Published in "Discrete Applied Mathematics 207(): 45–55, 2016"
which should be cited to refer to this work.

On the ratio between maximum weight perfect matchings and maximum weight matchings in grids

Guilherme D. da Fonseca [a,*], Bernard Ries [b], Diana Sasaki [c]

[a] Université d'Auvergne and LIMOS, France
[b] University of Fribourg, Fribourg, Switzerland
[c] Université Paris-Dauphine, Paris, France

Given a graph G that admits a perfect matching, we investigate the parameter $\eta(G)$ (originally motivated by computer graphics applications) which is defined as follows. Among all nonnegative edge weight assignments, $\eta(G)$ is the minimum ratio between (i) the maximum weight of a perfect matching and (ii) the maximum weight of a general matching. In this paper, we determine the exact value of η for all rectangular grids, all bipartite cylindrical grids, and all bipartite toroidal grids. We introduce several new techniques to this endeavor.

1. Introduction

All graphs in this paper are finite, undirected and connected. We refer to the textbook of Diestel [3] for any undefined graph terminology. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we define its *neighborhood* as $N(v) = \{u : uv \in E\}$ and the vertices in $N(v)$ are called the *neighbors* of v. The *degree* $d(v)$ of a vertex $v \in V$ is defined as $d(v) = |N(v)|$. The *minimum degree* of G is denoted by $\delta(G)$. The *average degree* of G is defined as follows: $\bar{d}(G) = \frac{1}{n}\sum_{i=1}^{n} d(v_i)$, where $V = \{v_1, \ldots, v_{|V|}\}$. As usual K_n, $n \geq 1$, (resp. $K_{n,m}$, $n, m \geq 1$) denotes the complete graph (resp. complete bipartite graph) on n vertices (resp. with n vertices in one partition and m in the other partition). Finally, C_n, $n \geq 3$, denotes the induced cycle on n vertices.

A *matching* in G is a set $M \subseteq E$ such that no two edges in M share a common vertex. Given a matching M in a graph G, we say that M *saturates* a vertex v and that vertex v is *M-saturated*, if some edge of M is incident to v. A matching M is *perfect* if $|M| = \frac{|V|}{2}$, i.e., all vertices in G are M-saturated. A matching M is *maximal* if there exists no other matching M' such that $M \subseteq M'$ and $|M'| > |M|$. A matching M is *maximum* if it has maximum cardinality.

Let $w : E \to \mathbb{R}^+$ be a *weight function* on the edges of G. We will refer to w as an *edge weighting* of G. Given a subset $E' \subseteq E$, the quantity $w(E') = \sum_{e \in E'} w(e)$ is called the *weight of* E'. A *maximum weight matching* in G, denoted by $M^*(G)$, is a matching of maximum total weight in G. A *maximum weight perfect matching* in G, denoted by $P^*(G)$, is a perfect matching of maximum total weight (among all perfect matchings in G). Given a graph $G = (V, E)$ which admits a perfect matching, the parameter $\eta(G)$ is defined as

$$\eta(G) = \min_{w:E\to\mathbb{R}^+} \frac{w(P^*(G))}{w(M^*(G))}.$$

* Corresponding author.
E-mail addresses: guilherme.dias_da_fonseca@udamail.fr (G.D. da Fonseca), bernard.ries@unifr.ch (B. Ries), diana.sasaki@gmail.com (D. Sasaki).

The study of the parameter η was initiated in [2] and motivated by applications in computer graphics (see [5,6]) where one seeks to convert a triangle mesh into a quadrangulation. Each triangle is represented by a vertex, two vertices are linked by an edge if the corresponding triangles are adjacent, and edge weights correspond to how "compatible" two triangles are (the definition of compatibility is largely dependent on the specific objective). Due to this application, the first study focused on cubic graphs, i.e. graphs in which all vertices have degree 3. Compared to triangulations, structured grids define a simpler but also widely used mesh [7]. By merging two adjacent grid cells, we obtain an unstructured grid with half as many cells. Thanks to the regular nature of structured grids we can calculate the exact values of η for each grid size, instead of only upper and lower bounds as in the case of cubic graphs.

Since the parameter η can be defined for any graph which admits a perfect matching, its study is of interest from a theoretical point of view. Furthermore, the value of the parameter η tells us how far or close the maximum weight of a perfect matching is from the maximum weight of a general matching in a given graph, considering any nonnegative edge weighting. Thus, it is natural to consider several different graph classes. Notice that the problem of deciding whether $\eta(G) = c$, for a given graph G and a nonnegative real c, is not known to be in P nor to be NP-hard.

It is easy to see that we necessarily have $0 \leq \eta(G) \leq 1$, for any graph G admitting a perfect matching. In [2], the authors characterize those graphs G for which $\eta(G) = 0$ as well as the graphs G for which $\eta(G) = 1$. Furthermore, they provide lower and upper bounds on η for several types of bridgeless cubic graphs, i.e. cubic graphs not containing any edge whose deletion disconnects the graph. Finally, the authors show that if a graph G admits a perfect matching, then the value of $\eta(G)$ is well defined.

The main technique available so far to prove a lower bound on η is the following. Suppose $G = (V, E)$ contains k perfect matchings $P_1, \ldots, P_k$ such that each edge of G belongs to at least r of these matchings. Consider a nonnegative edge weighting w of E. Then $rw(M^*(G)) \leq \sum_{i=1}^{k} w(P_i)$. Without loss of generality, we may assume that $w(P_1) \geq w(P_2) \geq \cdots \geq w(P_k)$. Hence, $rw(M^*(G)) \leq kw(P_1) \leq kw(P^*(G))$ which implies that $\eta(G) \geq \frac{r}{k}$. This proof technique has a major weakness. It cannot prove lower bounds on η higher than 1 over the average degree $\bar{d}(G)$ of G, since $\frac{r}{k}$ is upper bounded by $\frac{1}{\bar{d}(G)}$. Indeed, since the size of a perfect matching is $\frac{|V|}{2}$, it follows from the above that $k\frac{|V|}{2} \geq r|E|$, i.e. $k|V| \geq 2|E|r$. Now using the fact that $2|E| = \sum_{i=1}^{|V|} d(v_i)$, we deduce that $k \geq r\bar{d}(G)$ and so $\frac{r}{k} \leq \frac{1}{\bar{d}(G)}$.

In this paper, we introduce new techniques that break this barrier (see Section 2). These techniques allow us to compute the exact value of η for the following graph classes: (i) rectangular grids; (ii) bipartite cylindrical grids; (iii) bipartite toroidal grids (see Section 3 for the corresponding definitions and theorems). Section 4 is devoted to concluding remarks and open problems.

2. New techniques to determine lower and upper bounds

In this section, we introduce new techniques that enable us to obtain upper and lower bounds on the value of η. We start with a lemma allowing us to determine upper bounds. It generalizes an argument given in [2].

Lemma 1. *Let $G = (V, E)$ be a graph with $\delta(G) \geq 2$. Suppose G contains a perfect matching and there exists a maximal matching that does not saturate a vertex v of degree $\delta(G)$. Then, $\eta(G) \leq \frac{\delta-1}{\delta}$.*

Proof. Consider a vertex v of G with $d(v) = \delta(G)$ and let M be a maximal matching not saturating v. Let $u_1, u_2, \ldots, u_{\delta(G)}$ be the neighbors of v. Since M is maximal, all neighbors of v are necessarily M-saturated. Let u_iw_i, $i = 1, \ldots, \delta(G)$, be the edges of M saturating the neighbors of v. Notice that the neighbors of v may be adjacent, and that the edges of M saturating the neighbors of v are not necessarily distinct. Now any perfect matching P contains at most $\delta(G) - 1$ edges in $\{u_iw_i, i = 1, \ldots, \delta(G)\}$ since P saturates v. Define a nonnegative edge weighting w such that $w(u_iw_i) = 1$ for $i = 1, \ldots, \delta(G)$, and $w(e) = 0$ otherwise. It follows that $w(P) \leq \delta(G) - 1$ and $w(M) = \delta(G)$. Hence, $\eta(G) \leq \frac{\delta(G)-1}{\delta(G)}$. □

The next lemma allows us to obtain lower bounds on η.

Lemma 2. *Let $c, r \geq 0$ be two integers and $G = (V, E)$ be a graph which admits a perfect matching. Let $G_1 = (V, E_1), G_2 = (V, E_2), \ldots, G_k = (V, E_k)$ be k spanning subgraphs of G admitting each a perfect matching and such that $\eta(G_i) \geq c$ for $i = 1, \ldots, k$. If each edge of G is contained in at least r sets among $E_1, E_2, \ldots, E_k$, then $\eta(G) \geq \frac{cr}{k}$.*

Proof. First, notice that since $G_1, \ldots, G_k$ are spanning subgraphs of G, any perfect matching in G_i, $i \in \{1, \ldots, k\}$, is also a perfect matching in G. Now, let M be a maximum weight matching of G for some nonnegative edge weighting w. Since each edge of G is contained in at least r sets among $E_1, E_2, \ldots, E_k$, we have

$$\sum_{i=1}^{k} w(M \cap E_i) \geq r\ w(M). \tag{1}$$

Assume, without loss of generality, that $w(M \cap E_1) \geq w(M \cap E_i)$ for $i = 2, \ldots, k$. Then, inequality (1) implies that

$$w(M \cap E_1) \geq \frac{r\ w(M)}{k}. \tag{2}$$

Now let P_1 be a perfect matching of maximum weight in G_1. Since $\eta(G_1) \geq c$, the definition of η implies that

$$\frac{w(P_1)}{w(M \cap E_1)} \geq c. \tag{3}$$

Combining inequalities (2) and (3), we obtain

$$\frac{w(P_1)}{w(M)} \geq \frac{cr}{k}. \tag{4}$$

Let P be a perfect matching of maximum weight in G. Since $w(P_1) \leq w(P)$ (recall that a perfect matching in G_1 is also a perfect matching in G), we can rewrite inequality (4) as

$$\frac{w(P)}{w(M)} \geq \frac{cr}{k}. \tag{5}$$

Since inequality (5) holds for any edge weighting w, it follows that $\eta(G) \geq \frac{cr}{k}$. $\square$

Throughout the paper, we will use Lemma 2 repeatedly in order to compute lower bounds on η for various graph classes, which finally allow us to obtain the exact value of η. Notice that in order to use Lemma 2, the main challenge consists in finding the appropriate spanning subgraphs.

The following result, which was proven in [2], will also be an important tool for us.

Theorem 3 (*Brazil et al. [2]*)**.** *Let $G = (V, E)$ be a graph containing a perfect matching. Then,*

- *$\eta(G) = 0$ if and only if there is an edge $e \in E$ that is not contained in any perfect matching;*
- *$\eta(G) = 1$ if and only if all connected components of G are isomorphic to K_{2n} or $K_{n,n}$, for $n \geq 1$.*

From the previous theorem it follows immediately that $\eta(C_4) = \eta(K_2) = 1$. Also, a graph G which is the disjoint union of C_4's and K_2's satisfies $\eta(G) = 1$.

Even though it is often the case, Lemma 2 may not always provide lower bounds allowing to obtain the exact value of η. In other words, it may not always provide tight lower bounds. Surprisingly, it does give tight bounds for all graphs among the following graph classes: rectangular grids and bipartite cylindrical grids. Nevertheless, for the class of bipartite toroidal grids, there exists a single graph (on 16 vertices) for which Lemma 2 cannot be applied optimally, in the sense that it does not give us the lower bound needed to compute the exact value of η. For this special case, we need another technique which we introduce hereafter. This technique is much more intricate than the previous ones and can be applied not only to bipartite toroidal grids, but also to arbitrary bipartite graphs. We start with some definitions.

Given two sets S, S' the *symmetric difference* $S \ominus S'$ is the set of elements that are in S or in S' but not in both. Consider a graph $G = (V, E)$ and a matching M. An *alternating cycle* with respect to M is a cycle C of even length such that exactly half of the edges of C are in M. A matching M' is a *rotation* of M if there exists an alternating cycle C with respect to M such that $M' = M \ominus E(C)$, where $E(C)$ denotes the set of edges in C. The following lemma follows from the proof of Berge's lemma on augmenting paths [1].

Lemma 4. *Let $G = (V, E)$ be a graph. Given two matchings M_1, M_k in G saturating the same set of vertices, there is a sequence of k matchings $M_1, \ldots, M_k$, $k \geq 2$, such that M_{i+1} is a rotation of M_i for $i = 1, \ldots, k-1$.*

Proof. Consider the graph $G' = (V, M_1 \ominus M_k)$. Clearly, G' contains only vertices of degree 0 or 2 and is therefore a collection of cycles and isolated vertices. Since M_1 is a matching, the cycles are alternating with respect to M_1. By iteratively applying rotations on each of the cycles of G' we obtain M_k from M_1. $\square$

Let M be a matching in a graph G. An *alternating path* with respect to M is a path that alternates between edges in M and edges not in M. An *augmenting path* with respect to M is an alternating path with respect to M whose endvertices are not M-saturated.

Given a matching M that leaves exactly $2c$, $c \geq 1$, vertices unsaturated, an *augmenting path forest* with respect to M is a set of c vertex-disjoint augmenting paths with respect to M. A set of augmenting path forests is edge-disjoint if each edge of G is contained in at most one augmenting path forest. We obtain the following two lemmas.

Lemma 5. *Let M be a matching in a graph $G = (V, E)$ and let w be a nonnegative edge weighting of E. If there exist k edge-disjoint augmenting path forests, then there exists a perfect matching P with $w(P) \geq \frac{w(M)(k-1)}{k}$.*

Proof. Let $F_1, \ldots, F_k$ be the edge-disjoint augmenting path forests with respect to M. Let $F^* = \arg\min_{F_i,\ i=1,\ldots,k}(w(M \cap F_i))$. Since the forests are edge-disjoint, we have that $w(M \cap F^*) \leq w(M)/k$. It is easy to see that the matching $P = F^* \ominus M$ is perfect and satisfies $w(P) \geq \frac{w(M)(k-1)}{k}$. $\square$

Lemma 6. *Let $G = (V, E)$ be a bipartite graph and M a matching in G that saturates the vertex set $V' \subset V$. Suppose that $F_1, \ldots, F_k$ are k edge-disjoint augmenting path forests with respect to M. Then, for any matching M' saturating V' there exists a set of k edge-disjoint augmenting path forests with respect to M'.*

Proof. It follows from Lemma 4 that we may assume without loss of generality that M' is a rotation of M. Let C be the corresponding alternating cycle with respect to M. Let $S \subseteq V$ be the set of unsaturated vertices.

Denote by $A = (V_A, E_A)$ the graph formed by the union of all augmenting paths with respect to M in $F_1, \ldots, F_k$. If $E_A \cap E(C) = \emptyset$, then clearly $F_1, \ldots, F_k$ are k edge-disjoint augmenting path forests with respect to M', and hence the lemma holds. So we may assume that $E_A \cap E(C) \neq \emptyset$.

We denote by $A' = (V_{A'}, E_{A'})$ the graph defined as follows: $E_{A'} = (E_A \setminus E(C)) \cup (E(C) \setminus E_A)$ and $V_{A'}$ is the set of vertices incident with at least one edge in $E_{A'}$. We will show that A' can be decomposed into k edge-disjoint augmenting path forests with respect to M' and hence the lemma holds.

Claim. *Let $v \in V_{A'}$. If $v \in S$, $d_{A'}(v) = k$, otherwise $d_{A'}(v) = 2$.*

Proof. First notice that for any $v \in V_A$ we have $d_A(v) = k$ if $v \in S$ and $d_A(v) = 2$ if $v \notin S$. By the definition of A', we clearly have $d_{A'}(v) = d_A(v), \forall v \in V_{A'} \setminus V(C)$. Now consider some vertex $v \in V_{A'} \cap V(C)$. If $v \in V_A$, then necessarily v has some neighbor $w \notin V(C)$ in A. By definition of A', w is also a neighbor of v in A'. Furthermore, v has one neighbor $z \in V(C)$ in A such that $vz \in E(C)$. Hence, in A' vertex v has a neighbor $t \in V(C)$, $t \neq z$, such that $vz \in E(C)$ (since $vz \notin E_A$). Since v has no other neighbor in A', it follows that $d_{A'}(v) = 2$. Finally, suppose that $v \notin V_A$. Let w, t be the neighbors of v in C. Since $vw, vt \notin E_A$, it follows that $vw, vt \in E_{A'}$. Again, we conclude that $d_{A'}(v) = 2$, since v has no other neighbor in A'. □

It follows from the previous claim that A' can be decomposed into $\frac{|S|}{2} \cdot k$ edge-disjoint augmenting paths. Indeed, starting at some unsaturated vertex $v \in S$, we simply follow the edges of A' until we reach again an unsaturated vertex $w \in S$. Since G is bipartite, we necessarily have $v \neq w$. This gives us a first augmenting path P_1. By repeating the same procedure, we finally obtain $\frac{|S|}{2} \cdot k$ edge-disjoint augmenting paths $P_1, \ldots, P_{\frac{|S|}{2} \cdot k}$. It remains to show that these paths can be partitioned into k edge-disjoint augmenting path forests. To do this, we construct the following auxiliary graph $H = (V_H, E_H)$: with every vertex $s \in S$, we associate a vertex v_s in H; we add an edge $v_{s_1} v_{s_2}$ if there exists $i \in \{1, \ldots, \frac{|S|}{2} \cdot k\}$ such that P_i is an augmenting path from s_1 to s_2 in A'. Since $d_{A'}(s) = k, \forall s \in S$, H is a k-regular graph. Furthermore, H is bipartite. Indeed, if H contains an odd cycle $v_{s_1}, v_{s_2}, \ldots, v_{s_{2l+1}}, v_{s_1}$, it follows that in G, the union of the paths $P_{1,2}, P_{2,3}, \ldots, P_{2l+1,1}$, where $P_{i,j}$ is the augmenting path from s_i to s_j, forms a cycle of odd length since the length of every augmenting path is odd. But this contradicts the fact that G is bipartite. Thus, we conclude that H is a k-regular bipartite graph. Now, it follows from König's theorem on edge-coloring bipartite graphs [4], that the edges of H are k-colorable and each color class contains $\frac{V_H}{2} = \frac{|S|}{2}$ edges. It is easy to see that the edges of each color class corresponds to an augmenting path forest in A'. This completes the proof. □

Now by combining Lemmas 5 and 6, we may obtain new lower bounds on η. Indeed, let $G = (V, E)$ be a bipartite graph admitting a perfect matching and also containing at least one maximal matching which is not perfect. Let w be a nonnegative edge weighting of E. Let $S_1, \ldots, S_p$ be stable sets (a *stable set* is a set of pairwise nonadjacent vertices) in G of even cardinality. Suppose that for every maximal matching M in G which is not perfect, there exists $i \in \{1, \ldots, p\}$ such that the vertices in S_i are not M-saturated. Furthermore, suppose that for every $i \in \{1, \ldots, p\}$ there exists a maximal matching M which is not perfect such that the vertices in S_i are not M-saturated and such that there are k edge-disjoint augmenting path forests with respect to M. It follows from Lemma 6 that for any maximal matching M in G which is not perfect, there exist k edge-disjoint augmenting path forests with respect to M. Since in addition for any perfect matching P in G we have $\frac{w(P^*)}{w(P)} \geq 1$, we conclude from Lemma 5 that $\eta(G) \geq \frac{k-1}{k}$.

3. Rectangular, bipartite cylindrical and bipartite toroidal grids

In this section, we provide exact values of η for all rectangular grids, bipartite cylindrical grids, and bipartite toroidal grids.

A *rectangular grid* or simply a *grid* $G_{m,n}$ has vertex set $V = \{(i,j) : i = 1, \ldots, m, j = 1, \ldots, n\}$, and edge set $E = \{(i,j)(i',j') : |i - i'| + |j - j'| = 1, (i,j), (i',j') \in V\}$. The value m is called the *width* of the grid and the value n the *height* of the grid. A grid $G_{m,n}$ admits a perfect matching if and only if $|V| = mn$ is even, i.e. when the width or the height is even. Since $G_{m,n}$ is isomorphic to $G_{n,m}$, we consider only the cases in which n is even. An example of a perfect matching consists of the edges $(i,j)(i,j+1)$ for all i and all odd j. Notice that all rectangular grids are bipartite graphs.

A *cylindrical grid* $Y_{m,n}$ is a graph containing $G_{m,n}$ with the additional edges $(m,j)(1,j)$ for $j = 1, \ldots, n$. Cylindrical grids also admit a perfect matching if and only if mn is even, but $Y_{m,n}$ is *not* isomorphic to $Y_{n,m}$ unless $m = n$. A cylindrical grid is bipartite if and only if m is even.

A *toroidal grid* $T_{m,n}$ is a graph containing $Y_{m,n}$ with the additional edges $(i,n)(i,1)$ for $i = 1, \ldots, m$. Toroidal grids admit a perfect matching if and only if mn is even and $T_{m,n}$ is isomorphic to $T_{n,m}$. A toroidal grid is bipartite if and only if m and n are even.

Throughout the text, when we draw a grid graph we represent a vertex (i,j) by a point with horizontal coordinate i and vertical coordinate j where the vertex $(0,0)$ corresponds to the top-left corner.

3.1. Rectangular grids

In this section, we prove several lemmas that together give the following exact values of η for all rectangular grids.

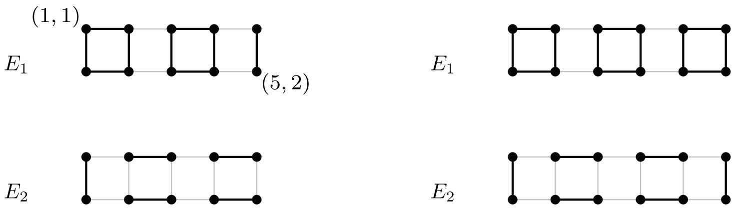

Fig. 1. Spanning subgraphs of $G_{5,2}$ and $G_{6,2}$ for Lemma 8.

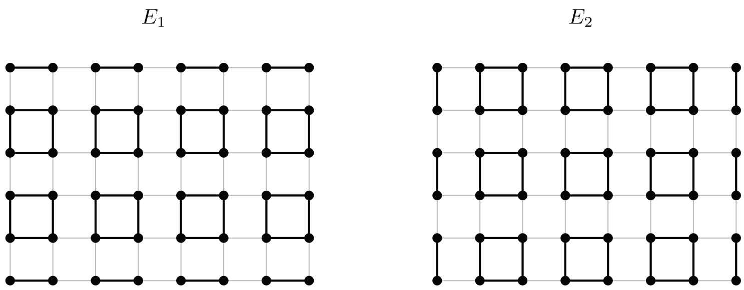

Fig. 2. Spanning subgraphs of $G_{8,6}$ for Lemma 9.

Theorem 7. *Consider the grid $G_{m,n}$ with $m, n \geq 1$ and mn even. Then*

$$\eta(G_{m,n}) = \begin{cases} 1 & \text{if } m \leq 2 \text{ and } n = 2, \\ \dfrac{1}{2} & \text{if } m \geq 3 \text{ and } n = 2, \\ 0 & \text{if } m = 1 \text{ and } n \geq 4, \\ \dfrac{1}{2} - \dfrac{1}{2m} & \text{if } m \geq 3 \text{ is odd and } n \geq 4, \\ \dfrac{1}{2} & \text{if } m, n \geq 4 \text{ are both even}, \\ \eta(G_{n,m}) & \text{otherwise.} \end{cases}$$

We start by establishing lower bounds on η.

Lemma 8. $\eta(G_{m,2}) \geq \frac{1}{2}$ *for all $m \geq 3$.*

Proof. We consider two spanning subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of $G_{m,2}$ with edge sets described below and illustrated in Fig. 1.

$$\begin{aligned} E_1 &= \{(i,j)(i+1,j) : i \text{ is } \textit{odd} \text{ from 1 to } m-1,\ j = 1, 2\} \\ &\cup \{(i,1)(i,2) : i = 1, \ldots, m\} \\ E_2 &= \{(i,j)(i+1,j) : i \text{ is } \textit{even} \text{ from 2 to } m-1,\ j = 1, 2\} \\ &\cup \{(1,1)(1,2)\} \cup \{(m,1)(m,2) \text{ if } m \text{ is } \textit{even}\}. \end{aligned}$$

Each of the spanning subgraphs is a disjoint union of C_4's and/or K_2's and therefore admits a perfect matching. Since $\eta(C_4) = \eta(K_2) = 1$ (see Theorem 3), it follows that $\eta(G_1) = \eta(G_2) = 1$. Furthermore, every edge of $G_{m,2}$ is contained in at least one set among E_1, E_2. Hence, we deduce from Lemma 2 that $\eta(G_{m,2}) \geq \frac{1}{2}$. $\square$

Using similar arguments, we obtain the same lower bound for grids with even width and height.

Lemma 9. $\eta(G_{m,n}) \geq \frac{1}{2}$ *for even $m, n \geq 2$.*

Proof. Consider two spanning subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of $G_{m,n}$ with edge sets as described below and illustrated in Fig. 2.

$$\begin{aligned} E_1 &= \{(i,j)(i+1,j) : i \text{ is } \textit{odd} \text{ from 1 to } m-1,\ j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 1, \ldots, m,\ j \text{ is } \textit{even} \text{ from 2 to } n-2\} \\ E_2 &= \{(i,j)(i+1,j) : i \text{ is } \textit{even} \text{ from 2 to } m-2,\ j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 1, \ldots, m,\ j \text{ is } \textit{odd} \text{ from 1 to } n-1\}. \end{aligned}$$

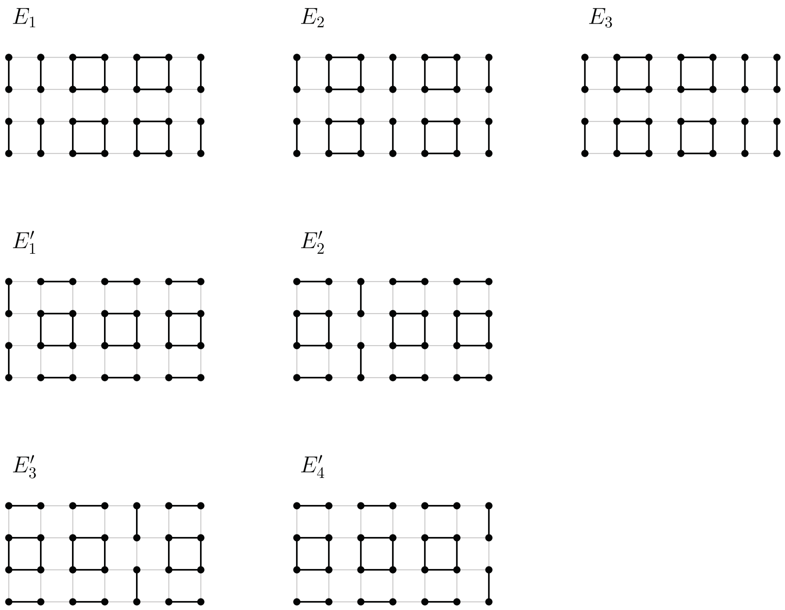

Fig. 3. Spanning subgraphs of $G_{7,4}$ for Lemma 10.

Each of the two spanning subgraphs is a disjoint union of C_4's and K_2's and therefore admits a perfect matching. As before, we deduce that $\eta(G_1) = \eta(G_2) = 1$. Furthermore, since every edge of $G_{m,n}$ is contained in at least one set of E_1, E_2, it follows from Lemma 2 that $\eta(G_{m,n}) \geq \frac{1}{2}$. $\square$

When the width is odd, the lower bound depends on the value of m.

Lemma 10. *$\eta(G_{m,n}) \geq \frac{1}{2} - \frac{1}{2m}$ for odd $m \geq 3$ and even $n \geq 2$.*

Proof. Consider m spanning subgraphs $G_k = (V, E_k)$ for $k = 1, \ldots, \frac{(m-1)}{2}$ and $G'_k = (V, E'_k)$ for $k = 1, \ldots, \frac{(m+1)}{2}$ with edge sets as described below and illustrated in Fig. 3.

$$\begin{aligned} E_k &= \{(i,j)(i+1,j) : \textit{even } i \text{ with } 2 \leq i \leq 2k-2 \text{ and } j = 1, \ldots, n\} \\ &\cup \{(i,j)(i+1,j) : \textit{odd } i \text{ with } 2k+1 \leq i \leq m-2 \text{ and } j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 1, \ldots, m \text{ and } \textit{odd } j \text{ with } 1 \leq j \leq n-1\} \\ E'_k &= \{(i,j)(i+1,j) : \textit{odd } i \text{ with } 1 \leq i \leq 2k-3 \text{ and } j = 1, \ldots, n\} \\ &\cup \{(i,j)(i+1,j) : \textit{even } i \text{ with } 2k \leq i \leq m-1 \text{ and } j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 2k-1 \text{ and } \textit{odd } j \text{ with } 1 \leq j \leq n-1\} \\ &\cup \{(i,j)(i,j+1) : i \neq 2k-1 \text{ and } \textit{even } j \text{ with } 2 \leq j \leq n-2\}. \end{aligned}$$

Each of the spanning subgraphs is a disjoint union of C_4's and K_2's and therefore admits a perfect matching. As before, we obtain that $\eta(G_k) = \eta(G'_k) = 1$. Since every edge of $G_{m,n}$ is contained in at least $\frac{(m-1)}{2}$ sets among

$$E_1, \ldots, E_{\frac{m-1}{2}}, E'_1, \ldots, E'_{\frac{m+1}{2}},$$

it follows from Lemma 2 that $\eta(G_{m,n}) \geq \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}$. $\square$

Let us now consider upper bounds on η.

Lemma 11. *$\eta(G_{m,n}) \leq \frac{1}{2}$ for $m \geq 3$ and even $n \geq 2$.*

Proof. Let M be an arbitrary maximal matching containing the edges $(2, 1)(3, 1)$ and $(1, 2)(2, 2)$. Then, vertex $v = (1, 1)$, which has degree 2, is not M-saturated. It follows from Lemma 1 that $\eta(G_{m,n}) \leq \frac{1}{2}$. $\square$

The upper bound can be improved for grids of odd width, matching the corresponding lower bound.

Lemma 12. *$\eta(G_{m,n}) \leq \frac{1}{2} - \frac{1}{2m}$ for odd $m \geq 3$, even $n \geq 4$.*

Proof. For $i = 1, \ldots, m$, let $s(i)$ denote the edge $(i, 1)(i, 2)$ if i is even and $(i, 2)(i, 3)$ if i is odd. We say that $s(i)$ is even (resp. odd) if i is even (resp. odd). Consider the matching $M = \{s(i) : i = 1, \ldots, m\}$ (see Fig. 4 for an example) and set $w(e) = 1$ if $e \in M$ and $w(e) = 0$ if $e \notin M$. Clearly, M is a maximum weight matching and its total weight is m.

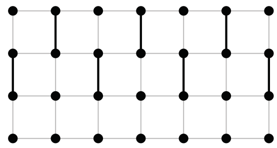

Fig. 4. Matching M in $G_{7,4}$ for Lemma 12.

Now, let P be a perfect matching in $G_{m,n}$. Let $i \in \{1, \ldots, m\}$ be even such that $s(i) \in P$. If $s(i+2) \in P$, set $i' = i + 1$. Otherwise, let $i' \in \{1, \ldots, m\}$ be odd and maximum such that $s(i^*) \notin P$, $\forall$ even $i < i^* < i'$. We claim that there exists odd $i+1 \leq j \leq i'$ such that $s(j) \notin P$. Indeed, if j does not exist, then the vertices $(i+1, 1), (i+2, 1), \ldots, (i', 1)$ must be saturated by edges from the path $(i+1, 1)$-$(i+2, 1)$-$\cdots$-$(i', 1)$, which is impossible since this path contains an odd number of vertices. Hence, for each edge $s(i) \in P$, i even, there exists an edge $s(j_i) \notin P$, j_i odd and $j_i > i$ and furthermore for $s(i_1), s(i_2) \in P$, $i_1 \neq i_2$ even, we have $j_{i_1} \neq j_{i_2}$.

Finally, we claim that there exists odd $j \in \{1, \ldots, m\}$ such that $s(j) \notin P$ and $j < i$, $\forall$ even $i \in \{1, \ldots, m\}$ with $s(i) \in P$. Indeed, if j does not exist, then vertices $(1, 1), \ldots, (i^* - 1, 1)$, where $i^* = \min\{i \text{ even} : s(i) \in P\}$ (resp. $i^* = m+1$ if $s(i) \notin P$, $\forall$ even $i \in \{1, \ldots, m\}$), must be saturated by edges from the path $(1, 1)$-$\cdots$-$(i^* - 1, 1)$, which is impossible since this path contains an odd number of vertices.

We conclude that if P contains k even edges from M, then there exist $k + 1$ odd edges of M not belonging to P. Thus, P contains at most $k + \frac{m+1}{2} - (k+1) = \frac{m-1}{2}$ edges of M and has total weight at most $\frac{m-1}{2}$. Hence, $\frac{w(P)}{w(M)} \leq \frac{(m-1)/2}{m} = \frac{1}{2} - \frac{1}{2m}$ and thus the result follows. $\square$

Proof of Theorem 7. Let $G_{m,n}$ be a rectangular grid with mn even. If $m = 1$, $G_{1,n}$ is a path which admits a perfect matching. By Theorem 3 we obtain that,

$$\eta(G_{1,n}) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 4. \end{cases}$$

Next, consider that case when $m = n = 2$. $G_{2,2}$ is isomorphic to C_4 and thus $\eta(G_{2,2}) = 1$. If $m \geq 3$ and $n = 2$, it follows from Lemmas 8 and 11 that $\eta(G_{m,2}) = \frac{1}{2}$. The case when $m \geq 3$ is odd and $n \geq 4$ immediately follows from Lemmas 10 and 12. When $m, n \geq 4$ are both even, then we deduce from Lemmas 9 and 11 that $\eta(G_{m,n}) = \frac{1}{2}$. Since $G_{m,n}$ is isomorphic to $G_{n,m}$, the result follows. $\square$

3.2. Bipartite cylindrical grids

In this section, we consider bipartite cylindrical grids. As before, we will prove several lemmas that together give the following exact values of η for all bipartite cylindrical grids.

Theorem 13. *Let $Y_{m,n}$ be a cylindrical grid with $m \geq 2$ even and $n \geq 1$. Then*

$$\eta(Y_{m,n}) = \begin{cases} 1 & \text{if } m = 2 \text{ and } n \leq 2, \\ 1 & \text{if } m = 4 \text{ and } n = 1, \\ \dfrac{2}{3} & \text{if } m = 4 \text{ and } n = 2, \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

Let us first consider lower bounds on η, starting by an alternative proof that the cube graph $(Y_{4,2})$ satisfies $\eta \geq \frac{2}{3}$. The original proof can be found in [2].

Lemma 14. $\eta(Y_{4,2}) \geq \frac{2}{3}$.

Proof. We consider three spanning subgraphs $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G_3 = (V, E_3)$ of $Y_{4,2}$ with edge sets as described below and illustrated in Fig. 5.

$$\begin{aligned} E_1 = \{&(1, 1)(2, 1),\ (3, 1)(4, 1),\ (1, 2)(2, 2),\ (3, 2)(4, 2), \\ &(1, 1)(1, 2),\ (2, 1)(2, 2),\ (3, 1)(3, 2),\ (4, 1)(4, 2)\} \\ E_2 = \{&(2, 1)(3, 1),\ (4, 1)(1, 1),\ (2, 2)(3, 2),\ (4, 2)(1, 2), \\ &(1, 1)(1, 2),\ (2, 1)(2, 2),\ (3, 1)(3, 2),\ (4, 1)(4, 2)\} \\ E_3 = \{&(1, 1)(2, 1),\ (2, 1)(3, 1),\ (3, 1)(4, 1),\ (4, 1)(1, 1), \\ &(1, 2)(2, 2),\ (2, 2)(3, 2),\ (3, 2)(4, 2),\ (4, 2)(1, 2)\}. \end{aligned}$$

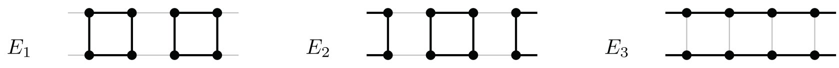

Fig. 5. Spanning subgraphs of $Y_{4,2}$ for Lemma 14.

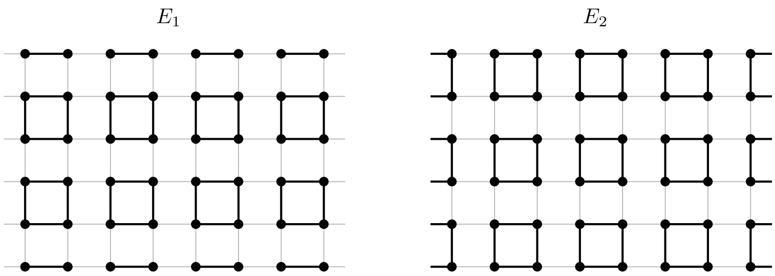

Fig. 6. Spanning subgraphs of $Y_{8,6}$ for Lemma 15.

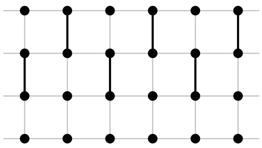

Fig. 7. Matching M in $Y_{6,4}$ for Lemma 16.

Each of the spanning subgraphs is a disjoint union of C_4's and therefore admits a perfect matching. It follows from Lemma 2 that $\eta(G_i) = 1$, for $i = 1, 2, 3$. Furthermore, every edge of $Y_{4,2}$ is contained in exactly two sets among E_1, E_2, E_3. Thus, we deduce from Lemma 2 that $\eta(Y_{4,2}) \geq \frac{2}{3}$. $\square$

Lemma 15. *$\eta(Y_{m,n}) \geq \frac{1}{2}$ for even $m \geq 2$ and $n \geq 1$.*

Proof. Consider two spanning subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of $Y_{m,n}$ with edge sets as described below and illustrated in Fig. 6. The first coordinate of the vertices is taken modulo m.

$$\begin{aligned} E_1 &= \{(i,j)(i+1,j) : i \text{ is } \textit{odd} \text{ from 1 to } m-1,\ j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 1, \ldots, m,\ j \text{ is } \textit{even} \text{ from 2 to } n-2\} \\ E_2 &= \{(i,j)(i+1,j) : i \text{ is } \textit{even} \text{ from 2 to } m,\ j = 1, \ldots, n\} \\ &\cup \{(i,j)(i,j+1) : i = 1, \ldots, m,\ j \text{ is } \textit{odd} \text{ from 1 to } n-1\}. \end{aligned}$$

Each of the two spanning subgraphs is a disjoint union of C_4's and/or K_2's and therefore admits a perfect matching. As before, we conclude that $\eta(G_1) = \eta(G_2) = 1$. Since every edge of $G_{m,n}$ is contained in exactly one set among E_1, E_2, it follows from Lemma 2 that $\eta(Y_{m,n}) \geq \frac{1}{2}$. $\square$

Let us now consider upper bounds on η.

Lemma 16. *$\eta(Y_{m,n}) \leq \frac{1}{2}$ for even $m \geq 4$ and $n \geq 3$.*

Proof. The proof is similar to the proof of Lemma 12. For $i = 1, \ldots, m$, let $s(i)$ denote the edge $(i, 1)(i, 2)$ if i is even and $(i, 2)(i, 3)$ if i is odd. We say that $s(i)$ is even (resp. odd) if i is even (resp. odd). Consider the matching $M = \{s(i) : i = 1, \ldots, m\}$ (see Fig. 7 for an example) and set $w(e) = 1$ if $e \in M$ and $w(e) = 0$ if $e \notin M$. Clearly, M is a maximum weight matching and its total weight is m.

Now, let P be a perfect matching in $Y_{m,n}$. Let $i \in \{1, \ldots, m\}$ be even such that $s(i) \in P$. Let us renumber the first coordinates $1, 2, \ldots, i-1$ by $m+1, m+2, \ldots, m+i-1$. If $s(i+2) \in P$, set $i' = i+1$. Otherwise, let $i' \in \{i+1, \ldots, m+i-1\}$ be odd and maximum such that $s(i^*) \notin P$, $\forall$ even $i < i^* < i'$. Using exactly the same arguments as in the proof of Lemma 12, we show that there exists odd $i+1 \leq j \leq i'$ such that $s(j) \notin P$. Hence, for each edge $s(i) \in P$, i even, there exists an edge $s(j_i) \notin P$, j_i odd and $j_i > i$ and furthermore for $s(i_1), s(i_2) \in P$, $i_1 \neq i_2$ even, we have $j_{i_1} \neq j_{i_2}$.

We conclude that if P contains k even edges from M, then there exist k odd edges of M not belonging to P. Thus, P contains at most $k + \frac{m}{2} - k = \frac{m}{2}$ edges of M and has total weight at most $\frac{m}{2}$. Hence, $\frac{w(P)}{w(M)} \leq \frac{m/2}{m} = \frac{1}{2}$ and thus the result follows. $\square$

Lemma 17. *$\eta(Y_{m,2}) \leq \frac{1}{2}$ for even $m \geq 6$.*

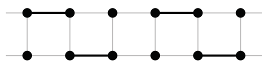

Fig. 8. Edge set E_1 in $Y_{6,2}$ for Lemma 17.

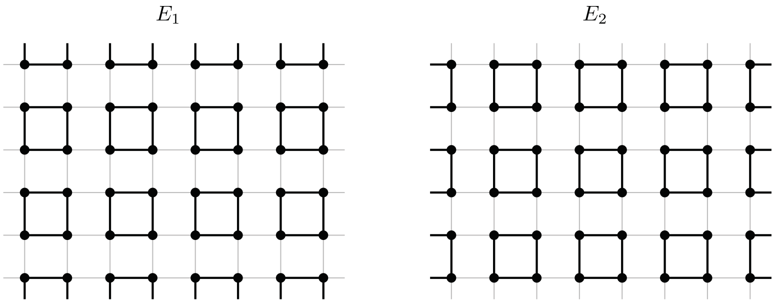

Fig. 9. Spanning subgraphs of $T_{8,6}$ for Lemma 19.

Proof. Let E_1 be the set of four edges described below and illustrated in Fig. 8.

$$E_1 = \{(1, 1)(2, 1),\ (2, 2)(3, 2),\ (4, 1)(5, 1),\ (5, 2)(6, 2)\}.$$

Clearly, at most two out of the four edges in E_1 can belong to a perfect matching. Now the result follows by setting $w(e) = 1$ for all $e \in E_1$ and $w(e) = 0$ otherwise. $\square$

Proof of Theorem 13. Consider a cylindrical grid $Y_{m,n}$ with $m \geq 2$ even and $n \geq 1$. Since $Y_{4,1}$ is isomorphic to C_4, we immediately conclude that $\eta(Y_{4,1}) = 1$. Next, consider $Y_{4,2}$. In [2], it was shown that $\eta(Y_{4,2}) \leq \frac{2}{3}$. It follows from Lemma 14 that $\eta(Y_{4,2}) = \frac{2}{3}$.

Since $Y_{2,n}$ is isomorphic to $G_{2,n}$, it immediately follows from Theorem 7 that $\eta(Y_{2,n}) = 1$ for $n \leq 2$ and $\eta(Y_{2,n}) = \frac{1}{2}$ for $n \geq 3$.

Finally, we deduce from Lemmas 15–17 that for all remaining cases we have $\eta(Y_{m,n}) = \frac{1}{2}$. $\square$

3.3. Bipartite toroidal grids

In this section, we obtain the following exact values of η for all bipartite toroidal grids.

Theorem 18. *Let $T_{m,n}$ be a toroidal grid with $m, n \geq 2$ both even. Then*

$$\eta(T_{m,n}) = \begin{cases} 1 & \text{if } m = n = 2, \\ \frac{2}{3} & \text{if } m = n = 4, \\ \frac{2}{3} & \text{if } m = 4 \text{ and } n = 2, \\ \frac{1}{2} & \text{if } m \geq 6 \text{ and } n \geq 2, \\ \eta(T_{n,m}) & \text{otherwise}. \end{cases}$$

The following lemma gives a lower bound for all bipartite toroidal grids.

Lemma 19. $\eta(T_{m,n}) \geq \frac{1}{2}$ *for even* $m \geq n \geq 2$.

Proof. Consider two spanning subgraphs $G_1 = (V, E_1), G_2 = (V, E_2)$ of $T_{m,n}$ with edge sets as described below and illustrated in Fig. 9. The first coordinate of the vertices is taken modulo m and the second modulo n.

$$\begin{aligned} E_1 &= \{(i, j)(i + 1, j) : i \text{ is } \textit{odd} \text{ from } 1 \text{ to } m - 1,\ j = 1, \dots, n\} \\ &\quad \cup \{(i, j)(i, j + 1) : i = 1, \dots, m,\ j \text{ is } \textit{even} \text{ from } 2 \text{ to } n - 2\} \\ E_2 &= \{(i, j)(i + 1, j) : i \text{ is } \textit{even} \text{ from } 2 \text{ to } m,\ j = 1, \dots, n\} \\ &\quad \cup \{(i, j)(i, j + 1) : i = 1, \dots, m,\ j \text{ is } \textit{odd} \text{ from } 1 \text{ to } n - 1\}. \end{aligned}$$

Each of the two spanning subgraphs is a disjoint union of C_4's and therefore admits a perfect matching. As previously, $\eta(G_1) = \eta(G_2) = 1$. Since every edge of $T_{m,n}$ is contained in exactly one set among E_1, E_2, it follows from Lemma 2 that $\eta(T_{m,n}) \geq \frac{1}{2}$. $\square$

We obtain the same upper bound for sufficiently large toroidal grids.

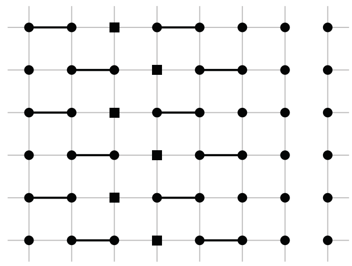

Fig. 10. Sets of edges of $T_{8,6}$ for Lemma 20, with vertices in V' represented as squares.

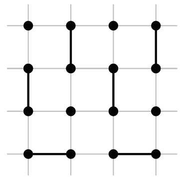

Fig. 11. Edge set E_1 for Lemma 21.

Lemma 20. *$\eta(T_{m,n}) \leq \frac{1}{2}$ for even $m \geq 6$ and even $n \geq 2$.*

Proof. For $i = 1, \ldots, n$, let $s(i)$ denote the edge $(1, i)(2, i)$ if i is odd and $(2, i)(3, i)$ if i is even. Similarly, for $i = 1, \ldots, n$, let $s'(i)$ denote the edge $(4, i)(5, i)$ if i is odd and $(5, i)(6, i)$ if i is even.

Consider the matching $M = \{s(i) : i = 1, \ldots, n\} \cup \{s'(i) : i = 1, \ldots, n\}$ (see Fig. 10 for an example) and set $w(e) = 1$ if $e \in M$ and $w(e) = 0$ if $e \notin M$. Clearly, M is a maximum weight matching and its total weight is $2n$.

Consider the set $V' = \{(3, i), \text{odd } i \in \{1, \ldots, n\}\} \cup \{(4, i), \text{even } i \in \{1, \ldots, n\}\}$. Notice that V' contains exactly n vertices and that these vertices are not saturated by M. Let P be a perfect matching in $T_{m,n}$. We will show that P contains at most n edges of M. Let v be a vertex in V' and let u_1, u_2, u_3, u_4 be its neighbors (not necessarily distinct). Notice that each vertex u_i, $i = 1, 2, 3, 4$, is saturated by M. Since P is perfect, it must contain exactly one of the edges vu_i, $i = 1, 2, 3, 4$. Hence, at least one edge of M saturating the vertices u_1, u_2, u_3, u_4 cannot belong to P. Also, only one endpoint of each edge in M is adjacent to some vertex in V'. Since every vertex in V' must be matched in P, it follows that there are at least n edges of M not belonging to P.

Thus, P contains at most n edges of M and has total weight at most n. Hence, $\frac{w(P)}{w(M)} \leq \frac{n}{2n} = \frac{1}{2}$ and thus the result follows. □

Lemma 21. $\eta(T_{4,4}) \leq \frac{2}{3}$.

Proof. Let E_1 be the set of four edges described below and illustrated in Fig. 11.

$$\begin{aligned} E_1 = \{&(2,1)(2,2),\ (4,1)(4,2),\ (1,2)(1,3),\\ &(3,2)(3,3),\ (1,4)(2,4),\ (3,4)(4,4)\}. \end{aligned}$$

Clearly, at most four out of the six edges in E_1 belong to a perfect matching. Now the result follows by setting $w(e) = 1$ for all $e \in E_1$ and $w(e) = 0$ otherwise. □

In order to prove the lower bound of $\eta(T_{4,4})$, we use the new technique introduced in Section 2.

Lemma 22. $\eta(T_{4,4}) \geq \frac{2}{3}$.

Proof. First notice that the graph $T_{4,4}$ has diameter 4, i.e. the shortest path between any two vertices has length at most 4. Consider a maximal matching M in $T_{4,4}$ that is not a perfect matching. Clearly, the number of unsaturated vertices in both sets of the bipartition must be the same. Therefore, the unsaturated vertices must be in pairs (u_i, v_i) such that the distance between u_i and v_i is 3. By testing all possibilities, we conclude that, up to isomorphisms, there are only three possible sets of unsaturated vertices, as illustrated in Fig. 12.

The figure also shows arbitrary maximal matchings with respect to each set of unsaturated vertices and 3 edge-disjoint augmenting path forests for each case. Now we conclude by applying Lemmas 5 and 6 as described at the end of Section 2 that $\eta(T_{4,4}) \geq \frac{2}{3}$. □

Proof of Theorem 18. Consider a bipartite toroidal grid $T_{m,n}$ with even $m \geq n \geq 2$. If $m = n = 2$, $T_{2,2}$ is isomorphic to C_4 and hence $\eta(T_{2,2}) = 1$. Next, if $m = 4$ and $n = 2$, $T_{4,2}$ is isomorphic to $Y_{4,2}$ and hence $\eta(T_{4,2}) = \frac{2}{3}$.

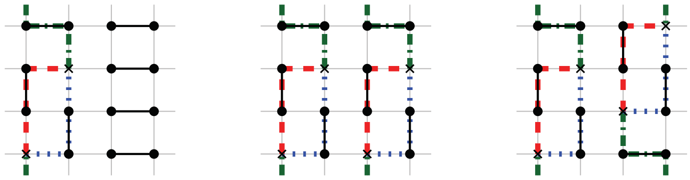

Fig. 12. Toroidal grid $T_{4,4}$ with the three possible sets of unsaturated vertices, corresponding maximal matchings, and augmenting path forests, for Lemma 22.

If $m = n = 4$, $\eta(T_{4,4}) = \frac{2}{3}$. So, we may assume now that $m \geq 6$ and $n \geq 2$. Then it follows from Lemmas 19 and 20 that $\eta(T_{m,n}) = \frac{1}{2}$.

For all remaining cases, we simply use the fact that $T_{m,n}$ is isomorphic to $T_{n,m}$. $\square$

4. Conclusion

In this paper, we considered the parameter η which represents the minimum ratio, over all possible nonnegative edge weightings, between the maximum weight of a perfect matching and the maximum weight of a general matching. We determined the exact value of η for the following graph classes: rectangular grids, bipartite cylindrical grids, and bipartite toroidal grids. Several open problems remain:

1. What is the complexity of deciding whether for a given graph $G = (V, E)$ and a nonnegative real c we have $\eta(G) = c$?
2. Determine the exact value of η for nonbipartite cylindrical grids and nonbipartite toroidal grids.

Here, we focused on grid graphs, but the study of the parameter η can be done for any graph admitting a perfect matching. Hence, it would be of interest to have more results on the exact value of η in additional graph classes. In particular, since we provide a new technique for bipartite graphs, it would be interesting to analyze additional subclasses of bipartite graphs.

Acknowledgment

Third author was partially supported by CNPq and CAPES/MathAmSud 021/14.

References

[1] C. Berge, Two theorems in graph theory, Proc. Natl. Acad. Sci. USA 43 (9) (1957) 842.

[2] E.V. Brazil, C.M.H. de Figueiredo, G.D. da Fonseca, D. Sasaki, The cost of perfection for matchings in graphs, Discrete Appl. Math. (2014) http://dx.doi.org/10.1016/j.dam.2014.12.006.

[3] R. Diestel, Graph Theory, fourth ed., in: Graduate Texts in Mathematics, vol. 173, Springer, 2012.

[4] D. König, Theorie der endlichen und unendlichen Graphen (in German), Akademische Verlagsgesellschaft, 1936.

[5] S. Ramaswami, P. Ramos, G. Toussaint, Converting triangulations to quadrangulations, Comput. Geom. 9 (1998) 257–276.

[6] J.-F. Remacle, J. Lambrechts, B. Seny, E. Marchandise, A. Johnen, C. Geuzainet, Blossom-quad: A non-uniform quadrilateral mesh generator using a minimum-cost perfect-matching algorithm, Int. J. Numer. Methods Eng. 89 (9) (2012) 1102–1119.

[7] J.F. Thompson, B.K. Soni, N.P. Weatherill, Handbook of Grid Generation, CRC Press, 1998.