçalves and Kochloukova detected that the line $\mathbb{R} \cdot \chi_0$ is fixed pointwise by $\operatorname{Aut} G$ if the homomorphism $\chi_0 : G \to \mathbb{R}$ has *rank* 1, i.e., if its image is infinite cyclic. Using this fact they were then able to prove that Thompson's group F, but also many other groups G, admit a rank 1 homomorphism that is fixed by $\operatorname{Aut} G$ and thus satisfy property R_∞.

1E. ***A generalization.*** In the second part of this paper we consider a collection of PL-homeomorphism groups G whose invariant $\Sigma^1(G)^c$ is finite but contains a point of rank greater than 1. One is then confronted with the following problem: Suppose $\mathbb{R}_{>0} \cdot \chi_0$ is a ray that is fixed by $\operatorname{Aut} G$ as a *set*. There may then exist an automorphism α which acts on the ray by multiplication by a positive real number

[4] Recall that $\Sigma^1(G)$ is a certain subset of the space of all half-lines $\mathbb{R}_{>0} \cdot \chi$ emanating from the origin of the real vector space $\operatorname{Hom}(G, \mathbb{R})$, and that $\operatorname{Aut} G$ acts canonically on this subset, as well as on its complement.

$s \neq 1$; if so, the 1-dimensional subspace $\mathbb{R} \cdot \chi_0$ in the real vector space $\mathrm{Hom}(G, \mathbb{R})$ is an eigenline with eigenvalue $s \neq 1$ of the linear transformation α^* induced by α on $\mathrm{Hom}(G, \mathbb{R})$. The existence of an eigenvalue $s \neq 1$ wrecks our attempt to extend the approach adopted in [Bleak et al. 2008] to more general classes of groups of PL-homomorphisms, but — as we shall see — it can be ruled out if the image B of the character χ_0 has only 1 and -1 as *units*, the definition of units being as follows:

Definition 1.6. Given a subgroup B of the additive group $\mathbb{R}_{\mathrm{add}}$ we set

$$U(B) = \{s \in \mathbb{R}^\times \mid s \cdot B = B\} \tag{1-4}$$

and call $U(B)$ the *group of units* of B (inside the multiplicative group of $\mathbb{R}$).

We next explain how the subgroups B will enter into the picture. The groups we shall be interested in will be subgroups of $\mathrm{PL}_o(\mathbb{R})$ with supports in a compact interval $[0, b]$; they are thus subgroups G of $G([0, b]; \mathbb{R}_{\mathrm{add}}, \mathbb{R}^\times_{>0})$. By restricting the homomorphisms σ_ℓ and σ_r, defined in Section 1C1, one obtains homomorphisms of G into the multiplicative group $\mathbb{R}^\times_{>0}$; by composing these with the natural logarithm function one arrives at the homomorphisms $\chi_\ell : G \to \mathbb{R}_{\mathrm{add}}$ and $\chi_r : G \to \mathbb{R}_{\mathrm{add}}$.

The next result lists conditions that allow one to infer that G admits a nontrivial homomorphism into $\mathbb{R}^\times_{>0}$ fixed by every automorphism of G.

Theorem 1.7. *Suppose G is a subgroup of $G([0, b]; \mathbb{R}_{\mathrm{add}}, \mathbb{R}^\times_{>0})$ that satisfies the following conditions*:

(i) *no interior point of the interval $I = [0, b]$ is fixed by G*;

(ii) *the homomorphisms χ_ℓ and χ_r are both nonzero*;

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group*; *and*

(iv) *at least one of the groups $U(\mathrm{im}\, \chi_\ell)$ or $U(\mathrm{im}\, \chi_r)$ is reduced to $\{1, -1\}$.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G. The group G has therefore property R_∞.

There remains the problem of finding subgroups $B \subset \mathbb{R}_{\mathrm{add}}$ that have only the units 1 and -1. This problem is addressed in Section 6E. We shall show that a subgroup $B = \ln P$ has this property if the multiplicative group $P \subset \mathbb{R}^\times_{>0}$ is free abelian and generated by algebraic numbers. In addition, we shall construct in Section 8A a collection $\mathcal{G}$ of pairwise nonisomorphic 3-generator groups G_s enjoying the properties that each group G_s satisfies the assumptions of Theorem 1.7 and that the cardinality of $\mathcal{G}$ is that of the continuum.

2. Preliminaries on automorphisms of the groups $G(I; A, P)$

The groups $G(I; A, P)$ form a class of subgroups of the group $\mathrm{PL}_o(\mathbb{R})$, the group of all orientation preserving, piecewise linear homeomorphisms of the real line.

They enjoy some special properties, in particular the following two: each group acts approximately[5] highly transitively on the interior of I, and all its automorphisms are induced by conjugation by homeomorphisms. It is, above all, this second property that will be exploited in the sequel.

In this section, we recall the basic representation theorem for automorphisms of the groups $G(I; A, P)$ and deduce then some consequences.

2A. ***Representation of isomorphisms.*** We begin by fixing the set-up of this section: P is a nontrivial subgroup of $\mathbb{R}^{\times}_{>0}$ and A a nonzero subgroup of $\mathbb{R}_{\text{add}}$ that is stable under multiplication by P. Next, I is a closed interval of positive length; we assume the left end point of I is in A if I is bounded from below and similarly for the right end point.

Remark 2.1. Distinct intervals I_1 and I_2 may give rise to isomorphic groups $G(I_1; A, P)$ and $G(I_2; A, P)$. In particular, it is true that every group $G(I_1; A, P)$ is isomorphic to one whose interval I_2 has one of the three forms

$$[0, b] \text{ with } b \in A, \quad [0, \infty[, \quad \text{and} \quad \mathbb{R}. \tag{2-1}$$

See [Bieri and Strebel 2016, Sections 2.4 and 16.4] for proofs.

We come now to the announced result about isomorphisms of groups $G(I; A, P)$ and $G(\bar{I}; \bar{A}, \bar{P})$. It asserts that each isomorphism of the first group onto the second one is induced by conjugation by a homeomorphism of the interior $\operatorname{int}(I)$ of I onto the interior of $\bar{I}$. This claim holds even for suitably restricted subgroups of $G(I; A, P)$ and of $G(\bar{I}; \bar{A}, \bar{P})$. In order to state the generalized assertion we need the subgroup of "bounded elements".

Definition 2.2. Let $B(I; A, P)$ be the subgroup of $G(I; A, P)$ consisting of all PL-homeomorphisms f that are the identity near the end points or, more formally, that satisfy the inequalities $\inf I < \inf \operatorname{supp} f$ and $\sup \operatorname{supp} f < \sup I$.

We are now in a position to state the representation theorem.

Theorem 2.3. *Assume G is a subgroup of $G(I; A, P)$ that contains the derived subgroup of $B(I; A, P)$, and $\bar{G}$ is a subgroup of $G(\bar{I}; \bar{A}, \bar{P})$ containing the derived group of $B(\bar{I}; \bar{A}, \bar{P})$. Then every isomorphism $\alpha : G \xrightarrow{\sim} \bar{G}$ is induced by conjugation by a unique homeomorphism φ_α of the interior $\operatorname{int}(I)$ of I onto the interior of $\bar{I}$; more precisely, the equation*

$$\alpha(g) \restriction \operatorname{int}(\bar{I}) = \varphi_\alpha \circ (g \restriction \operatorname{int}(I)) \circ \varphi_\alpha^{-1} \tag{2-2}$$

holds for every $g \in G$. Moreover, φ_α maps $A \cap \operatorname{int}(I)$ onto $\bar{A} \cap \operatorname{int}(\bar{I})$.

Proof. The result is a restatement of [Bieri and Strebel 2016, Theorem E16.4]. □

[5] See [Bieri and Strebel 2016, Chapter A] for details.

Remarks 2.4. (a) Theorem 2.3 has two simple, but important consequences. First of all, every homeomorphism of intervals is either increasing or decreasing; since the homeomorphism φ_α inducing an isomorphism $\alpha : G \xrightarrow{\sim} \bar{G}$ is uniquely determined by α, there exist therefore two types of isomorphisms: the *increasing* isomorphisms, induced by conjugation by an increasing homeomorphism, and the decreasing ones.

Assume now that $\bar{I} = I$. If the homeomorphism $\varphi_\alpha : \mathrm{int}(I) \xrightarrow{\sim} \mathrm{int}(I)$ is increasing, it extends uniquely to a homeomorphism of I, but this may not be so if it is decreasing. Indeed, φ_α extends if I is a compact interval or the real line, but not if I is a half-line. If the extension exists, it will be denoted by $\tilde{\varphi}_\alpha$.

(b) The increasing automorphisms of a group G form a subgroup $\mathrm{Aut}_+ G$ of $\mathrm{Aut}\, G$ of index at most 2. It will turn out that is often easier to find a nonzero homomorphism $\psi : G \to B$ that is fixed by the subgroup $\mathrm{Aut}_+ G$ than a nonzero homomorphism fixed by $\mathrm{Aut}\, G \smallsetminus \mathrm{Aut}_+ G$ (in case this set is nonempty). For this reason, it is useful to dispose of criteria guaranteeing that $\mathrm{Aut}\, G = \mathrm{Aut}_+ G$.

(c) The derived group of $B(I; A, P)$ is a simple, infinite group (see [Bieri and Strebel 2016, Proposition C10.2]), but $B(I; A, P)$ itself may not be perfect. To date, no characterization of the parameters (I, A, P) corresponding to perfect groups $B(I; A, P)$ is known. The quotient group $G(I; A, P)/B(I; A, P)$, on the other hand, is a metabelian group that can be described explicitly in terms of the triple (I, A, P); see [Bieri and Strebel 2016, Section 5.2]. In the sequel, we shall therefore restrict attention to subgroups G containing $B(I; A, P)$.

(d) The second important consequence of Theorem 2.3 is the fact that $B(I; A, P)$ is a characteristic subgroup of every subgroup G with $B(I; A, P) \subseteq G \subseteq G(I; A, P)$. (The proof is easy; see [Bieri and Strebel 2016, Corollary E16.5] or Corollary 2.7 below.)

In part (a) of the previous remarks the term *increasing isomorphism* has been introduced. In the sequel, this parlance will be used often, and so we declare:

Definition 2.5. Let $\alpha : G \xrightarrow{\sim} \bar{G}$ be an isomorphism induced by the (uniquely determined) homeomorphism $\varphi_\alpha : \mathrm{int}(I) \xrightarrow{\sim} \mathrm{int}(\bar{I})$. If φ is *increasing* then α will be called increasing, and similarly for *decreasing*.

2B. ***The homomorphisms λ and ρ.*** By Remark 2.4(d) the group $B = B(I; A, P)$ is a characteristic subgroup of every group G containing it. Now G has, in addition, subgroups containing B that are invariant under the subgroup $\mathrm{Aut}_+ G$, namely the kernels of the homomorphisms λ and ρ. To set these homomorphisms into perspective, we go back to the homomorphisms

$$\sigma_\ell : G(I; A, P) \to P \quad \text{and} \quad \sigma_r : G(I; A, P) \to P,$$

introduced in Section 1C1. Their images are abelian and coincide with the group of slopes P. If I is not bounded from below, there exist a homomorphism λ, related to σ_ℓ, whose image is contained in $\mathrm{Aff}(A, P)$, the group of all affine maps of $\mathbb{R}$ with slopes in P and displacements $f(0) \in A$. The definition of λ is this:

$$\lambda : G(I; A, P) \to \mathrm{Aff}(A, P), \qquad g \mapsto (\text{affine map coinciding with } g \text{ near } -\infty). \tag{2-3}$$

If the interval I is not bounded from above, then there exists a similarly defined homomorphism

$$\rho : G(I; A, P) \to \mathrm{Aff}(A, P), \qquad g \mapsto (\text{affine map coinciding with } g \text{ near } +\infty). \tag{2-4}$$

The images of λ and ρ are, in general, smaller than $\mathrm{Aff}(A, P)$. They are equal to the entire group $\mathrm{Aff}(A, P)$ if $I = \mathbb{R}$; if I is not bounded from below, but bounded from above, the image of λ is $\mathrm{Aff}(I P \cdot A, P)$ and the analogous statement holds for ρ. In the above, $I P \cdot A$ denotes the submodule of A generated by the products $(p-1) \cdot a$ with $p \in P$ and $a \in A$; see [Bieri and Strebel 2016, Section 4 and Corollary A5.3].

For uniformity of notation, we extend the definition of λ and ρ to compact intervals: if $I = [0, b]$ and $f \in G(I; A, P)$ then $\lambda(g)$ is the linear map $t \mapsto \sigma_\ell(g) \cdot t$ and $\rho(g)$ is the affine map $t \mapsto \sigma_r(f) \cdot (t - b) + b$. Similarly one defines $\lambda(g)$ if I is the half-line $[0, \infty[$.

The homomorphisms λ and ρ allow one to restate the definition of $B(I; A, P)$:

$$B(I; A, P) = \ker \lambda \cap \ker \rho. \tag{2-5}$$

Remark 2.6. In the sequel, we shall often deal with subgroups, denoted G, of a group $G(I; A, P)$ that contain $B(I; A, P)$. For ease of notation, we shall then denote the restrictions of λ and ρ to G again by λ and ρ.

2C. ***First consequences of the representation theorem.*** Let G be a subgroup of $G(I; A, P)$ that contains the derived subgroup of $B(I; A, P)$ and let $\overline{G}$ be a subgroup of $G(\bar{I}; \bar{A}, \bar{P})$ containing the derived subgroup of $B(\bar{I}; \bar{A}, \bar{P})$. Suppose φ_α is a homeomorphism of $\mathrm{int}(I)$ onto $\mathrm{int}(\bar{I})$ that induces an isomorphism $\alpha : G \xrightarrow{\sim} \overline{G}$. The map φ_α need not be piecewise linear. Theorem 2.3, however, has useful consequences even in such a case. One implication is recorded in this result:

Corollary 2.7. *Assume G and $\overline{G}$ are subgroups of $G(I; A, P)$ both of which contain $B(I; A, P)$, and let*

$$\lambda, \rho : G \to \mathrm{Aff}(A, P) \quad \textit{and} \quad \bar{\lambda}, \bar{\rho} : \overline{G} \to \mathrm{Aff}(A, P)$$

be the obvious restrictions of the homomorphisms λ, ρ *introduced in Section 2B. Consider now an isomorphism* $\alpha : G \xrightarrow{\sim} \overline{G}$ *that is induced by the homeomorphism* $\varphi_\alpha : \mathrm{int}(I) \xrightarrow{\sim} \mathrm{int}(I)$. *If* φ_α *is increasing then*

(i) α *maps* $\ker\lambda$ *onto* $\ker\bar{\lambda}$ *and induces an isomorphism* α_ℓ *of* $G/\ker\lambda$ *onto* $\overline{G}/\ker\bar{\lambda}$;

(ii) α *maps* $\ker\rho$ *onto* $\ker\bar{\rho}$ *and induces an isomorphism* α_r *of* $G/\ker\rho$ *onto* $\overline{G}/\ker\bar{\rho}$.

Proof. (i) If $g \in \ker\lambda$ then g is the identity near $\inf I$. As φ_α is *increasing*, the image $\alpha(g) = \varphi_\alpha \circ g \circ \varphi_\alpha^{-1}$ of g is therefore also the identity near $\inf I$. It follows that $\alpha(\ker\lambda) \subseteq \ker\bar{\lambda}$. This inclusion is actually an equality, for $\alpha^{-1} : \overline{G} \to G$ is an isomorphism and so $\alpha^{-1}(\ker\bar{\lambda}) \subseteq \ker\lambda$. Claim (ii) can be proved similarly. □

2D. ***Automorphisms induced by finitary PL-homeomorphisms.*** Suppose that the group $G \subseteq G(I; A, P)$ is as before, and let α be an automorphism of G. According to Theorem 2.3, α is induced by conjugation by a unique autohomeomorphism φ_α. This autohomeomorphism may not be piecewise linear, but the situation improves if P, the group of slopes, is not cyclic (and hence dense in $\mathbb{R}^\times_{>0}$).

Theorem 2.8. *Suppose* P *is not cyclic. For every automorphism* α *of* G *there exists then a nonzero real number* s *such that* $A = s \cdot A$ *and that the autohomeomorphism* $\varphi_\alpha : \mathrm{int}(I) \xrightarrow{\sim} \mathrm{int}(I)$ *is piecewise linear with slopes in the coset* $s \cdot P$ *of* P. *Moreover,* φ_α *maps the subset* $A \cap \mathrm{int}(I)$ *onto itself and has only finitely many breakpoints in every compact subinterval of* $\mathrm{int}(I)$.

Proof. The result is a special case of [Bieri and Strebel 2016, Theorem E17.1]. □

Theorem 2.8 indicates that automorphisms of groups with a noncyclic group of slopes P are easier to analyze than those of the groups with cyclic P. Note, however, that the conclusion of Theorem 2.8 does not rule out that φ_α has infinitely many breakpoints which accumulate in one or both end points[6] and so φ_α may not be differentiable at the end points. In Section 3A we shall therefore be interested in differentiability criteria.

3. Characters fixed by Aut $G([0, b]; A, P)$

In this section, we prove Theorem 1.4 for the case of a compact interval and various extensions of it. An important ingredient in the proofs of these results is a criterion that allows one to deduce that an autohomeomorphism φ_α inducing an automorphism α of the group is differentiable near one or both of its end points.

[6]The notion of end point is to be interpreted suitably if I is not bounded.

3A. ***A differentiability criterion.*** The proof of the criterion is rather involved. Prior to stating the criterion and giving its proof, we discuss therefore a result that explains the interest in the criterion.

Proposition 3.1. *Let G be a subgroup of $G([a,b];A,P)$ that contains the derived subgroup of $B(I;A,P)$. Suppose $\tilde{\varphi}:[0,b]\xrightarrow{\sim}[0,b]$ is an autohomeomorphism that induces, by conjugation, an automorphism α of G. Then the following are true*:

(i) *if $\tilde{\varphi}$ is increasing, differentiable at 0, and $\tilde{\varphi}'(0)>0$, then α fixes σ_ℓ;*

(ii) *if $\tilde{\varphi}$ is increasing, differentiable at b, with $\tilde{\varphi}'(b)>0$, then α fixes σ_r;*

(iii) *if $\tilde{\varphi}$ is differentiable both at 0 and at b, with nonzero derivatives, then α fixes the homomorphism $\psi: g\mapsto \sigma_\ell(g)\cdot\sigma_r(g)$.*

Proof. (i) and (ii) Suppose the extended autohomeomorphism $\tilde{\varphi}=\tilde{\varphi}_\alpha$ is increasing and fix $g\in G$. If $\tilde{\varphi}$ is differentiable at 0 and $\tilde{\varphi}'(0)>0$, the chain rule justifies the computation

$$\sigma_\ell(\alpha(g))=(\tilde{\varphi}\circ g\circ\tilde{\varphi}^{-1})'(0)=\tilde{\varphi}'(0)\cdot g'(0)\cdot(\tilde{\varphi}^{-1})'(0)=\sigma_\ell(g). \tag{3-1}$$

It follows that σ_ℓ is fixed by α. If $\tilde{\varphi}$ admits a left derivative at b and if $\tilde{\varphi}'(b)>0$, one sees similarly, that σ_r is fixed by α.

(iii) Assume now that $\tilde{\varphi}=\tilde{\varphi}_\alpha$ is differentiable, both at 0 and at b, and that both derivatives are different from 0. If $\tilde{\varphi}$ is *increasing*, parts (i) and (ii) guarantee that σ_ℓ and σ_r are fixed by α, whence so is their product ψ. If, on the other hand, $\tilde{\varphi}$ is *decreasing*, the calculation

$$\sigma_r(\alpha(g))=(\tilde{\varphi}\circ g\circ\tilde{\varphi}^{-1})'(b)=\tilde{\varphi}'(0)\cdot g'(0)\cdot(\tilde{\varphi}^{-1})'(b)=\sigma_\ell(g) \tag{3-2}$$

holds for every $g\in G$ and establishes the relation $\sigma_r\circ\alpha=\sigma_\ell$.

A similar calculation shows that the relation $\sigma_\ell\circ\alpha=\sigma_r$ is valid. The claim for ψ is then a consequence of the computation

$$(\psi\circ\alpha)(g)=\sigma_\ell(\alpha(g))\cdot\sigma_r(\alpha(g))=\sigma_r(g)\cdot\sigma_\ell(g)=\psi(g). \qquad \square$$

3A1. *Statement and proof of the criterion.* We now come to the criterion; we choose a formulation that is slightly more general than what is needed for the case at hand; the extended version will be used in Section 4.

Proposition 3.2. *Suppose I is an interval of one of the forms $[0,b]$ or $[0,\infty[$, and G as well as $\overline{G}$ are subgroups of $G(I;A,P)$ that contain $B(I;A,P)$. Assume $\tilde{\varphi}:I\xrightarrow{\sim}I$ is an* increasing *autohomeomorphism that induces, by conjugation, an isomorphism α of the group G onto the group $\overline{G}$.*

If the image of $\sigma_\ell: G\to P$ is not cyclic, *then $\tilde{\varphi}$ is linear on a small interval of the form $[0,\delta]$ and so $\tilde{\varphi}$ is differentiable at 0 with positive derivative.*

Proof. The following argument uses ideas from the proofs of Proposition E16.8 and Supplement E17.3 in [Bieri and Strebel 2016]. The proof will be divided into three parts. In the first one, we show that $\alpha : G \xrightarrow{\sim} \overline{G}$ induces an isomorphism $\alpha_\ell : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\bar{\sigma}_\ell$, that takes $p \in \operatorname{im}\sigma_\ell$ to $p^r = \mathrm{e}^{r\cdot\log p}$ for some positive real number r that does *not* depend on p. In the second part, we establish that $\tilde{\varphi}$ satisfies the relation

$$\tilde{\varphi}(p\cdot t) = p^r\cdot\tilde{\varphi}(t) \tag{3-3}$$

for every $p \in (\operatorname{im}\sigma_\ell \cap]0,1[)$ and t varying in some small interval $[0,\delta]$. In the last part, we deduce from this relation that $\tilde{\varphi}$ is linear near 0.

We now embark on the first part. Since φ is increasing, Corollary 2.7 applies and shows that α maps the kernel of $\sigma_\ell : G \to P$ onto the kernel of the homomorphism $\bar{\sigma}_\ell : \overline{G} \to P$, and thus induces an isomorphism $\alpha_\ell : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\bar{\sigma}_\ell$ that renders the square

$$\begin{array}{ccc} G & \xrightarrow{\ \alpha\ } & \overline{G} \\ \big\downarrow{\scriptstyle \sigma_\ell} & & \big\downarrow{\scriptstyle \bar{\sigma}_\ell} \\ \operatorname{im}\sigma_\ell & \xrightarrow{\ \alpha_\ell\ } & \operatorname{im}\bar{\sigma}_\ell \end{array}$$

commutative. We claim α_ℓ maps the set $(\operatorname{im}\sigma_\ell)\cap]0,1[$ onto $(\operatorname{im}\bar{\sigma}_\ell)\cap]0,1[$. Indeed, let $p \in \operatorname{im}\sigma_\ell$ be a slope with $p < 1$ and let $f_p \in G$ be a preimage of p. Then $\alpha(f_p)$ is linear on some interval $[0,\varepsilon_p]$ and has slope $\bar{\sigma}_\ell(\alpha(f_p)) = \alpha_\ell(p)$ there. Since $\tilde{\varphi}$ is continuous at 0, there exists $\delta_p > 0$ so that f_p is linear on $[0,\delta_p]$ and that $\tilde{\varphi}([0,\delta_p]) \subseteq [0,\varepsilon_p]$. Fix $t \in [0,\delta_p]$. The hypothesis that α is induced by conjugation by $\tilde{\varphi}$ then leads to the chain of equalities

$$\tilde{\varphi}(p\cdot t) = (\tilde{\varphi}\circ f_p)(t) = (\alpha(f_p)\circ\tilde{\varphi})(t) = \alpha(f_p)(\tilde{\varphi}(t)) = \alpha_\ell(p)\cdot\tilde{\varphi}(t). \tag{3-4}$$

Since $\tilde{\varphi}$ is increasing and as $p < 1$, the chain of equalities implies that $\alpha_\ell(p) < 1$. It follows that α_ℓ maps $(\operatorname{im}\sigma_\ell)\cap]0,1[$ into $\operatorname{im}\bar{\sigma}_\ell\cap]0,1[$ and then, by applying the preceding argument to φ^{-1}, that

$$\alpha_\ell(\operatorname{im}\sigma_\ell \cap]0,1[) = \operatorname{im}\bar{\sigma}_\ell \cap]0,1[.$$

We show next that $\alpha_\ell(p) = p^r$ for all $p \in \operatorname{im}\sigma_\ell$ and some positive real number r. We begin by passing from the multiplicative subgroup $\operatorname{im}\sigma_\ell \subset \mathbb{R}^\times_{>0}$ to a subgroup of $\mathbb{R}_{\mathrm{add}}$; to that end, we introduce the homomorphism

$$L_0 = \ln\circ\alpha_\ell\circ\exp : \ln(\operatorname{im}\sigma_\ell) \xrightarrow{\sim} \ln(\operatorname{im}\bar{\sigma}_\ell).$$

The previous verification implies that L_0 is an order preserving isomorphism; by the assumption on $\operatorname{im}\sigma_\ell$ the domain of L_0 is a dense subgroup of $\mathbb{R}_{\mathrm{add}}$. It follows that L_0 extends uniquely to an order preserving automorphism $L : \mathbb{R}_{\mathrm{add}} \to \mathbb{R}_{\mathrm{add}}$.

The homomorphism L is continuous, hence linear, and so given by multiplication by some positive real number r. The isomorphism α_ℓ has therefore the form

$$p \mapsto p^r = \exp(r \cdot \ln p) \quad \text{with } r > 0.$$

We come now to the second part of the proof. Fix a slope $p_1 < 1$ in $\operatorname{im}\sigma_\ell$. Formula (3-4) and the previously found formula for α_ℓ then imply that there exists a small positive number δ_{p_1} such that the equation

$$\tilde{\varphi}(p_1 \cdot t) = p_1^r \cdot \tilde{\varphi}(t) \tag{3-5}$$

holds for every $t \in [0, \delta_{p_1}]$. Consider next another slope $p < 1$. There exists then, as before, a real number $\delta_p > 0$ so that $\tilde{\varphi}(p \cdot t) = p^r \cdot \tilde{\varphi}(t)$ for $t \in [0, \delta_p]$. Choose now $m \in \mathbb{N}$ so large that $p_1^m \cdot \delta_{p_1} \le \delta_p$. The following chain of equalities then holds for each $t \in [0, \delta_{p_1}]$:

$$p_1^{m\cdot r} \cdot \tilde{\varphi}(p \cdot t) = \tilde{\varphi}(p_1^m \cdot p \cdot t) = \tilde{\varphi}(p \cdot p_1^m \cdot t) = p^r \cdot \tilde{\varphi}(p_1^m \cdot t) = p^r \cdot p_1^{m\cdot r} \cdot \tilde{\varphi}(t).$$

The calculation shows that $\tilde{\varphi}(p \cdot t) = p^r \cdot \tilde{\varphi}(t)$ for every $t \in [0, \delta_1]$. Upon setting $\delta = \delta_{p_1}$ one arrives at (3-3).

The proof is now quickly completed. By assumption, $\operatorname{im}\sigma_\ell$ is not cyclic and so (3-3) holds for a dense set of slopes p and a fixed argument t, say $t = \delta$. Since φ is continuous and increasing, (3-3) continues to hold for every real $x \in]0, 1[$. The formula

$$\varphi(x \cdot \delta) = \exp(r \cdot \ln x) \cdot \varphi(\delta) = x^r \cdot \varphi(\delta)$$

is therefore valid for every $x \in]0, \delta]$. By Theorem 2.8, on the other hand, φ is piecewise linear on $]0, \delta]$. So the exponent r must be equal to 1, whence φ is linear on $[0, \delta]$ with slope $\varphi(\delta)/\delta > 0$ and so, in particular, differentiable at 0. □

Remark 3.3. Assume I is a compact interval of the form $[0, b]$ with $b \in A_{>0}$ and the images of σ_ℓ and σ_r are both not cyclic. It follows then from Proposition 3.2 that every *increasing* automorphism $\alpha : G \xrightarrow{\sim} G$ is induced by an autohomeomorphism $\tilde{\varphi}$ that is affine near both end points. By [Bieri and Strebel 2016, Proposition E16.9] the homeomorphism $\tilde{\varphi}$ is thus *finitary* piecewise linear.

3A2. *First application.* As a further step towards the main results we give a corollary that combines Propositions 3.1 and 3.2.

Corollary 3.4. *Let G be a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Assume $I = [0, b]$ and let α be an automorphism of G that is induced by the autohomeomorphism $\tilde{\varphi} : I \xrightarrow{\sim} I$. Then the following statements hold*:

(i) *if α is increasing*[7] *and* $\operatorname{im}\sigma_\ell$ *not cyclic, then σ_ℓ is fixed by α*;

[7] See Definition 2.5.

(ii) *if α is increasing and* $\operatorname{im}\sigma_r$ *not cyclic, then σ_r is fixed by α*;

(iii) *if $\tilde{\varphi}$ is decreasing and* $\operatorname{im}\sigma_\ell$ *is not cyclic, then $\tilde{\varphi}$ is affine near both end points and the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is fixed by α.*

Proof. (i) The statement is a direct consequence of Proposition 3.2 and part (i) of Proposition 3.1.

(ii) We invoke Proposition 3.2 for an auxiliary group G_1. Let $\vartheta : I \xrightarrow{\sim} I$ be the reflection in the midpoint of I; set $G_1 = \vartheta \circ G \circ \vartheta^{-1}$ and $\varphi_1 = \vartheta \circ \tilde{\varphi}_\alpha \circ \vartheta^{-1}$. Since $G(I; A, P)$ and $B(I; A, P)$ are both invariant under conjugation by ϑ, and as the image of σ_r is not cyclic, Proposition 3.2 applies to the couple (G_1, φ_1) and shows that φ_1 is linear in a small interval $[0, \delta_1]$ of positive length. But if so, φ_α is affine in the interval $[b - \delta_1, b]$. Now use part (ii) in Proposition 3.1.

(iii) Since $\tilde{\varphi}$ is *decreasing*, the subgroups $\operatorname{im}\sigma_\ell$ and $\operatorname{im}\sigma_r$ are isomorphic by Lemma 3.6 below; the hypothesis on $\operatorname{im}\sigma_\ell$ therefore implies the image of σ_r is not cyclic either. Let $\vartheta : \operatorname{int}(I) \xrightarrow{\sim} \operatorname{int}(I)$ be the reflection in the midpoint of the interval I and set $\tilde{\varphi}_1 = \vartheta \circ \tilde{\varphi}$ and $\overline{G} = \vartheta \circ G \circ \vartheta^{-1}$. Conjugation by $\tilde{\varphi}_1$ induces then an increasing isomorphism $\alpha_1 : G \xrightarrow{\sim} \overline{G}$. Since $G(I; A, P)$ and $B(I; A, P)$ are both invariant under conjugation by ϑ, Proposition 3.2 applies to $\tilde{\varphi}_1$ in the rôle of $\tilde{\varphi}$ and shows that $\tilde{\varphi}_1$ is linear near 0. But if so, $\tilde{\varphi}$ is linear near 0. Consider now the autohomeomorphism $\tilde{\varphi}_2 = \tilde{\varphi} \circ \vartheta$ of I. It induces an isomorphism $\alpha_2 : \overline{G} \xrightarrow{\sim} G$ by conjugation; an argument similar to the preceding one then reveals that $\tilde{\varphi}$ is affine near b. The remainder of the claim follows from part (iii) in Proposition 3.1. □

3B. ***Construction of homomorphisms fixed by*** $\mathbf{Aut_+}\,G$**.** The first main result holds for all groups G with $B(I; A, P) \subsetneq G \subseteq G(I; A, P)$, but the exhibited homomorphisms may only be fixed by $\operatorname{Aut}_+ G$.

Theorem 3.5. *Suppose $I = [0, b]$ with $b \in A_{>0}$ and G is a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$ properly. Then the homomorphisms σ_ℓ and σ_r are fixed by* $\operatorname{Aut}_+ G$*, and at least one of them is nontrivial.*

Proof. Let α be an increasing automorphism of G and let $\tilde{\varphi}$ be the autohomeomorphism of I that induces α. (The map $\tilde{\varphi}$ exists by Theorem 2.3 and Remark 2.4(a).) Since the quotient group $G(I; A, P)/B(I; A, P)$ is isomorphic to the image of $\sigma_\ell \times \sigma_r : G(I; A, P) \to P \times P$ and as G contains $B(I; A, P)$ properly, at least one of the homomorphisms σ_ℓ and σ_r is nonzero.

Assume first that $\psi = \sigma_\ell$ is nonzero. Two cases then arise, depending on whether the image of ψ is cyclic or not. If the image of ψ is *not cyclic* then part (i) in Corollary 3.4 shows that α fixes ψ. If, on the other hand, ψ is cyclic, consider the generator $p \in \operatorname{im}\psi$ with $p < 1$ and pick a preimage $g_p \in G$ of p. Then g_p attracts points in every sufficiently small interval of the form $[0, \delta]$ towards 0; hence so does

$\alpha(g_p) = \tilde{\varphi} \circ g_p \circ \tilde{\varphi}^{-1}$ and thus $p' = (\alpha(g_p))'(0) < 1$. Now p' generates also $\operatorname{im}\sigma_\ell$; being smaller than 1, it therefore coincides with $p = \psi(g_p)$ and so $\psi = \psi \circ \alpha$.

Assume next that $\psi = \sigma_r$ is not zero. If its image is not cyclic, part (ii) of Corollary 3.4 allows us to conclude that α fixes ψ. If $\operatorname{im}\psi$ is cyclic, consider the generator $p \in \operatorname{im}\psi$ with $p < 1$ and pick a preimage $g_p \in G$. Then g_p attracts points in every sufficiently small interval $[b-\delta, b]$ towards b. It then follows, as before, that $\psi(\alpha(g_p)) = p = \psi(g_p)$, whence $\psi \circ \alpha = \alpha$. □

3C. ***Existence of decreasing automorphisms.*** Theorem 3.5 is very satisfactory in that it produces a nonzero homomorphism ψ onto an infinite abelian group whenever such a homomorphism is likely to exist, i.e., if G contains $B(I; A, P)$ properly. This homomorphism is, however, only guaranteed to be fixed by the subgroup $\operatorname{Aut}_+ G$ of $\operatorname{Aut} G$ which has index 1 or 2 in $\operatorname{Aut} G$. If the index is 1, the conclusion of Theorem 3.5 is as good as we can hope for. So the question arises whether there are useful criteria that force the index to be 1. Here is a very simple observation that leads to such a criterion:

Lemma 3.6. *Assume $I = [0, b]$ with $b \in A_{>0}$ and let G be a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\sigma_r$.*

Proof. The kernel of σ_ℓ consists of all elements in G that are the identity near 0. Since α is induced by conjugation by a homeomorphism of I that maps 0 onto b, the image of $\ker\sigma_\ell$ consists of elements that are the identity near b, so $\alpha(\ker\sigma_\ell) \subseteq \ker\sigma_r$. Since α^{-1} is also a decreasing automorphism, the preceding inclusion is actually an equality. So α induces an isomorphism $\alpha_* : \operatorname{im}\sigma_\ell \xrightarrow{\sim} \operatorname{im}\sigma_r$ that renders the square

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ \downarrow{\scriptstyle \sigma_\ell} & & \downarrow{\scriptstyle \sigma_r} \\ \operatorname{im}\sigma_\ell & \xrightarrow{\alpha_*} & \operatorname{im}\sigma_r \end{array} \tag{3-6}$$

commutative. □

Example 3.7. Suppose the slope group P is finitely generated and hence free abelian of finite rank r, say. Choose subgroups Q_ℓ and Q_r of P and set

$$G(Q_\ell, Q_r) = \{g \in G(I; A, P) \mid (\sigma_\ell(g), \sigma_r(g) \in Q_\ell \times Q_r\}. \tag{3-7}$$

Then $\operatorname{im}\sigma_\ell = Q_\ell$ and $\operatorname{im}\sigma_r = Q_r$, and the image of $(\sigma_\ell, \sigma_r) : G \to P \times P$ coincides with $Q_\ell \times Q_r$. (These claims follow from [Bieri and Strebel 2016, Corollary A5.5]).

Now assume that $G(Q_\ell, Q_r)$ admits a decreasing automorphism, say α. By Lemma 3.6 the groups Q_ℓ and Q_r are then isomorphic, and thus have the same rank. But more is true: if $Q_\ell = \operatorname{im}\sigma_\ell$ is *not* cyclic, then Proposition 3.2 and the last line of Proposition 3.1 show that $\sigma_r = \sigma_\ell \circ \alpha$, whence Q_r, the image of σ_r,

coincides with Q_ℓ, the image of σ_ℓ. The same conclusion holds if Q_r is not cyclic. Conversely, if $Q_\ell = Q_r$ then $G(Q_\ell, Q_r)$ admits decreasing automorphisms, for instance the automorphism induced by conjugation by the reflection about the mid point of I.

So the only case where the existence of a decreasing automorphism is neither obvious nor easy to rule out by the preceding arguments is that where Q_ℓ and Q_r are both cyclic, but distinct. We shall come back to this exceptional case in Example 3.13.

3D. ***Construction of a homomorphism fixed by*** **Aut** ***G.*** We move on to the construction of a homomorphism fixed by all of $\operatorname{Aut} G$. The following result is our main result.

Theorem 3.8. *Suppose I is a compact interval of the form $[0, b]$ with $b \in A_{>0}$. Let G be a subgroup of $G(I; A, P)$ containing $B(I; A, P)$ and let $\psi : G \to P$ be the homomorphism $g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$. Then ψ is fixed by* $\operatorname{Aut} G$*, except possibly when G satisfies the following three conditions*:

(a) $\operatorname{im}(\sigma_\ell : G \to P)$ *is cyclic*,

(b) G *admits a decreasing automorphism*,

(c) G *does not admit a decreasing automorphism induced by an autohomeomorphism $\vartheta : I \xrightarrow{\sim} I$ that is differentiable at both end points with nonzero values.*

Proof. Let α be an automorphism of G and let φ be the autohomeomorphism of $\operatorname{int}(I)$ that induces α by conjugation. If φ is *increasing* both σ_ℓ and σ_r are fixed by α (see Theorem 3.5) and hence so is ψ. If, on the other hand, α is *decreasing* and the image of σ_ℓ is *not cyclic* then part (iii) of Corollary 3.4 yields the desired conclusion.

Now suppose that G admits an automorphism β that is induced by a decreasing autohomeomorphism $\tilde{\varphi}_\beta$ of I that is differentiable at 0, as well as at b, and has there nonzero derivatives. Then part (iii) of Proposition 3.1 allows us to conclude that ψ is fixed by β. Since β represents the coset $\operatorname{Aut} G \setminus \operatorname{Aut}_+ G$ and as ψ is fixed by $\operatorname{Aut}_+ G$, it follows that ψ is fixed by every decreasing automorphism.

All taken together we have proved that the automorphism α fixes ψ except, possibly, if $\operatorname{im} \sigma_\ell$ is cyclic, α is decreasing and if there does not exists a decreasing automorphism β that is differentiable at the end points and has there nonzero derivatives. □

We state next some consequences of Theorems 3.5 and 3.8. We begin with the special case where G is all of $G(I; A, P)$. Then G is normalized by the reflection in the midpoint of I and so Theorem 3.8 leads to

Corollary 3.9. *If G coincides with $G([0, b]; A, P)$ the homomorphism $\psi : G \to P$ taking $g \in G$ to $\sigma_\ell(g) \cdot \sigma_r(g)$ is surjective, hence nonzero, and fixed by* $\operatorname{Aut} G$.

The second result is a consequence of the proof of Theorem 3.5.

Corollary 3.10. *Suppose that I is the half-line $[0, \infty[$ and G is a subgroup of $G(I; A, P)$ containing $B(I; A, P)$. If G does not admit a decreasing automorphism then $\psi = \sigma_\ell$ is fixed by* Aut G.

Proof. The claim follows from Proposition 3.2 and from the proof of part (i) in Proposition 3.1 upon noting that the cited proof does not presuppose that the interval I be bounded from above. □

3E. ***Some examples.*** We exhibit some specimens of groups G that possess a homomorphism $\psi : G \to P$ fixed by Aut G. The existence of ψ will be established by recourse to Theorems 3.5 and 3.8 and to Corollary 3.9.

Example 3.11. We begin with variations on Thompson's group F. Assume P is infinite cyclic and A is a (nontrivial) $\mathbb{Z}[P]$-submodule of $\mathbb{R}$. Set $G_0 = G([0, b]; A, P)$ with $b \in A_{>0}$ and consider the following subgroups of G_0:

$$G_1 = \{g \in G_0 \mid \sigma_\ell(g) = 1\}, \tag{3-8}$$

$$G_2 = \{g \in G_0 \mid \sigma_\ell(g) = \sigma_r(g)\}, \tag{3-9}$$

$$G_3 = \{g \in G_0 \mid \sigma_\ell(g) = \sigma_r(g)^{-1}\}. \tag{3-10}$$

The group G_0 is the entire group $G(I; A, P)$ and so Corollary 3.9 tells us that the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is nonzero and fixed by Aut G_0.

The group G_1 is an ascending union of subgroups $H_n = G([a_n, b]; A, P)$ given by a strictly decreasing sequence $n \mapsto a_n$ of elements in A that converges to 0, and so the group G_1 is infinitely generated. It does not admit a decreasing automorphism (for instance because of Lemma 3.6) and so Theorem 3.5 allows us to infer that the epimorphism $\sigma_r : G_1 \twoheadrightarrow P$ is fixed by all of Aut G_1.

The group G_2 is an ascending HNN-extension with a base group that is isomorphic to G_0 (see [Bieri and Strebel 2016, Lemma E18.8]). If G_0 is finitely generated or finitely presented, so is therefore G_2. The group is normalized by the reflection in the midpoint of I and so Theorem 3.8 implies that $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is fixed by Aut G_2. This homomorphism ψ is nonzero, for it coincides with σ_ℓ^2. (Actually, σ_ℓ and σ_r are also fixed by Aut G_2.)

Now to the group G_3. It differs from G_2 in several respects: it cannot be written as an ascending HNN-extension with a finitely generated base group contained in $B(I; A, P)$; it is finitely generated if G_0 is so, but, if finitely generated, it does not admit a finite presentation (see part (ii) of Lemma E18.8 and Remark E18.10 in [Bieri and Strebel 2016]). The group G_3 is normalized by the reflection in the mid point of I and so $\psi : G_3 \to P$ is fixed by Aut G_3; this conclusion, however, is of no interest as ψ is the zero map. Actually, more is true: every homomorphism $\psi' : G_3 \to P$ fixed by ρ and vanishing on the bounded subgroup B_3 of G_3 is the

zero-map: by definition (3-10) the group G_3/B_3 is infinite cyclic and so ψ' must be a multiple of σ_ℓ.

Remark 3.12. The previous discussion shows that G_0, G_1 and G_2 admit nontrivial homomorphisms into P that are fixed by the corresponding automorphism groups. This fact and the observation made in Section 1A imply that every automorphism of one of these groups has infinitely many corresponding twisted conjugacy classes. This reasoning does not hold for G_3, for $\psi : G_3 \to P$ is the zero homomorphism.

So the question whether or not an automorphism α of G_3 has infinitely many twisted conjugacy classes has to be tackled by another approach. Note first that the homomorphisms σ_ℓ and σ_r are both nonzero; as G_3 satisfies the assumptions of Theorem 3.5 these homomorphisms are therefore fixed by $\mathrm{Aut}_+ G_3$. It follows that every increasing automorphism α of G_3 has infinitely many α-twisted conjugacy classes. We are thus left with the coset of decreasing automorphisms of G_3.

Consider, for example, the automorphism β induced by conjugation by the reflection ϑ in the midpoint of the interval I. Our aim is to construct an infinite collection of elements $g_n \in G_3$ and to verify then that they represent pairwise distinct β-twisted conjugacy classes. This verification will be based on the fact that β has order 2 and a connection between twisted and ordinary conjugacy classes, available for automorphisms of finite order.[8]

Let f and g be elements of G_3 that lie in the same β-twisted conjugacy class. By definition, there exists then $h \in G_3$ that satisfies the equation $g = h \circ f \circ \beta(h^{-1})$. The calculation

$$\begin{aligned} g \circ \beta(g) &= \big(h \circ f \circ \beta(h^{-1})\big) \circ \beta\big(h \circ f \circ \beta(h^{-1})\big) \\ &= h \circ (f \circ \beta(f)) \circ \beta^2(h^{-1}) = {}^h(f \circ \beta(f)) \end{aligned}$$

shows then that the elements $f \circ \beta(f)$ and $g \circ \beta(g)$ are conjugate. It suffices therefore to find a sequence of elements $n \mapsto f_n$ with the property that the compositions $f_{n_1} \circ \beta(f_{n_1})$ and $f_{n_2} \circ \beta(f_{n_2})$ represent distinct conjugacy classes whenever $n_1 \neq n_2$.

To obtain such a sequence, we use the fact that G contains $B(I; A, P)$ and that $B(I; A, P)$ consists of all PL-homeomorphisms with slopes in P, breakpoints in the dense subgroup A, and which are the identity near the end points. For every positive integer n there exists therefore a nontrivial element $f_n \in B(I; A, P)$ whose support has n connected components, all contained in the interval $]0, b/2[$. Then

$$h_n = f_n \circ \beta(f_n) = f_n \circ (\vartheta \circ f_n \circ \vartheta^{-1})$$

has $2n$ connected components, so h_{n_1} is not conjugate to h_{n_2} for $n_1 \neq n_2$. It follows that G_3 has infinitely many β-twisted conjugacy classes.

[8] Compare with [Gonçalves and Sankaran 2016, Lemma 2.3].

The previous reasoning allows of some improvements, but it does not seem powerful enough to establish that G_3 has infinitely many α-twisted conjugacy classes for every decreasing automorphism α of G_3.

Example 3.13. Example 3.11 admits a generalization that is worth being brought to the attention of the reader. Assume P is a nontrivial subgroup of the positive reals, A is a (nontrivial) P-submodule of $\mathbb{R}$, and ν is an endomorphism of P. Now fix $b \in A_{>0}$, set $I = [0, b]$, and define

$$G_\nu = \{g \in G([0, b]; A, P) \mid \sigma_r(g) = \nu(\sigma_\ell(g))\}. \tag{3-11}$$

We are interested in finding a nonzero homomorphism $\psi : G_\nu \to P$ that is fixed by $\operatorname{Aut} G_\nu$. Theorem 3.8 implies that the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is fixed by $\operatorname{Aut} G_\nu$ whenever P is not cyclic; this homomorphism is nonzero unless ν is the map that sends $p \in P$ to its inverse p^{-1}.

Assume now that P is cyclic. Then G_ν is isomorphic to one of the groups G_1, G_2, or G_3 discussed in Example 3.11. This claim is clear if ν is the zero map, for G_ν coincides then with $\ker \sigma_r$ and is therefore isomorphic to G_1. Assume now that ν is not zero. The quotient group $G_\nu/B(I; A, P)$ is then an infinite cyclic subgroup of the quotient group $G(I; A, P)/B(I; A, P)$ which is free abelian group of rank 2. By the classification in Section 18.4b of [Bieri and Strebel 2016], the group G_ν is therefore isomorphic, either to G_2 or to G_3. Since the isomorphism $G_\nu \xrightarrow{\sim} G_2$, respectively $G_\nu \xrightarrow{\sim} G_3$, is induced by conjugation by an autohomeomorphism of $]0, b[$ and as conjugation by the reflection in $b/2$ induces decreasing automorphisms in G_2 and in G_3, *the group G_ν admits a decreasing automorphism, say β*; it induces an isomorphism $\beta_* : \operatorname{im} \sigma_\ell \xrightarrow{\sim} \operatorname{im} \sigma_r$ (see Lemma 3.6). Our next aim is to obtain a formula for β_*.

The definition of G_ν shows, first of all, that $\operatorname{im} \sigma_\ell = P$ and that $\operatorname{im} \sigma_r = \nu(P)$. Let p be the generator of P with $p < 1$. Then $\nu(p) = p^m$ for some nonzero integer m (recall that ν is not the zero map). Pick an element $g_p \in G_\nu$ with $\sigma_\ell(g_p) = p$. Then 0 is an attracting fixed point of g_p restricted to a sufficiently small interval of the form $[0, \delta]$, and hence b is an attracting fixed point for the restriction of $\beta(g_p)$ to a sufficiently small interval of the form $[b - \varepsilon, b]$. Thus $\beta(g_p) < 1$. Since $\beta(g_p)$ generates $\operatorname{im} \sigma_r = \nu(P) = \operatorname{gp}(p^m)$ it follows that β_* is given by the formula

$$\beta_* : P \to P, \quad p \mapsto p^{|m|}. \tag{3-12}$$

Consider now the commutative square (3-6), with α replaced by β. It shows that

$$(\sigma_r \circ \beta)(g_p) = \beta_*(\sigma_\ell(g_p)) = \beta_*(p) = p^{|m|} = (\sigma_\ell(g_p))^{|m|} \tag{3-13}$$

and so $\sigma_r \circ \beta = \sigma_\ell^{|m|}$. The preceding reasoning is also valid with β^{-1} in place of β, for β^{-1} is also a decreasing automorphism of G_ν, and so the relation $\sigma_r \circ \beta^{-1} = \sigma_\ell^{|m|}$ holds, hence also the relation $\sigma_\ell^{|m|} \circ \beta = \sigma_r$.

Consider next the homomorphism $\psi : G_\nu \to P$ that takes g to $\sigma_\ell(g)^{|m|} \cdot \sigma_r(g)$. The calculation

$$(\psi \circ \beta)(g) = \sigma_\ell^{|m|}(\beta(g)) \cdot \sigma_r(\beta(g)) = \sigma_r(g) \cdot \sigma_\ell^{|m|}(g) = \psi(g)$$

shows then that ψ is fixed by β. Note, however, that ψ is the zero homomorphism whenever m is negative, for in this case the definition of G_ν implies that

$$\psi(g) = (\sigma_\ell(g))^{|m|} \cdot \sigma_r(g) = (\sigma_\ell(g))^{|m|} \cdot (\sigma_\ell(g))^m = 1$$

for every $g \in G_\nu$, just as it happens with G_3 in Example 3.11.

Remark 3.14. Suppose P is cyclic and $\nu : P \to P$ is neither the identity nor the passage to the inverse. Then G_ν admits decreasing automorphisms β, but none of them can be induced by an autohomeomorphism $\tilde{\varphi} : I \xrightarrow{\sim} I$ that is differentiable at the end points; indeed, (3-13) shows that $\sigma_r \circ \beta \neq \sigma_\ell$, in contrast to what happens if the chain rule can be applied (see Proposition 3.1). It follows, in particular, that the three conditions (a), (b), and (c) stated in Theorem 3.8 can occur simultaneously.

4. Characters fixed by Aut $G([0, \infty[; A, P)$

The results in this section differ from those of Section 3 in two important respects: in many situations several candidates for $\psi : G \to P$ are available and one of the candidates may not be fixed by $\mathrm{Aut}_+ G$.

4A. ***Existence of decreasing automorphisms.*** Every compact interval of the form $[0, b]$, and also the line, is invariant under a reflection. It follows that the groups $G(I; A, P)$ with I one of these intervals, but also many of their subgroups, admit decreasing automorphisms. The case where I is a half-line, say $[0, \infty[$, is different: then $G([0, \infty[; A, P)$ does not admit a decreasing automorphism.

In this section, we first justify this claim and discuss then the extent to which it continues to be valid for subgroups of $G([0, \infty[; A, P)$. We begin with an analogue of Lemma 3.6 in which the homomorphism σ_r is replaced by the homomorphism ρ defined in (2-4).

Lemma 4.1. *Assume I is the half-line $[0, \infty[$ and G is a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \mathrm{im}\, \sigma_\ell \xrightarrow{\sim} \mathrm{im}\, \rho$.*

Proof. The proof is very similar to that of Lemma 3.6. The kernel of σ_ℓ consists of all elements in G that are the identity near 0, while the kernel of ρ is made up of the elements in G that are the identity near ∞. Since α is induced by conjugation by a

decreasing homeomorphism of $]0, \infty[$, the image of $\ker \sigma_\ell$ consists of elements $\alpha(g)$ that are the identity on a half-line of the form $[t(g), \infty[$, and so $\alpha(\ker \sigma_\ell) \subseteq \ker \rho$. Since α^{-1} is also a decreasing automorphism, the preceding inclusion is actually an equality. It follows that α induces an isomorphism $\alpha_* : \operatorname{im} \sigma_\ell \xrightarrow{\sim} \operatorname{im} \rho$ that renders the square

(4-1)
$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ \downarrow{\scriptstyle \sigma_\ell} & & \downarrow{\scriptstyle \rho} \\ \operatorname{im} \sigma_\ell & \xrightarrow{\alpha_*} & \operatorname{im} \rho \end{array}$$

commutative. □

The preceding lemma leads directly to a criterion for the nonexistence of decreasing automorphisms. Indeed, the image of σ_ℓ is abelian, while that of ρ is often a nonabelian, metabelian group, and so we obtain

Criterion 4.2. Assume I is the half-line $[0, \infty[$ and G is a subgroup of $G(I; A, P)$ that contains $B(I; A, P)$. If $\operatorname{im} \rho$ is *not* abelian then $\operatorname{Aut} G = \operatorname{Aut}_+ G$.

4B. ***Construction of homomorphisms: part I.*** We turn now to the construction of homomorphisms fixed by $\operatorname{Aut}_+ G$, or even by $\operatorname{Aut} G$. Several homomorphisms are at our disposal. The first of them is σ_ℓ. Corollary 3.10 tells us then:

Proposition 4.3. *Assume that G is a subgroup of $G([0, \infty[; A, P)$ containing $B(I; A, P)$. Then the homomorphisms σ_ℓ is fixed by* $\operatorname{Aut}_+ G$.

We move on to the homomorphism ρ. Here two cases arise, depending on whether its image is abelian or nonabelian. In the second case, a very satisfying conclusion holds. It is enunciated in

Theorem 4.4. *Assume $I = [0, \infty[$ and G is a subgroup of $G(I; A, P)$ containing $B(I; A, P)$. If $\operatorname{im} \rho$ is not abelian, σ_r is a nonzero homomorphism fixed by* $\operatorname{Aut} G$.

Proof. Suppose $\operatorname{im} \rho$ is nonabelian. Then Lemma 4.1 forces α to be increasing. Let $\varphi :]0, \infty[\xrightarrow{\sim}]0, \infty[$ be the autohomeomorphism that induces α by conjugation. As it is increasing, it is affine near ∞ by Proposition 4.5 below, and so the following calculation

$$\begin{aligned} \sigma_r(\alpha(g)) &= \lim_{t\to\infty} \big(\varphi \circ g \circ \varphi^{-1}\big)'(t) \\ &= \lim_{t\to\infty} \big(\varphi'(g \circ \varphi^{-1}(t)) \cdot g'(\varphi^{-1}(t)) \cdot (\varphi^{-1})'(t)\big) \\ &= \lim_{t\to\infty} \big(\varphi'(t) \cdot g'(t) \cdot (\varphi^{-1})'(t)\big) \\ &= \lim_{t\to\infty} g'(t) = \sigma_r(g) \end{aligned}$$

is valid for every $g \in G$, which shows that α fixes the homomorphism σ_r. This homomorphism is nonzero. Indeed, $G/\ker \rho \xrightarrow{\sim} \operatorname{im} \rho$ is not abelian by hypothesis,

while $\ker\sigma_r/\ker\rho$ is abelian and thus the third term of the extension

$$\ker\sigma_r/\ker\rho \rightarrowtail G/\ker\rho \twoheadrightarrow G/\ker\sigma_r$$

is not zero, whence $\ker\sigma_r \neq G$. □

We are left with proving an analogue of Proposition 3.2. For later use, we state it in greater generality than needed at this point, namely as

Proposition 4.5. *Assume G and $\bar{G}$ are subgroups of $G(I; A, P)$, both containing the subgroup $B(I; A, P)$, and that I is either the half-line $[0, \infty[$ or the line $\mathbb{R}$. Let $\alpha : G \xrightarrow{\sim} \bar{G}$ be an isomorphism and let φ_α be an autohomeomorphism of $\mathrm{int}(I)$ that induces α by conjugation.*

If $\mathrm{im}\,\rho$ is not abelian and φ_α is increasing then φ_α is affine near ∞.

Proof. We adapt the argument of Part 2 in the proof of [Bieri and Strebel 2016, Supplement E17.3] to the case at hand. By assumption, the image of

$$\rho_* : G \to \mathrm{Aff}(A, P) \xrightarrow{\sim} A \rtimes P$$

is not abelian; its derived group is therefore (isomorphic to) a nontrivial submodule A_1 of A which, being nontrivial, contains arbitrary small positive elements and so is dense in $\mathbb{R}$. Let $\bar{\rho}_* : \bar{G} \to A \rtimes P$ be the similarly defined homomorphism; the derived group of its image is then isomorphic to a nontrivial submodule $\bar{A}_1$ of A.

By part (ii) of Corollary 2.7, the isomorphism α induces an isomorphism α_* of $G/\ker\rho$ onto $\bar{G}/\ker\bar{\rho}$; hence an isomorphism of $\mathrm{im}\,\rho$ onto $\mathrm{im}\,\bar{\rho}$, and, finally, an isomorphism α_1 of A_1 onto $\bar{A}_1$. They render commutative the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{\rho} & \mathrm{im}\,\rho & \hookleftarrow & A_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\alpha_1} \\ \bar{G} & \xrightarrow{\bar{\rho}} & \mathrm{im}\,\bar{\rho} & \hookleftarrow & \bar{A}_1 \end{array} \tag{4-2}$$

We claim the automorphism $\alpha_1 : A_1 \xrightarrow{\sim} \bar{A}_1$ is strictly *increasing*.

Let $b \in A_1$ be an arbitrary positive element and let $f_b \in G$ be a PL-homeomorphisms that is a translation with amplitude b near ∞, say on $[t_{b,1}, \infty[$. Then $\alpha(f_b)$ is a PL-homeomorphism which is a translation with amplitude $\alpha_1(b)$ near ∞, say for $t \geq \varphi(t_{b,2})$. Since α is induced by conjugation by φ, one has $\alpha(f_b) = {}^{\varphi} f_b$; so $\varphi \circ f_b = \alpha(f_b) \circ \varphi$. By evaluating this equality at $t \geq \max\{t_{b,1}, t_{b,2}\}$ one obtains the chain of equations

$$\varphi(t + b) = (\varphi \circ f_b)(t) = (\alpha(f_b) \circ \varphi)(t) = \alpha_1(b) + \varphi(t).$$

It implies that $\alpha_1(b)$ is positive, for b is so by assumption and φ is increasing.

We show next that α_1 is given by multiplication by a positive real number s_1. As stated in the first paragraph of the proof, A_1 is a dense subgroup of $\mathbb{R}_{\text{add}}$. Since α_1 is strictly increasing it extends to a (unique) strictly increasing automorphism $\tilde{\alpha}_1 : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$. This automorphism is continuous and hence an $\mathbb{R}$-linear map, given by multiplication by some positive real number s_1.

We come now to the final stage of the analysis of φ. In it we show that the restriction of φ to a suitable interval of the form $[t_*, \infty[$ is *affine*. Choose a positive element $b_* \in A_1$ and let $f_{b_*} \in G$ be an element whose image under ρ is a translation with amplitude b_*. It then follows, as before, that there is a positive number t_* so that the equation

$$\varphi(t + b_*) = \alpha_1(b_*) + \varphi(t) = \varphi(t) + s_1 \cdot b_* \tag{4-3}$$

holds for every $t \geq t_*$. Consider now an arbitrary positive element $b \in A_1$. There exists then a positive number t_b such that the calculation

$$\varphi(t + b) = \alpha_1(b) + \varphi(t) = \varphi(t) + s_1 \cdot b$$

is valid for $t \geq t_b$. Choose a positive integer m which is so large that $t_b \leq t_* + m \cdot b_*$. For every $t \geq t_*$ the following calculation is then valid:

$$\begin{aligned} \varphi(t + b) + s_1 \cdot mb_* &= \varphi(t + b + m \cdot b_*) \\ &= \varphi(t + m \cdot b_*) + s_1 \cdot b \\ &= \varphi(t) + s_1 \cdot mb_* + s_1 \cdot b. \end{aligned}$$

It follows, in particular, that the equation

$$\varphi(t_* + b) = \varphi(t_*) + s_1 \cdot b \tag{4-4}$$

holds for every positive element $b \in A_1$ and $t \geq t_*$. Since φ is continuous and increasing and as A_1 is dense in $\mathbb{R}$, this equation allows us to deduce that φ is affine with slope s_1 on the half-line $[t_*, \infty[$, and so the proof is complete. □

The hypotheses of the Theorem 4.4 are satisfied if $G = G([0, \infty[; A, P)$; the theorem, Lemma 4.1 and Corollary 3.10 thus yield the pleasant

Corollary 4.6. *If G coincides with $G([0, \infty[; A, P)$ then both $\sigma_\ell : G \to P$ and $\sigma_r : G \to P$ are surjective homomorphisms fixed by* Aut G.

Corollary 4.6 is the analogue of Corollary 3.9, but with the compact interval I replaced by a half-line. Groups $G(I; A, P)$ with I a half-line have, so far, been investigated less often than groups with I a compact interval; they have, however, their own merits, in particular the following one: to date, finitely generated groups of the form $G(I; A, P)$ with I compact are only known for very special choices of

the parameters (A, P).[9] By contrast, finitely generated groups with I a half-line are far more common, as is shown by the following characterization:

Proposition 4.7 [Bieri and Strebel 2016, Theorem B8.2]. *The homeomorphism group $G([0, \infty[; A, P)$ is finitely generated if and only if the following conditions are satisfied*:

(i) *P is finitely generated*,

(ii) *A is a finitely generated $\mathbb{Z}[P]$-module, and*

(iii) *$A/(I P \cdot A)$ is finite.*

4C. ***Construction of homomorphisms: part II.*** Theorem 4.4 is very pleasing: it shows that the homomorphism σ_r is fixed by all automorphisms provided merely *the image of $\rho : G \to \mathrm{Aff}(I P \cdot A, P)$ is not abelian*. In this section, we discuss the remaining case.

The image of $G([0, \infty[; A, P)$ under ρ is the affine group

$$\mathrm{Aff}(I P \cdot A, P) \xrightarrow{\sim} (I P \cdot A) \rtimes P$$

(see Section 2B). This group is metabelian and contains two obvious kinds of abelian subgroups: those made up of translations, corresponding to the subgroups of $I P \cdot A$, and the subgroups consisting of homotheties $t \mapsto q \cdot t$ with ratio q varying in a subgroup Q of P. We begin by discussing the second type of abelian subgroups.

4C1. *Image of ρ is made up of homotheties.* Given a subgroup Q of P let G_Q be the subgroup of $G = G([0, \infty[; A, P)$ consisting of the products $f \circ g$ with $g \in B = B([0, \infty[; A, P)$ and f a homothety $t \mapsto q \cdot t$ with $q \in Q$; since B is normal in G the set so defined is actually a subgroup of G. We do not know which of these subgroups G_Q admit decreasing automorphisms, but those with Q cyclic have this peculiarity, as can be seen from

Lemma 4.8. *Assume I is the half-line $[0, \infty[$ and Q is a cyclic subgroup of P. Then the subgroup*

$$G_Q = \{f \circ g \mid f = (t \mapsto q \cdot t) \text{ with } q \in Q \text{ and } g \in B\} \tag{4-5}$$

of the group $G([0, \infty[; A, P)$ does admit a decreasing automorphism.

Proof. Let q_0 be the generator of Q with $q_0 > 1$ and choose a positive element $a_0 \in I P \cdot A$. For each $k \in \mathbb{Z}$ set $t_k = q^k \cdot a_0$ and define $\varphi :]0, \infty[\xrightarrow{\sim}]0, \infty[$ to be the affine interpolation of the assignment $(t_k \mapsto t_{-k})_{k \in \mathbb{Z}}$. Then φ is an infinitary PL-autohomeomorphism of $]0, \infty[$ whose interpolation points lie in $(I P \cdot A) \times (I P \cdot A)$.

[9] See [Bieri and Strebel 2016, p. vii] for the list of the groups known at the end of 2014.

The slopes of the segments forming the graph of φ are the negatives of powers of q_0; indeed,

$$t_{k+1} - t_k = q_0^{k+1} \cdot a_0 - q_0^k \cdot a_0 = (q_0 - 1) \cdot q_0^k \cdot a_0$$

and

$$\varphi(t_{k+1}) - \varphi(t_k) = (1/q_0)^{k+1} \cdot a_0 - (1/q_0)^k \cdot a_0 = (1 - q_0) \cdot q_0^{-k-1} \cdot a_0$$

and so φ has slope $(-1) \cdot q_0^{-2k-1}$ on the interval $[t_k, t_{k+1}]$.

It follows that φ maps $IP \cdot A$ onto itself. Consider now a conjugate ${}^{\varphi}h = \varphi \circ h \circ \varphi^{-1}$ of an element $h \in G_Q$. If $h \in B(I; A, P)$, then h has support contained in some interval of the form $I_{k(h)} = [t_{-k(h)}, t_{k(h)}]$ for some $k(h) > 0$ and so ${}^{\varphi}h$ has support in $\varphi(I_{k(h)}) = I_{k(h)}$, slopes in P, break points in $IP \cdot A$ and is thus an element of $B \subset G_Q$. If, on the other hand, h is the homothety with ratio q_0, then $h(t_k) = t_{k+1}$ for each index $k \in \mathbb{Z}$ and its conjugate ${}^{\varphi}h$ is the PL-function with interpolation points (t_k, t_{k-1}), hence the homothety with center 0 and ratio q_0^{-1} and thus ${}^{\varphi}h = h^{-1}$ lies in G_Q. As G_Q is generated by $B \cup \{(t \mapsto q_0 \cdot t)\}$, the previous reasoning shows that the decreasing autohomeomorphism φ induces by conjugation an automorphism of G_Q and so the lemma is established. □

Remark 4.9. Assume A, P and Q are as in the statement of the lemma. Then the bounded group $B = B([0, \infty[; A, P)$ may be perfect and hence simple; cf. [Bieri and Strebel 2016, Section 12.4]. In such a case, B is the only normal subgroup N of G_Q with G/N *infinite abelian* and so the lemma implies that no homomorphism of G_Q onto an infinite abelian group is fixed by all of $\operatorname{Aut} G_Q$. Note, however, that ρ is fixed by every increasing automorphism of G_Q.

4C2. *Image of ρ consists of translations.* We turn now to the other type of abelian subgroups of $\operatorname{Aff}(IP \cdot A, P)$, but concentrate on a special case. Given a subgroup Q of P and a subgroup $A_0 \subseteq IP \cdot A$, we set

$$G_{Q,A_0} = \big\{ g \in G([0, \infty[; A, P) \;\big|\; \sigma_\ell(g) \in Q \text{ and } \rho(g) \in A_0 \rtimes \{1\} \big\}. \tag{4-6}$$

The group G_{Q,A_0} is an extension of $B([0, \infty[; A, P)$ by the abelian group $Q \times A_0$.

The class of groups having the form G_{Q,A_0} is of interest for several reasons. Firstly, if Q and A_0 are *not* isomorphic, every automorphism of G_{Q,A_0} is increasing by Lemma 4.1. This case occurs frequently, as is brought home by the following kind of examples. Suppose Q is finitely generated and contains an integer $p > 1$, while A_0 is a nonzero submodule of $IP \cdot A$. Then A_0 is divisible by p and, in particular, not free abelian.

Some groups of the form G_{Q,A_0} admit decreasing automorphisms, in particular the following ones: Let P be a cyclic group generated by the real number $p > 1$, let A be a $\mathbb{Z}[P]$-submodule of $\mathbb{R}_{\text{add}}$ and choose a positive element $b \in A$. The

group $\bar{G} = G([0, b]; A, P)$ admits decreasing automorphisms, for instance the automorphism induced by conjugation by the reflection $\bar{\varphi}$ at the midpoint of $I = [0, b]$.

Consider now the group $G = G_{P,\mathbb{Z}\cdot(p-1)b} \subset G([0, \infty[; P, A)$. It is isomorphic to $\bar{G}$; there exists actually an isomorphism induced by an increasing, infinitary PL-homeomorphism $\varphi_b : [0, \infty[\xrightarrow{\sim} [0, b[$; see [op. cit., Lemma E18.2]. Then the composition $\varphi_b^{-1} \circ \bar{\varphi} \circ \varphi_b$ induces by conjugation a decreasing automorphism of G.

Thirdly, let $\tau_r : G_{Q,A_0} \to \mathbb{R}_{\text{add}}$ be the homomorphism that maps the PL-homeomorphism $g \in G_{Q,A_0}$ to the amplitude of the translation $\rho(g)$. This homomorphism seems to have a good chance of being fixed by $\text{Aut}_+ G_{Q,A_0}$, but this impression is mistaken. Indeed, let $\text{Aut}_P A_0$ be the set of elements $p \in P$ with $p \cdot A_0 = A_0$; this set is a subgroup of P and the semidirect product $A_0 \rtimes \text{Aut}_P A_0$ is a subgroup of $(I P \cdot A) \rtimes P$; let $\tilde{G}$ denote the preimage of $A_0 \rtimes \text{Aut}_P A_0$ under the epimorphism

$$\bar{\rho} : G([0, \infty[; A, P) \xrightarrow{\rho} \text{Aff}(I P \cdot A, P) \xrightarrow{\sim} (I P \cdot A) \rtimes P.$$

Then G_{Q,A_0} is a normal subgroup of $\tilde{G}$. The group $\tilde{G}$ contains the homothety $\vartheta_p : t \mapsto p \cdot t$ for every $p \in \text{Aut}_P A_0$, and so conjugation by such a homothety induces an automorphism α_p of G_{Q,A_0}. The calculation

$$\begin{aligned}(\tau_r \circ \alpha_p)(g) &= \tau_r(\vartheta_p \circ g \circ \vartheta_p^{-1}) = (\vartheta_p \circ g \circ \vartheta_p^{-1})(t) - t \\ &= \vartheta_p(g(p^{-1}t)) - t = p \cdot (p^{-1}t + \tau_r(g)) - t = (p \cdot \tau_r)(g),\end{aligned}$$

valid for every sufficiently large real number t, then shows that the formula

$$\tau_r \circ \alpha_p = p \cdot \tau_r \tag{4-7}$$

holds for each $p \in \text{Aut}_P A_0$. We conclude that τ_r can only be fixed by all of $\text{Aut}_+ G_{Q,A_0}$ if $\text{Aut}_P A_0$ is reduced to $1 \in \mathbb{R}^\times_{>0}$. This condition is fulfilled, for instance, if A_0 is infinite cyclic.

Example 4.10. Given a real number $p > 1$, set $P = \text{gp}(p)$ and $A = \mathbb{Z}[P] = \mathbb{Z}[p, p^{-1}]$. Choose $A_0 = A$ and set $G = G_{P,A_0}$. Then $\text{Aut}_P A_0 = P$. Concrete examples are rational integers $p \in \mathbb{N} \smallsetminus \{0, 1\}$, with $A_0 = \mathbb{Z}[1/p]$, or quadratic integers like $\sqrt{2}+1$ with $A_0 = A = \mathbb{Z}[\sqrt{2}\,]$. We shall come back to the second of these examples in Section 6E1.

5. Characters fixed by Aut $G(\mathbb{R}; A, P)$

Let I denote one of the intervals $[0, b]$, $[0, \infty[$ or $\mathbb{R}$, and let G be a subgroup of $G(I; A, P)$ containing $B(I; A, P)$. In Sections 3 and 4 groups with I a compact interval or a half-line have been studied. In this section we now turn to the line $I = \mathbb{R}$. Finding nonzero homomorphisms $\psi : G \to \mathbb{R}^\times_{>0}$ fixed by $\text{Aut}\, G$, is then harder than in the previously investigated cases, and this for two reasons. Firstly, subgroups of $G(\mathbb{R}; A, P)$ often admit decreasing automorphisms α, in contrast to

what happens if I is a half-line; in the case of a decreasing automorphism, λ (or ρ) is only fixed by α if λ coincides with ρ. Secondly, if the image of λ or that of ρ consists of translations, neither λ nor ρ need be fixed by $\mathrm{Aut}_+ G$.

The plan of our investigation will be similar to that adopted in Section 4. We begin by discussing the existence of decreasing automorphisms (in Section 5A), move on to the main results about the existence of homomorphisms fixed by $\mathrm{Aut}_+ G$ or $\mathrm{Aut}\, G$ (in Section 5B) and complement these results with more special findings in Section 5C. The layout of the middle Section 5B will resemble that of Section 3A.

5A. ***Existence of decreasing automorphisms.*** As in the cases of a compact interval or a half-line, the existence of a decreasing automorphism has an easily stated consequence, namely

Lemma 5.1. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ that contains $B(\mathbb{R}; A, P)$. Then every decreasing automorphism α induces an isomorphism $\alpha_* : \mathrm{im}\,\lambda \xrightarrow{\sim} \mathrm{im}\,\rho$ that renders commutative the following square.*

(5-1)
$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \rho} \\ \mathrm{im}\,\lambda & \xrightarrow{\alpha_*} & \mathrm{im}\,\rho \end{array}$$

Proof. The claim can be established as in the proofs of Lemmata 3.6 and 4.1. □

The images of λ and ρ are both subgroups of the affine group $Q = \mathrm{Aff}_o(I P \cdot A, P)$. It is easy to describe some pairs of subgroups (Q_1, Q_2) that are *not* isomorphic for obvious reasons, for instance if one is abelian, and the other is nonabelian. We are, however, not aware of a classification of the isomorphism types of subgroups of $\mathrm{Aff}_o(I P \cdot A, P)$ for parameters $A \neq \{0\}$ and $P \neq \{1\}$.

5B. ***Construction of homomorphisms: part I.*** We turn now to the construction of homomorphisms that are fixed by $\mathrm{Aut}_+ G$ or by $\mathrm{Aut}\, G$. The next result is an analogue of Corollary 3.4. The main ingredient in its proof is Proposition 4.5.

Proposition 5.2. *Let G be a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$ and let α be an automorphism of G that is induced by conjugation by the autohomeomorphism $\varphi_\alpha : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$. Then the following statements hold*:

(i) *if α is increasing*[10] *and* $\mathrm{im}\,\rho$ *is not abelian, then φ_α is affine near ∞;*

(ii) *if α is increasing and* $\mathrm{im}\,\lambda$ *is not abelian, then φ_α is affine near $-\infty$;*

(iii) *if α is decreasing and* $\mathrm{im}\,\rho$ *is not abelian, then $\tilde{\varphi}_\alpha$ is affine, both near $-\infty$ and near ∞.*

[10] See Definition 2.5.

Proof. Statement (i) is a restatement of the claim of Proposition 4.5. To establish (ii), we show that (ii) can be reduced to (i). Let $\vartheta : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ be the reflection in the origin 0, set $G_1 = \vartheta \circ G \circ \vartheta^{-1}$ and $\varphi_1 = \vartheta \circ \tilde{\varphi}_\alpha \circ \vartheta^{-1}$. We claim that Proposition 4.5 applies to the couple (G_1, φ_1). Indeed, the groups $G(\mathbb{R}; A, P)$ and $B(\mathbb{R}; A, P)$ are invariant under conjugation by ϑ and so G_1 is a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$. Next, Lemma 5.3 below shows that

$$\rho(G_1) = \rho\left(\vartheta \circ G \circ \vartheta^{-1}\right) = \vartheta \circ \lambda(G) \circ \vartheta^{-1}.$$

The group $\vartheta \circ \lambda(G) \circ \vartheta^{-1}$ is isomorphic to $\operatorname{im} \lambda$, which is nonabelian by hypothesis, and so $\rho(G_1)$ is nonabelian. Proposition 4.5 thus applies to G_1 and to φ_1 and implies that $\varphi_1 = \vartheta \circ \varphi_\alpha \circ \vartheta^{-1}$ is affine near $+\infty$, whence φ_α itself is affine near $-\infty$.

To establish (iii), note that since α is *decreasing*, the groups $\operatorname{im} \lambda$ and $\operatorname{im} \rho$ are isomorphic (see Lemma 5.1); the hypothesis on $\operatorname{im} \rho$ implies therefore that the image of λ is not abelian. The idea now is to reduce (iii) to the previously treated cases (i) and (ii). As before, let $\vartheta : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ denote the reflection in the origin 0, and set $\varphi_2 = \vartheta \circ \varphi_\alpha$. Then φ_2 is increasing and conjugation by φ_2 maps G onto $\bar{G} = \vartheta \circ G \circ \vartheta^{-1}$. Proposition 4.5 thus applies and guarantees that φ_2 is affine near ∞. But $\varphi_2 = \vartheta \circ \varphi_\alpha$ and so φ_α itself is affine near ∞. Consider, secondly, $\varphi_3 = \varphi_\alpha \circ \vartheta$. This map is again increasing, and conjugation by it maps $\bar{G} = \vartheta \circ G \circ \vartheta^{-1}$ onto G. Invoking Proposition 4.5 once more, we learn that φ_3 is affine near $+\infty$, and so φ_α itself is affine near $-\infty$. All taken together, we have shown that φ_α is affine, both near $-\infty$ and $+\infty$, as asserted by claim (iii). □

We are left with proving

Lemma 5.3. *Let $\vartheta : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ denote the reflection in 0. Then the formula*

$$\rho\left(\vartheta \circ g \circ \vartheta^{-1}\right) = \vartheta \circ \lambda(g) \circ \vartheta^{-1} \tag{5-2}$$

holds for every $g \in \mathrm{PL}_o(\mathbb{R})$.

Proof. Let μ and ν denote the functions of $\mathrm{PL}_o(\mathbb{R})$ into itself given by the left hand and the right hand side of (5-2); thus $\mu(g) = \rho(\vartheta \circ g \circ \vartheta^{-1})$ for $g \in \mathrm{PL}_o(\mathbb{R})$, and similarly for ν. Both functions are homomorphisms of $\mathrm{PL}_o(\mathbb{R})$ into $\mathrm{Aff}_o(\mathbb{R})$ that vanish on $\ker \lambda$. It suffices therefore to check (5-2) on a complement of $\ker(\lambda : \mathrm{PL}_o(\mathbb{R}) \to \mathrm{Aff}_o(\mathbb{R}))$. Such a complement is $\mathrm{Aff}_o(\mathbb{R})$ and for affine maps h the following calculation holds:

$$\rho(\vartheta \circ h \circ \vartheta^{-1}) = \vartheta \circ h \circ \vartheta^{-1} = \vartheta \circ \lambda(h) \circ \vartheta^{-1}.$$ □

5B1. *Some corollaries.* The first corollary of Proposition 5.2 deals with homomorphisms fixed by $\mathrm{Aut}_+ G$; the corollary is an analogue of Theorem 3.5.

Theorem 5.4. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ that contains $B(\mathbb{R}; A, P)$. If $\operatorname{im}\rho$ is not abelian, σ_r is a nonzero homomorphism fixed by $\operatorname{Aut}_+ G$. Similarly, σ_ℓ is a nonzero homomorphism fixed by $\operatorname{Aut}_+ G$ in case $\operatorname{im}\lambda$ is not abelian.*

Proof. Let α be an increasing automorphism of G and let φ_α be the increasing autohomeomorphism of $\mathbb{R}$ inducing α by conjugation. (The map exists thanks to Theorem 2.3.) Assume first that $\operatorname{im}\rho$ is not abelian. By part (i) of Proposition 5.2 the map φ_α is then affine near ∞. On the other hand, the image of ρ, being nonabelian, cannot consist merely of translations; so the homomorphism $\sigma_r : G \to P$ is nonzero. The following calculation then reveals that σ_r is fixed by α:

$$\begin{aligned}(\sigma_r \circ \alpha)(g) &= \sigma_r(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1}) \\ &= \lim_{t\to\infty}(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})'(t) \\ &= \lim_{t\to\infty}\big(\varphi_\alpha'(g(\varphi_\alpha^{-1}(t))) \cdot g'(\varphi_\alpha^{-1}(t)) \cdot (\varphi_\alpha^{-1})'(t)\big) \\ &= \lim_{t\to\infty} g'(\varphi_\alpha^{-1}(t)) = \sigma_r(g).\end{aligned}$$

In this calculation the facts that the derivatives of φ_α and of g are constant on a half-line of the form $[t_*, \infty[$ and that φ_α is an increasing homeomorphism, have been used.

Assume next that $\operatorname{im}\lambda$ is not abelian. By part (ii) of Proposition 5.2 the map φ_α is then affine near $-\infty$. and the homomorphism $\sigma_\ell : G \to P$ is nonzero. Since the derivatives of every element $g \in G$ and of φ_α are constant near $-\infty$, a calculation similar to the preceding one will show that λ is fixed by α. □

As a second application of Proposition 5.2, we present a result that furnishes a homomorphism ψ that is fixed by every automorphism. Note, however, that the hypotheses of the result do not imply that ψ is nontrivial.

Theorem 5.5. *Assume G is a subgroup of $G(\mathbb{R}; A, P)$ containing $B(\mathbb{R}; A, P)$ and let $\psi : G \to P$ be the homomorphism $g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$. If the images of λ and of ρ are both nonabelian, then the homomorphism $\psi : G \to P$ is fixed by $\operatorname{Aut} G$.*

Proof. Let α be an automorphism of G and let φ_α be the autohomeomorphism of $\mathbb{R}$ that induces α by conjugation. If φ_α is *increasing* both σ_ℓ and σ_r are fixed by α (see Theorem 5.4) and hence so is ψ.

Assume now that α is *decreasing*. Part (iii) of Corollary 3.4 then guarantees that φ_α is affine near $-\infty$ and also near ∞. These facts imply the relations

$$\sigma_\ell \circ \alpha = \sigma_r \quad \text{and} \quad \sigma_r \circ \alpha = \sigma_\ell \tag{5-3}$$

(see below) and so $\psi = \sigma_\ell \cdot \sigma_r$ is fixed by α.

We are left with verifying relations (5-3). The following calculation uses the fact that both φ_α and g have constant derivatives near $-\infty$ and $+\infty$:

$$\begin{aligned}(\sigma_\ell \circ \alpha)(g) &= \sigma_\ell(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1}) \\ &= \lim_{t\to-\infty}(\varphi_\alpha \circ g \circ \varphi_\alpha^{-1})'(t) \\ &= \lim_{t\to-\infty}\big(\varphi_\alpha'(g(\varphi_\alpha^{-1}(t))) \cdot g'(\varphi_\alpha^{-1}(t)) \cdot (\varphi_\alpha^{-1})'(t)\big) \\ &= \lim_{t\to-\infty} g'(\varphi_\alpha^{-1}(t)) = \sigma_r(g).\end{aligned}$$

A similar calculation establishes the second relation in (5-3). □

We continue with an easy consequence of Theorem 5.5. If the group G is all of $G(I; A, P)$ the homomorphism $\psi : g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$ is surjective; in addition, $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ both coincide with $\mathrm{Aff}_o(A, P)$ and thus are nonabelian. Therefore, Theorem 5.5 implies the following:

Corollary 5.6. *If $G = G(\mathbb{R}; A, P)$, the homomorphism $\psi : G \to P$, mapping $g \mapsto \sigma_\ell(g) \cdot \sigma_r(g)$, is nonzero and fixed by* Aut G.

Corollary 5.6 is an analogue of Corollaries 3.9 and 4.6. Groups of the form $G(\mathbb{R}; A, P)$ have been investigated, so far, less often than groups with I a compact interval; they have, however, their own merits if it comes to finite generation. There exists, first of all, a characterization of the finitely generated groups of the form $G(\mathbb{R}; A, P)$, namely the following result:

Proposition 5.7 [Bieri and Strebel 2016, Theorem B7.1]. *The group $G(\mathbb{R}; A, P)$ is finitely generated if and only if P is finitely generated and A is a finitely generated $\mathbb{Z}[P]$-module.*

Remark 5.8. Proposition 5.7 implies that *there are continuously many, pairwise nonisomorphic, finitely generated groups of the form $G(\mathbb{R}; A, P)$.*

To prove this assertion, we recall the following result: *if two groups of the form $G(\mathbb{R}; A, P)$ and $G(\mathbb{R}; \bar{A}, \bar{P})$ are isomorphic and if P is not cyclic, then $P = \bar{P}$.*[11]

It suffices therefore to find a collection of finitely generated, pairwise distinct subgroups $\{P_j \mid j \in J\}$ of $\mathbb{R}^\times_{>0}$ with J an index set having the cardinality of $\mathbb{R}$, and to set $A_j = \mathbb{Z}[P_j]$ for each $j \in J$. Such a collection of subgroups can be obtained as follows: first one constructs a family of irrational real numbers $\{x_j \mid j \in J\}$ such that the extended family $\{1\} \cup \{x_j \mid j \in J\}$ is linearly independent (over $\mathbb{Q}$) and then sets $P_j = \exp(\mathrm{gp}(\{1, x_j\})$. Then each group P_j is free abelian of rank two, hence not cyclic, and for indices $j_1 \neq j_2$ the groups P_{j_1} and P_{j_2} are distinct.

[11] see [Bieri and Strebel 2016, Theorem E17.1].

5C. ***Construction of homomorphisms: part II.*** In this final part of Section 5, we consider subgroups G of $G(\mathbb{R};\, A,\, P)$, containing $B(\mathbb{R};\, A,\, P)$, with $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ both abelian.[12] The most interesting subcase seems to be that where the images of λ and ρ consists only of translations. Then two homomorphisms τ_ℓ and τ_r of G into $\mathbb{R}_{\text{add}}$ can be defined: they associate to $g \in G$ the amplitudes of the translations $\lambda(g)$ and $\rho(g)$, respectively. One sees, as in Section 4C2, that neither of these homomorphisms need be fixed by $\operatorname{Aut}_+ G$.

An exception occurs if the image of ρ or of λ is *infinite cyclic*. Suppose, for instance, that $\operatorname{im}\rho$ is infinite cyclic, and let $f \in G$ be an element that maps onto the positive generator, say x_f, of $\operatorname{im}\tau_r$. Consider an increasing automorphism α of G and let φ_α be the homeomorphism of $\mathbb{R}$ that induces α by conjugation. Then $\tau_r(\alpha(f))$ generates $\operatorname{im}\tau_r$, too, and so $\tau_r(\alpha(f)) = \pm x_f$. Near $+\infty$, the map f is a translation with positive amplitude, hence so is $\alpha(f) = \varphi_\alpha \circ f \circ \varphi_\alpha^{-1}$, and so $\tau_r(\alpha(f)) > 0$. Thus $\tau_r(f) = (\alpha \circ \tau_r)(f)$. We conclude that τ_r is fixed by α. An analogous argument shows that τ_ℓ is fixed by every increasing automorphism of G.

All taken together we have thus established the following result:

Proposition 5.9. *Let G be a subgroup of $G(\mathbb{R};\, A,\, P)$ containing $B(\mathbb{R};\, A,\, P)$. Now assume that the images of λ and ρ contain only translations and that these images are infinite cyclic. Then τ_ℓ and τ_r are both nonzero homomorphisms that are fixed by $\operatorname{Aut}_+ G$.*

Example 5.10. Suppose P is an infinite cyclic group, A a (nonzero) $\mathbb{Z}[P]$-module and b a positive element of A. Set $\bar{G} = G([0, b];\, A,\, P)$. Then there exists a homeomorphism $\vartheta :]0, b[\xrightarrow{\sim} \mathbb{R}$ that induces, by conjugation, an embedding

$$\mu : G([0, b];\, A,\, P) \rightarrowtail G(\mathbb{R};\, A,\, P)$$

whose image contains $B(\mathbb{R};\, A,\, P)$.[13] Let G denote the image of μ. The images of $\lambda \restriction G$ and $\rho \restriction G$ are both infinite cyclic and consist of translations. The images of τ_ℓ and τ_r are therefore infinite cyclic, too, and so the previous lemma applies.

Let's now consider the special case where P is generated by an integer $n \geq 2$, where $A = \mathbb{Z}[P] = \mathbb{Z}[1/n]$ and $b = 1$. For a suitably chosen homeomorphism ϑ the image G of μ consists then of all elements $g \in G(\mathbb{R};\, \mathbb{Z}[1/n],\, \mathrm{gp}(n))$ fulfilling the conditions

$$\sigma_\ell(g) = \sigma_r(g) = 1 \quad \text{and} \quad \tau_\ell(g),\, \tau_r(g) \in \mathbb{Z}(n-1); \tag{5-4}$$

[12] If exactly one of $\operatorname{im}\lambda$ and $\operatorname{im}\rho$ is abelian, the group does not admit a decreasing automorphism (by Lemma 5.1) and so Theorem 5.4 yields a nonzero homomorphism fixed by $\operatorname{Aut} G$.

[13] In special cases, for instance if $\bar{G}$ is Thompson's group F, this fact is well known (see, e.g., [Belk and Brown 2005, Proposition 3.1.1]); the general claim is established in [Bieri and Strebel 2016] (see Lemma E18.4).

see [Bieri and Strebel 2016, Lemma E18.4]. This group G is called $F_{n,\infty}$ in [Brin and Guzmán 1998, p. 298].

By relaxing conditions (5-4) one obtains supergroups of $F_{n,\infty}$, in particular the group called F_n in [op. cit., p. 298] and defined by the requirements

$$\text{(5-5)}\quad \sigma_\ell(g) = \sigma_r(g) = 1 \quad \text{and} \quad \tau_\ell(g), \tau_r(g) \in \mathbb{Z}, \quad \tau_r(g) - \tau_\ell(g) \in \mathbb{Z}(n-1);$$

see [op. cit., Proposition 2.2.6]. Proposition 5.9 applies to the groups $F_{n,\infty}$, but also to the larger groups F_n. Now, the groups F_n and $F_{n,\infty}$ both admit decreasing automorphisms, in particular the automorphism induced by the reflection in the origin. The homomorphisms τ_ℓ and τ_r are therefore not fixed by the full automorphism group of the groups $F_{n,\infty}$ and F_n, but the difference $\tau_r - \tau_\ell$ is a nonzero homomorphism, with infinite cyclic image, that enjoys this property.

6. Characters fixed by Aut G with G a subgroup of $\mathrm{PL}_o([0, b])$

In this section we prove Theorem 1.7. For the convenience of the reader we restate this result here.

Theorem 6.1. *Suppose $I = [0, b]$ is a compact interval of positive length and G is subgroup of $\mathrm{PL}_o(I)$ that satisfies the following conditions*:

(i) *no interior point of the interval $I = [0, b]$ is fixed by G*;

(ii) *the characters χ_ℓ and χ_r are both nonzero*;

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group*; *and*

(iv) *at least one of the group of units $U(\operatorname{im} \chi_\ell)$ or $U(\operatorname{im} \chi_r)$ is reduced to $\{1, -1\}$.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G. The group G therefore has property R_∞.

Next we explain the layout of Section 6. We begin by recalling the definition of the invariant Σ^1 and stating some basic results concerning it. In Section 6C, we prove Theorem 6.1. The hypotheses of the theorem allow of variations that deserve some comments. This topic is taken care of in sections 6D through 6F.

6A. ***Review of* $\boldsymbol{\Sigma^1}$.** Given an infinite group G, consider the real vector space $\mathrm{Hom}(G, \mathbb{R})$ made up of all homomorphisms $\chi : G \to \mathbb{R}_{\mathrm{add}}$ into the additive group of $\mathbb{R}$. These homomorphisms will be referred to as *characters*. Two nonzero characters χ_1 and χ_2 are called equivalent, if one is a positive real multiple of the other. Geometrically speaking, the associated equivalence classes are (open) rays emanating from the origin. The space of all rays is denoted by $S(G)$ and called the *character sphere* of G. In case the abelianization $G_{\mathrm{ab}} = G/[G, G]$ of G is finitely generated, the vector space $\mathrm{Hom}(G, \mathbb{R})$ is finite dimensional and carries a unique topology, induced by its norms; the sphere $S(G)$ equipped with the quotient

topology is then homeomorphic to the spheres in a Euclidean vector space of dimension $\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) = \dim_{\mathbb{Q}}(G_{\mathrm{ab}} \otimes \mathbb{Q})$.

The invariant $\Sigma^1(G)$ is a subset of $S(G)$. It admits several equivalent definitions; in the sequel, we use the definition in terms of Cayley graphs.[14] Fix a generating set $\mathcal{X}$ of G and define $\Gamma = \Gamma(G, \mathcal{X})$ to be the associated Cayley graph of G. This graph can be equipped with G-actions; as we want to work with *left* G-actions we define the set of positive edges of the Cayley graph like this:

$$E_+(\Gamma) = \{(g, g \cdot x) \in G \times G \mid (g, x) \in G \times \mathcal{X}\}.$$

We move on to the *definition of* $\Sigma^1(G)$. Given a nonzero character χ, consider the submonoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ of G and define $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ to be the full subgraph of $\Gamma(G; \mathcal{X})$ with vertex set G_χ. Both the submonoid G_χ and the subgraph Γ_χ remain the same if χ is replaced by a positive multiple; so these objects depend only on the ray $[\chi] = \mathbb{R}_{>0} \cdot \chi$ represented by χ. The Cayley graph Γ is connected, but its subgraph Γ_χ may not be so; the invariant $\Sigma^1(G)$ records the rays for which the subgraph $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ *is connected*. In symbols,

$$\Sigma^1(G, \mathcal{X}) = \{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X})_\chi \text{ is connected}\}. \tag{6-1}$$

One now faces the problem, familiar from homological algebra, that the definition of $\Sigma^1(G, \mathcal{X})$ involves an arbitrary choice and that one wants to construct an object that does not depend on this choice.

Suppose, first, that G is *finitely generated* and let $\mathcal{X}_f$ be a *finite* generating set. Then the subgraph $\Gamma(G, \mathcal{X}_f)_\chi$ is connected if and only if all the subgraphs $\Gamma(G, \mathcal{X})_\chi$, are connected (see, e.g., [Strebel 2013, Lemma C2.1]) and so the following definition is licit:

Definition 6.2. Let G be a finitely generated group and $\mathcal{X}_f$ a *finite* generating set of G. Then $\Sigma^1(G)$ is defined to be the subset

$$\{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X}_f)_\chi \text{ is connected}\}. \tag{6-2}$$

The fact that the set (6-2) does not depend on the choice of the finite set $\mathcal{X}_f$, allows one to select $\mathcal{X}_f$ in accordance with the problem at hand; see [Strebel 2013, Sections A2.3a and A2.3b] for some consequences of this fact.

Now suppose that G is an arbitrary group. A useful subset of $S(G)$ can then be obtained by defining

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X})_\chi \text{ is connected for every generating set } \mathcal{X}\}; \tag{6-3}$$

see [Strebel 2013, Definition C2.2]. If G happens to be finitely generated, the sets (6-2) and (6-3) are equal; for an arbitrary group, the set $\Sigma^1(G)$ coincides with the

[14] See, e.g., [Strebel 2013, Chapter C] for alternate definitions.

invariant $\Sigma(G)$ defined by Ken Brown in [Brown 1987b, p. 489] *up to a sign*; in other words,

$$\Sigma(G) = -\Sigma^1(G). \tag{6-4}$$

The sign in this formula is caused by the fact that Brown uses right actions on $\mathbb{R}$-trees, whereas *left* actions are employed in our definition of Σ^1.

The subset $\Sigma^1(G)$ of $S(G)$ is traditionally called the Σ^1-*invariant*. The epithet "invariant" is justified by a fact that we explain next. Suppose $\alpha : G \xrightarrow{\sim} \overline{G}$ is an isomorphism of groups. Then α induces, first of all, a linear isomorphism of vector spaces $\mathrm{Hom}(\alpha, \mathbb{R}) : \mathrm{Hom}(\overline{G}, \mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}(G, \mathbb{R})$, and so an isomorphism of spheres

$$\alpha^* : S(\overline{G}) \xrightarrow{\sim} S(G), \qquad [\overline{\chi}] \mapsto [\overline{\chi} \circ \alpha]. \tag{6-5}$$

This second isomorphism maps the subset $\Sigma^1(\overline{G}) \subseteq S(\overline{G})$ onto $\Sigma^1(G) \subseteq S(G)$. [Strebel 2013, Section B1.2a] has more details.

Later, the case where α is *automorphism* will be crucial. The assignment

$$\Sigma^1(G) \xrightarrow{\sim} \Sigma^1(G), \qquad \alpha \mapsto (\alpha^{-1})^*, \tag{6-6}$$

defines a homomorphism from the automorphism group of G into the group of bijections of $\Sigma^1(G)$, and hence also one into that of its complement $\Sigma^1(G)^c$.

Remarks 6.3. (a) Historically speaking, the invariant Σ^1 is a descendent of the invariant $\Sigma_A(G)$, introduced by R. Bieri and R. Strebel [1980]. Here the group G is abelian, A is a finitely generated $\mathbb{Z}G$-module, and $\Sigma_A(G)$ is a subset of the sphere $S(G)$ depending both on A and on G. The motivation for introducing this invariant stems from a question posed by G. Baumslag [1974], namely: *Is there any way of discerning finitely presented metabelian groups from the other finitely generated metabelian groups?*

(b) The invariant Σ^1 is a member of a sequence of invariants Σ^m introduced by B. Renz in his thesis [1988]. The definition of these higher Σ-invariants is considerably more involved than that of Σ^1 and so we shall not give it here; we refer the interested reader to Section 8 of K.-U. Bux's paper [2004] for a survey of various equivalent definitions given in the literature. Suffice it to say here that these invariants form a descending chain

$$\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots \supseteq \Sigma^m(G) \supseteq \cdots$$

of open subsets in $S(G)$, and that, so far, there are very few groups whose higher Σ-invariants are completely known. In the case of PL-homeomorphism groups, the most general result known today is due to M. Zaremsky [2016]; it deals with the sequence of groups $G([0, 1]; \mathbb{Z}[1/n], \mathrm{gp}(n))$, with $n \geq 2$ an integer.

6B. Σ^1 *of subgroups of* $\mathbf{PL}_o([0, b])$. Given a subgroup G of $\mathrm{PL}_o([0, b])$, let σ_ℓ be the homomorphism that assigns to a function $g \in G$ the value of its (right) derivative in the *left* end point 0; similarly, define $\sigma_r : G \to \mathbb{R}^\times_{>0}$ to be the homomorphism given by the formula $\sigma_r(g) = \lim_{t\to b} g'(t)$. The homomorphisms σ_ℓ and σ_r generalize the maps with the same names studied in Section 3. By composing them with the natural logarithm function, one obtains characters of G, namely

$$\chi_\ell = \ln \circ \sigma_\ell \quad \text{and} \quad \chi_r = \ln \circ \sigma_r. \tag{6-7}$$

The invariant $\Sigma^1(G)^c$ turns out to consist of precisely two points, represented by the characters χ_ℓ and χ_r, provided G satisfies certain restrictions. The first of them rules out that G is a direct product of subgroups G_1, G_2 with supports in two disjoint open subintervals I_1, I_2, and more general decompositions; the second requires that χ_ℓ, χ_r be nonzero and hence represent points of $S(G)$; the third condition is natural in the sense that it holds for all groups of the form $G([0; b]; A, P)$ investigated in Section 3.

Theorem 6.4. *Let I be a compact interval of positive length and G a subgroup of* $\mathrm{PL}_o(I)$. *Assume the following requirements are satisfied*:

(i) *no interior point of I is fixed by G*;

(ii) *the characters χ_ℓ and χ_r are both nonzero*; *and*

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi)$ is a torsion group.*

Then $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$.

Remarks 6.5. (a) Theorem 6.4 generalizes [Bieri et al. 1987, Theorem 8.1]; in that work, G is assumed to be finitely generated and condition (iii) is sharpened to $G = \ker \chi_\ell \cdot \ker \chi_r$. The theorem improves also on a result stated in [Brown 1987b, Remark on p. 502]. A proof of Theorem 6.4, based on the Cayley graph definition of $\Sigma^1(G)$, can be found in [Strebel 2015, Theorem 1.1].

(b) We continue with a comment that seems overdue. In [Bieri et al. 1987] an invariant $\Sigma_{G'}(G)$ is introduced for finitely generated groups G; in the sequel, this invariant will be called $\Sigma^{BNS}(G)$. It is defined in terms of a generation property that uses *right* conjugation, while left action is employed in the definition of $\Sigma^1(G)$. There is, however, a close connection between the two invariants: if G is finitely generated, then

$$\Sigma^{BNS}(G) = -\Sigma^1(G), \tag{6-8}$$

similar to the what happens for Brown's invariant $\Sigma(G)$; see (6-4).

Now, PL-homeomorphism groups are examples of groups made up of permutations, and for such a group G the underlying set can be equipped with two familiar compositions. Suppose the composition in the group G is the one familiar to

analysts (and used in this paper); to emphasize this fact call the group temporarily G_{ana}. The assignment $g \mapsto g^{-1}$ defines then an antiautomorphism of G_{ana} and hence an isomorphism $\iota : G_{\text{ana}} \xrightarrow{\sim} G_{\text{gt}}$ onto the group obtained by equipping the set underlying G_{ana} with the composition defined by $f \circ g : t \mapsto f(t) \mapsto g(f(t))$ and preferred by many group theorists (hence, the subscript "gt"). The invariants of the groups G_{ana} and G_{gt} are then related by the formulae

$$\Sigma^1(G_{\text{ana}}) = -\Sigma^1(G_{\text{gt}}) \quad \text{and} \quad \Sigma^{BNS}(G_{\text{ana}}) = -\Sigma^{BNS}(G_{\text{gt}}).$$

The analogous formula holds for the invariant Σ studied in [Brown 1987b].

The two parts of the comment, taken together, lead to the following formulae for groups made up of bijections:

$$G_{\text{gt}} \text{ arbitrary} \implies \Sigma(G_{\text{gt}}) = \Sigma^1(G_{\text{ana}}), \tag{6-9}$$

$$G_{\text{gt}} \text{ is finitely generated} \implies \Sigma^{BNS}(G_{\text{gt}}) = \Sigma^1(G_{\text{ana}}). \tag{6-10}$$

6C. ***Proof of Theorem 6.1.*** Let $I = [0, b]$ be an interval of positive length and G a subgroup of $\mathrm{PL}_o(I)$ that satisfies hypotheses (i) through (iv) stated in Theorem 6.1. Hypotheses (i), (ii), and (iii) allow one to invoke Theorem 6.4 and so

$$\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}.$$

In view of the remarks made at the end of Section 6A, every automorphism α of G will therefore permute the set $\{[\chi_\ell], [\chi_r]\}$. Two cases now arise, depending on whether or not the automorphism group of G acts by the identity on $\Sigma^1(G)^c$.

Suppose first that $\mathrm{Aut}\, G$ *acts trivially on* $\Sigma^1(G)^c$. By hypothesis (iv), one of the characters χ_ℓ and χ_r, say χ_ℓ, has an image B with $U(B) = \{1, -1\}$. We assert that χ_ℓ is fixed by $\mathrm{Aut}\, G$. Consider an automorphism α of G. It fixes the ray $\mathbb{R}_{>0} \cdot \chi_\ell$ and so $\chi_\ell \circ \alpha = s \cdot \chi_\ell$ for some positive real s. The relation $\chi_\ell \circ \alpha = s \cdot \chi_\ell$ implies next that

$$\operatorname{im} \chi_\ell = \operatorname{im}(\chi_\ell \circ \alpha) = s \cdot \operatorname{im} \chi_\ell.$$

So s is a positive element of $U(\operatorname{im} \chi_\ell) = \{1, -1\}$ and thus $s = 1$.

So far we have assumed that $U(\chi_\ell)$ equals $\{1, -1\}$; if $U(\operatorname{im} \chi_r)$ is so, one proves in the same way that χ_r is fixed by $\mathrm{Aut}\, G$. The homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ can thus be chosen to be σ_ℓ if $U(\operatorname{im} \chi_\ell) = \{1, -1\}$ and to be σ_r if $U(\operatorname{im} \chi_r) = \{1, -1\}$.

Assume now that $\mathrm{Aut}\, G$ *interchanges the points* $[\chi_\ell]$ and $[\chi_r]$. Pick an automorphism, say α_-, that interchanges these points (and hence is decreasing) and denote, as in Remark 2.4(b), by $\mathrm{Aut}_+ G$ the subgroup of $\mathrm{Aut}\, G$ made up of the increasing automorphisms. Then $\chi_r \circ \alpha_- = s \cdot \chi_\ell$ for some positive real s and so $\operatorname{im} \chi_r = s \cdot \operatorname{im} \chi_\ell$. This relation implies that $U(\operatorname{im} \chi_\ell) = U(\operatorname{im} \chi_r) = \{1, -1\}$.

We claim that the homomorphism

$$\psi = \sigma_\ell \cdot (\sigma_\ell \circ \alpha_-) = \sigma_\ell \cdot (s \cdot \sigma_r)$$

is fixed by $\operatorname{Aut} G$. Two cases arise. If $\alpha \in \operatorname{Aut}_+ G$ then σ_ℓ is fixed by α in view of the first part of the proof. Moreover, $\alpha' = \alpha_- \circ \alpha \circ (\alpha_-)^{-1} \in \operatorname{Aut}_+ G$ and so the calculation

$$\psi \circ \alpha = (\sigma_\ell \circ \alpha) \cdot (\sigma_\ell \circ \alpha_-) \circ \alpha = \sigma_\ell \cdot (\sigma_\ell \circ \alpha') \circ \alpha_- = \sigma_\ell \cdot (\sigma_\ell \circ \alpha_-) = \psi$$

holds. If $\alpha = \alpha_-$ then $\alpha_-^2 \in \operatorname{Aut}_+ G$ and so $\psi \circ \alpha_- = (\sigma_\ell \circ \alpha_-) \cdot (\sigma_\ell \circ \sigma_-^2) = \psi$. It follows that ψ is fixed by $\operatorname{Aut}_+ G \cup \{\alpha_-\}$ and hence by $\operatorname{Aut} G$.

6D. ***Discussion of the hypotheses of Theorem 6.1.*** This section and the next two contain various remarks on the hypotheses of Theorem 6.1.

6D1. *Irreducibility.* Let G be a subgroup of $\mathrm{PL}_o([0, b])$. The union of the supports of the elements of G is then an open subset of $I = [0, b]$, and hence a union of disjoint intervals J_k for k running over some index set K. For each $k \in K$ the assignment $g \mapsto g \restriction J_k$ defines an epimorphism π_k onto a quotient group G_k so G itself is isomorphic to a subgroup of the cartesian product $\prod\{G_k \mid k \in K\}$; more precisely, G is a subdirect product of the quotient groups G_k. Hypothesis (i) requires that K be a singleton, and so the group G does not admit such obvious decompositions. This fact prompted the authors of [Bieri et al. 1987] to call a group G *irreducible* if $\operatorname{card}(K) = 1$.

If the group G is not irreducible it may be a direct product $G_1 \times G_2$ with each factor G_k an irreducible subgroup of $\mathrm{PL}_o(I_k)$ where I_k is the closure of J_k. Then $\Sigma^1(G)^c$ can contain more than 2 points (for more details, see [Strebel 2015, Section 4.1]).

6D2. *Nontriviality of the characters χ_ℓ and χ_r.* In Theorem 6.1 the characters χ_ℓ and χ_r are assumed to be nonzero. They represent therefore points of $S(G)$; the remaining hypotheses and Theorem 6.4 then guarantee that $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$ and so every automorphism of G must permute the points $[\chi_\ell]$ and $[\chi_r]$.

There exists a variant of Theorem 6.1 in which only one of the characters, say χ_ℓ, is nonzero, the remaining hypotheses being as before. Then $\Sigma^1(G)^c = \{[\chi_\ell]\}$ (see [Strebel 2015, Theorem 1.1]) and so the argument in the first part of the proof of Theorem 6.1 applies and shows that $\psi = \chi_\ell$ is fixed by every automorphism of G.

Note that hypothesis (iii) holds automatically if χ_ℓ or χ_r vanishes.

6D3. *Almost independence of χ_ℓ and χ_r.* Among the assumptions of [Bieri et al. 1987, Theorem 8.1], a sharper form of hypothesis (iii) is assumed, namely that $G = \ker \chi_\ell \cdot \ker \chi_r$; in addition, G is assumed to be finitely generated. The authors of that reference refer to this stronger condition by saying that "χ_ℓ and χ_r are independent". In what follows, we exhibit various versions of this stronger requirement and explain then the reason that led the authors to adopt the mentioned language.

We start out with a general result.

Lemma 6.6. *Let $\psi_1 : G \twoheadrightarrow H_1$ and $\psi_2 : G \twoheadrightarrow H_2$ be epimorphisms of groups. Then the following statements are equivalent*:

(i) $H_1 = \psi_1(\ker \psi_2)$,

(ii) $H_2 = \psi_2(\ker \psi_1)$,

(iii) $G = \ker \psi_1 \cdot \ker \psi_2$,

(iv) $(\psi_1, \psi_2) : G \to H_1 \times H_2$ *is surjective.*

Proof. Note first that the product $\ker \psi_1 \cdot \ker \psi_2$ is a normal subgroup of G. Next, note that ψ_1 maps G onto H_1 and $\ker \psi_1 \cdot \ker \psi_2$ onto $\psi_1(\ker \psi_2)$ and induces thus an isomorphism

$$(\psi_1)_* : G/(\ker \psi_1 \cdot \ker \psi_2) \xrightarrow{\sim} H_1/\psi_1(\ker \psi_2). \tag{6-11}$$

It follows, in particular, that statements (i) and (iii) are equivalent. By exchanging the rôles of the indices 1 and 2, one sees that statements (ii) and (iii) are equivalent.

Assume now that statements (i) and (ii) hold and consider $(h_1, h_2) \in H_1 \times H_2$. Since ψ_1 is surjective, h_1 has a preimage $g_1 \in G$; as statement (i) holds, this preimage can actually be chosen in $\ker \psi_2$. If this is done, one sees that $(\psi_1, \psi_2)(g_1) = (h_1, 1)$. One finds similarly that there exists $g_2 \in \ker \psi_1$ with $(\psi_1, \psi_2)(g_2) = (1, h_2)$. The product $g_1 \cdot g_2$ is therefore a preimage of (h_1, h_2) under (ψ_1, ψ_2).

The preceding argument proves that the conjunction of (i) and (ii) implies statement (iv). Assume, finally, that (iv) holds. Given $h_1 \in H_1$, there exists then $g_1 \in G$ with $(\psi_1, \psi_2)(g_1) = (h_1, 1)$; so g_1 is a preimage of h_1 lying in $\ker \psi_2$. The implication (iv) $\Rightarrow$ (i) is thus valid, and so the proof is complete. □

Remark 6.7. Lemma 6.6 allows one to understand why the phrase "χ_ℓ and χ_r are independent" is used in [Bieri et al. 1987] to express the requirement that $G = \ker \chi_\ell \cdot \ker \chi_r$, the group G being a finitely generated, irreducible subgroup of $\mathrm{PL}_o([0, b])$. Let ψ_1 denote the epimorphism $G \twoheadrightarrow \operatorname{im} \chi_\ell$ obtained by restricting the domain of $\chi_\ell : G \to \mathbb{R}$ to $\operatorname{im} \chi_\ell$, and let ψ_2 be defined analogously. If statement (iii) holds, then the implication (iii) $\Rightarrow$ (iv) of Lemma 6.6 shows that the image of $(\chi_\ell, \chi_r) : G \to \mathbb{R}_{\mathrm{add}} \times \mathbb{R}_{\mathrm{add}}$ is $\operatorname{im} \chi_\ell \times \operatorname{im} \chi_r$. This fact amounts to saying that the values of the characters χ_ℓ and χ_r can be prescribed *independently* (within $\operatorname{im} \chi_\ell \times \operatorname{im} \chi_r$), in contrast to what happens, for instance, if the characters satisfy a relation like $\chi_2 = -\chi_1$.[15]

By analyzing the proof of Theorem 8.1 in [op. cit.] one finds that it suffices to require that the normal subgroup $\ker \chi_\ell \cdot \ker \chi_r$ has finite index in the finitely generated group G; a condition that we shall paraphrase by saying that χ_ℓ and χ_r are *almost independent*. Theorem 6.4 extends this result to possibly infinitely generated groups G; the new form of hypothesis (iii) will likewise be referred to by saying

[15] Example 3.13 considers more general relations.

that χ_ℓ and χ_r are *almost independent*. This form of almost independence is used in the proof Theorem 6.4 to find commuting elements of a certain type; see, e.g., [Strebel 2015, Section 3.3]. It remains unclear what $\Sigma^1(G)^c$ looks like if χ_ℓ and χ_r are not almost independent.[16]

6E. ***Group of units.*** In Section 1E, the *group of units* $U(B)$ of a subgroup B of $\mathbb{R}_{\text{add}}$ is introduced. This notion allows one to state a very simple condition that implies, in conjunction with the hypotheses of Theorem 6.4, that $\operatorname{Aut} G$ fixes the character χ_ℓ if it fixes the ray $[\chi_\ell] = \mathbb{R} \cdot \chi_\ell$.

In this section, we discuss the group of units of some concrete examples of subgroups B of $\mathbb{R}_{\text{add}}$, then study two types of subgroups B of $\mathbb{R}_{\text{add}}$ where methods taken from the theory of transcendental numbers allow one to establish that B has only trivial units.

6E1. *Elementary examples.* We begin with an observation: *a subgroup B and a nonzero real multiple $s \cdot B$ of B have the same group of units*. If B is not reduced to 0, we may therefore assume that $1 \in B$.

(a) If B is infinite cyclic, it is a positive multiple of $\mathbb{Z}$. Clearly $U(\mathbb{Z}) = \{1, -1\}$.

(b) If B is free abelian of rank 2, we may assume that it is generated by 1 and an irrational number ϑ; so $B = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \vartheta$. If u is a unit of B then $u = u \cdot 1 \in B$, say $u = a + b \cdot \vartheta$ with $(a, b) \in \mathbb{Z}^2$. The condition $u \cdot B \subseteq B$ implies next that $u \cdot \vartheta = a \cdot \vartheta + b \cdot \vartheta^2$ lies in B. If $b \neq 0$, the real ϑ is thus a quadratic algebraic number; if $b = 0$, the condition that $u \cdot B = B$ forces a to 1 or -1. It follows that $U(B) = \{1, -1\}$ if ϑ is an irrational, but not a quadratic algebraic number.

(c) Let B be the additive group of a subring R of $\mathbb{R}$, for instance the additive group of the ring $\mathbb{Z}[P]$ generated by a subgroup P of $\mathbb{R}^\times_{>0}$ or of a ring of algebraic integers. Then $U(B)$ is nothing but the group of units $U(R)$ of R; if R is a ring of the form $\mathbb{Z}[P]$ its group of units contains, of course, $P \cup -P$, but it may be considerably larger; moreover, rings of algebraic integers have also often units of infinite order. Note, however, that not every subring $R \neq \mathbb{Z}$ of $\mathbb{R}$ has nontrivial units, an example being the polynomial ring $\mathbb{Z}[s]$ generated by a transcendental number s.

6E2. *Transcendental subgroups.* Many of the familiar examples of subgroups of $\mathrm{PL}_o([0, b])$ consist of PL-homeomorphisms with rational slopes; this is true for Thompson's group F, but also for its generalizations $G_m = G([0, 1]; \mathbb{Z}[1/m], \operatorname{gp}(m))$ with $m \geq 3$ an integer and for many of the groups studied by Stein [1992].

The values of the characters χ_ℓ are then natural logarithms of rational numbers, so either transcendental numbers or 0 (see, e.g., [Niven 1956, Theorem 9.11c]). We are thus led to study the unit groups $U(B)$ of subgroups $B \subset \mathbb{R}$ that contain

[16] Sections 4.2 and 4.3 in [Strebel 2015] have some preliminary results.

transcendental numbers; in view of the fact that $U(B) = U(s \cdot B)$ for every $s \neq 0$, it is not so much the nature of the elements of B that is important, but the nature of the quotients b_1/b_2 of nonzero elements in B. The following definition singles out a class of subgroups B that turn out to be significant.

Definition 6.8. (a) Let $B \neq \{0\}$ be a subgroup of the additive group $\mathbb{R}_{\text{add}}$ of the reals. We say B is *transcendental* if, for each ordered pair (b_1, b_2) of nonzero elements in B, the quotient b_1/b_2 is either rational or transcendental.

(b) We call a nonzero character $\chi : G \to \mathbb{R}$ *transcendental* if its image in $\mathbb{R}_{\text{add}}$ is transcendental.

The next result explains why transcendental subgroups are welcome in our study.

Proposition 6.9. *If B is a nontrivial, finitely generated, transcendental subgroup of $\mathbb{R}_{\text{add}}$, then $U(B) = \{1, -1\}$.*

Proof. Suppose u is a unit of B. Then $u \cdot B = B$. Pick $b \in B \smallsetminus 0$; this is possible since B is not reduced to 0. The assignment $1 \mapsto b$ extends to a homomorphism $\mathbb{Z}[u] \to B$ of $\mathbb{Z}[u]$-modules; it is injective since $\mathbb{R}$ has no zero-divisors. The fact that B is finitely generated implies next that the additive group of the integral domain $\mathbb{Z}[u]$ is finitely generated and so u is an algebraic integer; as B is transcendental by assumption, u must therefore be an algebraic integer and also a rational number, hence an integer. Finally, u^{-1} satisfies also the relation $u^{-1} \cdot B = B$, and so u^{-1} is an integer, too. □

We continue with a combination of Theorem 6.1 and Proposition 6.9.

Corollary 6.10. *Suppose $I = [0, b]$ is a compact interval of positive length and G is subgroup of $\mathrm{PL}_o(I)$ that satisfies the following conditions*:

(i) *no interior point of the interval $I = [0, b]$ is fixed by G*;

(ii) *the characters χ_ℓ and χ_r are both nonzero*;

(iii) *the quotient group $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group G; and*

(iv) *the image of σ_ℓ or that of σ_r is finitely generated and transcendental.*

Then there exists a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by every automorphism of G.

6E3. *Examples of transcendental subgroups of $\mathbb{R}_{\text{add}}$.* In order to make use of Proposition 6.9, one needs a supply of transcendental subgroups of $\mathbb{R}$. The simplest ones are the cyclic subgroups; noncyclic subgroups are harder to come by.

Example 6.12 below describes a first collection of transcendental subgroups. It is based on the following theorem, established independently by A. O. Gelfond in 1934 and by T. Schneider in 1935:

Theorem 6.11 (Gelfond–Schneider theorem). *If p_1 and p_2 are nonzero (real or complex) algebraic numbers and if $p_2 \neq 1$, then $\ln p_1 / \ln p_2$ is either a rational or a transcendental number.*

Proof. See, e.g., [Niven 1956, Theorem 10.2]. □

Example 6.12. Let P denote a subgroup of $\mathbb{R}^{\times}_{>0}$ generated by a set $\mathcal{P}$ of algebraic numbers and define $B = \ln P$ to be its image in $\mathbb{R}_{\text{add}}$ under the natural logarithm. Then every element in P is a positive algebraic number, so the Gelfond–Schneider theorem implies that every quotient $\ln p_1 / \ln p_2$ of elements in $P \smallsetminus \{1\}$ is either rational or transcendental.

In Example 6.12 the set $\mathcal{P}$ is allowed to be infinite; for such a choice, the group $B = \ln(\text{gp}(\mathcal{P}))$ is not finitely generated and so neither Proposition 6.9 nor its Corollary 6.10 applies. Now, in Proposition 6.9 the finite generation of B is only used to infer that a unit u of B — which, by the transcendence of B, is either rational or transcendental — is also an algebraic integer, and hence a rational integer.

Proposition 6.14 below furnishes examples of infinitely generated, transcendental groups that have only 1 and -1 as units. Its proof makes use of the following result, due to C. L. Siegel and rediscovered by S. Lang; see [Lang 1966, Theorem II.1] or [Lang 1971, Theorem (1.6)]:

Theorem 6.13 (Siegel–Lang theorem). *Suppose β_1, β_2 and z_1, z_2, z_3 are nonzero complex numbers. If the subsets $\{\beta_1, \beta_2\}$ and $\{z_1, z_2, z_3\}$ are both $\mathbb{Q}$-linearly independent then at least one of the six numbers*

$$\exp(\beta_i \cdot z_j), \text{ with } (i, j) \in \{1, 2\} \times \{1, 2, 3\},$$

is transcendental.

Here then is the announced result:

Proposition 6.14. *Suppose that $\mathcal{P}$ is a set of positive algebraic numbers and set $B = \ln \text{gp}(\mathcal{P})$. If B is free abelian of positive rank, then $U(B) = \{1, -1\}$.*

Proof. Note first that every element of $P = \text{gp}(\mathcal{P})$ is a positive algebraic number. Consider now a unit u of B. Since B has positive rank, it contains a nonzero element $b_1 = \ln q_1$. Then $u \cdot b \in B \smallsetminus \{0\}$; so $b_2 = u \cdot b_1$ has the form $\ln q_2$ and thus u is either rational or transcendental (by the Gelfond–Schneider theorem).

Assume first that u is rational, say $u = m/n$, where m and n are relatively prime integers. The hypothesis $(m/n) \cdot B = B$ implies then that $mB = nB$. As B is free abelian of positive rank this equality can only hold if $|m| = |n| = 1$. So $u \in \{1, -1\}$.

Assume now that u is transcendental. Fix $p \in P \smallsetminus \{1\}$. Then $u \cdot \ln p \in B = \ln P$; so there exists $q \in P$ with $\ln q = u \cdot \ln p$; put differently, $\exp(u \cdot \ln p)$ lies in P and

is thus an algebraic number. As the powers of u are again units of B it follows that $\exp(u^\ell \cdot \ln p) \in P$ for every $\ell \in \mathbb{N}$. Set

$$\beta_1 = \ln p, \quad \beta_2 = u \cdot \ln p, \quad \text{and} \quad z_j = u^j \text{ for } j = 1, 2, 3.$$

Then the sets $\{\beta_1, \beta_2\}$ and $\{z_1, z_2, z_3\}$ fulfill the hypotheses of Theorem 6.13; its conclusion, however, is contradicted by the previous calculation. This state of affairs shows that the unit u cannot be transcendental. □

Example 6.15. Let $\mathcal{P}$ be a nonempty set of (rational) prime numbers and let P denote the subgroup of $\mathbb{Q}^\times_{>0}$ generated by $\mathcal{P}$. Then P is free abelian with basis $\mathcal{P}$ (by the unique factorization in $\mathbb{N}^\times_{>0}$) and so $U(\ln P) = \{1, -1\}$.

More generally, every nontrivial subgroup P of $\mathbb{Q}^\times_{>0}$ is a free abelian group and hence $B = \ln P$ has only the units 1 and -1.

6E4. *Some properties of transcendental subgroups and transcendental characters.* The transcendence of a character is a property that has not yet been discussed in the literature on the invariant Σ^1. In this section, we therefore assemble a few useful properties of this notion.

Assume $B \subset \mathbb{R}_{\text{add}}$ is a transcendental subgroup. Then:

(a) every nontrivial subgroup $B' \subseteq B$ is transcendental (immediate from the definition);

(b) if $\chi : G \to \mathbb{R}$ is a character whose image is a nontrivial subgroup of B then χ is transcendental (by (a)), and so are all the compositions $\chi \circ \pi$ with $\pi : \tilde{G} \twoheadrightarrow G$ an epimorphism of groups (immediate from the first part);

(c) if χ, χ' are characters of G with images equal to B, the image of $\chi + \chi'$ is contained in B, and so the character $\chi + \chi'$ is transcendental, unless it is 0;

(d) if χ is transcendental and $\alpha_1, \ldots, \alpha_m$ are automorphisms of G the character

$$\eta = \chi \circ \alpha_1 + \cdots + \chi \circ \alpha_m$$

is transcendental, unless it is zero.

A further property is discussed in part (iv) of Proposition 6.16 below.

6F. ***Passage to subgroups of finite index.*** The next proposition shows that the hypotheses stated in Corollary 6.10 are inherited by subgroups of finite index.

Proposition 6.16. *Let G be a subgroup of $\mathrm{PL}_o([0, b])$ and $H \subseteq G$ a subgroup of finite index. Denote the restrictions of χ_ℓ and χ_r to H by χ'_ℓ and χ'_r. Then the following statements are valid*:

(i) *G is irreducible if and only if H is irreducible*;

(ii) *χ_ℓ is nonzero precisely if χ'_ℓ is nonzero, and similarly for χ_r and χ'_r*;

(iii) *the characters χ_ℓ and χ_r are almost independent if and only if χ'_ℓ and χ'_r are almost independent*;

(iv) *χ_ℓ is transcendental exactly if χ'_ℓ is transcendental, and similarly for χ_r and χ'_r.*

Proof. Claim (i) holds since the support of a PL-homeomorphism f coincides with that of its positive powers f^m. Assertion (ii) is valid since the image of a character is a subgroup of $\mathbb{R}_{\text{add}}$ and hence torsion-free. The fact that the quotient b_1/b_2 of nonzero real numbers coincides, for every positive integer m, with the quotient $(mb_1)/(mb_2)$ allows one to see that a nonzero character χ of G is transcendental if its restriction to H is so; the converse is covered by property (a) stated in Section 6E4. We are left with establishing statement (iii).

To achieve this goal, we compare the quotient groups $G/(\ker\chi_\ell\cdot\ker\chi_r)$ and $H/(\ker\chi'_\ell\cdot\ker\chi'_r)$. By (6-11), the first of them is isomorphic to the quotient group $A_1=\operatorname{im}\chi_\ell/\chi_\ell(\ker\chi_r)$; the second one is isomorphic to $A_2=\operatorname{im}\chi'_\ell/\chi'_\ell(\ker\chi'_r)$. Clearly, $A_2=\operatorname{im}\chi'_\ell/\chi_\ell(\ker\chi'_r)$. The groups A_1 and A_2 fit into the short exact sequences

$$A_2=\operatorname{im}\chi'_\ell/\chi_\ell(\ker\chi'_r)\hookrightarrow A=\operatorname{im}\chi_\ell/\chi_\ell(\ker\chi'_r)\twoheadrightarrow \operatorname{im}\chi_\ell/\operatorname{im}\chi'_\ell, \tag{6-12}$$

$$\chi_\ell(\ker\chi_r)/\chi_\ell(\ker\chi'_r)\hookrightarrow A=\operatorname{im}\chi_\ell/\chi_\ell(\ker\chi'_r)\twoheadrightarrow A_1=\operatorname{im}\chi_\ell/\chi_\ell(\ker\chi_r). \tag{6-13}$$

The claim now follows from the fact that $\operatorname{im}\chi_\ell/\operatorname{im}\chi'_\ell$ and $\chi_\ell(\ker\chi_r)/\chi_\ell(\ker\chi'_r)$ are finite groups with orders that divide the index of H in G. □

A first application of Proposition 6.16 is the following corollary:

Corollary 6.17. *Let G be a finitely generated, irreducible subgroup of $\mathrm{PL}_o(I)$. If the characters χ_ℓ and χ_r are almost independent and one of them is transcendental, then any group Γ commensurable[17] with G has property R_∞.*

Proof. Let $H_0\subset G$ be a finite index subgroup of G that is isomorphic to a finite index subgroup Γ_0 of Γ. There exists then a *finite index subgroup Γ_1 of Γ_0 that is characteristic in Γ*; see, e.g., [Lyndon and Schupp 1977, Theorem IV.4.7]. Let H_1 be the subgroup of H_0 that corresponds to Γ_1 under an isomorphism $H_0\xrightarrow{\sim}\Gamma_0$. Then H_1 has finite index in G and thus Proposition 6.16 allows us to infer that H_1 inherits the properties enunciated for G in the statement of Corollary 6.10. This corollary applies therefore to H_1 and shows that H_1 admits a nonzero homomorphism $\psi_1: H_1\to\mathbb{R}^\times_{>0}$ that is fixed by $\operatorname{Aut}H_1$. So H_1, and hence Γ_1, satisfy property R_∞. Use now the fact that Γ_1 is a characteristic subgroup of Γ and apply [Mubeena and Sankaran 2014a, Lemma 2.2(ii)] to infer that Γ satisfies property R_∞. □

[17] Two groups G_1 and G_2 are called *commensurable* if they contain subgroups H_1 and H_2 that are isomorphic and of finite indices in G_1 and in G_2, respectively.

Remark 6.18. If the group G_1 has property R_∞, then a group G_2 commensurable to G_1 need not have this property, as is shown by the fundamental group G_1 of the Klein bottle and the fundamental group G_2 of a torus: the group G_1 has property R_∞ by [Gonçalves and Wong 2009, Theorem 2.2]), while the automorphism $-\mathbb{1}$ of $G_2 = \mathbb{Z}^2$ has Reidemeister number 4.

7. Miscellaneous examples

In this section we illustrate by various examples the notions of irreducible subgroup, almost independence of χ_ℓ and χ_r, and the group of units.

7A. ***Irreducible subgroups.*** Let b be a positive real number and G a subgroup of $\mathrm{PL}_o([0, b])$. Recall that G is called *irreducible* if no interior point of $I = [0, b]$ is fixed by all of G (see Section 6D1 for the motivation that led to this name).

The group is irreducible if and only if the supports of the elements of G cover the interior $\mathrm{int}(I)$ of I or, equivalently, if the supports of the elements in a generating set $\mathcal{X}$ of G cover $\mathrm{int}(I)$; these claims are easily verified. If G is cyclic, generated by f, say, it is therefore irreducible if f fixes no point in $\mathrm{int}(I)$ or, equivalently, if $f^\varepsilon(t) < t$ for $t \in \mathrm{int}(I)$ and some sign ε. Such a function is often called a *bump*.

Example 7.1. Here is a very simple kind of PL-homeomorphism bump. Given a positive slope $s \neq 1$, set

$$f_s(t) = \begin{cases} \dfrac{1}{s} \cdot t, & \text{if } 0 \le t \le \dfrac{s}{s+1} \cdot b, \\ s\Big(t - \dfrac{s \cdot b}{s+1}\Big) + \dfrac{b}{s+1}, & \text{if } \dfrac{s}{s+1} \cdot b < t \le b. \end{cases} \tag{7-1}$$

Then f_s is continuous at $s/(s+1) \cdot b$; since $f_s(0) = 0$ and $f_s(b) = b$, the function f_s lies in $\mathrm{PL}_o([0, b])$. Let G_s denote the group generated by f_s and let α be the automorphism that sends f_s to its inverse f_s^{-1}. Then

$$(\chi_\ell \circ \alpha)(f_s) = \chi_\ell(f_s^{-1}) = -\chi_\ell(f_s);$$

similarly $(\chi_r \circ \alpha)(f_s) = -\chi_r(f_s)$, whence

$$\chi_\ell \circ \alpha = -\chi_\ell \quad \text{and} \quad \chi_r \circ \alpha = -\chi_r. \tag{7-2}$$

So neither χ_ℓ nor χ_r is fixed by $\mathrm{Aut}(G_s)$. However, Theorem 6.4 cannot be applied, as requirement (iii) is violated; indeed, $\ker \chi_\ell = \ker \chi_r = \{\mathbb{1}\}$, so $G_s/(\ker \chi_\ell \cdot \ker \chi_r)$ is infinite cyclic. The conclusion of Theorem 6.4 is likewise false, for $\Sigma^1(G)^c = \varnothing$ (this follows, e.g., from [Strebel 2013, Example A2.5a]). Property R_∞, finally, does not hold, either; for the Reidemeister number of the automorphism α is 2, as a simple calculation shows.

The groups in the previous example are cyclic; more challenging groups are considered in the following example:

Example 7.2. Let $d > 1$ be an integer and $s_1, \dots, s_d$ pairwise distinct, positive real numbers not equal to 1. For each index $i \in \{1, \dots, d\}$, define f_i by (7-1) with $s = s_i$, and set

$$G = G_{\{s_1,\dots,s_d\}} = \mathrm{gp}(f_1, \dots, f_d).$$

The group G inherits two properties from the group G_s in the previous example: it is irreducible (obvious), and the assignment $f_i \mapsto f_i^{-1}$ extends to an automorphism α; indeed, the special form of the elements f_i implies that conjugation by the reflection in the midpoint of $I = [0, b]$ sends f_i to its inverse. It follows, as before, that the relations (7-2) are valid; so neither χ_ℓ nor χ_r is fixed by $\mathrm{Aut}\, G$.

Now to another property of the automorphism α. The calculation

$$(\chi_\ell \circ \alpha)(f_i) = \chi_\ell(f_i^{-1}) = -\chi_\ell(f_i) = \chi_r(f_i) \tag{7-3}$$

is valid for every index i. It shows that α exchanges χ_ℓ and χ_r. It follows, in particular, that $\ker \chi_\ell = \ker \chi_r$ and so the quotient

$$G/(\ker \chi_\ell \cdot \ker \chi_r) = G/\ker \chi_\ell \xrightarrow{\sim} \mathrm{im}\, \chi_\ell = \mathrm{gp}(\ln s_1, \dots, \ln s_d)$$

is a nontrivial free abelian group of rank at most d. Requirement (iii) in Theorem 6.4 is thus violated and so we cannot use that result to determine $\Sigma^1(G)^c$. Actually, only the following meager facts are known about $\Sigma^1(G)^c$: both χ_ℓ and $\chi_r = -\chi_\ell$ represent points of $\Sigma^1(G)^c$ [Strebel 2015, Proposition 2.5]; moreover, the existence and form of the automorphism α and formula (6-6) imply that $\Sigma^1(G)^c$ is invariant under the antipodal map $[\chi] \mapsto [-\chi]$.

Computation (7-3) shows that $\chi_\ell \circ \alpha = -\chi_\ell$. This conclusion holds, actually, for every character $\chi : G \to \mathbb{R}$ and proves that no nonzero character of G is fixed by α.

7B. ***Independence of χ_ℓ and χ_r.*** As before, let G be a subgroup of $\mathrm{PL}_o([0, b])$ with b a positive real number. Recall that the characters χ_ℓ and χ_r are called *independent* if $G = \ker \chi_\ell \cdot \ker \chi_r$; see Section 6D1. It follows that χ_ℓ and χ_r are independent if and only if G admits a generating set $\mathcal{X} = \mathcal{X}_\ell \cup \mathcal{X}_r$ in which the elements of $\mathcal{X}_\ell$ have slope 1 near b and those of $\mathcal{X}_r$ have slope 1 near 0.

It is thus very easy to manufacture groups for which χ_ℓ and χ_r are independent. In the next example, some very particular specimens are constructed.

Example 7.3. Choose a real number $b_1 \in]b/2, b[$. Given a tuple of positive real numbers $s_1, \dots, s_{d_\ell}$ that are pairwise distinct and not equal to 1, let f_i be the bump defined by (7-1) but with $s = s_i$ and $b = b_1$. Next let $s'_1, \dots, s'_{d_r}$ be another sequence of positive reals that are pairwise distinct and different from 1. Use them to define bump functions g_j with supports in $]b - b_1, b[$ like this: let h_j be the function given by (7-1) but with $s = s'_j$ and $b = b_1$, and define then g_j to be h_j conjugated by the

translation with amplitude $b - b_1$. Finally set

$$G = G_{\{s_1,\dots,s_{d_\ell},s'_1,\dots s'_{d_r};b_1\}} = \mathrm{gp}(f_1, \dots, f_{d_\ell}, g_1, \dots, g_{d_r}). \tag{7-4}$$

From now on, we assume that d_ℓ and d_r are both positive. Then G is irreducible (since $b_1 > b - b_1$), the characters χ_ℓ, χ_r are nonzero and independent, and thus Theorem 6.4 allows us to conclude that $\Sigma^1(G)^c = \{[\chi_\ell], [\chi_r]\}$.

The character χ_ℓ is transcendental if all the positive reals $s_1, \dots, s_{d_\ell}$ are algebraic (see Example 6.12). Then G admits a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by $\mathrm{Aut}\, G$ (see Theorem 6.1). If G does not admit an automorphism α with $\chi_\ell \circ \alpha \in [\chi_r]$ the homomorphism ψ can be chosen to be σ_ℓ (see the second paragraph of Section 6C). The stated condition holds, in particular, if there does not exists a number s with $\mathrm{im}\, \chi_r = s \cdot \mathrm{im}\, \chi_\ell$. Similar remarks apply to χ_r.

7B1. *Independence versus almost independence.* The characters χ_ℓ and χ_r are called almost independent if $G/(\ker \chi_\ell \cdot \ker \chi_r)$ is a torsion group (see Remark 6.7). Statement (iii) of Proposition 6.16 shows that almost independence of χ_ℓ and χ_r is inherited by the restricted characters $\chi'_\ell = \chi_\ell \restriction H$ and $\chi'_r = \chi_r \restriction H$ whenever $H \subseteq G$ is a subgroup of finite index. The next result characterizes those ordered pairs (G, H), with χ_ℓ, χ_r independent whose restrictions χ'_ℓ and χ'_r are again independent.

Lemma 7.4. *Let G be a subgroup of $\mathrm{PL}_o([0, b])$ for which χ_ℓ and χ_r are independent and let $H \subset G$ be a subgroup of finite index. Then the restrictions χ'_ℓ and χ'_r of these characters are independent if and only if the homomorphism*

$$\zeta : \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) \longrightarrow \mathrm{im}\, \chi_\ell/\, \mathrm{im}\, \chi'_\ell, \tag{7-5}$$

induced by the inclusions, is injective.

Proof. The justification will be an assemblage of facts extracted from the proof of Lemma 6.6 and from that of Proposition 6.16. Firstly, χ_ℓ and χ_r are independent if and only if the abelian group $A_1 = \mathrm{im}\, \chi_\ell/\chi_\ell(\ker \chi_r)$ is 0. Similarly, χ'_ℓ and χ'_r are independent precisely if $A_2 = \mathrm{im}\, \chi'_\ell/\chi_\ell(\ker \chi'_r)$ is the zero group. The groups A_1 and A_2 occur among the groups in the short exact sequences (6-12) and (6-13). Since $A_1 = 0$, these exact sequences lead to the short exact sequence

$$\mathrm{im}\, \chi'_\ell/\chi_\ell(\ker \chi'_r) \hookrightarrow \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) = \mathrm{im}\, \chi_\ell/\chi_\ell(\ker \chi'_r) \twoheadrightarrow \mathrm{im}\, \chi_\ell/\, \mathrm{im}\, \chi'_\ell,$$

It shows that $A_2 = \mathrm{im}\, \chi'_\ell/\chi_\ell(\ker \chi'_r)$ is the kernel of the homomorphism ζ. □

It is now easy to construct independent characters χ_ℓ and χ_r of G whose restrictions to a subgroup of finite index are no longer independent.

Example 7.5. Let G be a subgroup of $\mathrm{PL}_o([0, b])$ and H a subgroup of finite index. Assume the characters χ_ℓ and χ_r are independent. According to Lemma 7.4 the

restricted characters χ'_ℓ and χ'_r of H are independent if and only if the obvious homomorphism

$$\zeta : \chi_\ell(\ker \chi_r)/\chi_\ell(\ker \chi'_r) \longrightarrow \operatorname{im} \chi_\ell / \operatorname{im} \chi'_\ell$$

is injective. The characters χ'_ℓ and χ'_r of H will therefore *not* be independent whenever

$$\operatorname{im} \chi'_\ell = \operatorname{im} \chi_\ell \quad \text{but} \quad \chi_\ell(\ker \chi_r \cap H) \subsetneq \chi_\ell(\ker \chi_r). \tag{7-6}$$

Now to some explicit examples. We begin with quotients of the groups we shall ultimately be interested in. Set $\overline{G} = \mathbb{Z}^2$, let $p \geq 2$ be an integer and set

$$\overline{H} = \mathbb{Z}(p, 0) + \mathbb{Z}(1, 1).$$

Then $\overline{H}$ has index p in $\overline{G}$.

Next, let χ_1, χ_2 denote the canonical projections of $\mathbb{Z}^2$ onto its factors. Then

$$\chi_1(\overline{G}) = \mathbb{Z} = \chi_1(\overline{H}), \quad \ker \chi_2 = \mathbb{Z}(1, 0), \text{ and } \ker \chi_2 \cap \overline{H} = \mathbb{Z}(1, 0) \cap \overline{H} = \mathbb{Z}(p, 0),$$

and thus

$$\chi_1(\ker \chi_2 \cap \overline{H}) = \mathbb{Z} \cdot p \subsetneq \chi_1(\ker \chi_2) = \mathbb{Z}.$$

The auxiliary groups $\overline{G}$ and $\overline{H}$ therefore satisfy the relations (7-6).

We are now ready to define the group G; it will be of the kind considered in Example 7.3 with $d_\ell = d_r = 1$. Fix $b > 0$ and $b_1 \in]b/2, b[$ and choose positive numbers s_1, s'_1, both different from 1. Define f_1 and g_1 as in Example 7.3 and set

$$G = \operatorname{gp}(f_1, g_1).$$

Then G is an irreducible subgroup of $\mathrm{PL}_0([0, b])$ and the characters χ_ℓ and χ_r of G are independent. Moreover, G_{ab} is free abelian of rank 2, freely generated by the canonical images of f_1 and g_1. Set $H = \operatorname{gp}(f_1^p, f_1 \circ g_1, [G, G])$. The above calculations then imply that

$$\chi_\ell(G) = \mathbb{Z} \cdot \ln s_1 = \chi_\ell(H) \quad \text{and} \quad \chi_\ell(\ker \chi_r \cap H) = \mathbb{Z} \cdot p \cdot \ln s_1 \subsetneq \chi_\ell(\ker \chi_r) = \mathbb{Z} \ln s_1.$$

7C. ***Eigenlines.*** Let G be an irreducible subgroup of $\mathrm{PL}_o([0, b])$. If the characters χ_ℓ and χ_r are nonzero and almost independent, then $\Sigma^1(G)^c$ consist of the two points $[\chi_\ell]$ and $[\chi_r]$ (by Theorem 6.4). Every automorphism α of G either fixes or exchanges them. Suppose we are in the first case. Then $\chi_r \circ \alpha = s \cdot \chi_r$ for some positive real s, and so $\mathbb{R} \cdot \chi_r$ is an eigenline, with eigenvalue s, in the vector space $\operatorname{Hom}(G, \mathbb{R})$ acted on by α^*. No example with $s \neq 1$ has been found so far.

If the compact interval $[0, b]$ is replaced by the half-line $[0, \infty[$, such examples exist, provided χ_r is replaced by a suitable analogue τ_r. In order to construct examples, we return to the set-up of Section 4. So P is a nontrivial subgroup of

$\mathbb{R}^{\times}_{>0}$ and A is a nontrivial $\mathbb{Z}[P]$ submodule of $\mathbb{R}_{\text{add}}$. Define G to be the kernel of the homomorphism $\sigma_r : G([0,\infty[;\, A,\, P) \to \mathbb{R}^{\times}_{>0}$; thus G consists of all the elements of $G([0,\infty[;\, A,\, P)$ that are translations near $+\infty$. The analysis in Section 4C2 shows that conjugation by the PL-homeomorphism $f_p : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$, given by $f_p(t) = p \cdot t$ for $t \geq 0$, induces, for every $p \in P$, an automorphism α_p of G that satisfies the relation

$$\tau_r \circ \alpha_p = p \cdot \tau_r; \tag{7-7}$$

here $\tau_r : G \to \mathbb{R}_{\text{add}}$ is the character that sends $g \in G$ to the amplitude of the translation that coincides with g near $+\infty$. This character τ_r shares an important property with the character χ_r: the invariant $\Sigma^1(G)^c$ consists of two points, one represented by χ_ℓ, the other by τ_r; see [Strebel 2015, Theorem 1.2].

The image of τ_r in $\mathbb{R}_{\text{add}}$ is a subgroup B of A, namely

$$B = I\,P \cdot A = \Big\{ \sum (p-1) \cdot a \;\Big|\; p \in P \text{ and } a \in A \Big\}$$

(see assertion (iii) of [Bieri and Strebel 2016, CorollaryeA5.3]). The group of units $U(B)$ of B contains the group P and so it is not reduced to $\{1, -1\}$.

The subgroup B is typically infinitely generated; if so, G is likewise infinitely generated. Examples of finitely generated groups $G = \ker \sigma_r$ are harder to find, and they are so far rare. Suppose the group $G([0, b];\, A,\, P)$ is finitely generated for some $b \in A_{>0}$. Then $G([b, 2b];\, A,\, P)$ is a finitely generated subgroup of the group of bounded elements $B([0,\infty[;\, A,\, P)$. Pick now an element $g_0 \in G$ that moves every point of the open interval $]0,\infty[$ to the right and satisfies the inequality $g_0(b) < 2b$. Then translates of the interval $]b, 2b[$ under the powers of g_0 will then cover $]0,\infty[$. It follows that the subgroup

$$N = \operatorname{gp}\big(\{g_0^j \circ G([b, 2b];\, A,\, P) \circ g_0^{-j} \mid j \in \mathbb{Z}\}\big)$$

coincides with the bounded group $B([0,\infty[;\, A,\, P)$ (use [Bieri and Strebel 2016, Lemma E18.9]). So the group $B([0,\infty[;\, A,\, P) \rtimes \operatorname{gp}(g_0)$ is finitely generated. The group G, finally, is finitely generated if $G/N \xrightarrow{\sim} \operatorname{im} \tau_r = I\,P \cdot A$ is finitely generated.

To show that finitely generated groups of the form $G = \ker \sigma_r$ exist we need thus an example of a group $G([0, b];\, A,\, P)$ where both $G([0, b];\, A,\, P)$ and the abelian group underlying $B = I\,P \cdot A$ are finitely generated. The parameters

$$P = \operatorname{gp}\big(\sqrt{2}+1\big), \quad A = \mathbb{Z}\big[\sqrt{2}\big] = \mathbb{Z}[P], \quad b = 1$$

lead to such a group; see [Cleary 1995].

7D. ***Variation on Theorem 6.1.*** Among the hypotheses of Theorems 6.1 and 6.4 figures the requirement that G acts irreducibly on the interval $[0, b]$. This requirement rules out, in particular, that G is a product $G_1 \times G_2$ with G_1 acting irreducibly

on some interval $I_1 = [0, b_1]$ and G_2 acting irreducibly on an interval $I_2 = [b_2, b]$ that is disjoint from I_1.

Now suppose we are in this excluded case and that the groups G_1, G_2 satisfy the assumptions of Theorem 6.4, suitably interpreted; more explicitly, suppose the characters $\chi_{1,\ell}$ and $\chi_{1,r}$ of G_1 are nonzero and almost independent, and similarly for the characters $\chi_{2,\ell}$ and $\chi_{2,r}$ of G_2. The question then arises whether $G = G_1 \times G_2$ admits a nonzero homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by $\operatorname{Aut} G$. We shall see that this is the case if at least one of the four groups $\operatorname{im} \chi_{1,\ell}$, $\operatorname{im} \chi_{1,r}$ and $\operatorname{im} \chi_{2,\ell}$, $\operatorname{im} \chi_{2,r}$ has a unit group that is reduced to $\{1, -1\}$.

The following proposition is a variation on Theorem 3.2 in [Gonçalves and Kochloukova 2010].

Proposition 7.6. *Let G be a group for which $\Sigma^1(G)^c$ is a nonempty finite set with m elements. Assume the rays $[\chi] \in \Sigma^1(G)^c$ span a subspace of $\operatorname{Hom}(G, \mathbb{R})$ having dimension m over $\mathbb{R}$ and that $U(\operatorname{im} \chi_1) = \{1, -1\}$ for some point $[\chi_1] \in \Sigma^1(G)^c$. Then G admits a nontrivial homomorphism $\psi : G \to \mathbb{R}^\times_{>0}$ that is fixed by $\operatorname{Aut} G$.*

Proof. The automorphism group $\operatorname{Aut} G$ acts on $\Sigma^1(G)^c$ via the assignment

$$(\alpha, [\chi]) \mapsto [\chi \circ \alpha^{-1}];$$

let $\{[\chi_1], \dots, [\chi_n]\}$ be the orbit in $\Sigma^1(G)^c$ containing $[\chi_1]$. If $n = 1$, the point $[\chi_1]$ is fixed by $\operatorname{Aut} G$; hence χ_1 itself is fixed by $\operatorname{Aut} G$ in view of the assumption that $U(\operatorname{im} \chi_1) = \{1, -1\}$, and so we can take $\psi = \exp \circ \chi_1$.

Now suppose that $n > 1$ and choose, for every $i \in \{1, \dots, n\}$, an automorphism α_i with $[\chi_i] = [\chi_1 \circ \alpha_i]$. Let α be an automorphism of G. For every index $i \in \{1, \dots, n\}$ there exists then an index j so that $[\chi_i \circ \alpha^{-1}] = [(\chi_1 \circ \alpha_i) \circ \alpha^{-1}]$ is equal to $[\chi_j] = |\chi_1 \circ \alpha_j]$. It follows that there exists a positive real number $s_{i,j}$ so that

$$\chi_1 \circ \alpha_i \circ \alpha^{-1} = s_{i,j} \cdot \chi_1 \circ \alpha_j.$$

But if so, $\beta = \alpha_i \circ \alpha^{-1} \circ \alpha_j^{-1}$ is an automorphism with $\chi_1 \circ \beta = s_{i,j} \cdot \chi_1$. The assumption that $U(\operatorname{im} \chi_1) = \{1, -1\}$ permits one then to deduce that $s_{i,j} = 1$. So $\operatorname{Aut} G$ permutes the set of characters

$$\chi_1 \circ \alpha_1, \quad \chi_1 \circ \alpha_2, \quad \dots, \quad \chi_1 \circ \alpha_n. \tag{7-8}$$

Their sum η is therefore fixed by $\operatorname{Aut} G$. It is nonzero since the characters displayed in (7-8) are linearly independent over $\mathbb{R}$. Set $\psi = \exp \circ \eta$. □

Corollary 7.7. *Let G_1 be a subgroup of $\mathrm{PL}_0([0, b_1])$ and let G_2 be a subgroup of $\mathrm{PL}_0([b_2, b])$ with $0 < b_1 < b_2 < b$. Assume G_1 and G_2 are irreducible, the characters $\chi_{1,\ell}$ and $\chi_{1,r}$ of G_1 are nonzero and almost independent, and that the characters $\chi_{2,\ell}$ and $\chi_{2,r}$ of G_2 have the same properties. If the image of at least one of the four characters $\chi_{1,\ell}$, $\chi_{1,r}$ and $\chi_{2,\ell}$, $\chi_{2,r}$ has a unit group that is reduced*

to $\{1, -1\}$ *then* $G = G_1 \times G_2$ *admits a nontrivial homomorphism* $\psi : G \to \mathbb{R}^\times_{>0}$ *that is fixed by* Aut G.

Proof. The hypothesis on G_1 and G_2 allow us to apply Theorem 6.4 and so

$$\Sigma^1(G_1)^c = \{[\chi_{1,\ell}], [\chi_{1,r}]\} \quad \text{and} \quad \Sigma^1(G_2)^c = \{[\chi_{2,\ell}], [\chi_{2,r}]\}.$$

The product formula for Σ^1 then implies that $\Sigma^1(G)^c$ consists of the four points represented by

$$\chi_{1,\ell} \circ \pi_1, \quad \chi_{1,r} \circ \pi_1, \quad \chi_{2,\ell} \circ \pi_2, \quad \chi_{1,\ell} \circ \pi_2; \tag{7-9}$$

here $\pi_i : G \twoheadrightarrow G_i$ denotes the canonical projection onto the i-th factor G_i (see, e.g., [Strebel 2013, Proposition C2.55]). These four characters are $\mathbb{R}$-linearly independent since all are nonzero, as $\ker \chi_{1,\ell} \neq \ker \chi_{1,r}$ by the almost independence of $\chi_{1,\ell}$ and $\chi_{1,r}$, as $\ker \chi_{2,\ell} \neq \ker \chi_{2,r}$ by the almost independence of $\chi_{2,\ell}$ and $\chi_{2,r}$, and since $\pi_1^*(\mathrm{Hom}(G_1, \mathbb{R}))$ and $\pi_2^*(\mathrm{Hom}(G_2, \mathbb{R}))$ are complementary subspaces of $\mathrm{Hom}(G, \mathbb{R})$. Finally, at least one of the images of the four characters displayed in (7-9) has an image B with $U(B) = \{1, -1\}$. All the assumptions of Proposition 7.6 are thus satisfied, so the contention of the corollary follows from that proposition. □

Remark 7.8. It is not known whether the direct product of groups G_1, G_2 each of which has property R_∞ has again property R_∞. The previous corollary implies that this will be so if the groups G_1 and G_2 satisfy the assumptions of the corollary.

8. Complements

By Remark 5.8 there exist continuously many pairwise nonisomorphic, finitely generated groups of the form $G(\mathbb{R}; A, P)$, and by Corollary 5.6 each of these groups admits a nonzero homomorphism ψ into P. These facts prompt the question whether there exist similarly large collections of finitely generated subgroups of $\mathrm{PL}_o(I)$ with I a compact interval, say $I = [0, 1]$. Since only countably many *finitely generated* groups of the form $G([0, 1]; A, P)$ have been found so far, we look for finitely generated groups that satisfy the assumptions of Theorem 6.1.

In Section 8A we exhibit a collection $\mathcal{G}$ of 3-generator groups with the desired properties. Checking that each group in $\mathcal{G}$ satisfies the assumptions of Theorem 6.1 is fairly easy; the verification that distinct groups in $\mathcal{G}$ are not isomorphic, however, is more demanding. We shall succeed by exploiting properties of the Σ^1-invariant of the groups in $\mathcal{G}$ in a roundabout manner. In Section 8B we describe then a collection of 2-generator groups which, despite appearances, turn out to be pairwise isomorphic. This indicates once more that criteria which allow one to prove that two given, similarly looking, groups are not isomorphic, are very useful. In the final section, we give such a criterion.

8A. ***A large collection of groups G with characters fixed by*** **Aut** ***G.*** In this section we construct a collection $\mathcal{G}$ of pairwise nonisomorphic groups G_s with the following properties:

(i) each $G_s \in \mathcal{G}$ is an irreducible subgroup of $\mathrm{PL}_o([0,1])$ generated by 3 elements;

(ii) the characters χ_ℓ, χ_r of G_s are independent and have ranks 1, respectively 2;

(iii) for each $G_s \in \mathcal{G}$ the character χ_ℓ is fixed by $\mathrm{Aut}\, G_s$; and

(iv) the cardinality of $\mathcal{G}$ is that of the continuum.

8A1. *Construction of the groups* G_s. The groups G_s are obtained by the recipe described in Example 7.3. Fix a triple $s = (s_1, s_2 = s_1', s_3 = s_2')$ of real numbers in $]1, \infty[$. Let f_s be the PL-homeomorphism defined by (7-1) with $s = s_1$ and $b = \frac{3}{4}$. Next, let g be the function obtained by putting $s = s_2$, $b = \frac{3}{4}$ and by then conjugating the function so obtained by translation with amplitude $\frac{1}{4}$. Similarly, let h_s be the function obtained by setting $s = s_3$, $b = \frac{3}{4}$ and by conjugating the function so obtained by the translation $t \mapsto t + \frac{1}{4}$. Finally, set

$$G_s = G_{\{s_1, s_2, s_3\}} = \mathrm{gp}(f_s, g_s, h_s) \tag{8-1}$$

The definition of G_s shows that it is an irreducible subgroup of $\mathrm{PL}_o([0,1])$ with nonzero and independent characters χ_ℓ and χ_r. By Theorem 6.4, the complement of $\Sigma^1(G_s)$ consists therefore of the two rays $[\chi_\ell]$ and $[\chi_r]$.

Consider now an automorphism α of G_s. It induces an autohomeomorphism α^* of the sphere $S(G_s)$ that maps the subset $\Sigma^1(G_s)^c$ onto itself. Suppose α^* is the identity on $\Sigma^1(G_s)^c$. Since χ_ℓ has rank 1 and is thus transcendental, the first two paragraphs of Section 6C apply and show that α fixes the character χ_ℓ and hence also the homomorphism $\sigma_\ell : G \to \mathbb{R}^\times_{>0}$. This homomorphism σ_ℓ will therefore be fixed by all of $\mathrm{Aut}\, G_s$ whenever the images of χ_ℓ and χ_r are not isomorphic.

8A2. *Additional assumptions.* Assume therefore that s_2 and s_3 are *multiplicatively independent*. Then the free abelian group

$$\mathrm{im}\, \chi_r = \ln \mathrm{gp}(\{s_2, s_3\}) = \mathbb{Z} \ln s_2 + \mathbb{Z} \ln s_3.$$

has rank 2.

Consider now two triples s and s' where $s_2' = s_2$ and where both pairs $\{s_2, s_3\}$ and $\{s_2, s_3'\}$ are multiplicatively independent. Suppose there exists an isomorphism $\beta : G_s \xrightarrow{\sim} G_{s'}$. Then β induces a homeomorphism

$$\beta^* : S(G_{s'}) \xrightarrow{\sim} S(G_s)$$
$$\beta^*\big(\{[\chi_\ell'], [\chi_r']\}\big) = \{[\chi_\ell], [\chi_r]\}.$$

The ranks of the involved characters imply that $\beta^*[\chi_r'] = [\chi_r]$; so there exists a positive real number u with $\chi_r' \circ \beta = u \cdot \chi_r$. It follows that $\mathrm{im}\, \chi_r' = u \cdot \mathrm{im}\, \chi_r$ or,

equivalently, that

$$\mathbb{Z}(\ln s_3') + \mathbb{Z}(\ln s_2) = u \cdot \big(\mathbb{Z}(\ln s_3) + \mathbb{Z}(\ln s_2)\big).$$

This equality amounts to saying that there exists a matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \ln s_3' \\ \ln s_2 \end{pmatrix} = u \cdot T \cdot \begin{pmatrix} \ln s_3 \\ \ln s_2 \end{pmatrix} = u \cdot \begin{pmatrix} a \cdot \ln s_3 + b \cdot \ln s_2 \\ c \cdot \ln s_3 + d \cdot \ln s_2 \end{pmatrix}.$$

It follows that

$$\frac{\ln s_3'}{\ln s_2} = \frac{a(\ln s_3 / \ln s_2) + b}{c(\ln s_3 / \ln s_2) + d};$$

alternatively put, the numbers $\ln s_3$, $\ln s_3'$ lie in the same orbit of the group

$$H_{s_2} = \begin{pmatrix} \ln s_2 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{GL}(2, \mathbb{Z}) \cdot \begin{pmatrix} \ln s_2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{8-2}$$

acting on the extended real line $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations.

8A3. *Consequences.* It is now easy to exhibit a collection of groups $\mathcal{G}$ that enjoy the properties stated at the beginning of Section 8A. Choose first a number $s_1 > 1$; for instance $s_1 = 2$, and select s_2 so that $\ln s_2$ is rational, for instance $s_2 = \exp 1$. The group H_{s_2} is then a subgroup of $\mathrm{GL}(2, \mathbb{Q})$; it acts on $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations. The set $\mathbb{Q} \cup \{\infty\}$ is an orbit; all other orbits are made up of irrational numbers. Use the axiom of choice to find a set of representative $\mathcal{T}$ of the orbits of H_{s_2} contained in $\mathbb{R} \smallsetminus \mathbb{Q}$. For every $t \in \mathcal{T}$ the numbers $\ln s_2$ and t are then $\mathbb{Q}$-linearly independent, and hence s_2 and $\exp t$ are multiplicatively independent. Since $\mathbb{R} \smallsetminus \mathbb{Q}$ has the cardinality of the continuum and H_{s_2} is countable, the set $\mathcal{T}$ likewise has the cardinality of the continuum. The collection

$$\mathcal{G} = \{G_{(s_1, s_2, \exp t)} \mid t \in \mathcal{T}\} \tag{8-3}$$

therefore enjoys properties (i) through (iv) stated at the beginning of Section 8A.

8B. ***Some unexpected isomorphisms.*** Let t_1, t_2 be distinct irrational numbers and consider the groups $G_1 = G_{(2, \exp 1, \exp t_1)}$ and $G_2 = G_{(2, \exp 1, \exp t_2)}$. We don't know under which conditions on t_1 and t_2 the groups G_1 and G_2 are isomorphic. In the construction of the collection $\mathcal{G}$, carried out in Section 8A, we proceeded therefore in a very cautious manner and required that distinct elements in the parameter space $\mathcal{T}$ fail to satisfy a certain condition. The question now arises whether this approach is overly pessimistic. The next example indicates that caution may have been appropriate. We begin with a simple, but surprising, lemma.[18]

[18] Strebel got word of this result in discussions with Matt Brin and Matt Zaremsky.

Lemma 8.1. *Suppose that G is a subgroup of $\mathrm{PL}_o([a,d])$ generated by two PL-homeomorphisms f and g with the following properties*:

(i) $\operatorname{supp} f =]a,c[$ *and* $f(t) < t$ *for* $t \in \operatorname{supp} f$,

(ii) $\operatorname{supp} g =]b,d[$ *and* $g(t) < t$ *for* $t \in \operatorname{supp} g$,

(iii) $a < b < c < d$ *and* $f(g(c)) \leq b$.

Then G is isomorphic to Thompson's group F.

Proof. Set $h = f \circ g$ and note that $h(t) < t$ for every $t \in]a,d[$. Property (iii) then implies that $h(c) \leq b$ and so the supports of g and that of ${}^h f = h \circ f \circ h^{-1}$ are disjoint, as are the supports of g and that of ${}^{h^2} f$. The first fact implies that g commutes with ${}^h f$ and leads to the chain of equations

$$ {}^{h \circ h} f = {}^f\big({}^{g \circ h} f\big) = {}^f\big({}^h f\big) = {}^{f \circ h} f. \tag{8-4} $$

The second fact leads to the equations

$$ {}^{h \circ h^2} f = {}^f\big({}^{g \circ h^2} f\big) = {}^f\big({}^{h^2} f\big) = {}^{f \circ h^2} f. \tag{8-5} $$

Thompson's group F, on the other hand, has the presentation

$$ \big\langle x, x_1 \;\big|\; {}^{x^2} x_1 = {}^{x_1 x} x_1,\ {}^{x^3} x_1 = {}^{x_1 x^2} x_1 \big\rangle; $$

see, e.g., [Bieri and Strebel 2016, Examples D15.11]. The assignments $x \mapsto h$, $x_1 \mapsto f$ extend therefore to an epimorphism $\rho : F \twoheadrightarrow G$. As the derived group of F is simple (see, e.g., [Cannon et al. 1996, Theorem 4.5]) and as G is nonabelian, ρ must be injective, hence an isomorphism, and so the proof is complete. □

Our next result shows that the assumptions of the previous lemma can be satisfied by PL-homeomorphisms with preassigned values for the slopes in the end points.

Lemma 8.2. *Let s_f, s_g be positive reals with $s_f < 1 < s_g$ and let a, b, c, d be real numbers with $a < b < c < d$. Then there exist PL-homeomorphisms f and g that satisfy properties* (i) *through* (iii) *listed in Lemma 8.1 and, in addition*,

(iv) $f'(a) = s_f \quad$ *and* $\quad g'(d) = s_g$.

Proof. The generators f and g will both be affine interpolations of 5 interpolation points. To define them fix numbers t_1, t_2, t_3, t_4 so that

$$ a < t_1 < b < t_2 \leq t_3 < c < t_4 < d. $$

Next choose $t_0 \in]a, t_1[$ so that $(t_0 - a)/(t_1 - a) = s_f$. Then $a < t_1 < t_3 < c < d$ and $a < t_0 < b < c < d$ and so the affine interpolation, given by the 5 points

$$ (a,a), \quad (t_1,t_0), \quad (t_3,b), \quad (c,c), \quad (d,d), $$

exists and is an increasing PL-homeomorphism, say f, with $f'(a) = s_f$. Next, there exists a number $t_5 \in]t_4, d[$ so that $(d - t_4)/(d - t_5) = s_g$. Then $a < b < c < t_5 < d$ and $a < b < t_2 < t_4 < d$ and so the affine interpolation, given by the 5 points

$$(a, a), \quad (b, b), \quad (c, t_2), \quad (t_5, t_4), \quad (d, d),$$

exists and is an increasing PL-homeomorphism, say g; the definition of t_5 implies, in addition, that $g'(d) = s_g$. Finally, $f(g(c)) = f(t_2) \leq f(t_3) = b$. □

Remarks 8.3. (a) In the statement of Lemma 8.2 the slopes s_f and s_g have been chosen so that $s_f < 1 < s_g$. This requirement can be weakened to $s_f \neq 1$ and $s_g \neq 1$; indeed the four pairs $\{f, g\}$, $\{f, g^{-1}\}$ and $\{f^{-1}, g\}$, $\{f^{-1}, g^{-1}\}$ generate the same group.

(b) The generators f_s, g_s and h_s of the groups G_s, constructed in Section 8A, are simpler then those used in Lemma 8.2 in that they are defined by affine interpolations of 3 rather than of 5 points. But a variant of the Lemma 8.2 holds even in this more restricted setup.

Suppose $s_1 = s_2 = 2$ and $s_3 \geq 2$. The function f_s is then given by the formula

$$f_s(t) = \begin{cases} \frac{1}{2}t, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2\big(t - \frac{1}{2}\big) + \frac{1}{4}, & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}, \end{cases} \tag{8-6}$$

and it is the identity outside of $]0, \frac{3}{4}[$, while g_s is defined by

$$g_s(t) = \begin{cases} \frac{1}{2}\big(t - \frac{1}{4}\big) + \frac{1}{4}, & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4}, \\ 2\big(t - \frac{3}{4}\big) + \frac{1}{2}, & \text{if } \frac{3}{4} \leq t \leq 1, \end{cases} \tag{8-7}$$

and it is the identity outside of $]\frac{1}{4}, 1[$. The function h_s, finally, is defined by

$$h_s(t) = \begin{cases} \frac{1}{s_3}\big(t - \frac{1}{4}\big) + \frac{1}{4}, & \text{if } \frac{1}{4} \leq t \leq \frac{3s_3}{4(s_3+1)} + \frac{1}{4}, \\ s_3\big(t - \frac{3s_3}{4(s_3+1)} - \frac{1}{4}\big) + \frac{3}{4(s_3+1)} + \frac{1}{4}, & \text{if } \frac{3s_3}{4(s_3+1)} + \frac{1}{4} \leq t \leq 1, \end{cases} \tag{8-8}$$

and is the identity outside of $]\frac{1}{4}, 1[$. The function g_s is not differentiable at $\frac{1}{4}$, $\frac{3}{4}$, and 1, while the function h_s has singularities at $\frac{1}{4}$, $t_* = \frac{3s_3}{4(s_3+1)} + \frac{1}{4}$, and 1. Since $s_3 \geq s_2 = 2$, the inequality $t_* \geq \frac{3}{4}$ holds, as one verifies easily. The calculation

$$f_s\big(h_s\big(\tfrac{3}{4}\big)\big) = f_s\big(\tfrac{1}{s_3}\big(\tfrac{3}{4} - \tfrac{1}{4}\big) + \tfrac{1}{4}\big) = f_s\big(\tfrac{1}{2s_3} + \tfrac{1}{4}\big) \leq f_s\big(\tfrac{1}{4} + \tfrac{1}{4}\big) \leq \tfrac{1}{4}.$$

then shows that the functions f_s and h_s fulfill the assumptions imposed on the functions f and g in Lemma 8.1. It follows that the groups $\mathrm{gp}(f_s, h_s)$ are isomorphic to each other for every $s_3 \geq s_2 = 2$.

8C. ***A criterion.*** The groups G_s studied in Section 8A are generated by 3 elements; in addition the image of χ_ℓ is infinite cyclic and that of χ_r is free abelian of rank 2. Any isomorphism $\beta : G_s \xrightarrow{\sim} G_{s'}$ between two such groups must therefore induce an homeomorphism $\beta^* : S(G_{s'}) \xrightarrow{\sim} S(G_s)$ with $\beta^*([\chi'_r]) = [\chi_r]$. This consequence amounts to say that there exists a positive real number u so that $\chi'_r \circ \beta = u \cdot \chi_r$ and this new condition implies the equality

$$\text{im}\, \chi'_r = u \cdot \text{im}\, \chi_r. \tag{8-9}$$

In Section 8A we did not study this condition in general; we dealt only with the special case where

$$\text{im}\, \chi_r = \mathbb{Z}(\ln s_3) + \mathbb{Z}(\ln s_2) \quad \text{and} \quad \text{im}\, \chi'_r = \mathbb{Z}(\ln s'_3) + \mathbb{Z}(\ln s_2)$$

and exploited then the fact that, in this particular case, condition (8-9) involves basically only the two numbers $\ln s_3$ and $\ln s'_3$. In this final section we shall investigate another special case. It is reminiscent of a situation considered in Section 6E.

Let B_1 and B_2 be finitely generated subgroups of $\mathbb{R}_{\text{add}}$ and suppose there exists a positive real number u with $B_2 = u \cdot B_1$. If B_2 coincides with B_1, then u is a unit of B_1 and the results of Section 6E apply. They show, in particular, that $u = 1$ whenever B is the image under ln of a subgroup P of $\mathbb{R}^\times_{>0}$ that is generated by finitely many algebraic numbers. The proof of this consequence relies on Theorem 6.11, the Gelfond–Schneider theorem. Below we give an analogue of this criterion, but dealing with the equation $B_2 = u \cdot B_1$. In the proof, both the Gelfond–Schneider theorem and the Siegel–Lang theorem will be used.

Lemma 8.4. *Let P_1 and P_2 be subgroups $\mathbb{Q}^\times_{>0}$ and set $B_1 = \ln P_1$ and $B_2 = \ln P_2$. Suppose there exists a prime number π that occurs with nonzero power in the factorization of an element in P_1, but not in that of an element of P_2. If the rank of B_1 is at least 3, then B_2 is distinct from $u \cdot B_1$ for every positive real number u.*

Proof. Let $p_1 \in P_1$ be an element with a prime factorization that involves the prime π, and let u be a positive real number. Assume first that $u \in \mathbb{Q}$. Then π occurs in the prime factorization of p_1^u, so $q_1^u \notin P_2$, and thus $u \cdot B_1 = u \ln P_1 \neq \ln P_2 = B_2$. Now suppose that u is irrational and that $u \cdot \ln p_1 \in B_2$. There exists then a rational number $p_2 \in P_2$ with $u = \ln p_2 / \ln p_1$ and so u is transcendental by the Gelfond–Schneider theorem. Choose, finally, three $\mathbb{Q}$-linearly independent elements z_1, z_2 and z_3 in B_1 (this is possible as the rank of B_1 is at least 3) and consider the six numbers

$$\exp(1 \cdot z_j) \text{ with } j = 1, 2, 3, \quad \text{and} \quad \exp(u \cdot z_j) \text{ with } j = 1, 2, 3.$$

The first three of them are in P_1, and hence rational. As the subsets $\{1, u\}$ and $\{z_1, z_2, z_3\}$ are both linearly independent over $\mathbb{Q}$, Theorem 6.13 implies therefore

that at least one the remaining three numbers, say $\exp(u \cdot z_{j_*})$, is transcendental. This number is therefore outside of P_2 and so $u \cdot z_{j_*} \in uB_1 \smallsetminus B_2$. □

We end with an application of the preceding lemma.

Example 8.5. Given a nonempty set of prime numbers $\mathcal{P}$, let $G_{\mathcal{P}}$ be a subgroup of $\mathrm{PL}_o([0,1])$ generated by a set $\{f_p, g_p \mid p \in \mathcal{P}\}$ of elements that satisfy the conditions

(i) $\sigma_\ell(f_p) = p$, $\sigma_\ell(g_p) = 1$ and $\sigma_r(f_p) = 1$, $\sigma_\ell(g_p) = 1$ for every $p \in \mathcal{P}$;

(ii) the union of the supports of the generators f_p and g_p is $]0, 1[$.

The group $G_{\mathcal{P}}$ admits then an epimorphism $\psi : G_{\mathcal{P}} \twoheadrightarrow \mathrm{gp}(\mathcal{P})$ that is fixed by every automorphism of $G_{\mathcal{P}}$ (use Corollary 6.10). Moreover, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct sets of primes of cardinality at least 3, the groups $G_{\mathcal{P}_1}$ and $G_{\mathcal{P}_2}$ are not isomorphic in view of Lemma 8.4 and the considerations at the beginning of Section 8C.

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