

SHAPES OF POLYHEDRA, MIXED VOLUMES AND HYPERBOLIC GEOMETRY

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Abstract. We generalize to higher dimensions the Bavard–Ghys construction of the hyperbolic metric on the space of polygons with fixed directions of edges. The space of convex d -dimensional polyhedra with fixed directions of facet normals has a decomposition into type cones that correspond to different combinatorial types of polyhedra. This decomposition is a subfan of the secondary fan of a vector configuration and can be analyzed with the help of Gale diagrams. We construct a family of quadratic forms on each of the type cones using the theory of mixed volumes. The Alexandrov–Fenchel inequalities ensure that these forms have exactly one positive eigenvalue. This introduces a piecewise hyperbolic structure on the space of similarity classes of polyhedra with fixed directions of facet normals. We show that some of the dihedral angles on the boundary of the resulting cone-manifold are equal to $\pi/2$.

§0. Introduction.

0.1. *Motivation.* In [42], Thurston put a complex hyperbolic structure on the space $C(\alpha)$ of Euclidean metrics on the sphere with fixed cone angles $\alpha = (\alpha_1, \dots, \alpha_n)$ by showing that the area of the metric is a Hermitian form with respect to some natural choices of coordinates on the space of metrics. This provided a more elementary approach to the Deligne–Mostow examples of non-arithmetic complex Coxeter orbifolds.

Bavard and Ghys [4] adapted Thurston’s construction to the planar case by considering the space of convex polygons with fixed angles or, equivalently, with fixed directions of edges. On this space, the area of a polygon turns out to be a real quadratic form (with respect to the edge lengths) of the signature $(+, -, \dots, -)$. This turns the space of polygons with fixed edge directions into a hyperbolic polyhedron. By computing the dihedral angles of this polyhedron, Bavard and Ghys were able to obtain all hyperbolic Coxeter orthoschemes from the list previously drawn by Im Hof [18]. For more details, see [11]. The Bavard–Ghys polyhedron can be viewed as a subset of Thurston’s space, since gluing together two copies of a polygon along the boundary yields a Euclidean cone-metric on the sphere.

In the present paper, we generalize the construction of Bavard and Ghys to higher dimensions. Namely, we consider the space of d -dimensional polyhedra

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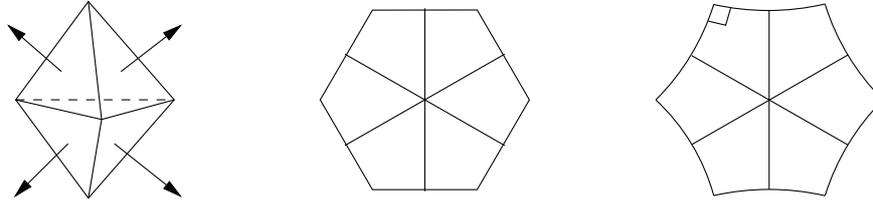


Figure 1: The space of polyhedra with face normals parallel to those of a triangular bipyramid is a right-angled hyperbolic hexagon.

with fixed directions V of the facet normals and exhibit a family of quadratic forms that makes it into a hyperbolic cone-manifold $M(V)$ with polyhedral boundary. We also obtain partial information about the cone angles in the interior and dihedral angles on the boundary of $M(V)$. Unlike the two-dimensional situations of Bavard–Ghys’ and Thurston’s works, in our situation it is, in general, difficult to control the angles. The only cases where we get complete information deal with simple combinatorial data or exploit the symmetries of a geometric configuration.

0.2. *Outline of the paper.* The study of the space of d -dimensional polyhedra with fixed facet normals poses some problems that are missing in the case of polygons $d = 2$. First, the facet normals do not determine the combinatorial structure of the polyhedron any more. In order to analyze the space of polyhedra, we employ machinery from discrete geometry, namely, Gale diagrams and secondary polyhedra [16]. This constitutes §1 of our paper.

Second, one needs to introduce a quadratic form and show that it has a hyperbolic signature. For $d = 3$ one can still take the surface area, but for $d > 3$ a new construction is needed. This is provided by mixed volumes, which yield a whole family of quadratic forms of hyperbolic signature, even for $d = 3$. The signature of the quadratic form is ensured by the Alexandrov–Fenchel inequalities. This aspect is discussed in §2.

Finally, the dihedral angles are now more difficult to compute. Positive and negative results in this direction are contained in §3.

Throughout the paper, we analyze several examples (see, in particular, §§1.3, 2.1 and 3.2). In fact, the reader can start from §§2.1 and 3.2 and then go to the other sections for a general theory and open questions.

Our main example is the space of three-dimensional polyhedra with six faces whose normals are parallel to those of a triangular bipyramid (see Figure 1, left). Translating each face independently yields six different combinatorial types, and the space of all polyhedra up to similarity forms a two-dimensional polyhedral complex, as shown in Figure 1, middle. The surface area is a quadratic form of signature $(+, -, -)$ on the space of polyhedra, which puts a hyperbolic metric on the complex making it a right-angled hyperbolic hexagon (Figure 1, right).

0.3. *Related work.* A different generalization (and dualization) of the Bavard–Ghys construction was given by Kapovich and Millson [22, 23] who considered the space of polygonal lines in \mathbb{R}^3 with fixed edge lengths.

By Alexandrov's theorem [2], every Euclidean metric with cone angles $\alpha_i < 2\pi$ is the intrinsic metric on the boundary of a unique convex polyhedron in \mathbb{R}^3 . Thus Thurston's space is the space of all convex polyhedra with fixed solid exterior angles at the vertices. A link between our construction and that of Thurston is discussed in §4, which contains also a number of other connections to discrete and hyperbolic geometry.

§1. *Shapes of polyhedra.*

1.1. *Normally equivalent polyhedra and type cones.*

1.1.1. *Convex polyhedra, polytopes and cones.* A convex polyhedron $P \subset \mathbb{R}^d$ is an intersection of a finite number of closed half-spaces. A bounded convex polyhedron is called a *polytope*; equivalently, a polytope is the convex hull of a finite number of points.

By $\text{aff}(P)$ we denote the affine hull of P , that is, the smallest affine subspace of \mathbb{R}^d containing P . The *dimension* of a convex polyhedron is the dimension of its affine hull.

A hyperplane H is called a *supporting hyperplane* of a convex polyhedron P if $H \cap P \neq \emptyset$ while P lies in one of the closed half-spaces bounded by H . The intersection $P \cap H$ is called a *face* of P (sometimes it makes sense to consider \emptyset and P as faces of P , too). Faces of dimension zero, one and $\dim P - 1$ are called vertices, edges and *facets*, respectively. A d -dimensional polytope is called *simple* if each of its vertices belongs to exactly d facets (equivalently, to exactly d edges).

A *convex polyhedral cone* is the intersection of a finite number of closed half-spaces whose boundary hyperplanes all pass through the origin. Equivalently, it is the positive hull

$$\text{pos}\{w_1, w_2, \dots, w_k\} := \left\{ \sum_{i=1}^k \lambda_i w_i \mid \lambda_i \geq 0, i = 1, \dots, k \right\}$$

of a finite number of vectors. A cone is called *pointed* if it contains no linear subspaces, except for $\{0\}$; equivalently, if it has $\{0\}$ as a face. A cone C is called *simplicial* if it is the positive hull of k linearly independent vectors. In this case, the intersection of C with an appropriately chosen hyperplane is a $(k - 1)$ -simplex.

See [10] or [46], for more details on polyhedra, cones and fans.

1.1.2. *Normal fans and normally equivalent polyhedra.*

Definition 1.1. A *fan* Δ in \mathbb{R}^d is a collection of convex polyhedral cones such that:

- (1) if $C \in \Delta$ and C' is a face of C , then $C' \in \Delta$; and
- (2) if $C_1, C_2 \in \Delta$, then $C_1 \cap C_2$ is a face of both C_1 and C_2 .

By $\Delta^{(k)}$ we denote the collection of k -dimensional cones of the fan Δ . The *support* of a fan is the union of all of its cones: that is,

$$\text{supp}(\Delta) := \bigcup_{\sigma \in \Delta} \sigma.$$

A fan Δ is called *complete* if $\text{supp}(\Delta) = \mathbb{R}^d$ and it is called *pointed* (respectively, *simplicial*) if all of its cones are pointed (respectively, simplicial). Complete simplicial fans in \mathbb{R}^d are in one-to-one correspondence with geodesic triangulations of \mathbb{S}^{d-1} .

For every proper face F of a convex polyhedron P there is a supporting hyperplane through F . The set of outward normals to all such hyperplanes spans a convex polyhedral cone, namely, the *normal cone* at F . Below is a more formal definition.

Definition 1.2. Let $P \subset \mathbb{R}^d$ be a convex polyhedron and let F be a non-empty face of P . The *normal cone* $N_F(P)$ of P at F is defined as

$$N_F(P) := \left\{ v \in \mathbb{R}^d \mid \max_{x \in P} \langle v, x \rangle = \langle v, p \rangle \text{ for all } p \in F \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^d . The *normal fan* of P is the collection of all normal cones of P : that is,

$$\mathcal{N}(P) := \{N_F(P) \mid F \text{ a proper face of } P\}.$$

Note that Definition 1.2 makes sense also when $\dim P < d$. In this case, every $N_F(P)$ contains the linear subspace $\text{aff}(P)^\perp$. The following are some simple facts about the normal fan.

- $\text{supp}(\mathcal{N}(P))$ is a convex polyhedral cone in \mathbb{R}^n , positively spanned by the normals to the facets of P .
- If F is a face of G , which is a face of P , then $N_G(P)$ is a face of $N_F(P)$.
- $\dim N_F(P) = d - \dim F$.
- P is a polytope $\Leftrightarrow \mathcal{N}(P)$ is complete.
- $\dim P = d \Leftrightarrow \mathcal{N}(P)$ is pointed.
- P is a simple d -polytope $\Leftrightarrow \mathcal{N}(P)$ is complete and simplicial.

Definition 1.3. Two convex polyhedra $P, Q \subset \mathbb{R}^d$ are called *normally equivalent* if they have the same normal fan: that is,

$$P \simeq Q \Leftrightarrow \mathcal{N}(P) = \mathcal{N}(Q).$$

For a fan Δ , we denote by $\tilde{T}(\Delta)$ the set of all convex polyhedra with the normal fan Δ : that is,

$$\tilde{T}(\Delta) := \{P \mid \mathcal{N}(P) = \Delta\}.$$

Also we denote by $T(\Delta)$ the set of all such polyhedra modulo translation: that is,

$$T(\Delta) := \tilde{T}(\Delta) / \sim \quad \text{where } P \sim P + x \text{ for all } x \in \mathbb{R}^d.$$

The set $T(\Delta)$ is called the *type cone* of Δ .

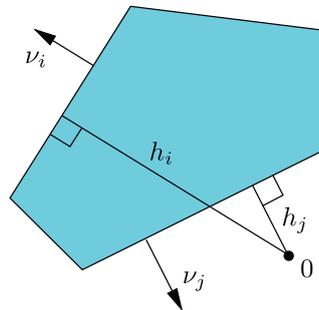


Figure 2: Support numbers: $h_i > 0, h_j < 0$.

The set $T(\Delta)$ is a cone in the sense that scaling a convex polyhedron does not change its normal fan. Later we will see that the closure of $T(\Delta)$ is, in fact, a convex polyhedral cone.

Not every fan Δ is the normal fan of some convex polyhedron; examples are shown in Figure 13. If it is, that is, if $T(\Delta) \neq \emptyset$, then Δ is called *polytopal*.

Normally equivalent polyhedra are also called “analogous” [1] or “strongly isomorphic” [27, 30]. The term “normally equivalent” is used in [9].

1.1.3. *Support numbers.* Let $P \subset \mathbb{R}^d$ be a d -dimensional convex polyhedron. Then one-dimensional cones of $\mathcal{N}(P)$ correspond to facets of P . Denote by $V := (v_1, \dots, v_n)$ the collection of unit vectors that generate $\mathcal{N}(P)^{(1)}$ and denote by F_i the facet of P with the outward unit normal v_i . Then P is the solution set of a system of linear inequalities $\langle v_i, x \rangle \leq h_i, i = 1, \dots, n$ for some $h \in \mathbb{R}^n$. We will express this as $P = P(V, h)$, where

$$P(V, h) := \{x \in \mathbb{R}^d \mid Vx \leq h\}, \quad V \in \mathbb{R}^{n \times d}, h \in \mathbb{R}^n. \quad (1)$$

Here, by abuse of notation, V denotes the $n \times d$ -matrix whose i th row is v_i . By a repeated abuse of notation, we will also write $V = \mathcal{N}(P)^{(1)}$. Thus V stands for any of the following objects:

- a collection of n rays in \mathbb{R}^d starting at the origin;
- a collection of n unit vectors in \mathbb{R}^d ; and
- an $n \times d$ -matrix with rows of norm one.

From $\|v_i\| = 1$, it follows that h_i is the signed distance from the coordinate origin to the affine hull of F_i (see Figure 2). The numbers h_i are called the *support numbers* of the polyhedron P and $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ is the *support vector* of P .

For any pointed polytopal fan Δ , the support vector determines an embedding

$$\tilde{T}(\Delta) \rightarrow \mathbb{R}^n, \quad P(V, h) \mapsto h,$$

where $V = \Delta^{(1)}$. Due to

$$P(V, h) + t = P(V, h + Vt), \quad (2)$$

the equivalence classes modulo translation correspond to points in $\mathbb{R}^n / \text{im } V$. Thus

$$T(\Delta) \subset \mathbb{R}^n / \text{im } V,$$

and $\widetilde{T}(\Delta) = \pi^{-1}(T(\Delta))$, where

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n / \text{im } V$$

is the canonical projection.

1.1.4. Support function.

Definition 1.4. Let $P \subset \mathbb{R}^d$ be a convex polyhedron. Its *support function* is defined as

$$\begin{aligned} h_P : \text{supp}(\mathcal{N}(P)) &\rightarrow \mathbb{R} \\ h_P(v) &:= \max_{x \in P} \langle v, x \rangle. \end{aligned}$$

In particular, if P is a polytope, then h_P is defined on the whole \mathbb{R}^d .

The support function is positively homogeneous and convex: that is, $h_P(\lambda v) = \lambda h_P(v)$ for $\lambda > 0$ and

$$h_P(\lambda v + (1 - \lambda)w) \leq \lambda h_P(v) + (1 - \lambda)h_P(w)$$

for all $\lambda \in [0, 1]$, $v, w \in \text{supp}(\mathcal{N}(P))$. The support function can be defined for any closed convex set and there is a one-to-one correspondence between closed convex sets and positively homogeneous convex functions defined on a convex cone in \mathbb{R}^d .

LEMMA 1.5. The support function of a convex polyhedron P is given by

$$h_P(v) = \langle v, x \rangle \quad \text{for } v \in N_F(P) \text{ and } x \in F.$$

Thus the support function of a polyhedron is linear on every normal cone.

The formula in Lemma 1.5 implies that the linearity domains of h_P are the normal cones at the vertices of P ; that is, the nonlinearity locus of h_P is the union of the normal cones at the edges of P .

Definition 1.6. Let $V = (v_1, \dots, v_n)$ be a vector configuration in \mathbb{R}^d and let Δ be a fan with $\Delta^{(1)} = V$. For every $h \in \mathbb{R}^n$, denote by

$$\widetilde{h}_\Delta : \text{supp}(\Delta) \rightarrow \mathbb{R}^d$$

the piecewise linear function obtained by extending the map $v_i \mapsto h_i$ linearly to each cone of Δ .

Note that if Δ is not simplicial, then \widetilde{h}_Δ is not defined for all h .

Let $P = P(V, h)$ be a d -dimensional convex polyhedron such that $\mathcal{N}(P)^{(1)} = V$. Since $h_P(v_i) = h_i$,

$$h_P = \widetilde{h}_{\mathcal{N}(P)}.$$

COROLLARY 1.7. *Let Δ be a pointed fan with a convex support. Then $h \in \widetilde{T}(\Delta)$ if and only if the piecewise linear function \widetilde{h}_Δ is defined, is convex and has $\Delta^{(d-1)}$ as its nonlinearity locus.*

Consequently, a pointed fan with convex support is polytopal if and only if there exists an h with the above properties.

Remark 1.8. Corollary 1.7 says that a polytopal fan is the same as a *regular subdivision* of a vector configuration (see [9, §9.5.1]). In a special case, when $\text{supp}(\Delta)$ is contained in an open half-space, the rays of Δ can be replaced by their intersection points with an affine hyperplane A . The convex hull of these points is subdivided by the cones of Δ into convex polytopes that are the linearity domains of a convex piecewise affine function on A . Such a subdivision is called a *regular subdivision* of a point configuration (see [46, Definition 5.3] and [9]).

1.1.5. Fans refinement and the Minkowski sum.

Definition 1.9. We say that a fan Δ' *refines* a fan Δ (or Δ *coarsens* Δ') and write $\Delta' \preceq \Delta$ if every cone of Δ is a union of cones of Δ' .

LEMMA 1.10. *Every non-simplicial polytopal fan Δ can be refined to a simplicial polytopal fan Δ' . In addition, for every such Δ' , $T(\Delta) \subset \text{cl } T(\Delta')$.*

Indeed, let $h \in \widetilde{T}(\Delta) \subset \mathbb{R}^n$. Choose a generic $y \in \mathbb{R}^n$ and consider $h' = h + ty$ with t sufficiently small. The extension \widetilde{h}'_Δ is not defined, but it becomes so if every $\sigma \in \Delta$ is appropriately subdivided (by a convex hull construction). This yields a simplicial fan Δ' such that $\widetilde{h}'_{\Delta'}$ is convex, (the convexity across $\Delta^{(d-1)}$ is preserved for small t). As for the second statement of the lemma, by choosing $h \in \widetilde{T}(\Delta)$ and $y \in \widetilde{T}(\Delta')$ we see, in a similar way, that $h + ty \in \widetilde{T}(\Delta')$ for all $t > 0$.

The geometric picture behind this argument is that translating facets of a non-simple polyhedron generically and by small amounts makes the polyhedron simple without destroying any of its faces.

Definition 1.11. The Minkowski sum of two sets $K, L \subset \mathbb{R}^d$ is defined as

$$K + L := \{x + y \mid x \in K, y \in L\}.$$

The additive structure on the space of support vectors is related to the Minkowski addition. However, one should be careful when the summands are not normally equivalent.

Definition 1.12. For two fans Δ_1 and Δ_2 with the same support, denote by $\Delta_1 \wedge \Delta_2$ their coarsest common refinement. Explicitly,

$$\Delta_1 \wedge \Delta_2 = \{\sigma_1 \cap \sigma_2 \mid \sigma_i \in \Delta_i, i = 1, 2\}.$$

LEMMA 1.13. *Let P and Q be two convex polyhedra such that $\text{supp}(\mathcal{N}(P)) = \text{supp}(\mathcal{N}(Q))$. Then*

$$\mathcal{N}(P + Q) = \mathcal{N}(P) \wedge \mathcal{N}(Q).$$

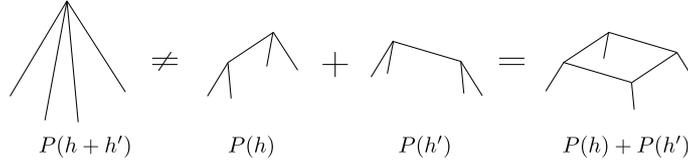


Figure 3: Minkowski addition does not always correspond to the addition of support numbers.

Fix V and define $P(h) := P(V, h)$. Lemma 1.13 implies that, in general,

$$P(h + h') \neq P(h) + P(h'),$$

because $P(h) + P(h')$ can have more facets than $P(h + h')$. See Figure 3, which shows fragments of three-dimensional polyhedra. (However, $P(h + h') \supset P(h) + P(h')$ always holds.)

Instead, we have the following.

LEMMA 1.14. *The support function of the Minkowski sum of two convex bodies is the sum of their support functions:*

$$h_{K+L} = h_K + h_L.$$

COROLLARY 1.15. *If $\mathcal{N}(P(h)) \preceq \mathcal{N}(P(h'))$, then $P(h) + P(h') = P(h + h')$.*

Also, clearly, $h_{\lambda K} = \lambda h_K$ and $P(\lambda h) = \lambda P(h)$ for $\lambda > 0$. This is false for $\lambda < 0$. Taking differences of the support functions embeds the space of convex polytopes in the vector space of the so-called virtual polytopes (see [29, 36]).

A subset of a Euclidean space is called *relatively open*, if it is open as a subset of its affine hull. For example, an open half-line in \mathbb{R}^2 is relatively open.

LEMMA 1.16. *For every polytopal fan Δ , the type cone $T(\Delta)$ is convex and relatively open.*

The convexity follows from

$$P(\lambda h + (1 - \lambda)h') = \lambda P(h) + (1 - \lambda)P(h')$$

for $\mathcal{N}(P(h)) \preceq \mathcal{N}(P(h'))$ and $\lambda \in [0, 1]$, and the relative openness follows from

$$h, h' \in \tilde{T}(\Delta) \Rightarrow h + \varepsilon h' \in \tilde{T}(\Delta)$$

for ε sufficiently small, whether positive or negative.

1.1.6. *Linear inequalities describing a type cone.* Let $P \subset \mathbb{R}^d$ be a d -polytope with the normal fan Δ . Then $P = P(h)$ for $h \in \tilde{T}(\Delta)$ (we omit $V = \Delta^{(1)}$ from

the notation $P(V, h)$). For every cone $\sigma \in \Delta$ let $F_\sigma(h)$ be the face of $P(h)$ with the normal cone σ .

If $\sigma \in \Delta^{(d-1)}$, then $F_\sigma(h)$ is an edge; denote its length by

$$\ell_\sigma^\Delta(h) := \text{vol}_1(F_\sigma(h)).$$

Of course, all functions $\ell_\sigma^\Delta: \tilde{T}(\Delta) \rightarrow \mathbb{R}$ descend through $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n / \text{im } V$ to functions on $T(\Delta)$. However, the proofs of the lemmas in this section are better written in terms of h than $\pi(h)$. Due to the fact that $T(\Delta) = \pi(\tilde{T}(\Delta))$, the statements of the lemmas are easily translated in terms of $T(\Delta)$.

LEMMA 1.17. (1) For every $\sigma \in \Delta^{(d-1)}$ and $h \in \tilde{T}(\Delta)$,

$$\ell_\sigma^\Delta(h) = \|\text{grad}(h_P|_{\rho_1}) - \text{grad}(h_P|_{\rho_2})\|,$$

where $\rho_1, \rho_2 \in \Delta^{(d)}$ are the two full-dimensional cones having σ as their facet and h_P is the support function of the polytope $P = P(h)$.

(2) $\ell_\sigma^\Delta: \tilde{T}(\Delta) \rightarrow \mathbb{R}$ extends to a linear function on $\text{span}(\tilde{T}(\Delta))$.

Proof. Denote by e_σ the unit vector orthogonal to $\text{aff}(\sigma)$ and directed from ρ_{i_2} into ρ_{i_1} . Then

$$p_1 - p_2 = \ell_\sigma^\Delta e_\sigma, \tag{3}$$

where p_1 and p_2 are vertices of P with normal cones ρ_1 and ρ_2 , respectively. By Lemma 1.5,

$$h_P(v) = \langle v, p_\alpha \rangle \quad \text{for all } v \in \rho_\alpha, \quad \alpha = 1, 2$$

and, therefore, $p_\alpha = \text{grad}(h_P|_{\rho_\alpha})$. Substituting this into (3) proves the first part of the lemma.

For the second part, choose from among the unit length generators of $\Delta^{(1)}$ linearly independent vectors

$$v_{j_1}, \dots, v_{j_{d-1}} \in \sigma.$$

Choose also $v_{i_1} \in \rho_1 \setminus \sigma$ and $v_{i_2} \in \rho_2 \setminus \sigma$. Then the function $h_P|_{\rho_\alpha}$, $\alpha = 1, 2$ is uniquely determined by its values $h_{j_1}, \dots, h_{j_{d-1}}, h_{i_\alpha}$.

As every set of $d + 1$ vectors in \mathbb{R}^d is linearly dependent,

$$\lambda_1 v_{i_1} + \lambda_2 v_{i_2} + \mu_1 v_{j_1} + \dots + \mu_{d-1} v_{j_{d-1}} = 0 \tag{4}$$

for some $\lambda_\alpha, \mu_\beta \in \mathbb{R}$. In addition, $\lambda_1, \lambda_2 \neq 0$. By definition of the support function,

$$h_{i_1} = \langle v_{i_1}, p_1 \rangle, \quad h_{j_\beta} = \langle v_{j_\beta}, p_1 \rangle = \langle v_{j_\beta}, p_2 \rangle, \quad h_{i_2} = \langle v_{i_2}, p_2 \rangle.$$

It follows that

$$\begin{aligned} & \lambda_1 h_{i_1} + \lambda_2 h_{i_2} + \mu_1 h_{j_1} + \dots + \mu_{d-1} h_{j_{d-1}} \\ &= \lambda_1 \langle v_{i_1}, p_1 \rangle + \lambda_2 \langle v_{i_2}, p_2 \rangle + \sum_{\beta=1}^{d-1} \mu_{j_\beta} \langle v_{i_2}, p_2 \rangle \\ &= \lambda_1 \langle v_{i_1}, p_1 - p_2 \rangle. \end{aligned}$$

By substituting (3), we obtain

$$\ell_\sigma^\Delta = \frac{1}{\lambda_1 \langle v_{i_1}, e_\sigma \rangle} (\lambda_1 h_{i_1} + \lambda_2 h_{i_1} + \mu_1 h_{j_1} + \cdots + \mu_{d-1} h_{j_{d-1}}),$$

which shows that $\ell_\sigma^\Delta: \tilde{T}(\Delta) \rightarrow \mathbb{R}$ is a restriction of a linear function. \square

Remark 1.18. The function ℓ_σ^Δ can be computed as follows. Choose a flag of faces

$$P \supset F_1 \supset \cdots \supset F_{d-1} = F_\sigma,$$

where $\dim F_i = d - i$. Project zero orthogonally to $\text{aff}(F_1)$. Then the support numbers of F_1 , with respect to the projection of zero, can be written as linear functions of the support numbers of P (the coefficients are trigonometric functions of the dihedral angles between F_1 and adjacent facets). Repeat this by expressing the support numbers of F_2 as linear functions of the support numbers of F_1 , and so on. At the end, we obtain the support numbers of F_{d-1} (there are two of them, as F_{d-1} is a segment) as linear functions of the support numbers of P . But the length of F_{d-1} is just the sum of these two numbers.

For $d = 3$, this computation is done in Appendix A.2.

Note that if Δ is not simplicial, then we might have several different choices for $v_{i_\alpha}, v_{j_\beta}$ in the proof of Lemma 1.17, and hence several formulas for ℓ_σ^Δ in terms of h_i . This is due to the fact that, for a non-simplicial Δ , the set $\tilde{T}(\Delta)$ does not span the space \mathbb{R}^n of support vectors. Thus different linear functions on \mathbb{R}^n have the same restrictions to $\text{span}(\tilde{T}(\Delta))$.

LEMMA 1.19. *Let Δ be a complete pointed polytopal fan with $|\Delta^{(1)}| = n$.*

(1) *If Δ is simplicial, then*

$$\text{span}(\tilde{T}(\Delta)) = \mathbb{R}^{\Delta^{(1)}}.$$

(2) *If Δ is not simplicial, then*

$$\text{span}(\tilde{T}(\Delta)) = \{h \in \mathbb{R}^{\Delta^{(1)}} \mid \ell_{\sigma'}^{\Delta'}(h) = 0 \text{ for all } \sigma' \in \Delta'^{(d-1)} \text{ such that } \text{relint } \sigma' \subset \text{relint } \rho \text{ for some } \rho \in \Delta^{(d)}\},$$

where Δ' is any simplicial refinement of Δ .

Proof. If $h \in \tilde{T}(\Delta)$ for a simplicial Δ , then, for every $y \in \mathbb{R}^n$, the piecewise linear extension of $h + ty$ with respect to Δ is convex and has $\Delta^{(d-1)}$ as its nonlinearity locus, provided that t is sufficiently small. This proves the first part of the lemma.

If Δ is not simplicial, then let us first show that

$$\tilde{T}(\Delta) \subset \{h \in \mathbb{R}^n \mid \ell_{\sigma'}^{\Delta'}(h) = 0\}, \tag{5}$$

with Δ' and σ' as in the statement of the lemma. Let $h \in \tilde{T}(\Delta)$. Then the support function $h_P = \tilde{h}_\Delta$ is linear on ρ , since ρ is a normal cone of $P(h)$. On the other hand, let $\rho'_1, \rho'_2 \in \Delta'$ be the d -cones adjacent to σ' . As $\rho'_1, \rho'_2 \subset \rho$, $\text{grad}(h_P|_{\rho'_1}) = \text{grad}(h_P|_{\rho'_2})$. As $h \in \text{cl } \tilde{T}(\Delta')$, by Lemma 1.10, the value $\ell_{\sigma'}^{\Delta'}(h)$ for $h \in \tilde{T}(\Delta)$ is given by the same formula as for $h \in T(\Delta')$. Thus, by the first part of Lemma 1.17, $\ell_{\sigma'}^{\Delta'}(h) = 0$.

In the other direction, let $h \in \tilde{T}(\Delta)$ and $y \in \ker \ell_{\sigma'}^{\Delta'}$ for all σ' , as in the lemma. We claim that then $h + ty \in T(\Delta)$ for t sufficiently small. This would mean that

$$\{h \in \mathbb{R}^n \mid \ell_{\sigma'}^{\Delta'}(h) = 0\} \subset \text{span}(T(\Delta)),$$

and thus, together with (5), it implies the second part of the lemma. So take any $\rho \in \Delta^{(d)}$ and consider the piecewise linear extension of y with respect to the subdivision of ρ induced by Δ' . This extension is, in fact, linear, since it is linear across all $(d-1)$ -cones of Δ' whose relative interiors lie in ρ . This shows that the extension \tilde{y}_Δ exists. For t small, the piecewise linear extension of $h + ty$ with respect to Δ is convex and has the same nonlinearity locus as \tilde{h}_Δ . Hence $h + ty \in \tilde{T}(\Delta)$ and the proof is complete. \square

Remark 1.20. For $d = 3$, the condition on σ' in the second part of Lemma 1.19 can be replaced by $\sigma' \notin \Delta$. This is not so for $d > 3$: if the cone $\sigma \in \Delta^{(d-1)}$ is not simplicial, then it gets subdivided into simplicial cones $\sigma'_i \in \Delta'^{(d-1)}$. Thus $\sigma'_i \notin \Delta$. However, the corresponding edge lengths are not zero. In fact, the edges $F_{\sigma'_i}(h)$, which are different for $h \in \tilde{T}(\Delta')$, become one edge $F_\sigma(h)$ for $h \in \tilde{T}(\Delta)$. This means that for $h \in \tilde{T}(\Delta)$

$$\ell_{\sigma'_i}^{\Delta'}(h) = \ell_{\sigma'_j}^{\Delta'}(h)$$

for any $\sigma'_i, \sigma'_j \subset \sigma$. Thus these linear equations hold also on $\text{span}(\tilde{T}(\Delta))$. However, they do not enter the description of $\text{span}(\tilde{T}(\Delta))$ given in the second part of Lemma 1.19 as they follow from the other ones (this simply follows from the statement of the lemma).

LEMMA 1.21. *Let Δ be a complete pointed fan in \mathbb{R}^d .*

If Δ is simplicial, then $\tilde{T}(\Delta)$ is the solution set of the system of linear inequalities given by

$$\tilde{T}(\Delta) = \{h \in \mathbb{R}^n \mid \ell_\sigma^\Delta(h) > 0 \text{ for all } \sigma \in \Delta^{(d-1)}\}.$$

If Δ is not simplicial, then $\tilde{T}(\Delta)$ is the solution set of the system of linear equations and inequalities given by

$$\begin{aligned} \tilde{T}(\Delta) = \{h \in \mathbb{R}^n \mid & \ell_\sigma^\Delta(h) > 0 \text{ for all } \sigma \in \Delta^{(d-1)} \text{ and} \\ & \ell_{\sigma'}^{\Delta'}(h) = 0 \text{ for all } \sigma' \in \Delta'^{(d-1)} \text{ such that} \\ & \text{relint } \sigma' \subset \text{relint } \rho \text{ for some } \rho \in \Delta^{(d)}\}, \end{aligned}$$

where Δ' is any simplicial fan that refines Δ .

In particular, the closure $\text{cl } T(\Delta)$ of the type cone is a convex polyhedral cone in $\mathbb{R}^n / \text{im } V$.

Note that the function $\ell_\sigma^\Delta(h)$ is well defined only on $\text{span}(\tilde{T}(\Delta))$. Due to Lemma 1.19, the above description of $\tilde{T}(\Delta)$ in the non-simplicial case is unambiguous.

Proof. First, let Δ be simplicial. Then the inequality $\ell_\sigma^\Delta(h) > 0$ just means that the piecewise linear function \tilde{h}_Δ is strictly convex across the $(d - 1)$ -cone σ . It follows that these inequalities are necessary and sufficient for h to belong to $\tilde{T}(\Delta)$.

Second, let Δ be non-simplicial. Then, similarly to the proof of Lemma 1.19, the equations $\ell_\sigma^{\Delta'}(h) = 0$ ensure that the linear extension \tilde{h}_Δ exists. And, as in the simplicial case, $\ell_\sigma^\Delta(h) > 0$ is equivalent to the strict convexity of this function across σ . The lemma is proved. \square

1.2. *Secondary fans.* Everywhere in this section, $V = (v_1, \dots, v_n)$ is a linearly spanning vector configuration in \mathbb{R}^d . As usual, V will also stand for the $n \times d$ matrix with rows v_i .

1.2.1. *The compatibility and the irredundancy domains.* As in §1.1.3, for an $h \in \mathbb{R}^n$, denote by $P(h)$ the solution set of the system $Vx \leq h$. It can happen that the set $P(h)$ is empty; and even when it is non-empty, it can happen that some of the inequalities $\langle v_i, x \rangle \leq h_i$ can be removed without changing $P(h)$. This leads us to consider the two subsets of \mathbb{R}^n

$$\tilde{\text{co}}(V) := \{h \in \mathbb{R}^n \mid Vx \leq h \text{ is compatible}\},$$

that is, all those h for which $P(h) \neq \emptyset$ and

$$\tilde{\text{ir}}(V) := \{h \in \mathbb{R}^n \mid Vx \leq h \text{ is compatible and irredundant}\},$$

where *irredundant* means that removing any inequality from the system makes the solution set bigger.

As in the case with a type cone and a lifted type cone, equation (2) implies that both $\tilde{\text{co}}(V)$ and $\tilde{\text{ir}}(V)$ are invariant under translation by Vt for any $t \in \mathbb{R}^d$. Therefore it suffices to study their quotients under the map

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n / \text{im } V. \tag{6}$$

Definition 1.22. Let $V \in \mathbb{R}^{n \times d}$ be of rank d . The set

$$\text{co}(V) := \tilde{\text{co}}(V) / \text{im } V$$

is called the *compatibility domain* and the set

$$\text{ir}(V) := \tilde{\text{ir}}(V) / \text{im } V$$

is the *irredundancy domain* for V . Furthermore, we denote by $\text{clir}(V)$ the closure of $\text{ir}(V)$.

For every polytopal fan Δ with $\Delta^{(1)} = V$, $T(\Delta) \subset \text{ir}(V)$ and we will see later that these types of cone decompose the interior of $\text{ir}(V)$. Moreover, this can be extended to a subdivision of $\text{co}(V)$, the so-called *secondary fan*.

Most of the material presented here can be found in [5, 6, 9, 27, 28]. The notation ir was introduced in [27], meaning “inner region”, but, in some later works by the same author, the terminology was changed to the “irredundancy domain”.

1.2.2. *Gale duality, linear dependencies and evaluations.* For the moment, we do not require all v_i to have norm one (in fact, we even allow $v_i = 0$ or $v_i = v_j$ for $i \neq j$).

Definition 1.23. Choose a linear isomorphism $\mathbb{R}^n / \text{im } V \cong \mathbb{R}^{n-d}$ and denote by \bar{V}^\top the matrix representation of the projection (6): that is,

$$\mathbb{R}^d \xrightarrow{V} \mathbb{R}^n \xrightarrow{\bar{V}^\top} \mathbb{R}^{n-d}.$$

Then the *Gale transform* or *Gale diagram* of (v_1, \dots, v_n) is the collection of n vectors $(\bar{v}_1, \dots, \bar{v}_n)$ that form the rows of the $n \times (n-d)$ -matrix \bar{V} .

LEMMA 1.24. *The vector configuration $(\bar{v}_1, \dots, \bar{v}_n)$ is well defined up to a linear transformation of \mathbb{R}^{n-d} . In addition, if $(\bar{v}_1, \dots, \bar{v}_n)$ is a Gale diagram for (v_1, \dots, v_n) , then (v_1, \dots, v_n) is also a Gale diagram of $(\bar{v}_1, \dots, \bar{v}_n)$.*

Indeed, the uniqueness up to a linear transformation follows from the freedom in the choice of the isomorphism $\mathbb{R}^n / \text{im } V \cong \mathbb{R}^{n-d}$. The involutivity of the Gale transform follows by transposing the short exact sequence in Definition 1.23. It allows us to also call the vector configuration (\bar{v}_i) *Gale dual* to (v_i) .

Remark 1.25. If V contains the vectors of the standard basis of \mathbb{R}^d , then its Gale dual is very easy to compute using

$$V = \begin{pmatrix} E_d \\ A \end{pmatrix} \Rightarrow \bar{V} = \begin{pmatrix} -A^\top \\ E_{n-d} \end{pmatrix},$$

where E_k denotes the $k \times k$ -unit matrix.

As an immediate consequence of Definition 1.23,

$$\ker V^\top = \text{im } \bar{V}. \tag{7}$$

The elements of $\ker V^\top$ are the *linear dependencies* between the vectors (v_1, \dots, v_n)

$$\lambda \in \ker V^\top \Leftrightarrow \sum_{i=1}^n \lambda_i v_i = 0,$$

while the elements of $\text{im } \bar{V}$ are the *evaluations of linear functionals* on $(\bar{v}_1, \dots, \bar{v}_n)$

$$\lambda \in \text{im } \bar{V} \Leftrightarrow \lambda_i = \langle \mu, \bar{v}_i \rangle \quad \text{for some } \mu \in \mathbb{R}^{n-d}.$$

Therefore (7) can be phrased as “dependencies of V equal evaluations on \bar{V} ”. Of course, the same holds with V and \bar{V} exchanged.

In the next lemma, the rank of a vector configuration means the dimension of its linear span, so that a rank d configuration is the same as a linearly spanning configuration. We also use the notation $[n] := \{1, \dots, n\}$ and, for any subset $I \subset [n]$, we denote by V_I the vector configuration $(v_i \mid i \in I)$.

LEMMA 1.26. *Let $V = (v_1, \dots, v_n)$ be a rank d configuration of n vectors in \mathbb{R}^d and let \bar{V} be its Gale dual.*

- (1) *The vector configuration V is positively spanning if and only if there is $\mu \in \mathbb{R}^{n-d}$ such that $\langle \mu, v_i \rangle > 0$ for all $i \in [n]$.*
- (2) *Subconfiguration V_I is linearly independent if and only if the subconfiguration $\bar{V}_{[n] \setminus I}$ has full rank. In particular, V_I is a basis of \mathbb{R}^d if and only if $\bar{V}_{[n] \setminus I}$ is a basis of \mathbb{R}^{n-d} .*
- (3) *Configuration \bar{V} contains a zero vector $\bar{v}_i = 0$ if and only if the subconfiguration $V \setminus \{v_i\}$ has rank $d - 1$. In particular, if V is positively spanning, then $\bar{v}_i \neq 0$ for all i .*
- (4) *Configuration \bar{V} contains two collinear vectors $\bar{v}_i = c\bar{v}_j$ (one or both of which may be zero) if and only if the subconfiguration $V \setminus \{v_i, v_j\}$ has rank at most $d - 1$.*

Proof. A configuration V is positively spanning if and only if it has rank d and is positively dependent: that is, if

$$\text{rk } V = d, \quad \sum_{i=1}^n \lambda_i v_i = 0 \quad \text{so that } \lambda_i > 0 \text{ for all } i. \quad (8)$$

Indeed, if (8) holds, then every linear combination of v_i can be made positive by adding a positive multiple of $\sum_i \lambda_i v_i$ so that $\text{rk } V = d$ implies that V is positively spanning. For the inverse implication, express $-v_1$ as a positive linear combination of v_i . On the other hand, by the dependence–evaluation duality (7), a positive dependence of V corresponds to a linear functional positively evaluating on \bar{V} . This proves the first part of the lemma.

The subconfiguration V_I is linearly dependent if and only if there is $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$ such that

$$\sum_{i=1}^n \lambda_i v_i = 0 \quad \text{and} \quad \lambda_i = 0 \quad \text{for } i \notin I.$$

By the dependence–evaluation duality, this is equivalent to the existence of a non-zero linear functional on \mathbb{R}^{n-d} that vanishes on all \bar{v}_i for $i \notin I$, which is equivalent to $\text{rk } \bar{V}_{[n] \setminus I} < n - d$. This proves the second part of the lemma.

The third part follows from the second: $\bar{v}_i = 0$ is equivalent to the set $\{\bar{v}_i\}$ being linearly dependent and, if $\text{rk } V = d$, then $\text{rk}(V \setminus \{v_i\}) \geq d - 1$. Also, if $\text{rk}(V \setminus \{v_i\}) = d - 1$, then $-v_i \notin \text{pos}(V)$.

The fourth part is a direct consequence of the second. □

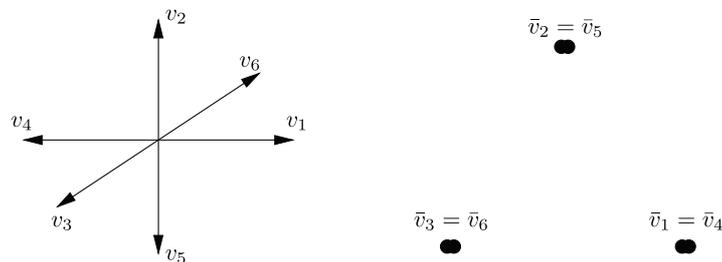


Figure 4: A vector configuration and its affine Gale diagram.

Let $V \in \mathbb{R}^{n \times d}$ be a vector configuration such that its Gale diagram $(\bar{v}_1, \dots, \bar{v}_n)$ contains no zero vectors (a necessary and sufficient condition for this is given in part 3 of Lemma 1.26). The *affine Gale diagram* is constructed by choosing an affine hyperplane $A \subset \mathbb{R}^{n-d}$ non-parallel to each of \bar{v}_i and scaling each \bar{v}_i so that

$$\bar{p}_i = \alpha_i v_i \in A.$$

In addition, a point \bar{p}_i is colored black if $\alpha_i > 0$ and white if $\alpha_i < 0$.

An affine Gale diagram determines the vector configuration V up to independent positive scalings of v_i . Indeed, the point \bar{p}_i , together with its color, determines \bar{v}_i up to a positive scaling; and positive scaling $\bar{v}_i \mapsto \beta_i \bar{v}_i$ of \bar{V} corresponds to positive scaling $v_i \mapsto \beta_i^{-1} v_i$ of V .

Remark 1.27. By part 1 of Lemma 1.26, a positively spanning vector configuration has an affine Gale diagram consisting of black points only.

By the dependencies–evaluations duality (7), if $\sum_{i=1}^n v_i = 0$, then the vectors \bar{v}_i lie already in an affine hyperplane.

Example 1.28. Let $V = (e_1, e_2, e_3, -e_1, -e_2, -e_3)$ be a configuration of six vectors in \mathbb{R}^3 . Its Gale dual is $\bar{V} = (e_1, e_2, e_3, e_1, e_2, e_3)$. The corresponding affine Gale diagram is shown in Figure 4.

For more details on Gale duality, see [9, 10, 46].

1.2.3. Positive circuits and $\text{co}(V)$.

Definition 1.29. A subset $C \subset [n]$ is called a *circuit* of a vector configuration $V = (v_1, \dots, v_n)$ if $V_C = (v_i \mid i \in C)$ is an inclusion-minimal linearly dependent subset of V .

An inclusion-minimal linearly dependent set is one that is linearly dependent, while each of its proper subsets is not.

The minimality condition implies that a linear dependence $\lambda^C \subset \mathbb{R}^C$ between vectors of a circuit C is unique up to a scaling and that all coefficients λ_i^C are non-zero. In particular, it makes sense to speak about the signs of the coefficients (up to a simultaneous change of all signs).

A circuit C is called *positive* if all coefficients of the corresponding linear dependence can be chosen to be positive: that is, if

$$\sum_{i \in C} \lambda_i^C v_i = 0 \quad \text{where } \lambda_i^C > 0 \text{ for all } i \in C.$$

By the dependence-evaluation duality (7), to a positive circuit C there corresponds a linear functional $\mu^C \in \mathbb{R}^{n-d}$ such that

$$\langle \bar{\mu}^C, \bar{v}_i \rangle \begin{cases} > 0 & \text{if } i \in C, \\ = 0 & \text{if } i \notin C. \end{cases}$$

Namely, $\lambda^C = \bar{V} \mu^C$. Since \bar{V} has full rank, μ^C is well defined up to a scaling. In addition, the minimality of C implies that $\bar{V}_{[n] \setminus C}$ is a *maximal non-spanning subconfiguration* of \bar{V} . Such subconfigurations are called *cocircuits*. Thus

$$C \text{ is a circuit for } V \Leftrightarrow [n] \setminus C \text{ is a cocircuit for } \bar{V}.$$

(See also the second part of Lemma 1.26.)

THEOREM 1.30. *Let (v_1, \dots, v_n) be a linearly spanning vector configuration in \mathbb{R}^d . Then the compatibility domain $\text{co}(V)$ is a convex polyhedral cone in \mathbb{R}^{n-d} . The compatibility domain and its lift $\tilde{\text{co}}(V)$ to \mathbb{R}^n can be described in the following ways:*

$$\tilde{\text{co}}(V) = \mathbb{R}_+^n + \text{im } V \quad \text{where } \mathbb{R}_+^n = \{(h_1, \dots, h_n) \mid h_i \geq 0 \text{ for all } i\}, \quad (9a)$$

$$\text{co}(V) = \text{pos}\{\bar{v}_1, \dots, \bar{v}_n\} \quad \text{where } \bar{V} \text{ is the Gale dual of } V, \quad (9b)$$

$$\tilde{\text{co}}(V) = \{h \in \mathbb{R}^n \mid \langle \lambda^C, h \rangle \geq 0 \text{ for all positive circuits } C\}, \quad (9c)$$

$$\text{co}(V) = \{y \in \mathbb{R}^{n-d} \mid \langle \mu^C, y \rangle \geq 0 \text{ for all positive circuits } C\}. \quad (9d)$$

Here $\lambda^C \in \mathbb{R}_+^n$ are the coefficients of the linear dependence and $\mu^C \in \mathbb{R}^{n-d}$ is the linear functional associated with the circuit C .

Moreover, V is positively spanning if and only if $\text{co}(V)$ is pointed.

Proof. Equation (9a) follows from

$$x \in P(h) \Leftrightarrow 0 \in P(h - Vx) \Leftrightarrow h - Vx \in \mathbb{R}_+^n.$$

Thus $P(h) \neq \emptyset$ if and only if $h \in \mathbb{R}_+^n + \text{im } V$.

(9a) \Rightarrow (9b). By definition of \bar{V} , $\pi(e_i) = \bar{v}_i$, where e_i is a standard basis vector of \mathbb{R}^n . Hence

$$\text{co}(V) = \pi(\tilde{\text{co}}(V)) = \pi(\mathbb{R}_+^n) = \text{pos}(\bar{v}_1, \dots, \bar{v}_n).$$

(9b) \Rightarrow (9d). Represent the cone $\text{pos}(\bar{v}_1, \dots, \bar{v}_n)$ as an intersection of half-spaces whose boundary hyperplanes pass through the origin. It suffices to take only those half-spaces whose boundaries are spanned by a subconfiguration

of \bar{V} . But every such half-space corresponds to a linear functional vanishing on a maximal non-spanning subset of \bar{V} and positive otherwise. This yields the representation (9d).

(9d) \Rightarrow (9c). Due to $\tilde{\text{co}}(V) = \pi^{-1}(\text{co}(V))$

$$\tilde{\text{co}}(V) = \{h \in \mathbb{R}^n \mid \langle \mu^C, \pi(h) \rangle \geq 0\}.$$

Since $\bar{V}: \mathbb{R}^{n-d} \rightarrow \mathbb{R}^n$ is the adjoint of π and $\lambda^C = \bar{V}\mu^C$, this corresponds to the description (9c).

Finally, by Lemma 1.26, V is positively spanning if and only if there exists a linear functional on \mathbb{R}^{n-d} taking on \bar{v}_i only positive values, which is equivalent to $\text{pos}(\bar{V})$ being pointed. \square

Note that if $\text{pos}(V)$ is pointed, then V has no positive circuits, and hence the description (9d) yields $\text{co}(V) = \mathbb{R}^{n-d}$. This is in full accordance with the dualization of the last statement of the theorem: \bar{V} is positively spanning if and only if $\text{pos}(v_1, \dots, v_n)$ is pointed.

Remark 1.31. Parts (9a) and (9b) of Theorem 1.30 are due to McMullen [27]. See also [9, Theorem 4.1.39].

Equation (9c) can be seen as a version of Farkas's lemma, [46, Ch. 1].

Still another interpretation of (9c) is in terms of the support function of $P(h)$. We have $P(h) \neq \emptyset$ if and only if there exists a convex positively homogeneous function $\tilde{h}: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\tilde{h}(v_i) \leq h_i$. The epigraph of \tilde{h} is a convex cone containing all points (v_i, h_i) . If $\lambda(h) \geq 0$ for all positive circuits, then $\text{conv}\{(v_i, h_i)\}$ "lies above" zero and, therefore, $\text{pos}\{(v_i, h_i)\}$ is the epigraph of a convex function.

1.2.4. *Hyperbolic circuits and $\text{ir}(V)$.* From now on, we assume that $v_i \neq 0$ and $v_i \neq \lambda v_j$ for $\lambda > 0$.

Our main goal here is to prove an analog of Theorem 1.30 for the space $\text{ir}(V)$. For this, we need some preliminary work.

Recall that $P(h) = \{x \in \mathbb{R}^d \mid Vx \leq h\}$ and that $h \in \tilde{\text{co}}(V) \Leftrightarrow P(h) \neq \emptyset$. Define

$$F_i(h) := \{x \in P(h) \mid \langle v_i, x \rangle = h_i\}.$$

LEMMA 1.32. *If $h \in \tilde{\text{ir}}(V)$, then $F_i(h) \neq \emptyset$.*

Proof. By definition, the i th inequality in $Vx \leq h$ is irredundant if and only if there exist $x_{\text{out}} \in \mathbb{R}^d$ such that

$$\langle v_i, x_{\text{out}} \rangle > h_i, \quad \langle v_j, x_{\text{out}} \rangle \leq h_j \quad \text{for all } j \neq i.$$

Pick $x_{\text{in}} \in P(h)$. Then, for an appropriate convex combination of x_{in} and x_{out} , $\langle v_i, x \rangle = h_i$ and $\langle v_j, x \rangle \leq h_j$ for all $j \neq i$. Thus $F_i(h) \neq \emptyset$ and the lemma is proved. \square

Remark 1.33. The inverse of Lemma 1.32 does not hold. If $F_i(h) \neq \emptyset$ and $\dim F_i(h) < \dim P(h) - 1$, then the i th inequality is still redundant. Even worse, the i th inequality can be redundant also when $\dim F_i(h) = \dim P(h) - 1$, although for this $\dim P(h) < d$ is necessary. A concrete example is the polytope in \mathbb{R}^2 given by

$$y \leq 0, \quad -y \leq 0, \quad -x \leq 0, \quad x + y \leq 1, \quad x - y \leq 1.$$

This is a segment with the endpoints $(0, 0)$ and $(1, 0)$ and $F_4 = F_5 = \{(1, 0)\}$, which is a facet of P . Nevertheless, both the fourth and the fifth inequalities are redundant.

Note that, in the last example, removing both redundant inequalities at once makes the solution set larger. Thus an inclusion-minimal irredundant subsystem is, in general, not unique.

Life becomes easier if we restrict our attention either to the interior $\text{int ir}(V)$ or to the closure $\text{clir}(V)$ of the irredundancy domain. Note that they are the images under the map (6) of the interior and of the closure of $\tilde{\text{ir}}(V)$, respectively.

LEMMA 1.34.

$$\text{int co}(V) = \{\pi(h) \mid \dim P(h) = d\}, \tag{10a}$$

$$\text{int ir}(V) = \{\pi(h) \mid \dim P(h) = d, \dim F_i(h) = d - 1 \text{ for all } i\}, \tag{10b}$$

$$\text{clir}(V) = \{\pi(h) \mid F_i(h) \neq \emptyset \text{ for all } i\}. \tag{10c}$$

Proof. From (9a), $\text{int } \tilde{\text{co}}(V) = \text{int}(\mathbb{R}_+^n + \text{im } V)$. It follows that

$$h \in \text{int co}(V) \Leftrightarrow \text{there exists } x : h - Vx \in \text{int } \mathbb{R}_+^n \Leftrightarrow \{x \mid Vx < h\} \neq \emptyset.$$

Since $\{x \mid Vx < h\} = \text{int } P(h)$ and $\text{int } P(h) \neq \emptyset \Leftrightarrow \dim P(h) = d$, we have (10a).

Let $h \in \text{int } \tilde{\text{ir}}(V)$. Since $\tilde{\text{ir}}(V) \subset \tilde{\text{co}}(V)$, $\text{int } \tilde{\text{ir}}(V) \subset \text{int } \tilde{\text{co}}(V)$ and thus $\dim P(h) = d$. Since every d -dimensional convex polyhedron in \mathbb{R}^d is the intersection of the half-spaces determined by its facets [46, Ch. 2], the i th inequality is irredundant only if $\dim F_i(h) = d - 1$. Thus the left-hand side in (10b) is a subset of the right-hand side.

Let us prove that the right-hand side of (10b) is a subset of the left-hand side. First, the right-hand side is a subset of $\text{ir}(V)$. Indeed, if $\dim F_i(h) = d - 1$, then, in a neighborhood of $x \in \text{relint } F_i(h)$, there are points for which all inequalities in $Vx \leq h$ hold except the i th: that is, the i th inequality is irredundant. Further, any small change of h preserves the properties $\dim P(h) = d$ and $\dim F_i(h) = d - 1$, and thus the right-hand side is a subset of $\text{int ir}(V)$.

By Lemma 1.32, $\text{ir}(V)$ is a subset of the right-hand side of (10c). Since the right-hand side is closed, it contains also $\text{clir}(V)$. To prove the inverse inclusion, let us show that any h such that $F_i(h) \neq \emptyset$ lies in the closure of $\text{int ir}(V)$. Indeed, for any $\varepsilon > 0$, put $h' = h + \varepsilon \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$. Then, for any $x \in F_i(h)$,

$$\begin{aligned} \langle v_i, x + \varepsilon v_i \rangle &= \langle v_i, x \rangle + \varepsilon = h_i + \varepsilon, \\ \langle v_j, x + \varepsilon v_i \rangle &= \langle v_j, x \rangle + \varepsilon \langle v_i, v_j \rangle < h_j + \varepsilon, \end{aligned}$$

(where we assume that $\|v_i\| = 1$ for all i). It follows that $x + \varepsilon v_i \in F_i(h')$ and that $\dim F_i(h') = d - 1$ for all i . Thus $h' \in \text{int ir}(V)$. This proves (10c). \square

LEMMA 1.35. *If (v_1, \dots, v_n) are different unit vectors, then $\mathbf{1} \in \text{int ir}(V)$. In particular, $\text{int ir}(V)$ is non-empty.*

The proof is similar to that of the last part of Lemma 1.34: $v_i \in F_i(\mathbf{1})$ and, since $\langle v_i, v_j \rangle < 1$, $\dim F_i(\mathbf{1}) = d - 1$. The polytope $P(\mathbf{1})$ is circumscribed about the unit sphere. Its normal fan is related to the Delaunay tessellation (see §4.2).

Note that the assumption $\|v_i\| = 1$ is not too restrictive: scaling all vectors v_i by positive factors scales the support numbers h_i correspondingly. In the context of Gale duality, if v_i is replaced by $\lambda_i v_i$, then it suffices to replace \bar{v}_i by $\lambda_i^{-1} \bar{v}_i$.

Two more definitions will be needed.

Definition 1.36. Let $W = (w_1, \dots, w_n)$ be a vector configuration in \mathbb{R}^m , where we allow $w_i = w_j$ for $i \neq j$. The k -core $\text{core}_k(W)$ of W consists of all $y \in \mathbb{R}^m$ such that any linear functional that takes a non-negative value on y takes a non-negative value on at least k entries of W .

(The k -core is also called the set of vectors of *depth* k ; see [45].)

If $w_i \neq 0$ for all i , then the 1-core is the positive hull

$$\text{core}_1(W) = \text{pos}(W).$$

Similarly, the 2-core can be expressed as

$$\text{core}_2(W) = \bigcap_{i=1}^n \text{pos}(W \setminus \{w_i\}).$$

For example, if, for every w_i , there is a $j \neq i$ such that $w_j = w_i$, then $\text{core}_2(W) = \text{core}_1(W) = \text{pos}(W)$.

Definition 1.37. A circuit C of a vector configuration V is called *hyperbolic* if one of the coefficients of the corresponding linear dependence is positive while the rest are negative. Equivalently, a hyperbolic circuit is an index subset $C = \{p(C)\} \cup C^-$ such that

$$v_{p(C)} = \sum_{i \in C^-} \lambda_i^{C^-} v_i \quad \text{where } \lambda_i^{C^-} > 0 \text{ for all } i \in C^- \quad (11)$$

and every proper subset of $V_C = \{v_i \mid i \in C\}$ is linearly independent.

A hyperbolic circuit of cardinality two consists of two non-zero vectors v_i and v_j such that $v_i = \lambda v_j$ for $\lambda > 0$. In this case, any of the indices i and j can be declared to be $p(C)$. As we assumed at the beginning of this section that $v_i \neq \lambda v_j$ for $\lambda > 0$, every hyperbolic circuit has cardinality at least three, and thus the positive index $p(C)$ is well defined.

Equation (11) means that we are scaling the coefficients of every hyperbolic circuit C so that

$$\lambda_{p(C)}^C = 1, \quad \lambda_i^C = \begin{cases} -\lambda_i^{C^-} & \text{if } i \in C^-, \\ 0 & \text{if } i \notin C. \end{cases}$$

Due to the dependence-evaluation duality (see (26) and the two equations following it), to every hyperbolic circuit there corresponds a unique vector $\mu^C \in \mathbb{R}^{n-d}$ such that

$$\langle \mu^C, \bar{v}_{p(C)} \rangle = 1, \quad \langle \mu^C, \bar{v}_i \rangle = \begin{cases} -\lambda_i^{C^-} & \text{if } i \in C^-, \\ 0 & \text{if } i \notin \{p(C)\} \cup C^-. \end{cases}$$

THEOREM 1.38. *Let (v_1, \dots, v_n) be a linearly spanning vector configuration in \mathbb{R}^d such that*

$$v_i \neq 0 \text{ for all } i \text{ and } v_i \neq \lambda v_j \text{ for all } i \neq j.$$

Then the closure $\text{clir}(V)$ of the irredundancy domain is a convex polyhedral cone in \mathbb{R}^n . The set $\text{clir}(V)$ and its lift $\widetilde{\text{clir}}(V)$ to \mathbb{R}^n can be described in the following ways.

$$\widetilde{\text{clir}}(V) = \bigcap_{i=1}^n (\mathbb{R}_+^{[n] \setminus i} + \text{im } V), \quad (12a)$$

where $\mathbb{R}_+^{[n] \setminus i} = \{(h_1, \dots, h_n) \mid h_i = 0, h_j \geq 0 \text{ for all } j \neq i\}$.

$$\text{clir}(V) = \bigcap_{i=1}^n \text{pos}(\bar{V} \setminus \{\bar{v}_i\}) = \text{core}_2(\bar{V}). \quad (12b)$$

$$\widetilde{\text{clir}}(V) = \{h \in \mathbb{R}^n \mid \langle \lambda^C, h \rangle \geq 0 \text{ for all positive circuits } C \text{ and } h_{p(C)} \leq \langle \lambda^{C^-}, h \rangle \text{ for all hyperbolic circuits } C\}. \quad (12c)$$

$$\text{clir}(V) = \{y \in \mathbb{R}^{n-d} \mid \langle \mu^C, y \rangle \geq 0 \text{ for all positive circuits } C \text{ and } \langle \mu^C, y \rangle \leq 0 \text{ for all hyperbolic circuits } C\}. \quad (12d)$$

Moreover, if V is positively spanning, then $\text{ir}(V)$ is pointed.

Proof. From the definition of $F_i(h)$, it follows that

$$x \in F_i(h) \Leftrightarrow h - Vx \in \mathbb{R}_+^{[n] \setminus i}.$$

Therefore

$$F_i(h) \neq \emptyset \text{ for all } i \Leftrightarrow h \in \bigcap_{i=1}^n (\mathbb{R}_+^{[n] \setminus i} + \text{im } V).$$

Since $\widetilde{\text{clir}}(V) = \{h \mid F_i(h) \neq \emptyset \text{ for all } i\}$, we have (12a).

(12a) \Rightarrow (12b). Since $\mathbb{R}_+^{[n]\setminus i}$ is the positive hull of all basis vectors except e_i , $\pi(\mathbb{R}_+^{[n]\setminus i}) = \text{pos}(\bar{V} \setminus \{\bar{v}_i\})$ and hence

$$\text{clir}(V) = \bigcap_{i=1}^n \pi(\mathbb{R}_+^{[n]\setminus i}) = \bigcap_{i=1}^n \text{pos}(\bar{V} \setminus \{\bar{v}_i\}),$$

(here \cap and π commute because $X + \text{im } V$ is a full preimage under π).

(12b) \Rightarrow (12c). By the definition of irredundancy, $h \in \text{ir}(V)$ if and only if, along with the system $Vx \leq h$, the n systems

$$\langle v_i, x \rangle > h_i, \langle v_j, x \rangle \leq h_j \quad \text{for all } j \neq i, \quad (13)$$

$i = 1, \dots, n$, are compatible. Consequently, $h \in \widetilde{\text{clir}}(V)$ if and only if all systems (13) are compatible when $>$ is replaced by \geq . The system (13) can be rewritten as $V^{(i)}x \leq h^{(i)}$, where

$$V^{(i)} = (v_1, \dots, -v_i, \dots, v_n), \quad h^{(i)} = (h_1, \dots, -h_i, \dots, h_n).$$

According to the criterion (9c) of compatibility, we have to consider all positive circuits of $V^{(i)}$. Each of them is either a positive circuit of $V \setminus \{v_i\}$ or a hyperbolic circuit of V with i as the positive index. This yields the system of linear inequalities in (12c).

(12c) \Rightarrow (12d). This follows from $\lambda^C = \bar{V}\mu^C$ for any circuit and from our convention for hyperbolic circuits (see the paragraph before the theorem).

Finally, if V is positively spanning, then, by Theorem 1.30, the cone $\text{co}(V)$ is pointed, and hence so is $\text{ir}(V) \subset \text{co}(V)$. \square

Remark 1.39. Again, part (12c) of Theorem 1.38 can be interpreted in terms of the support function. Due to (10c),

$$h \in \text{clir}(V) \Leftrightarrow h_i = \tilde{h}(v_i),$$

where \tilde{h} is the support function of $P(h)$ and $\|v_i\| = 1$ is assumed. Since \tilde{h} is convex, equality $h_i = \tilde{h}(v_i)$ implies the inequality $h_i \leq \sum_{j \in C^-} \lambda_j^C h_j$ for every hyperbolic circuit C with $p(C) = i$. In the opposite direction, if $h \notin \text{clir}(V)$, then $h_i > \tilde{h}(v_i)$ for some i . This yields a hyperbolic circuit C (where $p(C) = i$ and C^- are the indices of the extremal rays of the normal cone of $P(h)$ containing v_i in its relative interior) that violates the linear inequality in the second line of (12c).

1.2.5. *Chambers and type cones.* Let $V = (v_1, \dots, v_n)$ be a positively spanning configuration of n different unit vectors in \mathbb{R}^d . By results of the previous sections, every polytope $P(h)$ with outer facet normals v_i is represented by a point $y = \pi(h) \in \mathbb{R}^{n-d}$ lying in the interior of the 2-core of the Gale dual configuration $\bar{V} = (\bar{v}_1, \dots, \bar{v}_n)$. We now want to describe how the relative position of y with respect to \bar{v}_i determines the combinatorics of the polytope.

By slightly modifying the notation, we will distinguish between the *geometric* normal fan $\mathcal{N}(P(h))$ and the *abstract* fan Δ isomorphic to $\mathcal{N}(P(h))$. Here Δ is a collection of subsets of $[n]$ such that

$$\sigma \in \Delta \Leftrightarrow \text{pos}(V_\sigma) \in \mathcal{N}(P(h)),$$

where, as usual, $V_\sigma = \{v_i \mid i \in \sigma\}$.

LEMMA 1.40. *Let $y = \pi(h) \in \text{int core}_2(\bar{V})$. Then*

$$\text{pos}(V_\sigma) \in \mathcal{N}(P(h)) \Leftrightarrow y \in \text{relint pos}(\bar{V}_{[n]\setminus\sigma}). \quad (14)$$

In other words, $\{v_i \mid i \in \sigma\}$ span a normal cone of $P(h)$ if and only if $\pi(h)$ lies in the relative interior of the cone spanned by $\{\bar{v}_i \mid i \notin \sigma\}$.

Proof. Let $F_\sigma(h)$ denote the face of $P(h)$ with the normal cone $\text{pos}(V_\sigma)$. Then

$$x \in \text{relint } F_\sigma(h) \Leftrightarrow \langle v_i, x \rangle \begin{cases} = h_i & \text{if } i \in \sigma, \\ < h_i & \text{if } i \notin \sigma, \end{cases} \Leftrightarrow h - Vx \in \text{relint } \mathbb{R}_+^{[n]\setminus\sigma}.$$

By applying the projection π , we obtain

$$F_\sigma(h) \neq \emptyset \Leftrightarrow y \in \text{relint pos}(V_{[n]\setminus\sigma}).$$

But $F_\sigma(h) \neq \emptyset$ is equivalent to $\text{pos}(V_\sigma) \in \mathcal{N}(P(h))$ and the proof is complete. \square

Lemma 1.40 allows us to give a very concise description of the type cones of the vector configuration V .

Definition 1.41. Let $W = (w_1, \dots, w_n) \subset \mathbb{R}^m$ be a vector configuration. Two vectors y_1 and y_2 are said to lie in the same *relatively open chamber* if, for every $I \subset [n]$,

$$y_1 \in \text{pos}(W_I) \Leftrightarrow y_2 \in \text{pos}(W_I).$$

The closure of a relatively open chamber is called a *chamber*, and the collection of all chambers is called the *chamber fan* $\text{Ch}(W)$ of W .

Equivalently, a relatively open chamber is an inclusion-minimal intersection of relative interiors of cones generated by W and the chamber fan is the coarsest common refinement of all cones $\text{pos}(W_I)$, $I \subset [n]$.

COROLLARY 1.42. *Let Δ be a complete pointed fan with rays generated by V . Then $T(\Delta)$ is a relatively open chamber in the chamber fan $\text{Ch}(\bar{V})$.*

Indeed, two points $y_1, y_2 \in \mathbb{R}^{n-d}$ belong to the same type cone if and only if the corresponding polytopes have the same normal fan. By Lemma 1.40, this is equivalent to y_1 and y_2 belonging to the same collections of relatively open cones $\text{pos}(\bar{V}_I)$. Since relatively open and closed cones are related through the \cup and \setminus operations, this is equivalent to y_1 and y_2 belonging to the same collections of closed cones.

LEMMA 1.43. *Let Δ be a complete pointed fan with rays generated by V . Then*

$$T(\Delta) = \bigcap_{\sigma \in \Delta} \text{relint pos}(\bar{V}_{[n] \setminus \sigma}) = \bigcap_{\rho \in \Delta^{(d)}} \text{relint pos}(\bar{V}_{[n] \setminus \rho}). \quad (15)$$

In particular, the fan Δ is polytopal if and only if the intersection on the right-hand side is non-empty.

Proof. By Lemma 1.40, the left-hand side of (15) is a subset of the middle. Clearly, the middle is a subset of the right-hand side.

Let us show that the right-hand side is a subset of the left-hand side. Let $\pi(h) \in \text{relint pos}(\bar{V}_{[n] \setminus \rho})$ for all $\rho \in \Delta^{(d)}$. Since, for every i , there is $\rho \in \Delta^{(d)}$ such that $i \in \rho$,

$$\bigcap_{\rho \in \Delta^{(d)}} \text{relint pos}(\bar{V}_{[n] \setminus \rho}) \subset \bigcap_{i=1}^n \text{int pos}(\bar{V}_{[n] \setminus i}) = \text{int core}_2(\bar{V}) = \text{int ir}(V)$$

(here $\text{rk } \bar{V}_{[n] \setminus i} = n - d$ for all i allows to replace relint by the absolute interior). Hence, by (10b), $P(h)$ is a d -polytope with n facets having outer normals v_1, \dots, v_n and we are in a position to apply Lemma 1.40. It implies that $\text{pos}(V_\rho) \in \mathcal{N}(P(h))$ for all $\rho \in \Delta^{(d)}$. Since the cones $\text{pos}(V_\rho)$ cover \mathbb{R}^d , the polytope $P(h)$ has no other full-dimensional normal cones, and thus $\mathcal{N}(P(h)) = \Delta$. \square

Note that if the fan Δ is simplicial, then $|\rho| = d$ for all $\rho \in \Delta^{(d)}$, and hence $|[n] \setminus \rho| = n - d$. According to Lemma 1.26, the vector configuration $\bar{V}_{[n] \setminus \rho}$ is linearly independent, so that (15) represents $T(\Delta)$ as a finite intersection of open simplicial d -cones.

Remark 1.44. Chambers of $\text{Ch}(\bar{V})$ contained in $\partial \text{clir}(V) \cap \text{int co}(V)$ can be identified with type cones $T(\Delta)$ for complete pointed fans whose rays are generated by proper subsets of V . Chambers contained in $\text{int co}(V) \setminus \text{clir}(V)$ are linearly isomorphic to $T(\Delta) \times \text{int } \mathbb{R}_+^I$, where $I \subset [n]$ is the index set of redundant inequalities.

Chambers outside $\text{int ir}(V)$ can be studied using the operations of *contraction* and *deletion* on vector configurations. It is easy to see that, as a Gale dual of $V_{[n] \setminus i}$ (configuration obtained by deletion), one can take projections of vectors of $\bar{V}_{[n] \setminus i}$ along \bar{v}_i (configuration obtained by contraction). According to that, the

type cones $T(\Delta)$ with $\Delta^{(1)}$ generated by $V_{[n]\setminus i}$ are the chambers of the contracted Gale dual $\bar{V}_{[n]\setminus i}$. The contraction can be seen as a central projection in \mathbb{R}^{n-d} from \bar{v}_i . This projection maps those chambers of $\text{Ch}(\bar{V}) \cap \partial \text{clir}(V)$ visible from \bar{v}_i to the chambers of $\text{Ch}(\bar{V}_{[n]\setminus i}) \cap \text{clir}(V_{[n]\setminus i})$.

Chambers on the boundary of $\text{co}(V)$ correspond to type cones of polytopes of dimension smaller than d . The facet normals of such a polytope P are obtained from V by contraction along vectors v_i orthogonal to $\text{aff}(P)$.

Remark 1.45. The chambers fan $\text{Ch}(\bar{V})$ is polytopal. Lemma 1.13 and the observation after Definition 1.41 imply that $\text{Ch}(\bar{V})$ is the normal fan of the Minkowski sum of representatives of all normal equivalence types with facet normals in \bar{V} . The fan $\text{Ch}(\bar{V})$ is called the *secondary fan* of the vector configuration V and any polyhedron that has $\text{Ch}(\bar{V})$ as its normal fan is called a *secondary polyhedron* of V (see [5, 6]).

If the cone $\text{pos}(V)$ is pointed, then, by Lemma 1.26, \bar{V} is positively spanning; and thus a secondary polyhedron is compact and is called a *secondary polytope*. The interest in secondary polytopes was aroused by their applications to algebraic geometry (see the book [16] by Gel'fand *et al*).

Remark 1.46. Polytopes whose facet normals determine their normal equivalence class are called monotypic. For such polytopes,

$$\text{clir}(V) = \text{cl } T(\Delta),$$

for a unique polytopal fan Δ with $\Delta^{(1)} = V$. For example, polygons and polygonal prisms (including parallelepipeds) are monotypic. Monotypic polytopes are studied in [32].

Remark 1.47. One can show that any convex polyhedral cone is combinatorially (and even affinely) isomorphic to a chamber of some chamber fan $\text{Ch}(W)$. By duplicating each vector in W , we obtain $\text{core}_2(W) = \text{pos}(W)$ so that all chambers lie in the 2-core. It follows that every convex polyhedral cone can be realized as a type cone of some vector configuration.

Remark 1.48. When the vector configuration V is positively spanning, its Gale dual \bar{V} can be represented by a point configuration in an affine hyperplane $A \subset \mathbb{R}^{n-d}$ (see the paragraphs preceding Remark 1.27). Then the linear functionals μ on \mathbb{R}^{n-d} are replaced by affine functions on A , in particular, in the definition of the k -core.

1.2.6. *Facets of $\text{clir}(V)$ and truncated polytopes.* Recall that $C \subset [n]$ is a circuit of V if and only if $[n]\setminus C$ is a cocircuit of \bar{V} : that is, the index set of a maximal non-spanning subconfiguration. This subconfiguration spans a hyperplane which is the kernel of the linear functional μ^C (see the beginning of §1.2.3).

By (12d), every facet of $\text{clir}(V) = \text{core}_2(\bar{V})$ is determined by a positive or a hyperbolic circuit of V . The corresponding hyperplane in \mathbb{R}^{n-d} spanned by

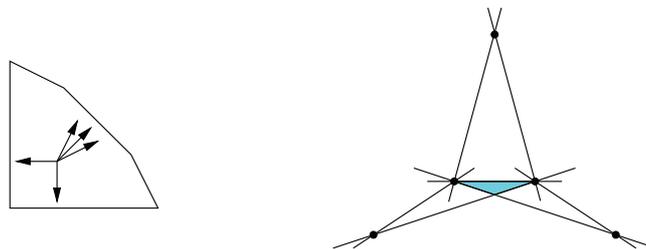


Figure 5: A configuration of five vectors in \mathbb{R}^2 and its affine Gale dual with a shaded 2-core.

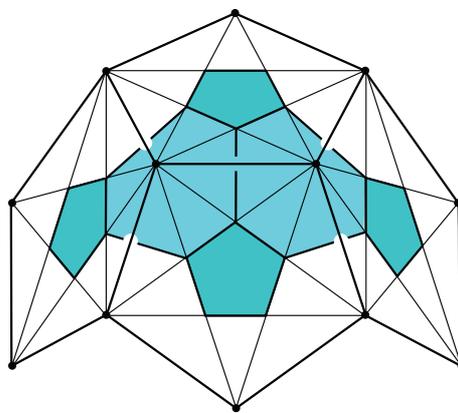


Figure 6: A fragment of the 2-core of the vertices of the dodecahedron.

$\bar{V}_{[n]\setminus C}$ has all of \bar{V}_C on one of its sides, if C is positive, and separates $\bar{v}_{p(C)}$ from \bar{V}_{C^-} , if C is hyperbolic.

Although every facet corresponds to a cocircuit, not every cocircuit (not even every hyperbolic one) corresponds to a facet of $\text{core}_2(\bar{V})$, as the following example shows.

Example 1.49. The point configuration shown in Figure 5, right has three positive and seven hyperbolic cocircuits. None of the positive cocircuits and only three of the hyperbolic cocircuits determine a facet of its 2-core.

This point configuration is the affine Gale dual of the five vectors in \mathbb{R}^2 shown in Figure 5, left. Thus the points of the shaded triangle on the right parametrize the space of pentagons with normals as those on the left.

In the next example, every positive and every hyperbolic cocircuit determines a facet of the 2-core.

Example 1.50. The 2-core of the vertices of a dodecahedron is the truncated icosahedron. Every hyperbolic cocircuit gives rise to a hexagonal facet and every positive cocircuit gives rise to a pentagonal facet (see Figure 6).

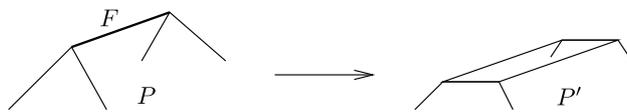


Figure 7: Truncating an edge of a 3-polytope.

Vertices of a dodecahedron correspond to a configuration of 12 vectors in \mathbb{R}^4 , which is Gale dual to a configuration of 12 vectors in \mathbb{R}^8 .

Definition 1.51. Let $P \subset \mathbb{R}^d$ be a d -polytope and let F be a proper face of P . We say that a polytope P' is obtained from P by *truncating* the face F if

$$P' = P \cap \{x \in \mathbb{R}^d \mid \langle v, x \rangle \leq h_P(v) - \varepsilon\},$$

where v is any unit vector in $\text{relint } N_F(P)$ and $\varepsilon > 0$ is sufficiently small (so small that $\text{Vert}(P) \setminus \text{Vert}(F) \subset \text{Vert}(P')$, Vert denotes the set of vertices).

Note that $P \cap \{x \in \mathbb{R}^d \mid \langle v, x \rangle = h_P(v)\} = F$. Thus, geometrically, truncation means pushing a supporting hyperplane of P at F inwards. The operation of truncating an edge of a 3-polytope is shown in Figure 7.

For any positive or hyperbolic circuit C of V , define

$$\text{clir}^C(V) = \text{clir}(V) \cap \{y \in \mathbb{R}^{n-d} \mid \langle \mu^C, y \rangle = 0\}. \quad (16)$$

LEMMA 1.52. Let $C = \{p(C)\} \cup C^-$ be a hyperbolic circuit of V such that $\text{clir}^C(V)$ is a facet of $\text{clir}(V)$. Then, for every $h \in \mathbb{R}^n$ such that $\pi(h) \in \text{relint } \text{clir}^C(V)$, the following hold.

- (1) $P(h)$ is a d -dimensional polytope with outer facet normals $V_{[n] \setminus p(C)}$.
- (2) $\text{pos}(V_{C^-}) \in \mathcal{N}(P(h))$.
- (3) $P(h - \varepsilon e_{p(C)})$ is obtained from $P(h)$ by truncating the face F_{C^-} , provided that $\varepsilon > 0$ is sufficiently small.

Proof. It is easy to show that, for a hyperbolic circuit C , $\text{relint } \text{clir}^C(V) \subset \text{int } \text{co}(V)$. Thus $h \in \text{relint } \text{clir}^C(V)$ implies that $\dim P(h) = d$.

Next, if $\dim P(h) = d$, then $\dim F_i(h) = d - 1$ is equivalent to the compatibility of the system

$$\langle v_i, x \rangle = h_i, \quad \langle v_j, x \rangle < h_j \quad \text{for all } j \neq i,$$

which, by a standard argument, is equivalent to

$$\pi(h) \in \text{int } \text{pos}(\bar{V}_{[n] \setminus i}). \quad (17)$$

The assumption $\pi(h) \in \text{relint } \text{clir}^C(V)$ implies that (17) holds for all $i \neq p(C)$, but does not hold for $i = p(C)$. Hence $F_i(h)$ is a facet of $P(h)$ if and only if $i \neq p(C)$. This finishes the proof of part (1).

For part (2), observe that $\text{clir}^C(V) \subset \text{pos}(\bar{V}_{[n]\setminus C})$, which implies that

$$\text{relint clir}^C(V) \subset \text{relint pos}(\bar{V}_{[n]\setminus C}).$$

Then the argument from the proof of Lemma 1.40 implies that $\text{pos}(V_{C^-})$ belongs to the normal fan.

Finally, the hyperbolic circuit relation (11) implies that $v_{p(C)} \in \text{relint pos}(V_{C^-})$. Thus, by Definition 1.51, $P(h - \varepsilon e_{p(C)})$ is obtained from $P(h)$ by truncating

F_{C^-} . □

LEMMA 1.53. *Let C_1 and C_2 be two hyperbolic circuits such that $\text{clir}^{C_1}(V)$ and $\text{clir}^{C_2}(V)$ are two facets of $\text{clir}(V)$ intersecting along a codimension two face of $\text{clir}(V)$. Then, for every $h \in \mathbb{R}^n$ such that $\pi(h) \in \text{relint}(\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V))$, the following hold.*

- (1) $P(h)$ is a d -dimensional polytope with outer facet normals $V_{[n]\setminus\{p_1, p_2\}}$, where p_i is the positive index of C_i , $i = 1, 2$.
- (2) If $p_1 \notin C_2$ and $p_2 \notin C_1$ (in particular, if $p_1 \neq p_2$), then $\text{pos}(V_{C_i^-}) \in \mathcal{N}(P(h))$ for $i = 1, 2$.
- (3) Under the assumptions

$$p_1 \notin C_2, p_2 \notin C_1, \quad \text{pos}(V_{C_1^- \cup C_2^-}) \notin \mathcal{N}(P(h)),$$

the polytope $P(h - \varepsilon_1 e_{p_1} - \varepsilon_2 e_{p_2})$ for $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small is obtained from $P(h)$ by independent truncation of $F_{C_1^-}$ and $F_{C_2^-}$. (Truncation is called independent if the truncated parts are disjoint.)

Proof. The first part is similar to that of Lemma 1.52, if one notes that the relative interior of a codimension two face belongs to two faces only.

For the second part, we have to prove that

$$\text{relint}(\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V)) \subset \text{relint pos}(\bar{V}_{[n]\setminus C_i}) \quad \text{for } i = 1, 2. \quad (18)$$

In other words, for two cones $\text{clir}^{C_1}(V) \subset \text{pos}(\bar{V}_{[n]\setminus C_1})$ of the same dimension, we have to show that the facet $\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V)$ of the former is not contained in a facet of the latter (and the same with indices 1 and 2 exchanged). The ray $\mathbb{R}_+ \bar{v}_{p_2}$ is an extreme ray of $\text{pos } \bar{V}$ and, since $p_2 \in [n]\setminus C_1$, also of $\text{pos}(\bar{V}_{[n]\setminus C_1})$. Since $\mathbb{R}_+ v_{p_2}$ and $\text{clir}^{C_1}(V)$ lie on different sides from the facet $\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V)$, this facet cannot be contained in a facet of $\text{pos}(\bar{V}_{[n]\setminus C_i})$, and the proof is complete.

Finally, the third part is true because $\text{pos}(V_{C_1^- \cup C_2^-}) \notin \mathcal{N}(P(h))$ implies that the faces $F_{C_1^-}$ and $F_{C_2^-}$ of $P(h)$ are disjoint. Hence all small truncations of those faces are independent. The lemma is proved. □

1.3. *Examples.* In the following examples of vector configurations V , we analyze the closure $\text{clir}(V)$ of the irredundancy domain and its decomposition into type cones. We are using both the direct approach (“what happens when the facets of a polytope are translated”) and the more formal one, through Gale diagram, circuits and the chamber fan.

1.3.1. *Parallelepipeds with fixed face directions.* Let (v_1, v_2, v_3) be a basis of \mathbb{R}^3 , put $v_{i+3} = -v_i, i = 1, 2, 3$ and consider the resulting configuration V of six vectors in \mathbb{R}^3 .

All polyhedra with facet normals V are normally equivalent: they are parallelepipeds with face normals $\pm v_i$. The normal fan Δ is generated by a hyperplane arrangement spanned on the vectors v_1, v_2, v_3 .

The lifted type cone $\tilde{T}(\Delta)$ consists of $h \in \mathbb{R}^6$ that satisfy

$$h_1 + h_4 > 0, \quad h_2 + h_5 > 0, \quad h_3 + h_6 > 0. \quad (19)$$

By identifying $\mathbb{R}^6 / \text{im } V$ with $\{h \in \mathbb{R}^6 \mid h_4 = h_5 = h_6 = 0\}$, we obtain

$$\text{cl } T(\Delta) \cong \{(h_1, h_2, h_3) \mid h_i \geq 0, i = 1, 2, 3\}.$$

The extreme rays of $\text{cl } T(\Delta)$ correspond to degeneration of a parallelepiped into a segment parallel to one of the vectors $v_1 \times v_2, v_2 \times v_3, v_3 \times v_1$.

The Gale diagram of V has the property $\bar{v}_{i+3} = \bar{v}_i, i = 1, 2, 3$. This fits together with the fact that $\text{co}(V) = \text{ir}(V)$: the second core of a vector configuration where each vector is repeated twice is its convex hull (see Definition 1.36 and equation (12b)). The facets of the cone (19) correspond to the three positive circuits $\{1, 4\}, \{2, 5\}$ and $\{3, 6\}$.

1.3.2. *Polygons.* Let $\alpha := (\alpha_1, \dots, \alpha_n)$ with $0 < \alpha_i < \pi$ be an n -tuple of real numbers such that $\sum_i \alpha_i = 2\pi$. Let $v_1, \dots, v_n \in \mathbb{R}^2$ be unit vectors such that the angle from v_i to v_{i+1} equals α_{i+1} (of course, the indices are taken modulo n). This determines a fan Δ in \mathbb{R}^2 . The fan is polytopal since, for every collection of positive numbers a_1, \dots, a_n such that $\sum_{i=1}^n a_i v_i = 0$, there is a convex polygon with edge lengths a_i and edge normals v_i . We will denote $T(\Delta)$ by $T(\alpha)$. Note that $\text{cl } T(\alpha)$ is a pointed $(n - 2)$ -dimensional cone. To avoid dealing with trivial cases, below, we assume that $n \geq 5$.

By Lemma 1.21, every facet of $T(\alpha)$ corresponds to vanishing of an edge: $\ell_i = 0$. This is possible without making any other edges disappear if and only if $\alpha_i + \alpha_{i+1} \leq \pi$. It follows that $\text{cl } T(\alpha)$ has $n, n - 1$ or $n - 2$ facets. Denote by T_i the facet $\ell_i = 0$ of $\text{cl } T(\alpha)$. If $j \notin \{i, i - 1, i + 1\}$, then T_i and T_j meet along a codimension two facet. Otherwise, T_i and T_{i+1} meet if and only if $\alpha_i + \alpha_{i+1} + \alpha_{i+2} < \pi$.

An extreme ray e of $\text{cl } T(\alpha)$ corresponds to triangles with fixed edge normals $v_{i_1}, v_{i_2}, v_{i_3}$. This means that $\ell_j = 0$ for all $j \notin \{i_1, i_2, i_3\}$ while $\ell_{i_\alpha} \neq 0$. Thus e is contained in exactly $n - 3$ facets, and hence $T(\alpha)$ is simple. It follows that $\text{cl } T(\alpha)$ is the cone over an $(n - 3)$ -dimensional polytope that is either a simplex, a truncated simplex or a doubly truncated simplex. See [4] for more details and the case of non-convex polygons.

In the Gale diagrams language, the cone $\text{cl } T(\alpha)$ is the 2-core of \bar{V} . Figure 5 shows an example for $n = 5$, where this cone has three facets.

1.3.3. *Polygonal prisms.* Embed \mathbb{R}^2 in \mathbb{R}^3 and add to the vectors $v_1, \dots, v_n \in \mathbb{R}^2$ from the preceding example the third basis vector and its inverse: $v_{n+1} = e_3$

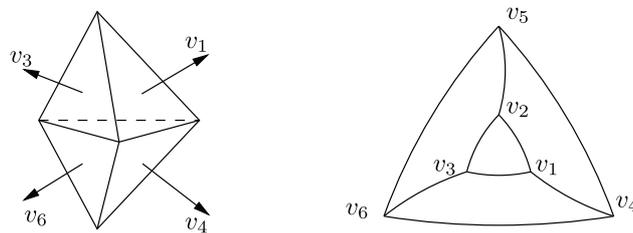


Figure 8: The bipyramid over a triangle and its normal fan.

and $v_{n+2} = -e_3$. The new vector configuration V^+ determines only one pointed fan, namely, the normal fan of a prism over an n -gon. Denote this fan by Δ^+ and denote its type cone by $T^+(\alpha)$. Then

$$(h, h_{n+1}, h_{n+2}) \in T^+(\alpha) \Leftrightarrow h \in T(\alpha) \quad \text{and} \quad h_{n+1} + h_{n+2} > 0.$$

Thus $T^+(\alpha)$ is a product of $T(\alpha)$ with a half-space. It follows that

$$T^+(\alpha) = T(\alpha) \times \mathbb{R}^+.$$

The new extreme ray $\{0\} \times \mathbb{R}^+$ corresponds to degeneration of the prism into a segment parallel to e_3 .

The Gale diagram of V^+ lives in the space \mathbb{R}^{n-1} , which is one dimension higher than \bar{V} . It is easy to see that \bar{V}^+ is obtained from \bar{V} by adding two equal vectors $\bar{v}_{n+1} = \bar{v}_{n+2} = e_{n-1}$. It follows that the 2-core of $\bar{V} \cup \{e_{n-1}, e_{n-1}\}$ is the pyramid over the 2-core of \bar{V} , which yields the same result as above.

1.3.4. *Triangular bipyramid.* Let $u_1, u_2, u_3 \in \mathbb{R}^2$ be outward unit normals to the edges of a regular triangle. Choose $\lambda, \mu > 0$ such that $\lambda^2 + \mu^2 = 1$ and consider the six vectors in \mathbb{R}^6 given by

$$\begin{aligned} v_1 &= \lambda u_1 + \mu e_3, & v_2 &= \lambda u_2 + \mu e_3, & v_3 &= \lambda u_3 + \mu e_3, \\ v_4 &= \lambda u_1 - \mu e_3, & v_5 &= \lambda u_2 - \mu e_3, & v_6 &= \lambda u_3 - \mu e_3. \end{aligned} \tag{20}$$

The 3-polytope $Vx \leq \mathbf{1}$ is a bipyramid over a triangle (see Figure 8). The right half of the picture schematically shows the normal fan of the bipyramid (as a stereographic projection of the intersection of the normal fan with the sphere).

By translating the faces of the bipyramid, we can split its four-valent vertices into pairs of three-valent vertices. For the spherical section of the normal fan, this means subdividing quadrilaterals by their diagonals. Let us apply the Gale diagram technique to study the arrangement of the type cones that correspond to different combinatorial types.

Let V be the 6×3 -matrix with rows v_i . We have to find a 6×3 -matrix \bar{V} of rank three whose columns are orthogonal to those of V . The matrix \bar{V} is unique up to a multiplication from the right with an element of $\text{GL}(\mathbb{R}, 3)$. Since the vectors v_4, v_5, v_6 form a basis of \mathbb{R}^3 , the vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3$ must do the same

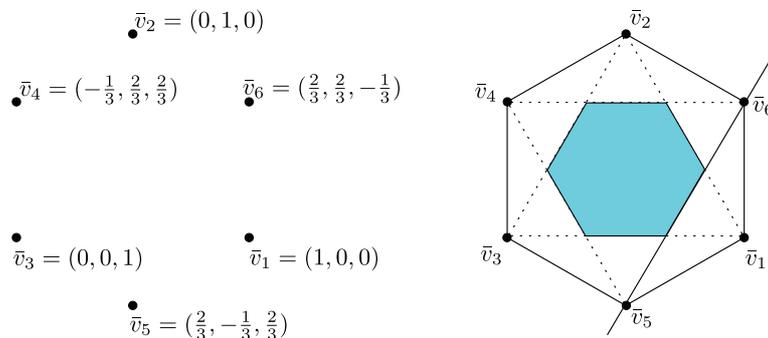


Figure 9: The Gale diagram of the face normals of a triangular bipyramid (left); the compatibility and the irredundancy domains (right).

(see Lemma 1.26). Thus we may assume that

$$\bar{v}_1 = (1, 0, 0), \quad \bar{v}_2 = (0, 1, 0), \quad \bar{v}_3 = (0, 0, 1).$$

The remaining entries of \bar{V} can be easily determined from the orthogonality condition between the columns of V and \bar{V}

$$V = \begin{pmatrix} \lambda u_1 & \mu \\ \lambda u_2 & \mu \\ \lambda u_3 & \mu \\ \lambda u_1 & -\mu \\ \lambda u_2 & -\mu \\ \lambda u_3 & -\mu \end{pmatrix} \quad \bar{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}.$$

Recall that the map

$$\bar{V}^\top : \mathbb{R}^n \rightarrow \mathbb{R}^{n-d} \cong \mathbb{R}^n / \text{im } V$$

projects the space of support vectors h to its quotient by translations of the polytope $P(h)$ (see equation (2) and Definition 1.23). We identify \mathbb{R}^{n-d} with a subspace of \mathbb{R}^n via a right inverse ι of \bar{V}^\top . In our case, we can put $\iota(e_i) = e_i$ for $i = 1, 2, 3$ so that $\mathbb{R}^{n-d} = \mathbb{R}^3$ is identified with the subspace $h_4 = h_5 = h_6 = 0$ of \mathbb{R}^6 . Geometrically, this corresponds to fixing the lower vertex of the bipyramid at the origin and varying only the heights h_1, h_2, h_3 .

The rows of \bar{V} are the coordinates of six vectors forming the Gale diagram of V . Since all of them lie in the subspace $h_1 + h_2 + h_3 > 0$, we can conveniently draw the *affine Gale diagram* by intersecting the cones generated by \bar{V} with the affine hyperplane $h_1 + h_2 + h_3 = 1$ (see Figure 9, left). (By a lucky coincidence, the vectors \bar{v}_i lie in this plane.)

Figure 9, right, shows the quotients $\text{co}(V)$ and $\text{ir}(V)$ of the compatibility domain and of the irredundancy domain. According to (9b), $\text{co}(V)$ is the positive

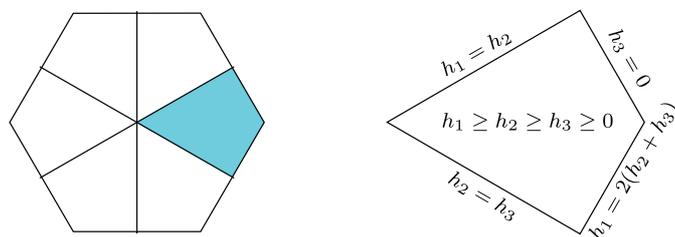


Figure 10: A type cone.

hull of \bar{V} , which, in the affine Gale diagram, becomes the convex hull. According to (12b), $\text{ir}(V)$ is the 2-core of \bar{V} , which is shown as a shaded hexagon in Figure 9.

It is also possible to interpret Figure 9, right, in terms of positive and hyperbolic circuits (see equations (9c) and (12c)). The six positive circuits of the vector configuration V are obtained from

$$v_1 + 2v_2 + v_4 + 2v_6 = 0$$

by the action of the dihedral group. This particular circuit leads, according to (9b), to the inequality $h_1 + 2h_2 \geq 0$ in the $h_4 = h_5 = h_6 = 0$ space, which determines the half-space containing vectors \bar{v}_3 and \bar{v}_5 in its boundary. The other five edges of the big hexagon shown in Figure 9 correspond to the other five positive circuits.

The lines bounding the irredundancy domain correspond to hyperbolic circuits (in this example, the inequalities in (12c) generated by the positive circuits turn out to be redundant). For example, the line highlighted in Figure 9 corresponds to the circuit

$$v_1 = 2v_2 + 2v_3 + 3v_4. \tag{21}$$

The principle “evaluations on \bar{V} correspond to dependencies in V ” (see (7)) and allow us to read off the signature of the circuit from the position of the line. Since the line separates \bar{v}_1 from \bar{v}_2, \bar{v}_3 and \bar{v}_4 , the coefficient at v_1 has the sign opposite to those at \bar{v}_2, \bar{v}_3 and \bar{v}_4 ; the points lying on the line correspond to zero coefficients.

In order to obtain the chamber fan of the vector configuration \bar{V} , one has to draw the diagonals $\bar{v}_1\bar{v}_4, \bar{v}_2\bar{v}_5$ and $\bar{v}_3\bar{v}_6$ in addition to those drawn already in Figure 9. The chambers in the interior of the 2-core are the type cones of V . Figure 10 shows the subdivision of $\text{clir}(V)$ into chambers and describes the faces of one of the full-dimensional type cones.

For any point $h \in \text{clir}(V)$ (recall that we identified \mathbb{R}^3 with a subspace of \mathbb{R}^6 by putting $h_4 = h_5 = h_6$), the combinatorics of the corresponding polytope can be read off from the diagram. By Lemma 1.40, the normal fan of $P(h)$ contains the cone $\text{pos}(V_I)$ (equivalently, facets with normals $\{v_i \mid i \in I\}$ intersect along a face) if and only if h lies in the relative interior of the positive hull of $\bar{V}_{[6] \setminus I}$. For example, since the type cone highlighted in Figure 10 lies in

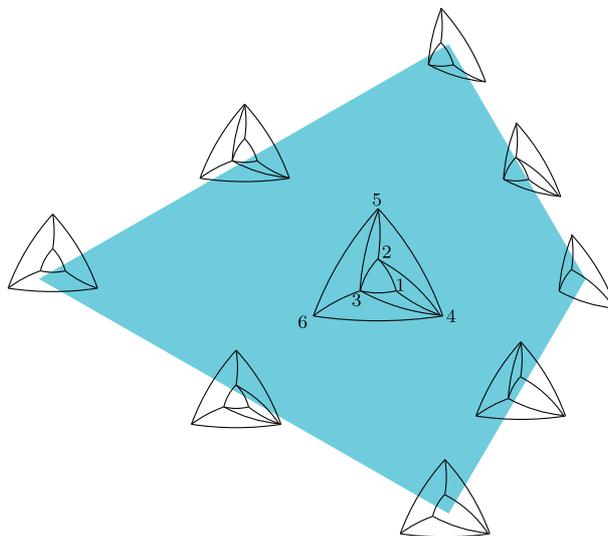


Figure 11: The fans corresponding to the faces of a type cone.

$\text{relint pos}\{\bar{v}_1, \bar{v}_2, \bar{v}_5\}$, the corresponding fan Δ contains the cone spanned by v_3, v_4, v_6 . The normal fan corresponding to this type cone is shown in the center of Figure 11. This normal fan is simplicial, since the type cone is full-dimensional (compare Lemma 1.19).

The other fans in Figure 11 are associated with the faces of the depicted type cone. Note that the fans corresponding to boundary points of $\text{clir}(V)$ are not using all of the vertices v_i (compare Lemma 1.52).

Crossing from one fully dimensional type cone to an adjacent one corresponds to a “flip” (see Figure 12). The edge F_{35} becomes replaced by the edge F_{26} (compare the description of the faces of the type cones through vanishing edge lengths in Lemma 1.21). From a different point of view, boundaries between full-dimensional type cones correspond to (non-positive and non-hyperbolic) circuits of the vector configuration V . A circuit of this form corresponds to contracting an edge of the polytope (see equation (4)). There are three such circuits

$$v_1 + v_5 = v_2 + v_4, \quad v_2 + v_6 = v_3 + v_5, \quad v_3 + v_4 = v_1 + v_6,$$

and they correspond to the hyperplanes

$$h_1 + h_5 = h_2 + h_4, \quad h_2 + h_6 = h_3 + h_5, \quad h_3 + h_4 = h_1 + h_6,$$

or, in our picture, to the lines

$$h_1 = h_2, \quad h_2 = h_3, \quad h_3 = h_1.$$

The fan shown in Figure 13, left, is not polytopal. Indeed, since it contains $\text{pos}\{v_1, v_6\}$, $\text{pos}\{v_2, v_4\}$ and $\text{pos}\{v_3, v_5\}$, the corresponding point $\pi(h)$ must lie

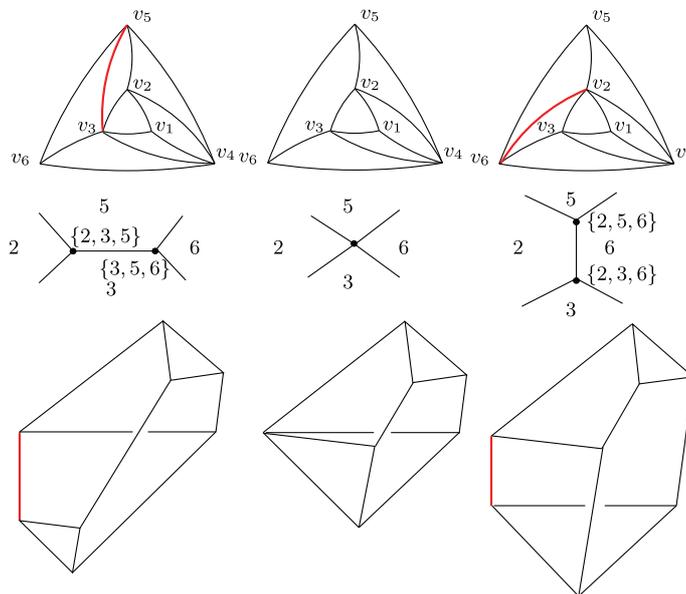


Figure 12: The flip $\{\{2, 3, 5\}, \{3, 5, 6\}\} \rightsquigarrow \{\{2, 3, 6\}, \{2, 5, 6\}\}$.

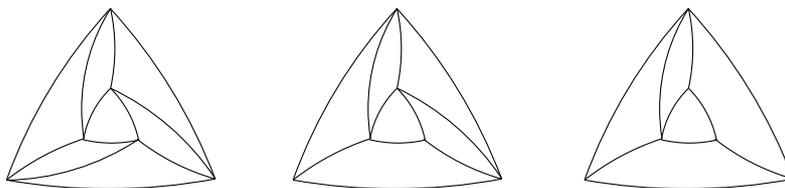


Figure 13: Examples of non-polytopal fans.

in the intersection

$$\text{relint pos}\{\bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5\} \cap \text{relint pos}\{\bar{v}_1, \bar{v}_3, \bar{v}_5, \bar{v}_6\} \cap \text{relint pos}\{\bar{v}_1, \bar{v}_2, \bar{v}_4, \bar{v}_6\},$$

which is empty. Similarly, the other two fans in Figure 13 are also non-polytopal. (One can also refer to Figure 11, where all, up to symmetry, polytopal fans with the 1-skeleton in V are shown.)

§2. *Mixed volumes.*

2.1. *The examples, continued.* We continue the examples considered in §1.3. Motivated by the observation of Bavard and Ghys [4] that the area of convex polygons as a function of their support numbers h_i is a quadratic form of hyperbolic signature, we are looking at the surface area of three-dimensional polytopes.

2.1.1. *Parallelepipeds with fixed face directions.* The surface area of the parallelepiped $\{x \mid 0 \leq \langle x, v_i \rangle \leq h_i, i = 1, 2, 3\}$ equals

$$\text{area}(h) = \frac{2}{D}(h_1h_2 + h_2h_3 + h_3h_1),$$

where $D = |\det(v_1, v_2, v_3)|$. This is a quadratic form of signature $(+, -, -)$.

2.1.2. *Polygons.* The area of a polygon with support vector $h \in \tilde{T}(\alpha) \subset \mathbb{R}^n$ equals

$$\text{area}(h) = \frac{1}{2} \sum_i h_i \ell_i(h),$$

where the edge length $\ell_i(h)$ is a linear function of h . Thus $\text{area}(h)$ is a quadratic form. The associated symmetric bilinear form is

$$\text{area}(h, k) = \frac{1}{2} \sum_i h_i \ell_i(k) = \frac{1}{2} \sum_i k_i \ell_i(h), \quad (22)$$

due to $(\partial \text{area}(h)/\partial h_i) = \ell_i(h)$, which follows from a simple geometric argument.

The quadratic form $\text{area}(h)$ has signature $(+, 0, 0, -, \dots, -)$. This can be proved by induction on the number of edges of the polygon, as in [4], or using Minkowski's inequality [38, p. 321]. See also [21, Lemma 2.14].

2.1.3. *Polygonal prisms.* Denote by $\text{area}^+(h^+)$ the surface area of a prism (h, h_{n+1}, h_{n+2}) over the n -gon. Here v_{n+1} and v_{n+2} are as in §1.3.3. Then

$$\begin{aligned} \text{area}^+(h^+) &= 2 \text{area}(h) + (h_{n+1} + h_{n+2}) \text{per}(h) \\ &= 2 \text{area}(h) + 2(h_{n+1} + h_{n+2}) \text{area}(\mathbf{1}, h) \end{aligned}$$

with the associated symmetric bilinear form

$$\begin{aligned} \text{area}^+(h^+, k^+) &= 2 \text{area}(h, k) \\ &\quad + (h_{n+1} + h_{n+2}) \text{area}(\mathbf{1}, k) + (k_{n+1} + k_{n+2}) \text{area}(\mathbf{1}, h). \end{aligned}$$

The restriction of area^+ to \mathbb{R}^n is 2area , which, by the above results, has signature $(+, 0, 0, -, \dots, -)$. The vector $(\mathbf{0}, -1, 1)$ belongs to the kernel of area^+ (it corresponds to vertical translation). The vector $(\mathbf{1}, -1, -1)$ is orthogonal to $\mathbb{R}^n \subset \mathbb{R}^{n+2}$ with respect to area^+ and is negative. This implies that area^+ has signature $(+, 0, 0, 0, -, \dots, -)$.

2.1.4. *Triangular bipyramid.* Let $v_1, \dots, v_6 \in \mathbb{R}^3$ be as in 1.3.4. The combinatorics of a polyhedron $P(h)$ with face normals (v_i) depend on h . Let us compute $\text{area}(\partial P(h))$ for the type cone Δ shaded in Figure 10.

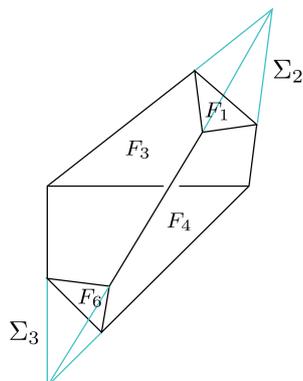


Figure 14: Representing $P(h)$ as a truncated tetrahedron.

The normal fan of $P(h)$ (the triangulation in the center of Figure 11) shows that $P(h)$ is a doubly truncated tetrahedron (see Figure 14). So

$$P(h) = \overline{(\Sigma_1 \setminus \Sigma_2) \setminus \Sigma_3},$$

where

$$\Sigma_1 := \{x \in \mathbb{R}^3 \mid \langle v_i, x \rangle \leq h_i, i \in \{2, 3, 4, 5\}\},$$

$$\Sigma_2 := \{x \in \mathbb{R}^3 \mid \langle v_i, x \rangle \leq h_i, i \in \{2, 3, 4\}, \langle v_1, x \rangle \geq h_1\},$$

$$\Sigma_3 := \{x \in \mathbb{R}^3 \mid \langle v_i, x \rangle \leq h_i, i \in \{3, 4, 5\}, \langle v_6, x \rangle \geq h_6\}.$$

The surface area of a tetrahedron with fixed face normals is proportional to the squared length of any of its edges and the edge length is a linear function of the support vector. Thus

$$\text{area}(\partial \Sigma_1) = f_1^2(h)$$

for some linear function f_1 . We have $f_1(h) = 0$ if and only if the hyperplanes $H_i, i \in \{2, 3, 4, 5\}$ pass through a common point, that is, if and only if the system $\langle v_i, x \rangle = h_i$ has a solution. Since $v_2 + 2v_3 + 2v_4 + v_5 = 0$, a solution exists if and only if $h_2 + 2h_3 + 2h_4 + h_5 = 0$. By restricting to $h_4 = h_5 = h_6 = 0$ as we have done in §1.3.4, we obtain

$$\text{area}(\partial \Sigma_1) = c_1(h_2 + 2h_3)^2$$

for some $c_1 > 0$.

Next, observe that

$$\begin{aligned} \text{area}(\partial \overline{\Sigma_1 \setminus \Sigma_2}) &= \text{area}(\partial \Sigma_1) - \text{area}(F_2(\Sigma_2)) - \text{area}(F_3(\Sigma_2)) \\ &\quad - \text{area}(F_4(\Sigma_2)) + \text{area}(F_1(\Sigma_2)). \end{aligned}$$

Each of the quantities $\text{area}(F_i(\Sigma_2))$ is proportional to the square of a linear function $f_2(h)$ that vanishes when Σ_2 degenerates. Similarly to the previous paragraph, using (21), we find that $f_2(h) = h_1 - 2h_2 - 2h_3 - 3h_4$, and hence

$$\text{area}(\partial \overline{\Sigma_1 \setminus \Sigma_2}) = c_1(h_2 + 2h_3)^2 - c_2(h_1 - 2h_2 - 2h_3)^2.$$

Here $c_2 > 0$ because the sum of areas of three faces of a tetrahedron is bigger than the area of its fourth face.

Finally, cutting off the tetrahedron Σ_3 yields

$$\text{area}(\partial P(h)) = q_\Delta(h) := c_1 f_1^2 - c_2 f_2^2 - c_3 f_3^2, \quad c_1, c_2, c_3 > 0,$$

where

$$f_1(h) = h_2 + 2h_3, \quad f_2(h) = h_1 - 2h_2 - 2h_3, \quad f_3(h) = h_3.$$

Since f_1, f_2 and f_3 are linearly independent, $q_\Delta(h)$ has signature $(+, -, -)$.

2.2. Mixed volumes and quadratic forms.

2.2.1. *Definition and basic properties of mixed volumes.* Minkowski [33] has shown that the volume behaves polylinearly with respect to the Minkowski addition and positive scaling. Namely, for any compact convex bodies $K_1, \dots, K_m \subset \mathbb{R}^d$, there exist real numbers $c_{i_1 \dots i_d}$, $1 \leq i_\alpha \leq m$ such that

$$\text{vol}(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_\alpha \in [m]} c_{i_1 \dots i_d} \lambda_{i_1} \dots \lambda_{i_d} \quad (23)$$

holds for all $\lambda_1, \dots, \lambda_m \geq 0$. The coefficients $c_{i_1 \dots i_d}$ are uniquely determined by the bodies K_{i_1}, \dots, K_{i_d} if we require that they are symmetric with respect to permutations of indices: $c_{\varphi \circ I} = c_I$ for all $\varphi \in S_m$.

Definition 2.1. The coefficient $c_{i_1 \dots i_d}$ in (23) is called a *mixed volume* and is denoted by $\text{vol}(K_{i_1}, \dots, K_{i_d})$.

Clearly, $\text{vol}(K, \dots, K) = K$. For more details on mixed volumes, see [38, Ch. 5] and [10, Ch. IV].

Example 2.2. A special case of (23) is the Steiner formula

$$\text{vol}(K + \rho B) = \sum_{i=0}^d \binom{d}{i} \rho^i W_i(K),$$

where B is the unit ball. The coefficients $W_i(K) = \text{vol}(\underbrace{K, \dots, K}_{d-i}, \underbrace{B, \dots, B}_i)$ are called *quermassintegrals* of K . We have $W_0(K) = \text{vol}(K)$, $W_1(K) = (1/d) \text{area}(\partial K)$ and $W_d(K) = \text{vol}(B)$. For a polytope P ,

$$W_i(P) = c_{d,i} \sum_{F \in \mathcal{F}^{d-i}(P)} \text{vol}_{d-i}(F) \cdot |N_F(P)|$$

for some constant $c_{d,i}$ independent of P . Here the sum ranges over all $(d-i)$ -faces of P and $|N_F(P)|$ denotes the angular measure of $N_F(P) \subset \mathbb{R}^d$.

The following properties of mixed volumes will be needed in the subsequent work.

- Mixed volume is multilinear with respect to the Minkowski addition

$$\text{vol}(\lambda K + \mu L, \mathcal{K}) = \lambda \text{vol}(K, \mathcal{K}) + \mu \text{vol}(L, \mathcal{K}) \quad \text{for } \lambda, \mu \geq 0,$$

where $\mathcal{K} = (K_1, \dots, K_{d-1})$.

- Mixed volume is monotone under inclusion: $\text{vol}(K, \mathcal{K}) \geq \text{vol}(L, \mathcal{K})$ if $K \supset L$. In particular,

$$\text{vol}(K_1, \dots, K_d) \geq 0. \tag{24}$$

More precisely, the following theorem holds [38, Theorem 5.1.7].

THEOREM 2.3. *The inequality in (24) is strict if and only if there are segments $s_i \subset K_i$, $i = 1, \dots, d$ with linearly independent directions.*

In particular, the inequality (24) is strict if $\dim K_i = d$ for all i .

2.2.2. *Alexandrov–Fenchel inequalities and signatures of quadratic forms.* Fix convex bodies $K_1, \dots, K_{d-2} \subset \mathbb{R}^d$ and denote $\mathcal{K} := (K_1, \dots, K_{d-2})$. Then the function

$$\text{vol}_{\mathcal{K}} : K \mapsto \text{vol}(K, K, \mathcal{K}) \tag{25}$$

on the set of convex bodies in \mathbb{R}^d possesses the valuation property

$$\text{vol}_{\mathcal{K}}(K \cup L) + \text{vol}_{\mathcal{K}}(K \cap L) = \text{vol}_{\mathcal{K}}(K) + \text{vol}_{\mathcal{K}}(L), \tag{26}$$

provided that $K \cup L$ is convex. In addition, it is homogeneous of degree two: that is,

$$\text{vol}_{\mathcal{K}}(\lambda K) = \lambda^2 \text{vol}_{\mathcal{K}}(K). \tag{27}$$

This follows quite easily from the corresponding properties of the volume (see [31]).

Let Δ be a complete polytopal fan in \mathbb{R}^d . Due to the multilinearity of the mixed volume and the compatibility of the Minkowski addition with the linear structure of $\tilde{T}(\Delta)$ (Corollary 1.15), the function $\text{vol}_{\mathcal{K}}$ on $\tilde{T}(\Delta)$ is a restriction of a quadratic form.

Definition 2.4. Given a collection $\mathcal{K} = (K_1, \dots, K_{d-2})$ of convex bodies and a complete polytopal fan Δ , denote by $q_{\mathcal{K}, \Delta}$ the unique quadratic form on $\text{span}(\tilde{T}(\Delta))$ such that

$$q_{\mathcal{K}, \Delta}(h) = \text{vol}(P(h), P(h), \mathcal{K}) \quad \text{for } h \in \tilde{T}(\Delta).$$

The following properties of $q_{\mathcal{K}, \Delta}$ are immediate.

- The kernel of $q_{\mathcal{K}, \Delta}$ contains $\text{im } V$ and thus has dimension at least d .
- If $\dim K_i = d$ for all $i = 1, \dots, d - 2$, then $q_{\mathcal{K}, \Delta}(h) > 0$ for all $h \in \tilde{T}(\Delta)$.

The following theorem [1, 38] tells us more about the signature of q_Δ .

THEOREM 2.5 (Alexandrov–Fenchel inequalities). *The inequality*

$$\text{vol}(K, L, \mathcal{K})^2 \geq \text{vol}(K, K, \mathcal{K}) \text{vol}(L, L, \mathcal{K}) \quad (28)$$

holds for all compact convex bodies $K, L, K_1, \dots, K_{d-2}$.

Equation (28) basically says that the Gram matrix of the restriction of $q_{\mathcal{K}, \Delta}$ to any two-dimensional subspace spanned by two vectors from $\tilde{T}(\Delta)$ has a non-positive determinant. This is used in the proof of the following lemma (compare with [20, Appendix A.3]).

LEMMA 2.6. *Let $\mathcal{K} = (K_1, \dots, K_{d-2})$ consist of d -dimensional convex bodies. Then, for every polytopal fan Δ , the quadratic form $q_{\mathcal{K}, \Delta}$ has the following properties.*

- (1) *The positive index of $q_{\mathcal{K}, \Delta}$ equals one.*
- (2) *$\dim \ker q_{\mathcal{K}, \Delta} = d$ if and only if the Alexandrov–Fenchel inequality for*

$$K = P(h), \quad L = P(h'), \quad 0 \neq h, h' \in \tilde{T}(\Delta)$$

holds with equality only when $h' - \lambda h \in \text{im } V$ for some λ .

Proof. Due to $q_{\mathcal{K}, \Delta}(h) > 0$ for all $h \in \tilde{T}(\Delta)$, $\text{ind}_+(q_{\mathcal{K}, \Delta}) \geq 1$. If $\text{ind}_+(q_{\mathcal{K}, \Delta}) \geq 2$, then there exists a positive vector x in the orthogonal complement to h with respect to $q_{\mathcal{K}, \Delta}$. For a sufficiently small ε ,

$$h' := h + \varepsilon x \in \tilde{T}(\Delta),$$

since $\tilde{T}(\Delta)$ is relatively open (see Lemma 1.16). It follows that the restriction of $q_{\mathcal{K}, \Delta}$ to $\text{span}(h, x) = \text{span}(h, h')$ is positive definite, and hence the Gram matrix of $q_{\mathcal{K}, \Delta}$ with respect to h, h' has a positive determinant. Thus

$$q_{\mathcal{K}, \Delta}(h, h)q_{\mathcal{K}, \Delta}(h', h') - q_{\mathcal{K}, \Delta}(h, h')^2 > 0,$$

which contradicts the Alexandrov–Fenchel inequality. Hence $\text{ind}_+(q_{\mathcal{K}, \Delta}) = 1$ and the first part of the lemma is proved.

For $0 \neq h, h' \in \tilde{T}(\Delta)$, the condition $h' - \lambda h \notin \text{im } V$ for all λ is equivalent to

$$\dim E = 2 \quad \text{and} \quad E \cap \text{im } V = \{0\}, \quad (29)$$

where $E = \text{span}(h, h')$. Assume that $\dim \ker q_{\mathcal{K}, \Delta} = d$, that is, that $\ker q_{\mathcal{K}, \Delta} = \text{im } V$. Then (29) implies that the restriction of $q_{\mathcal{K}, \Delta}$ to E is non-degenerate, and thus the determinant of its Gram matrix does not vanish. It follows that the Alexandrov–Fenchel inequality for h, h' is strict.

In the opposite direction, if $\dim \ker q_{\mathcal{K}, \Delta} \geq d + 1$, then

$$\dim \text{span}\{h, \ker q_{\mathcal{K}, \Delta}\} \geq d + 2,$$

which allows us to choose $E \subset \text{span}\{h, \ker q_{\mathcal{K}, \Delta}\}$ transversal to $\text{im } V$. Then the restriction of $q_{\mathcal{K}, \Delta}$ to E is degenerate and, representing E as $\text{span}(h, h')$ for $h' \in \tilde{T}(\Delta)$, we see that the Alexandrov–Fenchel inequality for h, h' holds with equality. The lemma is proved. \square

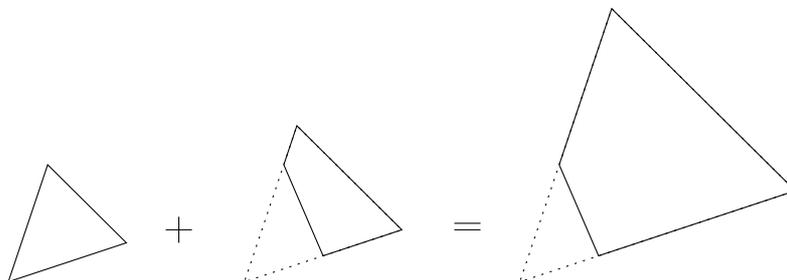


Figure 15: Illustration for Example 2.7.

2.2.3. *Quadratic forms of nullity d .* The condition $h' - \lambda h \in \text{im } V$ means that $K = P(h)$ and $L = P(h')$ are homothetic. But equality in (28) can take place also for non-homothetic K and L .

Example 2.7. For $d = 3$, let $K = \Sigma$ be a tetrahedron and let $L = \overline{\Sigma \setminus \Sigma_1}$ be a truncated tetrahedron, where we assume that the common vertex of Σ and Σ_1 lies at the coordinate origin. Then it can be shown (for example, with the help of the support function) that

$$\lambda K + \mu L = \overline{(\lambda + \mu)\Sigma \setminus \mu\Sigma_1}.$$

See also Figure 15 for the two-dimensional case.

It follows that

$$\begin{aligned} \text{vol}(\lambda K + \mu L) &= (\lambda + \mu)^3 \text{vol}(\Sigma) - \mu^3 \text{vol}(\Sigma_1) \\ &= \lambda^3 \text{vol}(\Sigma) + 3\lambda^2\mu \text{vol}(\Sigma) + 3\lambda\mu^2 \text{vol}(\Sigma) + \mu^3 \text{vol}(\overline{\Sigma \setminus \Sigma_1}), \end{aligned}$$

and hence

$$\text{vol}(L, L, K) = \text{vol}(L, K, K) = \text{vol}(K).$$

Therefore $\text{vol}(K, L, K)^2 = \text{vol}(K, K, K) \text{vol}(L, L, K)$, although K and L are not homothetic.

A complete characterization of the equality case in the Alexandrov–Fenchel inequality is still missing (see [38, §6.6]), but there are some partial results. Roughly speaking, if equality holds for some non-homothetic K and L , then their normal fans are more complicated than those of K_1, \dots, K_{d-2} .

THEOREM 2.8. *If $\mathcal{K} = (B, \dots, B)$, where B is the unit ball, and if $\dim K = \dim L = d$, then the Alexandrov–Fenchel inequality (28) holds with equality only if K and L are homothetic.*

This was proved by Kubota [25] (see also [38, Theorem 6.6.2]). More generally, $K_i, i = 1, \dots, d - 2$ may be any smooth convex bodies.

In view of Example 2.2, Theorem 2.8 implies the following corollary.

COROLLARY 2.9. For $d = 3$, the function $h \mapsto \text{area}(\partial P(h))$ restricted to any type cone $\tilde{T}(\Delta)$ is a quadratic form q_Δ of signature $(+, 0, 0, 0, -, \dots, -)$.

For $d > 3$, the $(d - 2)$ -nd quermassintegral

$$h \mapsto \sum_{\sigma \in \Delta^{(d-2)}} |\sigma| \cdot \text{area}(F_\sigma)$$

restricted to any type cone $\tilde{T}(\Delta)$ is a quadratic form q_Δ of signature $(+, d \cdot 0, -, \dots, -)$.

The form q_Δ in this corollary is proportional to $q_{\mathcal{K}, \Delta}$, where $\mathcal{K} = (B, \dots, B)$. The next result is a reformulation of [38, Theorem 6.6.20].

THEOREM 2.10. Let $\mathcal{K} = \{K_1, \dots, K_{d-2}\}$ be a collection of normally equivalent simple polytopes with the normal fan Δ_0 . Then the Alexandrov–Fenchel inequality (28) holds with equality if and only if, after applying a suitable homothety to K or L ,

$$h_K(v) = h_L(v) \quad \text{for all } v \in \Delta_0^{(2)}.$$

COROLLARY 2.11. Let $\mathcal{K} = \{K_1, \dots, K_{d-2}\}$ be a collection of normally equivalent simple polytopes with the normal fan Δ_0 . Then the quadratic form $q_{\mathcal{K}, \Delta}$ has signature $(+, d \cdot 0, -, \dots, -)$ if $\Delta^{(1)} \subset \Delta_0^{(2)}$.

In particular, the assumption of the Corollary 2.11 is fulfilled when $\Delta^{(1)} = \Delta_0^{(1)}$, that is, when K_i and $K = P(h)$ have the same sets of the outward facet normals.

COROLLARY 2.12. Let V be a vector configuration in \mathbb{R}^3 and let $h^0 \in \text{int ir}(V)$. Then the weighted sum of face areas

$$h \mapsto \sum_{i=1}^n h_i^0 \text{area}(F_i(h))$$

restricts on every type cone $\tilde{T}(\Delta)$ with $\Delta^{(1)} = V$ to a quadratic form of signature $(+, 0, 0, 0, -, \dots, -)$.

Proof. Since

$$\text{vol}(P(h), P(h), P(h^0)) = \frac{1}{3} \sum_{i=1}^n h_i^0 \text{area}(F_i(h)),$$

(see [10, 38]), the quadratic form in the corollary is proportional to $q_{P(h^0), \Delta}$. Polytopes $P(h^0)$ and $P(h)$ have the same sets of face normals: $\Delta_0^{(1)} = \Delta^{(1)} = V$. If $P(h^0)$ is simple, then the remark after Corollary 2.11 applies. If $P(h^0)$ is not simple, then it can be made simple by truncating vertices that have more than three adjacent edges. The quadratic form does not change (similarly to Example 2.7), while the normal fan of K_1 becomes only richer. Thus, by Corollary 2.11, the form has the right signature. \square

Example 2.13. We know (from Lemma 1.35) that $\mathbf{1} \in \text{int ir}(V)$. For $h^0 = \mathbf{1}$, the quadratic form in Corollary 2.12 is simply the surface area of P , that is, it coincides with the quadratic form from the first part of Corollary 2.9. Geometrically, $P(\mathbf{1})$ is the circumscribed polytope. Thus

$$\text{vol}(P(h), P(h), P(\mathbf{1})) = \text{vol}(P(h), P(h), B),$$

which is another example of non-strict monotonicity of mixed volumes (compare Theorem 2.3 and the paragraph before it).

Remark 2.14. For every simplicial polytopal fan Δ in \mathbb{R}^d there exists a homogeneous polynomial Z_Δ of degree d such that

$$\text{vol}(P(h)) = Z_\Delta(h) \quad \text{for all } h \in \tilde{T}(\Delta).$$

The mixed volume of polytopes with the normal fan Δ is given by the polarization of the polynomial Z_Δ . (In fact, the existence of development (23) is usually proved through approximation of K_i by normally equivalent simple polytopes.)

It follows that the quadratic form $q_{\mathcal{K}, \Delta}$ for $\mathcal{K} = (P(h^0), \dots, P(h^0))$ with $h^0 \in \Delta$ is proportional to the Hessian of $\text{vol}(P(h))$ at $h = h^0$. By Corollary 2.11, this Hessian is non-degenerate modulo translations. This is related to the uniqueness part of the Minkowski problem for polytopes; in addition, knowing the signature of the Hessian allows to prove the existence part. By duality, this is related to the infinitesimal rigidity of convex polytopes (see [20, 21]).

§3. Hyperbolic geometry.

3.1. *From type cones to hyperbolic polyhedra.* Let Δ be a simplicial polytopal fan with $\Delta^{(1)} = V$ and let $T(\Delta)$ be the corresponding type cone. Let $\mathcal{K} = (K_1, \dots, K_{d-2})$ be a collection of convex bodies such that the quadratic form $q_{\mathcal{K}, \Delta}$ from Definition 2.4 has signature $(+, d \cdot 0, -, \dots, -)$ (examples of such \mathcal{K} are given in Corollaries 2.9, 2.11 and 2.12). Since $\ker q_{\mathcal{K}, \Delta} = \text{im } V$, the form $q_{\mathcal{K}, \Delta}$ descends from \mathbb{R}^n to $\mathbb{R}^n / \text{im } V$. By an abuse of notation, this form will also be denoted by $q_{\mathcal{K}, \Delta}$; its signature is $(+, -, \dots, -)$.

Thus $q_{\mathcal{K}, \Delta}$ is a Minkowski scalar product on $\mathbb{R}^n / \text{im } V$. The upper half of the hyperboloid $\{\pi(h) \mid q_{\mathcal{K}, \Delta}(h) = 1\}$ becomes a model of the hyperbolic space \mathbb{H}^{n-d-1} . On the convex polyhedral cone $\text{cl } T(\Delta)$, the form $q_{\mathcal{K}, \Delta}$ takes non-negative values, due to the non-negativity of mixed volumes. Thus

$$H_{\mathcal{K}}(\Delta) := \text{cl } T(\Delta) \cap \{\pi(h) \mid q_{\mathcal{K}, \Delta}(h) = 1\}$$

(recall that $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n / \text{im } V$ is the projection map) becomes a convex hyperbolic polyhedron. More exactly, $H_{\mathcal{K}}(\Delta)$ is the convex hull of a finite number of points, some of which can be ideal.

An ideal vertex of $H_{\mathcal{K}}(\Delta)$ corresponds to $P(h)$ degenerating into a segment (at least in the situations of Corollaries 2.9 and 2.12). Such a degeneration is

possible if and only if the orthogonal complement of the segment is positively spanned by a subset of V .

A non-simplicial fan Δ gives rise to a hyperbolic polyhedron $H_{\mathcal{K}}(\Delta)$ of dimension smaller than $n - d - 1$. If $\Delta' \preceq \Delta$, then $H_{\mathcal{K}}(\Delta)$ is a face of $H_{\mathcal{K}}(\Delta')$.

In general, the irreducibility domain $\text{ir}(V)$ is composed of several type cones, which are quadratic forms on which there are different extensions to \mathbb{R}^n : $q_{\mathcal{K},\Delta} \neq q_{\mathcal{K},\Delta'}$. However, at the common boundary points of the type cones, these forms coincide: that is,

$$q_{\mathcal{K},\Delta}(h) = q_{\mathcal{K},\Delta'}(h) \quad \text{for } h \in \text{cl } \tilde{T}(\Delta) \cap \text{cl } \tilde{T}(\Delta'),$$

which follows, for example, from the continuity of the mixed volumes with respect to the Hausdorff metric. Thus the closure of the irredundancy domain $\text{clir}(V)$ becomes equipped with a piecewise hyperbolic metric. Denote by

$$M_{\mathcal{K}}(V) := \bigcup_{\Delta} H_{\mathcal{K}}(\Delta) \tag{30}$$

the corresponding metric space.

Recall that the combinatorial structure of $M_{\mathcal{K}}(V)$ is that of the chamber fan $\text{Ch}(\bar{V})$ intersected with $\text{clir}(V)$ and that $\text{clir}(V) = \text{core}_2(V)$ (see §1.2.5). (Strictly speaking, $M_{\mathcal{K}}(V)$ is combinatorially isomorphic to the chamber complex of the *affine* Gale diagram, intersected with the *affine* 2-core.) In addition, each facet of $\text{clir}(V)$ corresponds to a positive or hyperbolic circuit of V (see (12d) and (16)). Let

$$M_{\mathcal{K}}^C(V) \cong \text{clir}^C(V) \cap A$$

be the subset of $M_{\mathcal{K}}(V)$ that corresponds to the facet $\text{clir}^C(V)$ under the isomorphism of polyhedral complexes

$$M_{\mathcal{K}}(V) \cong \text{clir}(V) \cap A,$$

where A is an affine hyperplane in \mathbb{R}^{n-d} whose intersection with $\text{clir}(V)$ is bounded.

Let us now look at the polytopes $H_{\mathcal{K}}(\Delta)$ and the metric space $M_{\mathcal{K}}(V)$ in our standard examples.

3.2. *The examples, continued.* In each of the examples below, the quadratic form $q(h)$ (or $q_{\Delta}(h)$) is the area of a polygon or the surface area of a three-dimensional polytope, already studied in §2.1. It is proportional to $q_{\mathcal{K},\Delta}(h)$ with $\mathcal{K} = \emptyset$ for polygons and $\mathcal{K} = B$ for 3-polytopes.

In the first three examples, the vector configurations are monotypic (there is only one full-dimensional type cone $T(\Delta)$), so that it suffices to study the corresponding hyperbolic polytope $H(\Delta)$. In the last example, we have several $H(\Delta)$, and we are also studying the space (30) obtained by gluing them together.

3.2.1. *Parallelepipeds with fixed face directions form an ideal hyperbolic triangle.* We identified $\mathbb{R}^6 / \text{im } V$ with $\mathbb{R}^3 = \{h_4 = h_5 = h_6 = 0\}$. The

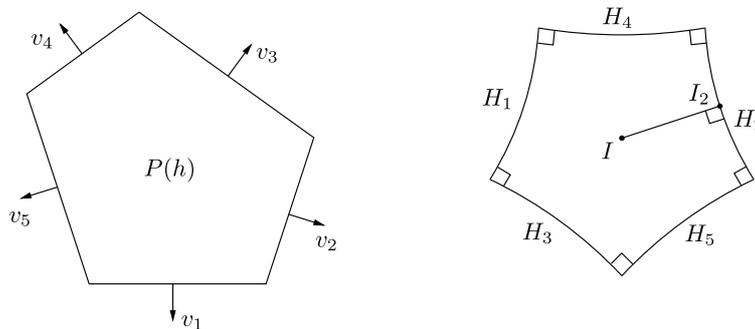


Figure 16: The space of equiangular pentagons is the regular right-angled hyperbolic pentagon.

cone $\text{cl}T(\Delta) = \{h \in \mathbb{R}^3 \mid h_i \geq 0, i = 1, 2, 3\}$ is spanned by three vectors e_1, e_2, e_3 , which are light-like with respect to the quadratic form $q(h) = (2/D)(h_1h_2 + h_2h_3 + h_3h_1)$. Thus $H(\Delta)$ is an ideal hyperbolic triangle.

3.2.2. *Polygons.* We have $\dim T(\alpha) = n - 2$, so that $H(\alpha)$ is an $(n - 3)$ -dimensional hyperbolic polyhedron. Recall, from §1.3.2, that the facet T_i of the cone $T(\alpha)$ corresponds to vanishing of the i th edge of the polygon: $\ell_i(h) = 0$. At the same time, (22) implies that $\text{area}(e_i, h) = (1/2)\ell_i(h)$. It follows that $\text{aff}(T_i)$ is orthogonal to e_i with respect to the quadratic form q . In other words, the point corresponding to e_i is polar dual to the corresponding facet H_i of $H(\alpha)$. (This point lies in the de Sitter space if T_i intersects the interior of the light cone.) This leads to the formula for the dihedral angle Θ_{i-1i} between H_{i-1} and H_i given by

$$\cos^2(\Theta_{i-1i}) = \frac{\sin(\alpha_{i-1}) \sin(\alpha_{i+1})}{\sin(\alpha_{i-1} + \alpha_i) \sin(\alpha_i + \alpha_{i+1})}.$$

If $j \notin \{i - 1, i, i + 1\}$, then this formula implies that H_i and H_j meet orthogonally. Thus $H(\alpha)$ is an *orthoscheme*. All hyperbolic orthoschemes (including truncated and doubly truncated ones) can be constructed in this way. For example, $H(2\pi/5, 2\pi/5, 2\pi/5, 2\pi/5, 2\pi/5)$ is the regular right-angled hyperbolic pentagon (see Figure 16). For more details, see [4, 11].

The polyhedron $H(\alpha)$ contains a distinguished point I that corresponds to a circumscribed polygon. In other words, $I := (\text{area}(\mathbf{1}))^{-1/2}\mathbf{1}$. For further reference, we want to compute the hyperbolic distance from I to the facet H_i of $H(\alpha)$. Let I_i be the orthogonal projection of I onto H_i . As e_i is orthogonal to H_i , the polygon corresponding to I_i is obtained from a circumscribed polygon by moving its i th edge until it disappears. Thus I_i corresponds to a circumscribed polygon with one fewer edge. In other words, its support vector is proportional to

$$\hat{\mathbf{1}}_i := (1, \dots, 0, \dots, 1),$$

with the only zero at the i th place. See Figure 17.

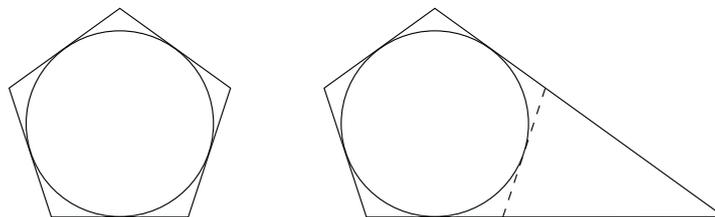


Figure 17: The polygons $P(\mathbf{1})$ and $P(\hat{\mathbf{1}}_i)$.

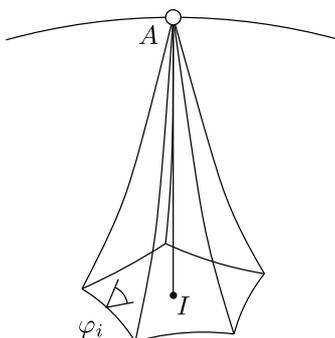


Figure 18: The space of prisms is the pyramid over the space of polygons.

By standard hyperbolic geometry,

$$\cosh \text{dist}(I, I_i) = \frac{\text{area}(\mathbf{1}, \hat{\mathbf{1}}_i)}{\sqrt{\text{area}(\mathbf{1}) \text{area}(\hat{\mathbf{1}}_i)}}.$$

By (22), $\text{area}(\mathbf{1}, \hat{\mathbf{1}}_i)$ is half of the perimeter of $P(\hat{\mathbf{1}}_i)$. Since $P(\hat{\mathbf{1}}_i)$ is circumscribed about a unit circle, its perimeter equals $2 \text{area}(\hat{\mathbf{1}}_i)$, so

$$\cosh \text{dist}(I, I_i) = \sqrt{\frac{\text{area}(\hat{\mathbf{1}}_i)}{\text{area}(\mathbf{1})}}. \tag{31}$$

3.2.3. *Polygonal prisms.* Denote by $H^+(\alpha)$ the hyperbolic polyhedron that is the section of $T^+(\alpha)$. Then $H^+(\alpha)$ is a pyramid over $H(\alpha)$ with apex A that corresponds to prisms degenerating into a segment. Since this degeneration nullifies the surface area, A is an ideal point. In coordinates, A is given by the light-like vector $(\mathbf{0}, 1, 1)$ (proportional to $(\mathbf{0}, 0, 1)$ and $(\mathbf{0}, 1, 0)$ modulo $\text{im } V$). The orthogonal projection of the apex A onto the hyperplane $h_{n+1} = h_{n+2} = 0$ that contains the basis $H(\alpha)$ is a linear combination of $(\mathbf{0}, 1, 1)$ and the normal $(\mathbf{1}, -1, -1)$ to that hyperplane. We obtain the vector $(\mathbf{1}, 0, 0)$. In other words, the orthogonal projection of the apex A is the point $I \in H(\alpha)$ which corresponds to a circumscribed polygon. See Figure 18.

Let us compute the dihedral angle φ_i between the facets $\text{conv}\{A, H_i\}$ and $H(\alpha)$ of the pyramid $H^+(\alpha)$. Applying the hyperbolic Pythagorean theorem [41]

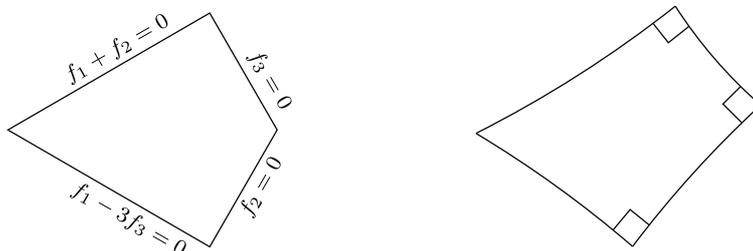


Figure 19: The hyperbolic quadrilateral corresponding to the type cone.

to the triangle AI_i , we obtain, from (31),

$$\sin \varphi_i = \sqrt{\frac{\text{area}(\mathbf{1})}{\text{area}(\hat{\mathbf{1}}_i)}}.$$

The formulas for the other dihedral angles of $H^+(\alpha)$ do not look nice.

3.2.4. *Triangular bipyramid.* The space $M_{\mathcal{K}}(V)$ is glued from six quadrilaterals as shown in Figure 10. Each of these quadrilaterals is equipped with a hyperbolic metric that arises from the corresponding quadratic form q_{Δ} . *A priori*, the hexagon shown in Figure 10 can become a hyperbolic 12-gon with a conic singularity at the point where all six quadrilaterals meet. Let us understand what its shape is, in reality.

Consider the hyperbolic quadrilateral $H(\Delta)$, where Δ is the cone over the quadrilateral shaded in Figure 10. The equations of the boundary hyperplanes in Figure 10, right, can be rewritten in coordinates f_1, f_2, f_3 , as shown in Figure 19. Since

$$q_{\Delta}(h) = c_1 f_1^2 - c_2 f_2^2 - c_3 f_3^2, \tag{32}$$

as was proved in §2.1.4, it follows that the quadrilateral $H(\Delta)$ has three right angles. The same holds for all of the other type cones. For two adjacent type cones, the two adjacent right angles sum up to π . It follows that the union of all six quadrilaterals is a right-angled hyperbolic hexagon, possibly with a conic singularity in the interior.

To proceed further, one needs to know the coefficients c_1, c_2, c_3 in (32). This can be done with the help of formulas from Appendix A.2, but, in our case, we can exploit the symmetry.

Let Δ' be the type cone $h_1 \geq h_3 \geq h_2$ sharing with Δ the facet $h_2 = h_3$. *A priori*, $q_{\Delta} \neq q_{\Delta'}$: indeed, the surface areas of polytopes with pairwise parallel faces but different combinatorial structure are, in general, given by different quadratic forms. However, we claim that, in our case, $q_{\Delta'} = q_{\Delta}$.

The fans Δ and Δ' are related by a flip (see Figure 12). It follows that

$$\begin{aligned} q_{\Delta'}(h) = & \text{area}(\partial P(h)) + \text{area}(F_3(\Sigma)) + \text{area}(F_5(\Sigma)) \\ & - \text{area}(F_2(\Sigma)) - \text{area}(F_6(\Sigma)), \end{aligned}$$

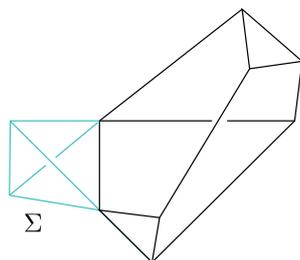


Figure 20: The difference $q_{\Delta'} - q_{\Delta}$ for fans related by a flip.

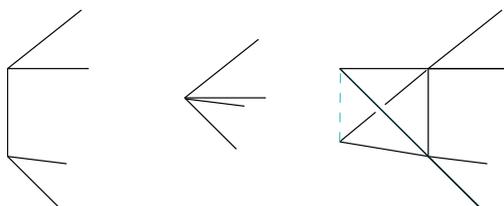


Figure 21: Changing a height of a convex polytope with combinatorics Δ until going outside $\tilde{T}(\Delta)$, but keeping the combinatorics Δ . The shaded edge has negative length (in the sense of Lemma 1.17). Compare “butterfly moves” [42].

where Σ is the tetrahedron bounded by the planes H_2, H_3, H_5, H_6 (see Figures 20 and 21). On the other hand, $v_3 + v_5 - v_2 - v_6 = 0$ and $-v_3, -v_5, v_2, v_6$ are the outer unit normals to the faces of Σ . Together with the Minkowski identity $\sum \text{area}(F_i)v_i = 0$, this implies that $\text{area}(F_3(\Sigma)) + \text{area}(F_5(\Sigma)) - \text{area}(F_2(\Sigma)) - \text{area}(F_6(\Sigma)) = 0$, and thus $q_{\Delta'} = q_{\Delta}$.

The same holds for any other pair of adjacent type cones. Thus, for all $h \in \text{ir}(V)$, the surface area of $P(h)$ is given by the same quadratic form $q(h)$, independent of the combinatorics of $P(h)$.

It follows that the right-angled hyperbolic hexagon $\bigcup_{\Delta} H(\Delta)$ has no cone singularity in the interior. By exploiting the symmetry of the vector configuration further, it can be shown that the hexagon is equilateral and that the lines subdividing it into the type polygons are its axes of symmetry. Indeed, the maps

$$(h_1, h_2, h_3) \mapsto (h_1, h_3, h_2), \quad (h_1, h_2, h_3) \mapsto \left(\frac{2}{3} - h_1, \frac{2}{3} - h_2, \frac{2}{3} - h_3\right)$$

send the hexagon to itself and preserve the value of $q(h)$: the first map corresponds to reflection of $P(h)$ in the coordinate xz -plane and the second one to reflection in the xy -plane. Thus both maps are hyperbolic isometries. It follows that all of the type quadrilaterals are congruent and symmetric.

The invariance of the form $q(h)$ under permutations of coordinates allows us to quickly compute the coefficients c_i in (32) up to a common factor. Namely, $c_1 = 3c_2, c_3 = 3c_1$ and

$$q(h) = c(-h_1^2 - h_2^2 - h_3^2 + 4h_1h_2 + 4h_2h_3 + 4h_3h_1).$$

The factor c depends on the parameter λ in (20).

3.3. *Dihedral angles at the boundary.* Let C_1 and C_2 be positive or hyperbolic circuits determining two facets of $\text{clir}(V)$ and let $M_{\mathcal{K}}^{C_1}(V)$ and $M_{\mathcal{K}}^{C_2}(V)$ be the corresponding subcomplexes of $\partial M_{\mathcal{K}}(V)$ (see §3.1). If $M_{\mathcal{K}}^{C_1}(V)$ and $M_{\mathcal{K}}^{C_2}(V)$ intersect along a codimension two (with respect to $M_{\mathcal{K}}(V)$) subcomplex, then one may ask what are the dihedral angles of $M_{\mathcal{K}}(V)$ along this intersection. Note that the dihedral angles may be different at different cells $H_{\mathcal{K}}(\Delta)$ of $M_{\mathcal{K}}^{C_1}(V) \cap M_{\mathcal{K}}^{C_2}(V)$.

The following theorem describes a special case when $M_{\mathcal{K}}^{C_1}(V)$ and $M_{\mathcal{K}}^{C_2}(V)$ intersect at a right angle. The proof generalizes an argument from [4]. Example in §3.2.4 can serve as an illustration.

THEOREM 3.1. *Let $C_1, C_2 \subset [n]$ be hyperbolic circuits with the positive indices p_1 and p_2 such that*

$$|C_1| = |C_2| = d + 1 \quad \text{and} \quad p_1 \notin C_2, p_2 \notin C_1$$

(in particular, $p_1 \neq p_2$) and such that $\text{clir}^{C_1}(V)$ and $\text{clir}^{C_2}(V)$ are facets of $\text{clir}(V)$ intersecting along a codimension two face. Then the dihedral angle between the corresponding subsets of $\partial M_{\mathcal{K}}(V)$ equals $\pi/2$ at every point.

Proof. Let h^0 be such that $\pi(h^0) \in \text{relint}(\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V))$. By Lemma 1.53, the polytope $P(h)$ has dimension d and outer normals $V_{[n] \setminus \{p_1, p_2\}}$. For a generic choice of h^0 , the polytope $P(h^0)$ is simple. Since $|C_1^-| = |C_2^-| = d$, the faces $F_{C_1^-}$ and $F_{C_2^-}$ are two different vertices of $P(h^0)$. Also by Lemma 1.53, for all $h \in \text{ir}(V)$ sufficiently close to h^0 , the polytopes $P(h)$ have the same combinatorics. Namely, $P(h)$ is obtained by an independent truncation of the vertices $F_{C_1^-}$ and $F_{C_2^-}$ of the polytope $P(h^0)$, where $h - h^0 = \varepsilon_1 \bar{v}_{p_1} + \varepsilon_2 \bar{v}_{p_2}$. Let Δ be the corresponding complete simplicial fan. It suffices to compute the dihedral angle of the hyperbolic polytope $H_{\mathcal{K}}(\Delta)$ at $\text{clir}^{C_1}(V) \cap \text{clir}^{C_2}(V)$.

$$P(h) = \overline{P(h^0) \setminus (\Sigma_1 \cup \Sigma_2)},$$

where Σ_i is a simplex with outward facet normals $V_{C_i^-} \cup \{-v_{p_i}\}$, $i = 1, 2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$. The valuation property (26) implies that

$$\begin{aligned} \text{vol}_{\mathcal{K}}(P(h^0)) &= \text{vol}_{\mathcal{K}}(P(h)) - \text{vol}_{\mathcal{K}}(\Sigma_1) - \text{vol}_{\mathcal{K}}(\Sigma_2) \\ &\quad + \text{vol}_{\mathcal{K}}(F_{p_1}(h^0)) + \text{vol}_{\mathcal{K}}(F_{p_2}(h^0)), \end{aligned} \quad (33)$$

where F_{p_i} is a common facet of $P(h)$ and Σ_i . It follows that

$$q_{\mathcal{K}, \Delta}(h^0) = q(h_{[n] \setminus \{p_1, p_2\}}) + c_1 f_1^2(h) + c_2 f_2^2(h), \quad (34)$$

where the quadratic form q on the right-hand side does not depend on h_{p_1} and h_{p_2} and where

$$f_i(h) = \langle \lambda^{C_i}, h \rangle, \quad i = 1, 2$$

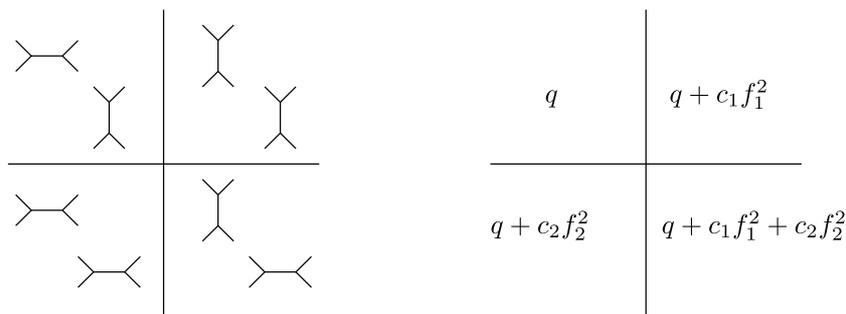


Figure 22: Two independent flips.

is the linear function corresponding to the circuit C_i . Indeed, the linear measurements of both $\Sigma_1(h)$ and $F_{p_1}(h)$ are proportional to $f_1(h)$, and thus, by homogeneity (27), both $\text{vol}_{\mathcal{K}}(\Sigma_1(h))$ and $\text{vol}_{\mathcal{K}}(F_{p_1}(h))$ are proportional to $f_1^2(h)$.

As the decomposition (34) is orthogonal, the hyperplanes

$$\{h \mid f_1(h) = 0\} \quad \text{and} \quad \{h \mid f_2(h) = 0\}$$

are orthogonal with respect to the quadratic form $q_{\mathcal{K},\Delta}$. It follows that the corresponding facets of $\text{cl}T(\Delta')$ are orthogonal, and hence the dihedral angle of $M_{\mathcal{K}}(V)$ at $\pi(h)$ equals $\pi/2$. \square

Remark 3.2. The monotonicity of mixed volumes under inclusion implies that $c_1, c_2 \leq 0$ in (34). Similarly to §2.1.4, this can be used to determine the signature of the quadratic form $q_{\mathcal{K},\Delta}$ if Δ is the normal fan of a (multiply) truncated simplex.

3.4. Cone angles in the interior. In this section, we provide evidence that the metric space $M_{\mathcal{K}}(V)$ is, in general, a cone-manifold, that is, that some of its interior codimension two strata can have total angles different from 2π around them.

Let $\Delta, \Delta^{(1)} = V$ be a polytopal fan in \mathbb{R}^d , all of whose cones are simplicial except for two, each of which is spanned by $d + 1$ vectors in general position (non-positive circuits of full rank). Such a cone can be triangulated in two ways and perturbations of $h \in T(\Delta)$ allow us to obtain any of the four combinations of these two pairs of triangulations. This results in four simplicial fans that we denote by $\Delta_{00}, \Delta_{01}, \Delta_{10}$ and Δ_{11} . For $d = 3$, we have two quadrilaterals and subdivide each of them by a diagonal; it is reflected in the face structure of $P(h)$ by flipping two edges (see Figure 22, left).

Locally, the arrangement of the type cones $T(\Delta_{ij})$ around $T(\Delta)$ is that of the intersections of half-spaces determined by linear functionals f_1 and f_2 on \mathbb{R}^n . Here f_i is proportional to the length of any edge in a triangulation of the i th circuit, $i = 1, 2$. Denote by q the quadratic form $q_{\mathcal{K},\Delta_{00}}$. Crossing the hyperplane $f_i(h) = 0$ changes the quadratic form by a multiple of f_i^2 so that we have the situation shown in Figure 22, right.

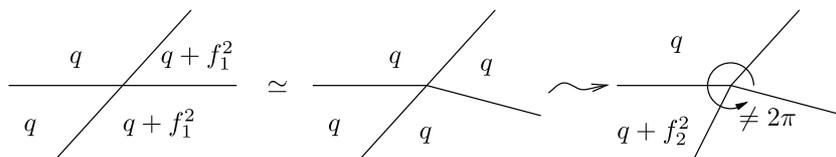


Figure 23: Angles around a generic codimension two stratum.

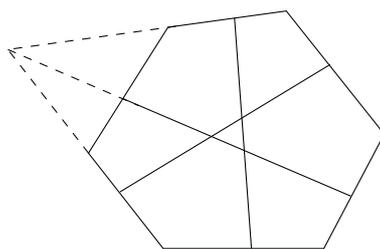


Figure 24: Decomposition of $\text{ir}(V)$, with V obtained by perturbing the normals of a triangular bipyramid.

The total angle around $H_{\mathcal{K}}(\Delta)$ in the metric space $M_{\mathcal{K}}(V)$ equals 2π if one of the following conditions is fulfilled.

- (1) One of the coefficients c_1 or c_2 equals zero.
- (2) The dihedral angle of $H_{\mathcal{K}}(\Delta_{00})$ at $H_{\mathcal{K}}(\Delta)$ equals $\pi/2$.

Figure 23 shows how the geometry around $T(\Delta)$ changes if we change the quadratic form q in two steps: first, by adding a summand in one half-space, then in the other. It indicates that if neither of the above conditions is satisfied, then the total angle around $H_{\mathcal{K}}(\Delta)$ is different from 2π .

Example 3.3. The normals of the triangular bipyramid produce six type cones shown in Figure 10. There is only one codimension two stratum, namely, the point in the center, and it corresponds to vanishing of three edges rather than two. The symmetry can be broken by perturbing the Gale diagram \bar{V} in Figure 9 (which is equivalent to perturbing the normal vectors of the bipyramid). Figure 24 shows the type cones arising from a generic perturbation.

In the metric space $M_{\mathcal{K}}(V)$, the three intersection points in Figure 24 are singular, in general. Indeed, if the perturbation is small, then the angles of the polygons $H(\Delta)$ at these points are close to $\pi/3$ or $2\pi/3$. Thus the second of the conditions of non-singularity above is not fulfilled. The first condition means that crossing one of the one-dimensional strata in the interior does not change the quadratic form q . As we have seen in §3.2.4, this is equivalent to the coefficient sum of the corresponding circuit being zero. For a generic perturbation, this is not the case.

3.5. *Questions.* There is definitely more to say about the metric space $M_{\mathcal{K}}(V)$, in general.

CONJECTURE 3.4. *The subsets $M_{\mathcal{K}}^C$ of $\partial M_{\mathcal{K}}(V)$, defined in §3.3, are orthogonal to all singular strata. This would allow us to call them facets of $M_{\mathcal{K}}(V)$.*

Let $C_1, C_2 \subset [n]$ be two circuits of the vector configuration V and let Δ be a polytopal fan that contains $\text{pos}(V_{C_1})$ and $\text{pos}(V_{C_2})$ and is otherwise simplicial. In §3.4, we have shown that the angle around $H_{\mathcal{K}}(\Delta)$ may be different from 2π . The following conjecture suggests that the situation is even worse.

CONJECTURE 3.5. *Even if the circuits C_1 and C_2 are disjoint (and even if they are “sufficiently far apart”), the four dihedral angles at $H_{\mathcal{K}}(\Delta)$ may be different from $\pi/2$. Moreover, these angles are not determined by local data. That is, changing a vector v_i with $i \notin C_1 \cup C_2$ can change the values of these angles.*

A generalization of the triangular bipyramid example is the configuration of the $2d$ vectors

$$v_i^{\pm} = u_i \pm e_d, \quad i = 1, \dots, d, \tag{35}$$

where u_i are normals to a regular simplex in \mathbb{R}^{d-1} . The corresponding affine Gale diagram has the properties $\bar{v}_i^- = -\bar{v}_i^+$ and $\sum_i \bar{v}_i = 0$. It follows that

$$\text{clir}(V) = \text{pos}(S \cap -S),$$

where S is a $(d - 1)$ -dimensional simplex with the barycenter at the coordinate origin. The case $d = 3$ was dealt with in §1.3.4. For $d = 4$ the Gale diagram is formed by the vertices of a 3-cube and $\text{clir}(V)$ is the cone over the octahedron. The corresponding secondary polytope is related to the permutahedron (see [9, §6.2.1], where a similar point configuration is analyzed).

Problem 3.6. For the vector configuration (35), describe the metric structure of the space $M_{\mathcal{K}}(V)$ in the case when $\mathcal{K} = (B, \dots, B)$: that is, when the quadratic form is given by the $(d - 2)$ -nd quermassintegral.

For example, $d = 4$ should yield the right-angled ideal octahedron.

The hyperbolic polyhedron $H_{\mathcal{K}}(\Delta)$ has an especially nice structure if the fan Δ is obtained by a sequence of stellar subdivisions from the normal fan of a tetrahedron. The corresponding polyhedron $P(h)$ is a multiply truncated tetrahedron (see §2.1.4).

Problem 3.7. Analyze the combinatorics and geometry of $H_{\mathcal{K}}(\Delta)$, where Δ is a multiple stellar subdivision of the normal fan of a tetrahedron.

In the following two problems, we assume that $\mathcal{K} = (B, \dots, B)$ and $d = 3$, and thus the quadratic form is the surface area of a three-dimensional polytope.

Problem 3.8. What is $M(V)$ when V consists of normals to an octahedron? (The quadratic form seems to be independent of the choice of the type cone, and thus $M(V)$ should be a four-dimensional hyperbolic polytope with eight facets.)

We ask the same question when V consists of normals to a regular bipyramid over an n -gon.

Problem 3.9. Let Δ be the normal fan of the dodecahedron. Describe the corresponding hyperbolic eight-dimensional polyhedron $H(\Delta)$ and compute its dihedral angles.

Problem 3.10. Describe all vector configurations in \mathbb{R}^3 for which the surface area is given by a same quadratic form independent of a choice of a type cone.

A trivial class of examples are monotypic polyhedra (see Remark 1.46). For vectors in general position, the necessary and sufficient condition is probably that, for all $(2, 2)$ -circuits, the sum of coefficients is zero.

Finally, inspired by the results of [4], we pose the following problem.

Problem 3.11. Describe some of the known examples or construct new examples of hyperbolic Coxeter polyhedra that appear as $H_{\mathcal{K}}(\Delta)$ or $M_{\mathcal{K}}(V)$.

Note that both $H_{\mathcal{K}}(\Delta)$ and $M_{\mathcal{K}}(V)$ can belong to any combinatorial type (see Remark 1.47). It is harder to determine the possible values of their dihedral angles.

§4. *Related work.*

4.1. *The first weight space and the discrete Christoffel problem.* Here we explain the relation of §1.1.6 with the first weight space of McMullen [30]. Definitions and propositions below are taken from [30].

Let Δ be a complete polytopal fan in \mathbb{R}^d . Pick a cone $\tau \in \Delta^{(d-2)}$ and consider all $\sigma \in \Delta^{(d-1)}$ such that $\tau \subset \sigma$. Denote by $v_{\sigma/\tau} \in \text{aff}(\sigma)$ the inner unit normal to the facet τ of the cone σ (see Figure 25, left).

LEMMA 4.1. *For every $h \in \tilde{T}(\Delta)$ and every $\tau \in \Delta^{(d-2)}$, the following equality holds*

$$\sum_{\sigma:\sigma \supset \tau} \ell_{\sigma}(h)v_{\sigma/\tau} = 0. \tag{36}$$

Proof. Indeed, $v_{\sigma/\tau}$ is the outer unit normal to the edge $F_{\sigma}(h)$ of the 2-face $F_{\tau}(h)$. Thus equation (36) follows from

$$\sum_{\sigma:\sigma \supset \tau} \ell_{\sigma}(h)e_{\sigma} = \sum_{i=1}^n (p_{i+1}(h) - p_i(h)) = 0,$$

where $p_1, \dots, p_n, p_{n+1} = p_1$ are the vertices of the polygon $F_{\tau}(h)$ in a cyclic order. Figure 25 illustrates the case $d = 3$. □

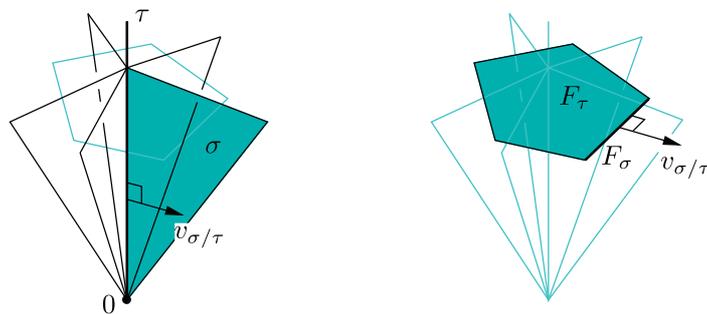


Figure 25: The star of a $(d - 2)$ -cone $\tau \in \mathcal{N}(P)$ corresponds to a 2-face $F_\tau \subset P$.

Definition 4.2. A 1-weight on Δ is a map $\sigma \mapsto a_\sigma$ on the set of $(d - 1)$ -cones of Δ such that

$$\sum_{\sigma:\sigma \supset \tau} a_\sigma v_{\sigma/\tau} = 0 \quad \text{for all } \tau \in \Delta^{(d-2)}.$$

The set of all 1-weights is denoted by $\Omega_1(\Delta)$. Denote by

$$\Omega_1^+(\Delta) := \{a \in \Omega_1(\Delta) \mid a_\sigma > 0 \text{ for all } \sigma\}$$

the set of positive 1-weights.

Since Δ is polytopal, $\Omega_\Delta^+ \neq \emptyset$. In addition, since Ω_Δ^+ is open in Ω_Δ , $\text{aff}(\Omega_\Delta^+) = \Omega_\Delta$.

THEOREM 4.3. *For any $a \in \Omega_1^+(\Delta)$, there exists a convex polytope $P \in T(\Delta)$ with $\ell_\sigma(P) = a_\sigma$. Moreover, P is unique up to translation.*

Sketch of proof. Choose an arbitrary point in \mathbb{R}^d as a vertex of P and follow the graph of P to construct the other vertices. The fan Δ gives us the directions of the edges, and the weight a gives their lengths. The weight condition ensures that this construction is well defined. One needs to check that the result is a convex polytope with the normal fan Δ . This can be done with the help of Lemma 1.17: the construction yields a conewise linear function h with the gradient jump across σ equal to a_σ and the positivity of a is equivalent to the convexity of h , which thus becomes a support function of a polytope with the normal fan Δ . \square

In terms of weights, Lemma 1.21 has the following reformulation.

LEMMA 4.4. *The linear map*

$$\ell^\Delta: \text{span}(T(\Delta)) \rightarrow \mathbb{R}^{\Delta^{(d-1)}}$$

is injective and has image $\Omega_1(\Delta)$. In addition,

$$\ell^\Delta(T(\Delta)) = \Omega_1^+(\Delta).$$

Remark 4.5. Theorem 4.3 is [30, Lemma 8.1]. It first appeared in [39] (see also [17, Theorem 15.1.2]), in the context of decomposition of convex polytopes into Minkowski sum.

An element of $\Omega_1^+(\Delta)$ can be viewed as a Borel measure on \mathbb{S}^{d-1} . In this context, Theorem 4.3 solves the polyhedral version of the Christoffel problem (find a convex body with a given first area measure), and was proved in [37] (see also [38]).

Theorem 4.3 appears as a part of [35, Theorem 15.5], which also contains other characterizations of $T(\Delta)$.

The space $\Omega_1(\Delta)$ is the linear space of virtual polytopes with the “normal fan” Δ (consult Figure 21).

Remark 4.6. More generally, an r -weight on Δ is a map $a: \Delta^{(d-r)} \rightarrow \mathbb{R}$ such that

$$\sum_{\sigma: \sigma \supset \tau} a_\sigma v_{\sigma/\tau} = 0 \quad \text{for all } \tau \in \Delta^{(d-r-1)}.$$

McMullen [30] defined product of weights (modeled on mixed volumes) that gives rise to a graded algebra

$$\Omega(\Delta) := \bigoplus_{r=0}^d \Omega_r(\Delta).$$

He defined quadratic forms that generalize the form $q_{\mathcal{K}, \Delta}$ (see Definition 2.4) and proved a generalization of the signature theorem 2.11. This led him to a proof of the so called g -theorem for simple polytopes (previously proved in [40] using heavy machinery from algebraic geometry).

4.2. Regular subdivisions of constant curvature surfaces with cone singularities.

A configuration V of unit vectors in \mathbb{R}^d is a finite set of points on the unit sphere \mathbb{S}^{d-1} , and a polytopal fan Δ with $\Delta^{(1)} = V$ yields a subdivision of \mathbb{S}^{d-1} with the vertex set V . If Δ is a polytopal fan, then the corresponding subdivision is called regular. For every V , there is a distinguished fan Δ_V , which is the normal fan of the circumscribed polytope with v_i as tangent points between the facets and the sphere. (Equivalently, this is the central fan of the convex hull of V .) It is easy to show that the subdivision corresponding to Δ_V is the *Delaunay subdivision* of the sphere with the vertex set V . Here a Delaunay subdivision is one in which every cell is an inscribed polygon and the circumcircle of every cell contains no vertices in its interior.

Delaunay subdivisions can be constructed for (Euclidean, spherical or hyperbolic) surfaces with cone singularities. (In the spherical case, there is a restriction on the metric; see [15, Lemma 2.11].) As a vertex set V , one can choose any finite set containing the cone points. In the Euclidean case, the proof was sketched in [42, Proposition 3.1] and a more detailed treatment was given in [8, 19, 26].

Assigning to every point $v_i \in V$ a weight $w_i \in \mathbb{R}$ allows us to define a *weighted Delaunay subdivision* $\Delta_V(w)$ by requiring that the extension of the

map $v_i \mapsto w_i$, piecewise linear with respect to $\Delta_V(w)$, is convex. Compare Corollary 1.7 and Remark 1.8, where the role of weights is played by the support numbers h_i . The weighted Delaunay subdivision with equal weights is the usual Delaunay subdivision.

Weighted Delaunay triangulations of Euclidean cone-surfaces were introduced in [7] and those of hyperbolic and spherical cone-surfaces were introduced in [14, 15]. In the non-Euclidean case, instead of piecewise linear extensions one uses functions of a different sort.

4.3. *Relation to Thurston's space of shapes of polyhedra.* Let $P \subset \mathbb{R}^3$ be a simple polytope with n facets. From the Euler formula, it has $s := 2n - 4$ vertices. Then the metric on the boundary of P , induced from the ambient Euclidean space, is a Euclidean metric on the sphere \mathbb{S}^2 with conical singularities of positive curvature. The cone angle around a vertex of P is the sum of the adjacent face angles.

Let us denote by $C(\alpha)$, where $\alpha = (\alpha_1, \dots, \alpha_s)$, the set of Euclidean metrics on the sphere with (marked) cone singularities of angles α_i , up to orientation-preserving similarity. If we fix a polytopal simplicial fan Δ , then all polytopes from $T(\Delta)$ have the same cone angles. Thus we have a map

$$T(\Delta) \rightarrow C(\alpha). \tag{37}$$

Together with the combinatorics Δ , the induced metric determines the edge lengths of a polytope $P(h) \in T(\Delta)$ and this determines the polytope, according to Theorem 4.3. Therefore the map (37) is injective on any affine slice of the type cone $T(\Delta)$.

The space $C(\alpha)$ can be endowed with a structure of a complex manifold of dimension $s - 3$, as follows. Any $m \in C(\alpha)$ can be geodesically triangulated so that the singularities are exactly the vertices of the triangulation. After fixing the position of a vertex and the direction of one of its adjacent edges, the triangulation can be developed in \mathbb{C} . In this way, to each edge a complex number is associated, and it can be shown that certain $s - 2$ of these numbers suffice to recover the triangulation. Modulo scaling we have $s - 3$ complex parameters. This gives a local chart for $C(\alpha)$ around m . Changes of charts correspond to flip of the triangulation and are linear maps in the coordinates. Any chart can be endowed with the restriction of a Hermitian form on \mathbb{C}^{s-2} , which is given by the area of the Euclidean metrics (the sum of the area of each triangle). This makes $C(\alpha)$ into a complex hyperbolic manifold of dimension $s - 3$. See [42] for details.

It follows that the interior of the hyperbolic polyhedron $H(\Delta)$ associated with $T(\Delta)$ (see §3.1) embeds isometrically in $C(\alpha)$: that is,

$$\text{int } H(\Delta) \subset C(\alpha).$$

Note that $\text{int } H(\Delta)$ has real dimension $s/2 - 2$, while $C(\alpha)$ has complex dimension $s - 3$.

Also note that changing the type cone $T(\Delta)$ while preserving the set of facet normals changes, in general, the collection of angles α (see Figure 26).

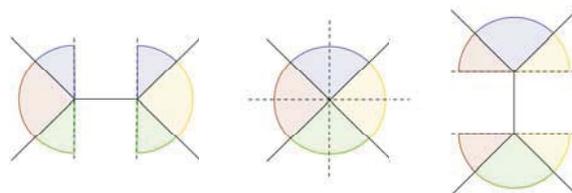


Figure 26: Cone angles of a simple polytope depend not only on the facet normals but also on the combinatorics.

4.4. *Shape of co-convex polyhedra, mixed covolumes and spherical geometry.* The following definitions and results are from [24], restricted to the polyhedral case.

Let C be a simple pointed convex polyhedral cone. A C -convex (or *co-convex*) polyhedron P is a convex polyhedral subset of C such that $x + C \subset P$ for all $x \in P$. If the volume of $C \setminus P$ is finite, then it is called the *covolume* of P . Multiplication by positive scalars and Minkowski addition are well defined for C -convex polyhedra. This leads to the notion of the mixed covolume for simple normally equivalent C -convex polyhedra. As shown in [24], the mixed covolume satisfies the inverse Alexandrov–Fenchel inequalities.

It follows that certain quadratic forms associated with the covolume in the way described in §2.2.2 are positive definite. Thus, in the co-convex case the type cones give rise to convex spherical polyhedra.

A different point of view on C -convex sets is presented in [12]. View \mathbb{R}^d as the Lorentzian Minkowski space (that does not change notions of convexity, polyhedrality or volume). Take a convex polyhedral cone C that is a fundamental domain for a cocompact action of a group of linear isometries (in other words, of a subgroup of the isometry group of the hyperbolic space \mathbb{H}^{d-1}). The case $d = 2$, that is, a Lorentzian analog of the Bavard–Ghys construction [4] was analyzed in [13]. The case $d = 3$ is of a special interest, since the boundary of a C -convex polyhedron quotiented by the group action is a closed surface of genus greater than one, and the induced metric on it is a Euclidean metric with cone singularities of negative curvature. Spaces of such metrics were studied, for example, in [34, 43, 44]. The construction of the present paper would yield spherical convex polyhedra isometrically embedded in spaces of flat metrics on compact surfaces.

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A. *Appendix. Computing the surface area of a three-dimensional polytope.*

A.1. *Face areas of a tetrahedron.* Let $P \subset \mathbb{R}^3$ be a polytope whose facet normals belong to a set V . Then $\text{area}(\partial P)$ can be represented as a sum of face

areas of certain tetrahedra with facet normals in the set V . Indeed, this is true for a truncated tetrahedron; any other combinatorial type can be obtained by a sequence of flips, and a flip adds two faces of a tetrahedron and subtracts the other two. For an illustration, see §2.1.4. Lemma A.1 below expresses the face areas of a tetrahedron in terms of its face normals and support numbers.

Let v_0, v_1, v_2, v_3 be positively spanning vectors in \mathbb{R}^3 . (We are not assuming them to have length one.) There is a unique, up to scaling, positive linear dependency

$$\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0, \quad \lambda_i > 0 \text{ for all } i.$$

Denote by

$$\Delta_0 := \{x \in \mathbb{R}^3 \mid \langle v_0, x \rangle \leq \|v_0\|, \langle v_i, x \rangle \leq 0 \text{ for } i = 1, 2, 3\}$$

the tetrahedron with facet normals v_0, v_1, v_2, v_3 such that its altitude with respect to the face F_0 has length one.

LEMMA A.1. *The face areas of Δ_0 satisfy the following relations.*

$$\begin{aligned} \frac{\text{area}(F_0)}{\|v_0\|} : \frac{\text{area}(F_1)}{\|v_1\|} : \frac{\text{area}(F_2)}{\|v_2\|} : \frac{\text{area}(F_3)}{\|v_3\|} &= \lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 \\ &= \det(v_1, v_2, v_3) : -\det(v_0, v_2, v_3) : \det(v_0, v_1, v_3) : -\det(v_0, v_1, v_2), \end{aligned} \tag{A.1}$$

$$\text{area}(F_0) = \frac{\|v_0\|^3 (\det(v_1, v_2, v_3))^2}{2|\det(v_0, v_1, v_2) \det(v_0, v_2, v_3) \det(v_0, v_1, v_3)|}. \tag{A.2}$$

Proof. Equation (A.1) follows from the Minkowski formula

$$\text{area}(F_0) \frac{v_0}{\|v_0\|} + \text{area}(F_1) \frac{v_1}{\|v_1\|} + \text{area}(F_2) \frac{v_2}{\|v_2\|} + \text{area}(F_3) \frac{v_3}{\|v_3\|} = 0$$

and from the elementary linear algebra.

To prove (A.2), let e_{12} denote the vector along the edge F_{12} of Δ_0 , directed towards the face F_0 . Then

$$e_{12} = \frac{\|v_0\|(v_1 \times v_2)}{\det(v_0, v_1, v_2)}.$$

It is easy to show that $(v_1 \times v_2) \times (v_1 \times v_3) = \det(v_1, v_2, v_3)v_1$. This implies that

$$\det(v_1 \times v_2, v_2 \times v_3, v_3 \times v_1) = (\det(v_1, v_2, v_3))^2.$$

Therefore

$$\begin{aligned} \text{vol}(\Delta_0) &= \frac{1}{6} |\det(e_{12}, e_{23}, e_{31})| \\ &= \frac{\|v_0\|^3 (\det(v_1, v_2, v_3))^2}{6|\det(v_0, v_1, v_2) \det(v_0, v_2, v_3) \det(v_0, v_1, v_3)|}. \end{aligned}$$

The formula for the area of F_0 follows from $\text{vol}(\Delta_0) = (1/3)h_0 \text{area}(F_0)$. \square

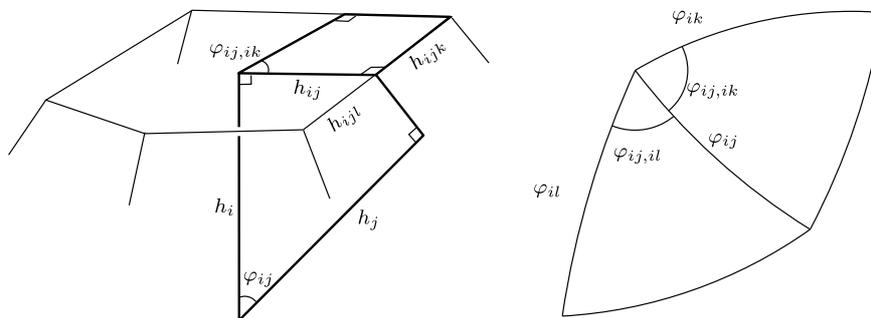


Figure A.1: Lengths and angles in a three-dimensional polytope.

Formulas of Lemma A.1 imply that the formula for the area of the face F_1 is given by

$$\text{area}(F_1)|_{h_0=1} = \frac{\|v_0\|^2 \|v_1\| \det(v_1, v_2, v_3)}{2 \det(v_2, v_1, v_0) \det(v_1, v_3, v_0)}.$$

In practice, if the coefficients λ_i of the linear dependency are known, it is easier to compute one of the areas by the above formulas for F_0 and F_1 and then the others using the proportions from the first part of the Lemma.

A.2. *Formula in terms of the angles in the normal fan.* Denote by q_Δ the quadratic form in variables h_1, \dots, h_n that computes the area of a polyhedron with normal fan Δ . So

$$q_\Delta(h) = \sum_{i,j} a_{ij} h_i h_j,$$

where $a_{ij} \neq 0$ if and only if $i = j$ or faces F_i and F_j are adjacent (in other words, $\text{pos}\{v_i, v_j\} \in \Delta$). To determine the coefficients a_{ij} , compute the partial derivatives of $q_\Delta(h)$. We are using this notation in Figure A.1.

Let q_i be the area of the i th face. Then [7]

$$\frac{\partial q_j}{\partial h_i} = \frac{\ell_{ij}}{\sin \varphi_{ij}} \quad \text{for } i \neq j, \quad \frac{\partial q_i}{\partial h_i} = - \sum_{j \neq i} \ell_{ij} \cot \varphi_{ij},$$

where $\ell_{ij} = h_{ijk} + h_{ijl}$ is the length of the ij -edge. Hence

$$\frac{\partial q}{\partial h_i} = \sum_{j \neq i} \ell_{ij} \frac{1 - \cos \varphi_{ij}}{\sin \varphi_{ij}} = \sum_{j \neq i} \ell_{ij} \tan \frac{\varphi_{ij}}{2}.$$

Differentiating again, we obtain

$$\frac{\partial^2 q}{\partial h_i \partial h_j} = \frac{\partial \ell_{ik}}{\partial h_j} \tan \frac{\varphi_{ik}}{2} + \frac{\partial \ell_{il}}{\partial h_j} \tan \frac{\varphi_{il}}{2} + \frac{\partial \ell_{ij}}{\partial h_j} \tan \frac{\varphi_{ij}}{2}.$$

Using

$$h_{ijk} = h_{ik} \csc \varphi_{ij,ik} - h_{ij} \cot \varphi_{ij,ik}, \quad h_{ij} = h_j \csc \varphi_{ij} - h_i \cot \varphi_{ij}$$

we compute

$$\frac{\partial \ell_{ik}}{\partial h_j} = \csc \varphi_{ij} \csc \varphi_{ij,ik} \quad \frac{\partial \ell_{ij}}{\partial h_j} = -\csc \varphi_{ij} (\cot \varphi_{ij,ik} + \cot \varphi_{ij,il}).$$

Therefore

$$\begin{aligned} 2a_{ij} &= \frac{\partial^2 q}{\partial h_i \partial h_j} = \tan \frac{\varphi_{ik}}{2} \csc \varphi_{ij} \csc \varphi_{ij,ik} + \tan \frac{\varphi_{il}}{2} \csc \varphi_{ij} \csc \varphi_{ij,il} \\ &\quad - \tan \frac{\varphi_{ij}}{2} \csc \varphi_{ij} (\cot \varphi_{ij,ik} + \cot \varphi_{ij,il}) \\ 2a_{ii} &= \frac{\partial^2 q}{\partial h_i^2} = - \sum_{j \neq i} \tan \frac{\varphi_{ij}}{2} \csc \varphi_{ij} (\cot \varphi_{ji,jk} + \cot \varphi_{ji,jl}). \end{aligned}$$

Example A.2. For a dodecahedron, $\cos \varphi_{ij} = 1/\sqrt{5}$, $\cos \varphi_{ij,ik} = (\sqrt{5} - 1)/4$. The coefficients of the quadratic form are

$$a_{ij} = \tan \frac{\varphi_{ij}}{2} \csc \varphi_{ij} (\csc \varphi_{ij,ik} - \cot \varphi_{ij,ik}), \quad a_{ii} = -5 \tan \frac{\varphi_{ij}}{2} \csc \varphi_{ij} \cot \varphi_{ij,ik}.$$

Hence $a_{ij} = C(1 - \cos \varphi_{ij,ik})$, $a_{ii} = -5C \cos \varphi_{ij,ik}$ and $a_{ij}/a_{ii} = -1/\sqrt{5}$. Up to scaling, the quadratic form is thus equal to

$$q(h) = - \sum_i h_i^2 + \frac{2}{\sqrt{5}} \sum_{\{i,j\}} h_i h_j.$$

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