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On the inclusion of the quasiconformal Teichmüller space into the length-spectrum Teichmüller space

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Abstract This paper is about surfaces of infinite topological type. Unlike the case of surfaces of finite type, there are several deformation spaces associated with a surface S of infinite topological type. Such spaces depend on the choice of a basepoint (that is, the choice of a fixed conformal structure or hyperbolic structure on S) and they also depend on the choice of a distance on the set of equivalence classes of marked hyperbolic structures. We address the question of the comparison between two deformation spaces, namely, the quasiconformal Teichmüller space and the length-spectrum Teichmüller space. There is a natural inclusion map of the quasiconformal space into the length-spectrum space, which is not always surjective. We work under the hypothesis that the basepoint (a hyperbolic surface) satisfies a condition we call "upper-

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boundedness”. This means that this surface admits a pants decomposition defined by curves whose lengths are bounded above. The theory under this upper-boundedness hypothesis shows a dichotomy. On the one hand there are surfaces satisfying what we call Shiga’s condition, i.e. they admit a pants decomposition defined by curves whose lengths are bounded above and below. If the base point satisfies Shiga’s condition, then the inclusion of the quasiconformal space into the length-spectrum space is surjective, and it is a homeomorphism. In this paper we concentrate on the other kind of upper-bounded surfaces, which we call “upper-bounded with short interior curves”. This means that the corresponding hyperbolic surface admits a pants decomposition defined by curves whose lengths are bounded above, and such that the lengths of some interior curves approach zero. We show that in this case the behavior is completely different. Under this hypothesis, the image of the inclusion between the two Teichmüller spaces is nowhere dense in the length-spectrum space. As a corollary of the methods used, we obtain an explicit parametrization of the length-spectrum Teichmüller space in terms of Fenchel–Nielsen coordinates and we prove that the length-spectrum Teichmüller space is path-connected.

Keywords Length-spectrum metric · Quasiconformal metric · Quasiconformal map · Teichmüller space · Fenchel–Nielsen coordinates

Mathematics Subject Classification 32G15 · 30F30 · 30F60

1 Introduction

In this paper, by a *surface*, we mean a connected orientable surface of finite or infinite topological type. In most cases of interest, this surface will be of infinite type. We shall specify this whenever needed. We obtain results on the Teichmüller space of the surface, and the main theme is the interplay between the complex analytic point of view (quasiconformal mappings) and the hyperbolic geometry one. In this introduction, we first recall more precisely the setting and then state the results. To motivate these results, we start by reviewing some basic facts about the Teichmüller spaces of surfaces of infinite topological type.

1.1 The quasiconformal Teichmüller space

Given a surface S, its Teichmüller space is a parameter space for some homotopy classes of marked complex structures on S. There are several possible ways for defining the set of homotopy classes that we want to parametrize, and there are several possibilities for the topology we put on this set. It is often not necessary to worry about these details, because in the most common case, i.e. the case when S is a closed surface, the set is just the set of all possible homotopy classes of complex structures, and all “reasonable” possible definitions of a topology on that set are equivalent. Therefore, in the case of closed surfaces, this freedom of choice between several possible definitions is not a problem, and it is, instead, a very useful tool in the theory: one can choose the definition that best suits the problem studied.

As soon as we leave the setting of closed surfaces, it is necessary to be more careful with the definitions. In this paper, we deal with surfaces of infinite topological type, and in this case the different possible definitions do not always agree. First of all, it is necessary to choose a basepoint, i.e. a base complex structure R on the surface S, and then to consider only the set of homotopy classes of complex structures on S that are "comparable" to R in a suitable sense. This notion of comparability usually suggests a good definition of the topology. For example, the most commonly used definition is the so-called *quasiconformal* Teichmüller space $\mathcal{T}_{qc}(R)$, a set which parametrizes the homotopy classes of complex structures X on S that are quasiconformally equivalent to R, i.e. such that there exists a quasiconformal homeomorphism between R and X that is homotopic to the identity of S. (Note that the space $\mathcal{T}_{qc}(R)$ we consider here is the *reduced* Teichmüller space i.e. homotopies need not fix the ideal boundary pointwise.)

The topology on $\mathcal{T}_{qc}(R)$ is given by the quasiconformal distance d_{qc}, also called the Teichmüller distance, defined using quasiconformal dilatations of quasiconformal homeomorphisms: for any two homotopy classes of complex structures $X, Y \in \mathcal{T}_{qc}(R)$, their quasiconformal distance $d_{qc}(X, Y)$ is defined as

$$d_{qc}(X, Y) = \frac{1}{2} \log \inf_{f} K(f)$$

where $K(f)$ is the quasiconformal dilatation of a quasiconformal homeomorphism $f : X \to Y$ which is homotopic to the identity.

1.2 Fenchel–Nielsen coordinates

In [2], we studied the quasiconformal Teichmüller space of a surface of infinite topological type using pair of pants decompositions and Fenchel–Nielsen coordinates. We will also use this technique in the present paper, and we first recall some of the main facts we need here. The definition of these coordinates depends on the representation of every complex structure as a hyperbolic metric on the surface. To do so, we use the so-called intrinsic hyperbolic metric on a complex surface, defined by Bers. For complex structures of the first type (i.e. if the ideal boundary is empty), this metric is just the Poincaré metric, but for complex structures of the second type it is different from the Poincaré metric. We refer the reader to the paper [2] for the detailed definition and for explanations about how the intrinsic metric may differ from the Poincaré metric. The intrinsic metric has the property that every puncture of the surface shows one of the following behaviors:

1. It has a neighborhood isometric to a cusp, i.e., the quotient of $\{z = x + iy \in \mathbb{H}^2 \mid a < y\}$, for some $a > 0$, by the group generated by the translation $z \mapsto z+1$.
2. It is possible to glue to the puncture a boundary component that is a simple closed geodesic for the hyperbolic metric. Punctures of this kind will be called boundary components, and the boundary geodesic will be considered as part of the surface.

For infinite-type surfaces, we proved in [2] that given a topological pair of pants decomposition $\mathcal{P} = \{C_i\}$ of S and a complex structure X on S, it is always possible to find closed geodesics $\{\gamma_i\}$ for the intrinsic metric of X such that each γ_i is homotopic to C_i and the set $\{\gamma_i\}$ is again a pair of pants decomposition of S. To every curve C_i of the topological decomposition, we can associate two numbers $(\ell_X(C_i), \tau_X(C_i))$, where $\ell_X(C_i)$ is the length in X of the geodesic γ_i and $\tau_X(C_i)$ is the twist parameter between the two pairs of pants (which can be the same) with geodesic boundary adjacent to γ_i. In the present paper, the twist parameter is a length. We note that this is a slight change in notation with reference to the paper [2], where this parameter was an "angle" parameter (length measured along the curve divided by the total length of the curve).

Let R be a complex structure equipped with its intrinsic metric and with a geodesic pants decomposition $\mathcal{P} = \{C_i\}$. We say that the pair $(R, \mathcal{P})$ is *upper-bounded* if $\sup_{C_i} \ell_R(C_i) < \infty$. We say that the pair $(R, \mathcal{P})$ is *lower-bounded* if $\inf_{C_i} \ell_R(C_i) > 0$. If $(R, \mathcal{P})$ is both upper-bounded and lower-bounded, then we say that $(R, \mathcal{P})$ satisfies *Shiga's property*. Note that if R is of finite type, then any pants decomposition of R satisfies Shiga's property. Shiga's property was used for the first time in [16], and we used it in our papers [3,4]. We note however that this property is used in a weaker form in the papers [3,4], and this is also the form which will be useful in the present paper. In fact, we shall say from now on that Shiga's property holds for the pair $(R, \mathcal{P})$ if the pants decomposition $\mathcal{P}$ is upper-bounded and if there exists a positive constant δ such that $\ell_R(C_i) > \delta$ for any $C_i \in \mathcal{P}$ which is in the *interior* of the surface.

In this paper, we will often use the following condition: we say that $(R, \mathcal{P})$ admits *short interior curves* if there is a sequence of curves of the pair of pants decomposition $\alpha_k = C_{i_k}$ $(k = 1, 2, \ldots)$ such that the curves α_k are not boundary components of S and such that $\ell_R(\alpha_k)$ tends to zero as $k \to \infty$.

In the paper [2], we proved, using Fenchel–Nielsen coordinates, that if $(R, \mathcal{P})$ is upper-bounded, then the quasiconformal Teichmüller space $(\mathcal{T}_{qc}(R), d_{qc})$ is locally bi-Lipschitz equivalent to the sequence space ℓ^∞. We note that an analogous result, in the case of the non-reduced Teichmüller space, is due to Fletcher; cf. [7] and the survey [8] by Fletcher and Markovic.

1.3 The length-spectrum Teichmüller space

In this paper, we study a different kind of deformation space, which we call the length-spectrum Teichmüller space. The definition of this space and of its distance depend on a measure of how the lengths of essential curves change when we modify the complex structure. We recall that a simple closed curve on a surface is said to be *essential* if it is not homotopic to a point or to a puncture (but it can be homotopic to a boundary component). We denote by $\mathcal{S}$ the set of homotopy classes of essential simple closed curves on the surface S.

Given a complex structure X on S and an essential simple closed curve γ on S, we denote by $\ell_X(\gamma)$ the length, for the intrinsic metric on X, of the unique geodesic that is homotopic to γ. The value $\ell_X(\gamma)$ does not change if we take another complex structure homotopic to X, hence this function is well defined on homotopy classes of complex structures on S.

Given two homotopy classes X, Y of complex structures on S, we define the functional

$$L(X, Y) = \sup_{\gamma \in \mathcal{S}} \left\{ \frac{\ell_X(\gamma)}{\ell_Y(\gamma)}, \frac{\ell_Y(\gamma)}{\ell_X(\gamma)} \right\} \leq \infty.$$

Given a base complex structure R on S, the *length-spectrum* Teichmüller space $\mathcal{T}_{ls}(R)$ is the space of homotopy classes of complex structures X on S satisfying $L(R, X) < \infty$.

For any two distinct elements $X, Y \in \mathcal{T}_{ls}(R)$, we have $1 < L(X, Y) < \infty$. We define a metric d_{ls} on $\mathcal{T}_{ls}(R)$, called the *length-spectrum distance*, by setting

$$d_{ls}(X, Y) = \frac{1}{2} \log L(X, Y).$$

Chronologically, the length-spectrum distance was defined before the length-spectrum Teichmüller space but initially people considered this distance as a distance on the quasiconformal Teichmüller space: they studied the metric space $(\mathcal{T}_{qc}(R), d_{ls})$. For finite type surfaces this is a perfectly fine distance on $\mathcal{T}_{qc}(R)$; it makes this space complete, it induces on it the ordinary topology, and it is suitable for studying problems related to lengths of geodesics. For an example of a paper studying this space, see [6], some of whose results we will use in the following. The first paper dealing with the space $(\mathcal{T}_{qc}(R), d_{ls})$ in the case of surfaces of infinite type is [16].

We proved in [3] that the metric space $(\mathcal{T}_{qc}(R), d_{ls})$ is, in general, not complete. More precisely, this happens if there exists a pair of pants decomposition $\mathcal{P} = \{C_i\}$ such that the pair $(R, \mathcal{P})$ admits short interior curves, i.e. if there is a sequence of curves of the pair of pants decomposition $\alpha_k = C_{i_k}$ contained in the interior of R with $\ell_R(\alpha_k) \to 0$. The idea was to construct a sequence of hyperbolic metrics by large twists along short curves. Let us be more precise.

For any $X \in \mathcal{T}_{qc}(R)$, we have again $\ell_X(\alpha_k) = \epsilon_k \to 0$. Denote by $\tau_\alpha^t(X)$ the surface obtained from X by a twist of magnitude t along α and let $X_k = \tau_{\alpha_i}^{t_k}(X)$, where $t_k = \log|\log \epsilon_k|$. Then we proved that $d_{qc}(X, X_k) \to \infty$, while $d_{ls}(X, X_k) \to 0$.

For $n \geq 0$, if we set $Y_n = \tau_{\alpha_n}^{t_n} \circ \cdots \circ \tau_{\alpha_2}^{t_2} \circ \tau_{\alpha_1}^{t_1}(X)$ and if we define Y_∞ to be the surface obtained from X by a twist of magnitude t_i along α_i for every i, then a similar argument shows that $Y_\infty \in \mathcal{T}_{ls}(R) \setminus \mathcal{T}_{qc}(R)$ and $\lim_{n\to\infty} d_{ls}(Y_n, Y_\infty) = 0$; see [3,4] for more details. We shall give another proof of the last result in Sect. 5 below.

The fact that the metric space $(\mathcal{T}_{qc}(R), d_{ls})$ is, in general, not complete is an indication of the fact that $\mathcal{T}_{qc}(R)$ is not the right underlying space for this distance. The length-spectrum Teichmüller space was defined in [12], with the idea that it is the most natural space for that distance. This space was studied in [3,4]. We proved in [3] that for every base complex structure R, the metric space $(\mathcal{T}_{ls}(R), d_{ls})$ is complete. This result answered a question raised in [12] (Question 2.22).

Other properties, such as connectedness and contractibility, are unknown in the general case for surfaces of infinite type. If the basepoint R satisfies Shiga's condition, then it follows from the main result of [4] that $(\mathcal{T}_{ls}(R), d_{ls})$ is homeomorphic to the sequence space ℓ^∞ with a homeomorphism that is locally bi-Lipschitz, and, in particular, the space is contractible. One of the results we prove in the present paper is the following.

Theorem 6.3 *If* $(R, \mathcal{P})$ *is upper-bounded and admits short interior curves, then* $(\mathcal{T}_{ls}(R), d_{ls})$ *is path-connected.*

To obtain this result, we will use some results on the comparison between the quasiconformal and the length-spectrum spaces, and we will also need the following explicit characterization of the length-spectrum Teichmüller space in terms of Fenchel–Nielsen coordinates, which is interesting in itself:

Theorem 6.2 *Assume* $X = (\ell_X(C_i), \tau_X(C_i))$. *Then* X *lies in* $\mathcal{T}_{ls}(R)$ *if and only if there is a constant* $N > 0$ *such that for each* i,

$$\left|\log \frac{\ell_X(C_i)}{\ell_R(C_i)}\right| < N$$

and

$$|\tau_X(C_i) - \tau_R(C_i)| < N \max\{|\log \ell_R(C_i)|, 1\}.$$

1.4 Comparison between the two spaces

It is interesting to compare the two spaces $(\mathcal{T}_{ls}(R), d_{ls})$ with $(\mathcal{T}_{qc}(R), d_{qc})$.

A classical result of Sorvali [17] and Wolpert [18] states that for any K-quasiconformal map $f : X \to Y$ and any $\gamma \in \mathcal{S}$, we have

$$\frac{1}{K} \leq \frac{\ell_Y(f(\gamma))}{\ell_X(\gamma)} \leq K.$$

It follows from this result that there is a natural inclusion map

$$I : (\mathcal{T}_{qc}(R), d_{qc}) \to (\mathcal{T}_{ls}(R), d_{ls})$$

and that this map is 1-Lipschitz.

In [12] we proved that if R satisfies Shiga's condition, then this inclusion is surjective, showing that under this hypothesis we have $\mathcal{T}_{ls}(R) = \mathcal{T}_{qc}(R)$ as sets. In the same paper we also gave an example of a complex structure R whose associated inclusion map I is not surjective.

The inverse map of I (defined on the image set $\mathcal{T}_{qc}(R)$) is not always continuous. Shiga gave in [16] an example of a hyperbolic structure R on a surface of infinite type and a sequence (R_n) of hyperbolic structures in $\mathcal{T}_{ls}(R) \cap \mathcal{T}_{qc}(R)$ which satisfy

$$d_{ls}(R_n, R) \to 0, \text{ while } d_{qc}(R_n, R) \to \infty.$$

In particular, the metrics d_{ls} and d_{qc} do not induce the same topology on $\mathcal{T}_{qc}(R)$. A more general class of surfaces with the same behavior was described in the paper [13] by Liu, Sun and Wei.

In the [16], Shiga also showed that if the hyperbolic metric R carries a geodesic pants decomposition that satisfies Shiga's condition, then d_{ls} and d_{qc} induce the same topology on $\mathcal{T}_{qc}(R)$. In the paper [4] we strengthened this result by showing that under Shiga's condition the inclusion map is locally bi-Lipschitz.

Several natural problems arise after this, for instance:

1. Give necessary and sufficient conditions under which the inclusion map I is surjective.
2. Under what conditions is the inverse map (defined on the image set) continuous? Under what conditions is it Lipschitz? bi-Lipschitz?
3. Are the two spaces $(\mathcal{T}_{qc}(R), d_{qc})$ and $(\mathcal{T}_{ls}(R), d_{ls})$ in the general case locally isometric to the infinite sequence space ℓ^∞? What are the "local model spaces" to which these spaces are locally isometric in the general case?
4. How does the image $I\left((\mathcal{T}_{qc}(R), d_{qc})\right)$ sit in the space $(\mathcal{T}_{ls}(R), d_{ls})$? Is it dense? Is it nowhere dense?

Some of these problems are solved in the present paper, and others are solved in the paper [15] by Šarić which was motivated by a first version of it. We now state some of the results.

Consider $\mathcal{T}_{qc}(R)$ as a subset in $\mathcal{T}_{ls}(R)$. A natural question which is asked in [3] is whether the subset $\mathcal{T}_{qc}(R)$ is dense in $(\mathcal{T}_{ls}(R), d_{ls})$. A positive answer would tell us that $\mathcal{T}_{ls}(R)$ is the metric completion of $\mathcal{T}_{qc}(R)$ with reference to the distance d_{ls}, and it would also imply that $(\mathcal{T}_{ls}(R), d_{ls})$ is connected (since the closure of a connected subset is also connected). In this paper, we give a negative answer to this question. More precisely, we prove the following:

Theorem 5.8 If R admits a geodesic pants decomposition which is upper-bounded and if it admits short interior geodesics, then the space $(\mathcal{T}_{qc}(R), d_{ls})$ is nowhere dense in $(\mathcal{T}_{ls}(R), d_{ls})$.

The proof of Theorem 5.8 involves some estimates between quasiconformal dilatation and hyperbolic length under the twist deformation. Some of the techniques used in this paper are developed in our papers [2,4]. The upper-boundedness assumption on the pair of pants decomposition is used here so that we can get a lower bound estimate of the length-spectrum distance under a twist. Without this assumption, the density of $\mathcal{T}_{qc}(R)$ in $\mathcal{T}_{ls}(R)$ is an open question.

It is interesting to study the closure of $\mathcal{T}_{qc}(R)$ in $\mathcal{T}_{ls}(R)$. This space is the completion of $\mathcal{T}_{qc}(R)$ with reference to the length-spectrum metric. Under the upper-boundedness condition, Šarić gave in [15] a characterization of the closure of $\mathcal{T}_{qc}(R)$ in $\mathcal{T}_{ls}(R)$.

We prove (Proposition 5.4) that if the surface R admits a pants decomposition $\mathcal{P}$ such that $(R, \mathcal{P})$ is upper-bounded and admits short interior curves, then there exists a point in $\mathcal{T}_{ls}(R) \setminus \mathcal{T}_{qc}(R)$ which can be approximated by a sequence in $\mathcal{T}_{qc}(R)$ with the length-spectrum metric. This gives in particular a new proof of the fact (obtained in [3]) that the space $\mathcal{T}_{qc}(R)$ equipped with the restriction of the metric d_{ls} is not complete.

Regarding Problem (2), under the geometric conditions of Theorem 5.8, we showed in [3] (Example 5.1) that the inverse of the inclusion map I restricted to $\mathcal{T}_{qc}(R)$: $(\mathcal{T}_{qc}(R), d_{ls}) \to (\mathcal{T}_{qc}(R), d_{qc})$ is nowhere continuous. We provide another proof of this fact in the present paper (Proposition 5.2).

Regarding Problem (3), Šarić proved in [15], using the Fenchel-Nielsen-coordinates and again under the upper-boundedness condition, that $\mathcal{T}_{ls}(R)$ is bi-Lipschitz homeomorphic to the sequence space ℓ^∞. This shows that the space $\mathcal{T}_{ls}(R)$ is contractible. The paper [15] settles some questions which were raised in a first version of the present paper.

2 Preliminaries

Let R be a base topological surface equipped with a hyperbolic structure X and with a geodesic pants decomposition $\mathcal{P} = \{C_i\}$. The pieces of the decomposition (completions of connected components of the complements of the curves C_i) are spheres with three holes equipped with hyperbolic metrics, where a hole is either a cusp or a geodesic boundary component. We call such a piece a *generalized* pair of pants to stress on the fact that it is not necessarily a hyperbolic pair of pants with three geodesic boundary components. To each $C_i \in \mathcal{P}$, we consider its length parameter $\ell_X(C_i)$ and its twist parameter $\tau_X(C_i)$. Recall that the latter is only defined if C_i is not a boundary component of R, and it is a measure of the relative twist amount along the geodesic C_i between the two generalized pairs of pants (which may be the same) that have this geodesic in common. The twist amount per unit time along C_i is chosen so that a complete positive Dehn twist along C_i changes the twist parameters on C_i by addition of $\ell_X(C_i)$. For any hyperbolic metric X, its Fenchel–Nielsen parameters relative to $\mathcal{P}$ is the collection of pairs

$$\{(\ell_X(C_i), \tau_X(C_i))\}_{i=1,2,\ldots}$$

where it is understood that if C_i is a boundary component of R, then there is no twist parameter associated to it, and instead of a pair $(\ell_X(C_i), \tau_X(C_i))$ we have a single parameter $\ell_X(C_i)$.

There is an injective mapping from $\mathcal{T}_{qc}(R)$ or $\mathcal{T}_{ls}(R)$ to an infinite-dimensional real parameter space:

$$X \mapsto \left(\left(\log \frac{\ell_X(C_i)}{\ell_R(C_i)}, \tau_X(C_i) - \tau_R(C_i)\right)\right)_{i=1,2,\ldots}.$$

If the image of X belongs to ℓ^∞, then we say that X is *Fenchel–Nielsen bounded* (with respect to $(R, \mathcal{P})$).

For each C_i in the interior of X, there is a simple closed curve β_i satisfying the following (see Fig. 1):

1. β_i and C_i intersect minimally, that is, $i(C_i, \beta_i) = 1$ or 2;
2. β_i does not intersect any C_j, $j \neq i$.

The following result is proved in [2].

Lemma 2.1 *Suppose that the pants decomposition $\{C_i\}$ is upper-bounded by M, that is, $\sup_i\{\ell_X(C_i)\} \leq M$. Then there exists a positive constant ρ depending only on M such that for each i, β_i can be chosen so that the intersection angle(s) θ_i of C_i and β_i (there are one or two such angles for each curve C_i) satisfy $\sin\theta_i \geq \rho$.*

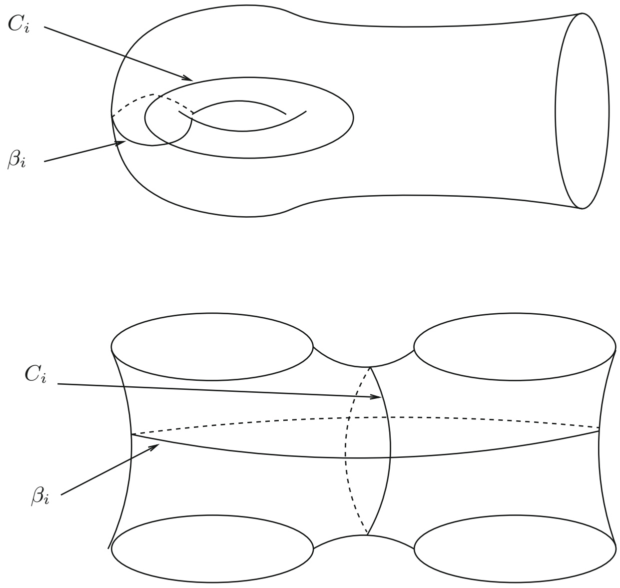

Fig. 1 In each case, we have represented the curve C_i and its dual curve β_i

We shall need some background material on the conformal moduli of quadrilaterals. Recall that a *quadrilateral* $Q(z_1, z_2, z_3, z_4)$ consists of a Jordan domain Q in the complex plane and a sequence of ordered vertices z_1, z_2, z_3, z_4 on the boundary of Q. The vertices of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ divide its boundary into four Jordan arcs, called the sides of the quadrilateral. When we use this notation, we shall call the arcs $\overline{z_1 z_2}$ and $\overline{z_3 z_4}$ the a-sides, and the other two arcs the b-sides of Q. Two quadrilaterals $Q(z_1, z_2, z_3, z_4)$ and $Q'(w_1, w_2, w_3, w_4)$ are said to be conformally equivalent if there is a conformal map from Q to Q' which carries each z_i to w_i.

Every quadrilateral $Q(z_1, z_2, z_3, z_4)$ is conformally equivalent to a rectangle

$$R(0, a, a + ib, ib) = \{x + iy : 0 < x < a, 0 < y < b\}.$$

It is easy to see that two rectangles $R(0, a, a+ib, ib)$ and $R'(0, a', a'+ib', ib')$ are conformally equivalent if and only if there is a similarity transformation between them. Therefore, we can define the (conformal) modulus of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ by

$$\mathrm{mod}(Q(z_1, z_2, z_3, z_4)) = \frac{a}{b}.$$

It follows from the definition that the modulus of a quadrilateral is a conformal invariant and that $\mathrm{mod}(Q(z_1, z_2, z_3, z_4)) = 1/\mathrm{mod}(Q(z_2, z_3, z_4, z_1))$.

The modulus of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be described in terms of extremal length in the following way. Let $\mathcal{F} = \{\gamma\}$ be the family of curves in Q joining the a-sides. The extremal length of the family $\mathcal{F}$, denoted by $\mathrm{Ext}(\mathcal{F})$, is defined by

$$\mathrm{Ext}(\mathcal{F}) = \sup_{\rho} \frac{\inf_{\gamma\in\mathcal{F}} \ell_\rho(\gamma)^2}{\mathrm{Area}_\rho}$$

where the supremum is taken over all conformal metrics ρ on Q of finite positive area. Then it can be shown [1] that

$$\mathrm{mod}(Q(z_1, z_2, z_3, z_4)) = \frac{1}{\mathrm{Ext}(\mathcal{F})}.$$

3 A lower bound for the quasiconformal dilatation under a twist

The main result of this section is Theorem 3.2 which gives a lower bound for the quasiconformal distance between a surface X and its image X_t under a Fenchel–Nielsen multi-twist along curves in the pants decomposition in terms of the amount of twisting and some other quantity which is a sequence of lower bounds for the angles between the curves of the pants decomposition and a collection of dual curves. We also prove Corollary 3.3, which is important for Sect. 5. For a hyperbolic metric X and a simple closed geodesic α on X, we denote by $\tau_\alpha^t(X)$, $t \in \mathbb{R}$ the hyperbolic metric obtained from X by a Fenchel–Nielsen twist of magnitude t along α. We fix the simple closed curve α and, to simplify the notation, we set $X_t = \tau_\alpha^t(X)$.

For a small $\epsilon > 0$, let N_ϵ be an ϵ-neighborhood of α. We denote by $g_\epsilon^t : X \to X_t$ a homeomorphism that is the natural isometry outside of N_ϵ and that is homotopic to the identity.

It is convenient to work in the universal cover of the surface. In the case where X has no boundary, its universal cover is the hyperbolic plane $\mathbb{H}^2$. We let $f_\epsilon^t : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of g_ϵ^t. In the case where X has non-empty boundary, we take the double of X and X_t, extend the map g_ϵ^t to $X^d \to X_t^d$ and we then let f_ϵ^t be the lift of the extended map to $\mathbb{H}^2$. Thus, in any case, the map f_ϵ^t is defined on the plane $\mathbb{H}^2$.

Let $\tilde{\alpha}$ be the lift of the closed geodesic α to the universal cover. Then $\tilde{\alpha}$ can be seen as a lamination with discrete leaves.

When ϵ tends to zero, the maps f_ϵ^t converges pointwise on $\mathbb{H}^2\backslash\tilde{\alpha}$ to a map f^t that is an isometry on every connected component of $\mathbb{H}^2 \setminus \tilde{\alpha}$. We choose an orientation on α. Now every connected component of $\tilde{\alpha}$ divides the plane into two parts, a left part and a right part. We extend f^t to $\tilde{\alpha}$ by requiring that this extension is continuous on the left part. We denote the extended map again by f^t; it is a piecewise isometry from $\mathbb{H}^2$ to $\mathbb{H}^2$.

To make some explicit computations, we work in the upper-half model of the hyperbolic plane $\mathbb{H}^2$. Up to conjugation in the domain and in the range, we can assume that the geodesic $i\mathbb{R}^+$ is a leaf of $\tilde{\alpha}$ that is fixed pointwise by f^t. In particular, f^t fixes 0, i and ∞.

Lemma 3.1 *For any bi-infinite geodesic in the upper-half plane model of $\mathbb{H}^2$ with endpoints $x_1 < 0 < x_2$ on $\mathbb{R}$ and intersecting $i\mathbb{R}^+$ at i (see Fig. 2), we have*

$$f^t(x_1) < -\sqrt{e^{2t} + \left(\frac{x_1+x_2}{2}\right)^2} + \left(\frac{x_1+x_2}{2}\right) < 0 \textit{ and } 0 < f^t(x_2) < x_2, \ \forall\, t > 0.$$

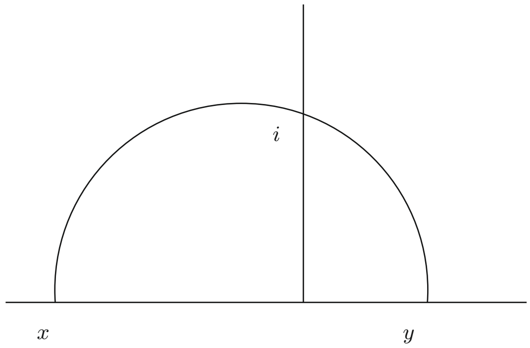

Fig. 2 A bi-infinite geodesic in the upper half-space model of the hyperbolic plane intersecting the imaginary axis at the point i

Proof This follows from the construction of the Fenchel–Nielsen twist deformation. See, for example, the proof of Lemma 3.6 in Kerckhoff [9]. We give here the proof for the sake of completeness.

Let γ be the bi-infinite geodesic connecting x_1 and x_2. By assumption, $i\mathbb{R}^+$ and γ intersect at the point i. Under the twist deformation, γ is deformed into a sequence of disjoint geodesic arcs $\{A_i\}$, each coming from γ under the twist deformation. Let $\bar{\gamma}$ be the infinite piecewise geodesic arc which is the union of $\{A_i\}$ and of pieces of leaves of $\tilde{\alpha}$. See Fig. 3 in the case where $x_1 = -1$ and $x_2 = 1$.

Note that one such arc A_0 passes through the point i. If A_0 is continued to a bi-infinite geodesic, its endpoints will be precisely those of γ. Move along $\bar{\gamma}$ in the left direction, running along the leaf $i\mathbb{R}^+$ (by a hyperbolic distance t) until coming to the next arc A_1. If the arc is continued in the forward direction, one of its endpoints is $-\sqrt{e^{2t} + (\frac{x_1+x_2}{2})^2} + (\frac{x_1+x_2}{2})$ (this can be shown by the cosine formula for triangles). Similarly, the forward endpoint of the next arc, A_2, is strictly to the left of $-\sqrt{e^{2t} + (\frac{x_1+x_2}{2})^2} + (\frac{x_1+x_2}{2})$. In fact, the forward endpoint of each arc A_{i+1} is strictly

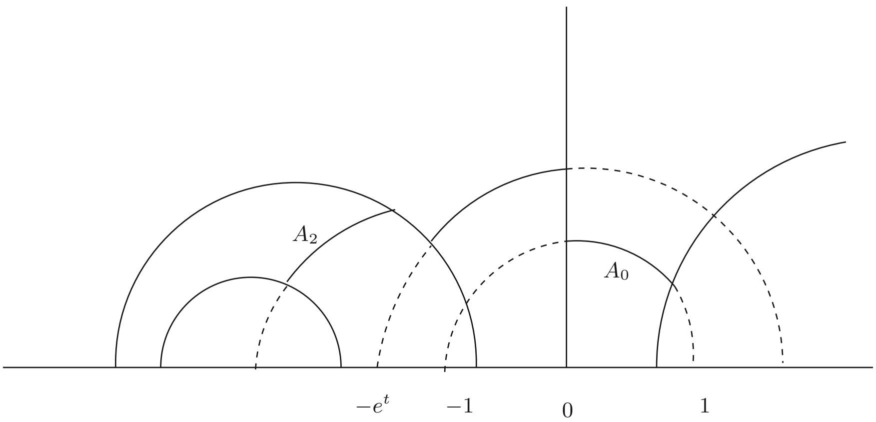

Fig. 3 The image of γ under a twist

to the left of A_i. Since the forward endpoints of the $A_i^{,}s$ converge to $f^t(x_1)$, we see that $f^t(x_1)$ is strictly less than $-\sqrt{e^{2t}+(\frac{x_1+x_2}{2})^2}+(\frac{x_1+x_2}{2})$.

An analogous (and simpler) argument shows that $0 < f^t(x_2) < x_2$. □

As before, assume that $\mathcal{P} = \{C_i\}$ is a geodesic pants decomposition of the hyperbolic surface R.

Let $t = (t_1, t_2, \ldots)$ be a sequence of real numbers. Fix $X \in \mathcal{T}_{qc}(R)$. We say that X_t is a *multi-twist deformation* of X along $\mathcal{P}$ if X_t is obtained from X by the composition of t_i-twists along C_i. Let us set $\|t\| = \sup_i |t_i|$. In the following theorem, the simple closed curves (β_i) are chosen as in Sect. 2. We assume that the intersection angle(s) θ_i of C_i and β_i satisfy $\sin\theta_i \geq \rho_i$.

Since each β_i intersects C_i and no C_j, $j \neq i$, then under the multi-twist deformation, the image of β_i depends only on the twist along C_i.

We now denote by $H(\infty, -1, 0, t)$, $t > 0$ the quadrilateral where the underlying Jordan domain H is the upper half-plane. Let $h(t) = \mathrm{mod}(H(\infty, -1, 0, t))$. Then, $h(t)$ is related to the modulus of the *Grötzch ring* (the ring domain obtained by deleting the interval $[0, r]$ from the unit disk) $\mu(r)$ by the following equality (see Page 60–61 in [11]):

$$h(t) = \frac{2}{\pi}\mu\left(\sqrt{\frac{1}{1+\lambda}}\right), \text{ where } \lambda = t.$$

From the known properties of $\mu(r)$, it follows that $h(t)$ is a strictly increasing function and $\lim_{t\to+\infty} h(t) = \infty$.

Theorem 3.2 *For the hyperbolic surface X_t defined above, we have*

$$d_{qc}(X, X_t) \geq \frac{1}{2}\log\sup_i \frac{h(K_i e^{|t_i|})}{h\left(\frac{1+\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)},$$

where

$$K_i = \frac{1-\sqrt{1-\rho_i^2}}{\left(1+\sqrt{1-\rho_i^2}\right)\left(\sqrt{1+(\frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}})^2}+\frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)}.$$

Proof For each C_i, consider the twist t_i. Without loss of generality, we assume that $t_i > 0$.

Assume that $i\mathbb{R}^+$ is a lift of C_i to the universal cover and that β_i has a lift γ which intersects $i\mathbb{R}^+$ at the point i. Denote by $x_1 < 0 < x_2$ the two endpoints of γ. The intersection θ of $i\mathbb{R}^+$ and γ satisfies:

$$\sin\theta = \frac{2}{|x_1| + |x_2|}. \tag{1}$$

This is because θ is the intersection angle determined by two lines, one is the line through i and $\frac{|x_1|+|x_2|}{2}$, and the other is the real line (in negative direction). The sine of the angle is equal to

$$\frac{2}{|x_1| + |x_2|}.$$

Note that Formula (1) for the angle may also been deduced from Eq. (4) below.
Applying Lemma 3.1, we have

$$f^t(x_1) < -\sqrt{e^{2t_i} + \left(\frac{x_1+x_2}{2}\right)^2} + \left(\frac{x_1+x_2}{2}\right) < 0, 0 < f^t(x_2) < x_2. \tag{2}$$

Let us set $-\sqrt{e^{2t_i} + (\frac{x_1+x_2}{2})^2} + (\frac{x_1+x_2}{2}) = -Ae^{t_i}$, where

$$A = \frac{1}{\sqrt{1 + e^{-2t_i}\left(\frac{x_1+x_2}{2}\right)^2} + e^{-t_i}\left(\frac{x_1+x_2}{2}\right)}.$$

By the geometric definition of quasiconformal maps,

$$K(f^t) \geq \frac{\operatorname{mod}\left(H(f^t(x_1), f^t(0), f^t(x_2), f^t(\infty))\right)}{\operatorname{mod}(H(x_1, 0, x_2, \infty))}.$$

By (2) and the monotonicity of conformal modulus, we have

$$\operatorname{mod}(H(f^t(x_1), 0, f^t(x_2), \infty)) \geq \operatorname{mod}(H(-Ae^{t_i}, 0, x_2, \infty)).$$

Note that

$$\begin{aligned} \operatorname{mod}(H(-Ae^{t_i}, 0, x_2, \infty)) &= \operatorname{mod}(H(\infty, -x_2, 0, Ae^{t_i})) \\ &= \operatorname{mod}\left(H\left(\infty, -1, 0, \frac{Ae^{t_i}}{|x_2|}\right)\right). \end{aligned}$$

Therefore

$$K(f^t) \geq \frac{\operatorname{mod}\left(H\left(\infty, -1, 0, \frac{Ae^{t_i}}{|x_2|}\right)\right)}{\operatorname{mod}\left(H\left(\infty, -1, 0, |\frac{x_1}{x_2}|\right)\right)}. \tag{3}$$

We can use the cross ratio to estimate $|\frac{x_1}{x_2}|$ in terms of the lower bound $\rho_i(X)$ of $\sin\theta_i$. Let $\chi(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ be the cross ratio of $a, b, c, d \in \mathbb{R} \cup \{\infty\}$. We can map $\mathbb{H}^2$ conformally to the unit disc, and $i\mathbb{R}^+$ and γ to the geodesics with endpoints ± 1 and $\pm e^{i\theta}$ respectively. It is easy to show that

$$\cos^2(\theta/2) = \chi(1, e^{i\theta}, -e^{i\theta}, -1).$$

By the conformal invariance of the cross ratio, we have

$$\cos^2(\theta/2) = \chi(0, x_2, x_1, \infty) = \frac{|x_1|}{|x_1| + |x_2|}. \tag{4}$$

Since $\sin\theta \geq \rho_i$, we have

$$\frac{1 - \sqrt{1 - \rho_i^2}}{1 + \sqrt{1 - \rho_i^2}} \leq \frac{|x_1|}{|x_2|} \leq \frac{1 + \sqrt{1 - \rho_i^2}}{1 - \sqrt{1 - \rho_i^2}}. \tag{5}$$

There are other restrictions on the values x_1, x_2. Since the geodesic γ passes through the point i, we can show that $|x_2||x_1| = 1$.

If $|x_1| \geq 1 \geq |x_2|$, then

$$\frac{A}{|x_2|} \geq A \geq \frac{1}{\sqrt{1 + \left(\frac{|x_1|/|x_2| - 1}{2}\right)^2} + \frac{|x_1|/|x_2| - 1}{2}}.$$

Combined with the inequality (5), this gives

$$\frac{A}{|x_2|} \geq \frac{1}{\sqrt{1 + \left(\frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)^2} + \frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}}.$$

If $|x_2| \geq 1 \geq |x_1|$, then (using again Inequality (5)) we obtain

$$\frac{A}{|x_2|} \geq A\frac{|x_1|}{|x_2|} \geq \frac{1 - \sqrt{1 - \rho_i^2}}{\left(1 + \sqrt{1 - \rho_i^2}\right)\left(\sqrt{1 + (\frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}})^2} + \frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)}.$$

Denote

$$K_i = \frac{1 - \sqrt{1 - \rho_i^2}}{\left(1 + \sqrt{1 - \rho_i^2}\right)\left(\sqrt{1 + (\frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}})^2} + \frac{\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)}.$$

We conclude that

$$\frac{A}{|x_2|} \geq K_i.$$

It follows from Inequality (3) that

$$K(f^t) \geq \frac{\operatorname{mod}(H(\infty, -1, 0, K_i e^{t_i}))}{\operatorname{mod}\left(H\left(\infty, -1, 0, \frac{1+\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)\right)}.$$

Using the function $h(t)$ already introduced for moduli of quadrilaterals, we get

$$K(f^t) \geq \frac{h(K_i e^{t_i})}{h\left(\frac{1+\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)}.$$

By Teichmüller's Theorem, there is an extremal quasiconformal map from X to X_t that realizes the Teichmüller distance $d_{\mathrm{qc}}(X_t, X)$. Lifting this map to the universal cover, since it is homotopic to f^t, it has the same boundary value as f^t. It follows that

$$2d_T(X, X_t) \geq \log \sup_i \frac{h(K_i e^{t_i})}{h\left(\frac{1+\sqrt{1-\rho_i^2}}{1-\sqrt{1-\rho_i^2}}\right)}.$$

□

Corollary 3.3 *If* $\sup_i l_X(C_i) < \infty$ *and if* X_t *is a multi-twist deformation of* X *along* $\mathcal{P}$*, then* $d_{qc}(X, X_t) \to \infty$ *as* $\|t\| \to \infty$*.*

Proof By assumption, there is a positive constant M such that $\sup_i \ell_X(C_i) < M$. By Lemma 2.1, there is a positive constant ρ (depending on M) such that $\inf_i \rho_i \geq \rho$. It is easy to see that $\frac{(1-\sqrt{1-\rho^2})^2}{1+\sqrt{1-\rho^2}}$ is an increasing function of ρ. For each i, we have

$$\frac{\left(1+\sqrt{1-\rho_i^2}\right)^2}{1-\sqrt{1-\rho_i^2}} \geq \frac{\left(1-\sqrt{1-\rho^2}\right)^2}{1+\sqrt{1-\rho^2}}.$$

As a result, if follows from Theorem 3.2 that

$$d_{qc}(X, X_t) \geq \frac{1}{2} \log \sup_i \frac{h\left(\frac{(1-\sqrt{1-\rho^2})^2}{1+\sqrt{1-\rho^2}} e^{t_i}\right)}{h\left(\frac{1+\sqrt{1-\rho^2}}{1-\sqrt{1-\rho^2}}\right)}. \tag{6}$$

As $\|t\| \to \infty$, the properties of the function $h(t)$ tell us that $d_{qc}(X, X_t) \to \infty$. $\square$

4 Estimation of hyperbolic length under a twist deformation

The twist deformation is an important tool to understand the difference between the quasiconformal metric and the length-spectrum metric. As in the previous section, we fix a simple closed geodesic α and we set $X_t = \tau_\alpha^t(X)$. In this section, we give a upper bound and a lower bound for $d_{ls}(X, X_t)$.

Proposition 4.1 *For every t in $\mathbb{R}$, we have*

$$d_{ls}(X, X_t) \leq \frac{1}{2} \max \left\{ \sup_{\gamma, i(\alpha,\gamma)\neq 0} \frac{i(\alpha, \gamma)|t|}{\ell_X(\gamma)}, \sup_{\gamma, i(\alpha,\gamma)\neq 0} \frac{i(\alpha, \gamma)|t|}{\ell_{X_t}(\gamma)} \right\}.$$

Proof Without loss of generality, we can assume that $t > 0$. For any simple closed curve γ intersecting α, let $\ell_t(\gamma)$ denote the hyperbolic length of γ in X_t. We have

$$\ell_X(\gamma) - i(\alpha, \gamma)t \leq \ell_t(\gamma) \leq \ell_X(\gamma) + i(\alpha, \gamma)t.$$

Recall that the length-spectrum distance is given by

$$d_{ls}(X, X_t) = \max \left\{ \frac{1}{2} \log \sup_\gamma \frac{\ell_t(\gamma)}{\ell_X(\gamma)}, \frac{1}{2} \log \sup_\gamma \frac{\ell_X(\gamma)}{\ell_t(\gamma)} \right\},$$

where the supremum is taken over all essential simple closed curves.

For a simple closed curve γ satisfying $i(\alpha, \gamma) = 0$, the hyperbolic length of γ is invariant under the twist along α. As a result, we have

$$d_{ls}(X, X_t) = \max \left\{ \frac{1}{2} \log \sup_{\gamma, i(\alpha,\gamma)\neq 0} \frac{\ell_t(\gamma)}{\ell_X(\gamma)}, \frac{1}{2} \log \sup_{\gamma, i(\alpha,\gamma)\neq 0} \frac{\ell_X(\gamma)}{\ell_t(\gamma)} \right\}.$$

For any simple closed curve γ with $i(\alpha, \gamma) \neq 0$, we have

$$\log \frac{\ell_t(\gamma)}{\ell_X(\gamma)} \leq \log \frac{\ell_X(\gamma) + i(\alpha, \gamma)t}{\ell_X(\gamma)} \leq \frac{i(\alpha, \gamma)t}{\ell_X(\gamma)}$$

(using, for the right-hand side, the inequality $\log(1 + x) \leq x$ for $x > 0$), and likewise

$$\log \frac{\ell_X(\gamma)}{\ell_t(\gamma)} \leq \left| \log \frac{\ell_t(\gamma) + i(\alpha, \gamma)t}{\ell_t(\gamma)} \right| \leq \frac{i(\alpha, \gamma)t}{\ell_X(\gamma)}.$$

The result is thus proved. $\square$

Note that if $\ell_X(\alpha) \le L$, then it follows from the Collar Lemma that there is a constant C depending on L such that for any simple closed geodesic γ with $i(\alpha, \gamma) \neq 0$, we have $\ell_X(\gamma) \ge Ci(\alpha, \gamma)|\log \ell_X(\alpha)|$ and $\ell_{X_t}(\gamma) \ge Ci(\alpha, \gamma)|\log \ell_X(\alpha)|$. We deduce from Proposition 4.1 the following

Corollary 4.2 *If $\ell_X(\alpha) \le L$, then there is a constant C depending on L such that*

$$d_{ls}(X, X_t) \le \frac{|t|}{2C|\log \ell_X(\alpha)|}.$$

Now we need a lower bound. We use the idea of an (ϵ_0, ϵ_1)-decomposition of a hyperbolic surface (cf. Minsky [14, sec.2.4] and Choi-Rafi [6, sec.3.1]).

Consider a hyperbolic metric X on a surface of finite type (we will need only the case where X is homeomorphic to a one-holed torus or to a four-holed sphere).

Choose two numbers $\epsilon_1 < \epsilon_0$ less than a Margulis constant of the hyperbolic surface. Assume that α is a closed geodesic in the interior of X with $\ell_X(\alpha) \le \epsilon_1$. Let A be an annular (collar) neighborhood of α such that the two boundary components of A have length ϵ_0. We can choose ϵ_1 and ϵ_0 small enough such that any simple closed geodesic on X that intersects α is either the core curve of A or crosses A (this upper bound for ϵ_1, ϵ_0 can be chosen in a way that is independent on the surface X).

Let $Q = X \setminus A$. For any simple closed geodesic γ on X, its restriction to Q is homotopic (relative to ∂Q) to a shortest geodesic, which we denote by γ_Q.

Lemma 4.3 (Choi-Rafi [6, prop. 3.1]) *There is a constant C depending on ϵ_0, ϵ_1 and on the topology of X, such that*

$$|\ell_X(\gamma \cap Q) - \ell_X(\gamma_Q)| \le Ci(\gamma, \partial Q),$$

$$|\ell_X(\gamma \cap A) - [2 \log \frac{\epsilon_0}{\ell_X(\alpha)} + \ell_X(\alpha)|\mathrm{tw}_X(\gamma, \alpha)|]i(\gamma, \alpha)| \le Ci(\gamma, \alpha).$$

In the second formula, the quantity $\mathrm{tw}_X(\gamma, \alpha)$ is called the twist of γ around α. This quantity is defined in [14, sec. 3]. Its difference with the Fenchel-Nielsen twist coordinate is given by the following estimate (For the proof, see Minsky [14, Lemma 3.5]).

Lemma 4.4 (Minsky [14, lemma 3.5]) *Suppose that X_t is the twist deformation of X by a twist of magnitude $t = \tau_{X_t}(\alpha) - \tau_X(\alpha)$ along α. We normalize the twist coordinate by setting $s(X_t) = \frac{\tau_{X_t}(\alpha)}{\ell_X(\alpha)}$ and $s(X) = \frac{\tau_X(\alpha)}{\ell_X(\alpha)}$. Then*

$$|\mathrm{tw}_{X_t}(\gamma, \alpha) - \mathrm{tw}_X(\gamma, \alpha) - (s(X_t) - s(X))| \le 4.$$

Now assume that X is homeomorphic to a one-holed torus or to a four-holed sphere, with a pair of pants decomposition α. Consider the simple closed curve β we constructed in Lemma 2.1. Note that $|\mathrm{tw}_X(\beta, \alpha)|$ is less than 4, and $i(\beta, \alpha)$ is 1 or 2. The following is a direct corollary of Lemmas 4.3 and 4.4.

Lemma 4.5 *If X_n is the hyperbolic metric obtained from X by a twist deformation along α with normalized twist coordinates $n = \frac{\tau_{X_n}(\alpha) - \tau_X(\alpha)}{\ell_X(\alpha)}$, then*

$$\frac{\ell_{X_n}(\beta)}{\ell_X(\beta)} \geq \frac{2\log\frac{\epsilon_0}{\ell_X(\alpha)} + \ell_X(\alpha)(n-8) + \frac{1}{2}\ell_{X_n}(\beta_Q) - 2C}{2\log\frac{\epsilon_0}{\ell_X(\alpha)} + 2\ell_X(\alpha) + \ell_X(\beta_Q) + 2C},$$

where C is the same constant as in Lemma 4.3; hence it only depends on ϵ_0, ϵ_1 and on the topology of X.

Proof We know that $i(\beta, \alpha)$ and $i(\beta, \partial Q)$ are 1 or 2, and that $|\mathrm{tw}_X(\beta, \alpha)| \leq 4$. By Lemma 4.4 we see that $n - 8 \leq |\mathrm{tw}_X(\beta, \alpha)| \leq n + 8$. We also know that $\ell_{X_n}(\alpha) = \ell_X(\alpha)$. We can express $\ell_X(\beta)$ as $\ell_X(\beta \cap Q) + \ell_X(\beta \cap A)$, and we do the same for $\ell_{X_n}(\beta)$. We then use Lemma 4.3 to estimate separately $\ell_X(\beta \cap Q)$, $\ell_X(\beta \cap A)$, $\ell_{X_n}(\beta \cap Q) and \ell_{X_n}(\beta \cap A)$. Finally, we estimate the ratio, and we get the desired formula.

Consider now a hyperbolic metric X with a geodesic pants decomposition $\mathcal{P} = \{C_i\}$ satisfying $\sup_{C_i} \ell_X(C_i) \leq M$. Suppose there exists a geodesic $\alpha \in \mathcal{P}$ which is in the interior of X and which has length less than ϵ_1. Let $X_t = \tau_\alpha^t(X)$. Here we set $t = \ell_X(C_i)n$, for a sufficiently large number n. With these assumptions, we now apply Lemma 4.5 to give a lower bound for the length-spectrum distance $d_{ls}(X, X_t)$.

As we have done before, we choose a simple closed curve β which intersects α once or twice but does not intersect any other curves in $\mathcal{P}$. Under the upper-boundedness assumption on the pants decomposition, there is a constant K depending on $\epsilon_0, \epsilon_1, M$ such that the length $\ell_X(\beta_Q)$ is bounded as

$$1/K \leq \ell_X(\beta_Q) \leq K.$$

Moreover, we may choose the constant ϵ_0 with upper and lower bounds which only depend on M. Then an analysis of the formula in Lemma 4.5 leads to the following

Theorem 4.6 *If $\sup_{C_i} \ell_X(C_i) \leq M$ and $\ell_X(\alpha) \leq \epsilon_1$, then there is a constant D depending on ϵ_1, M such that*

$$\begin{aligned} d_{ls}(X, X_t) &\geq \frac{1}{2}\log\frac{\ell_{X_t}(\beta)}{\ell_X(\beta)} \\ &\geq \frac{1}{2}\log\frac{2|\log\ell_X(\alpha)| + |t| - D}{2|\log\ell_X(\alpha)| + D}. \end{aligned}$$

Proof We can choose ϵ_0 and ϵ_1 such that they satisfy $\epsilon_0 = 2\epsilon_1$. Then $\log(\epsilon_0)$ can be included in the constant. We remark that the constant ϵ_0 is less than the Margulis constant that is less than 1. In particular $\ell_X(\alpha)$ is less than 1 and it can be included in the constant. The terms $\ell_X(\beta_Q)$ and $\ell_{X_n}(\beta_Q)$ can be estimated with K as above. Moreover, $n\ell_X(\alpha) = t$. Transforming the formula of Lemma 4.5 in this way, we get the conclusion. □

In fact, Theorem 4.6 is a particular case of Choi-Rafi's product region formula for the length-spectrum metric. Their proof requires a more detailed and complicated analysis, see Lemma 3.4 and Theorem 3.5 of [6]. Since we only need to consider a particular curve β to give a lower bound, we don't need the general formula.

5 Structure of the length-spectrum Teichmüller space

The goal of this section is to show how bad can be the inclusion $(T_{qc}, d_{qc}) \to (T_{ls}, d_{ls})$. In Sect. 6, we will use some of the results of this section to prove connectedness of (T_{ls}, d_{ls})

Given a surface S, we define a proper subsurface S' of S to be an open subset of S such that the frontier $\partial S'$ is a union of simple closed essential curves of S. If X is a complex structure on S and if S' is a proper subsurface of S, the restriction of X to S' is the complex structure that we have on the proper subsurface X' that is homotopic to S and such that the boundary curves of X' are geodesics for the intrinsic hyperbolic metric of X. The intrinsic metric of X' is the restriction of the intrinsic metric of X.

Proposition 5.1 *Consider an exhaustion of S by a sequence of subsurfaces with boundary:*

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots \text{ and } S = \cup_{n=1}^{\infty} S_n.$$

Given two complex structures X, Y on S, let X_n and Y_n be the complex structures obtained by restriction of X and Y to S_n respectively. Then

$$d_{ls}(X, Y) = \lim_{n\to\infty} d_{ls}(X_n, Y_n). \tag{7}$$

Proof From the definition, for any $\epsilon > 0$, there exists a simple closed curve γ on Σ such that

$$d_{ls}(X, Y) < \frac{1}{2}\left|\log \frac{\ell_X(\gamma)}{\ell_Y(\gamma)}\right| + \epsilon.$$

Such a curve γ must lie in some subsurface Σ_{n_0}. As a result,

$$d_{ls}(X, Y) < d_{ls}(X_{n_0}, Y_{n_0}) + \epsilon.$$

Since $d_{ls}(X_n, Y_n)$ is increasing and ϵ is arbitrary, we conclude that

$$d_{ls}(X, Y) \le \lim_{n\to\infty} d_{ls}(X_n, Y_n).$$

The other side of (7) is obvious. □

Let R be a hyperbolic surface of infinite type with a geodesic pants decomposition $\mathcal{P} = \{C_i\}$. In the rest of this section, we always assume that the pair $(R, \mathcal{P})$ is upper-bounded and that it admits short interior curves, i.e. that there is a sequence of curves of the decomposition $\alpha_k = C_{i_k}$ contained in the interior of R with $\ell_R(\alpha_i) \to 0$. Note that for any point $X \in \mathcal{T}_{ls}(R)$, the geodesic representative of $\mathcal{P}$ satisfies the same upper-boundedness property and admits short interior curves.

The following result was proved in [13] (see also [3, Example 5.1]). The proof here is simpler.

Proposition 5.2 *Under the above assumptions, the inverse of the inclusion map I restricted to $\mathcal{T}_{qc}(R)$: $(\mathcal{T}_{qc}(R), d_{ls}) \to (\mathcal{T}_{qc}(R), d_{qc})$ is nowhere continuous. More precisely, for any $X \in \mathcal{T}_{qc}(R)$, there is a sequence $X_n \in \mathcal{T}_{qc}(R)$ with $d_{qc}(X, X_n) \to \infty$ while $d_{ls}(X, X_n) \to 0$.*

Proof Let $X \in \mathcal{T}_{qc}(R)$. Assume that $\ell_X(\alpha_n) = C_{i_n} = \epsilon_n \to 0$. Let $X_n = \tau_{\alpha_n}^{t_n}(X)$, with $t_n = \log|\log \epsilon_n| \to \infty$.

By Corollary 4.2,

$$d_{ls}(X, X_n) \leq \frac{\log|\log \epsilon_n|}{2C|\log \epsilon_n|},$$

which tends to 0 as $n \to \infty$. On the other hand, Corollary 3.3 shows that $d_{qc}(X, X_n) \to \infty$. □

Remark 5.3 There exist hyperbolic surfaces R with no pants decomposition satisfying Shiga's property, but where the space $(\mathcal{T}_{qc}(R), d_{qc})$ is topologically equivalent to the space $(\mathcal{T}_{qc}(R), d_{ls})$, see Kinjo [10]. It would be interesting to know whether the two metrics in Kinjo's examples are locally bi-Lipschitz.

Proposition 5.4 (Boundary point) *Under the above assumptions, there exists a point in $\mathcal{T}_{ls}(R)\backslash\mathcal{T}_{qc}(R)$ that can be approximated by a sequence in $\mathcal{T}_{qc}(R)$ with the length-spectrum metric.*

Proof Let $X \in \mathcal{T}_{qc}(R)$. By assumption, there exists a sequence of simple closed curves α_n such that $\ell_X(\alpha_n) = \epsilon_n \to 0$. Let $X_n = \tau_{\alpha_n}^{t_n}(X_{n-1}) = \tau_{\alpha_n}^{t_n} \circ \cdots \circ \tau_{\alpha_2}^{t_2} \circ \tau_{\alpha_1}^{t_1}(X)$, with $t_n = \log|\log \epsilon_n| \to \infty$. Define X_∞ as the surface obtained from X by a twist of magnitude t_i along α_i for every i.

By Inequality (6), $d_{qc}(X, X_n) \to \infty$ and $d_{qc}(X, X_\infty) = \infty$.

For any simple closed curve γ on X, by an argument similar to that of the proof of Proposition 4.1 and Corollary 4.2, we have

$$d_{ls}(X, X_\infty) \leq \sup_{\gamma \in \mathcal{S}} \frac{\sum_{n=1}^{\infty} i(\gamma, \alpha_n) \log|\log \epsilon_n|}{2C \sum_{n=1}^{\infty} i(\gamma, \alpha_n)|\log \epsilon_n|}.$$

To see that the right hand side is uniformly bounded (independently of γ), we can use the inequality

$$\frac{\sum_{n=1}^{\infty} x_n}{\sum_{n=1}^{\infty} y_n} \leq \sup_{n \geq 1} \left\{ \frac{x_n}{y_n} \right\},$$

that holds for positive values of x_i and y_i. With this inequality we can see that

$$\frac{\sum_{n=1}^{\infty} i(\gamma, \alpha_n) \log|\log \epsilon_n|}{\sum_{n=1}^{\infty} i(\gamma, \alpha_n)|\log \epsilon_n|} \leq \max_{n \geq 1} \left\{ \frac{\log|\log \epsilon_n|}{|\log \epsilon_n|} \right\}.$$

Note that

$$d_{ls}(X_n, X_\infty) \leq \sup_{\gamma \in \mathcal{S}} \frac{\sum_{k=n+1}^{\infty} i(\gamma, \alpha_k) \log|\log \epsilon_k|}{2C \sum_{k=n+1}^{\infty} i(\gamma, \alpha_k)|\log \epsilon_k|} \leq \max_{k \geq n+1} \left\{ \frac{\log|\log \epsilon_k|}{|\log \epsilon_k|} \right\}.$$

Since $t_n = \log|\log \epsilon_n| \to \infty$, $\frac{t_n}{e^{t_n}} \to 0$ as $n \to \infty$. We have $d_{ls}(X_n, X_\infty) \to 0$. □

Proposition 5.5 (Nowhere open) *With the above assumptions, in the metric space $(\mathcal{T}_{ls}(R), d_{ls})$, any open neighborhood of a point $X \in \mathcal{T}_{qc}(R)$ contains a point in $\mathcal{T}_{ls}(R) \backslash \mathcal{T}_{qc}(R)$.*

Proof As above, assume that $\ell_X(\alpha_n) = \epsilon_n \to 0$. Let $t_n = \log|\log \epsilon_n|$, which tends to ∞ as $n \to \infty$. We let Y_n be the surface obtained from X by a multi-twist of magnitude t_i along α_i for every $i \geq n$. As in the proof of Proposition 5.4, we have

$$d_{ls}(X, Y_n) \leq \max_{i \geq n} \{\frac{\log|\log \epsilon_i|}{|\log \epsilon_i|}\} \to 0.$$

It is obvious that $Y_n \in \mathcal{T}_{ls}(R) \backslash \mathcal{T}_{qc}(R)$. □

Proposition 5.5 is also a consequence of Theorem 5.8 below.

Theorem 5.6 *With the above assumptions, the space $(\mathcal{T}_{qc}(R), d_{ls})$ is not dense in the space $(\mathcal{T}_{ls}(R), d_{ls})$.*

Proof We will show that there exists a point in $\mathcal{T}_{ls}(R)$ which is not a limit point of $\mathcal{T}_{qc}(R)$ equipped with the length-spectrum metric.

Start with a point $X \in \mathcal{T}_{qc}(R)$. Assume that $\ell_X(\alpha_n) = \epsilon_n \to 0$. Let $Y_n = \tau_{\alpha_n}^{T_n}(X_{n-1}) = \tau_{\alpha_n}^{T_n} \circ \cdots \circ \tau_{\alpha_2}^{T_2} \circ \tau_{\alpha_1}^{T_1}(X)$, with $T_n = N|\log \epsilon_n| \to \infty$ where N is a fixed positive constant. (In this proof, we can take $N = 1$, but taking a general N is important for the proof of Proposition 5.7 below.) Define Y_∞ as the surface obtained from X by a twist of magnitude T_i along α_i for every i. It is not hard to see that $Y_\infty \in \mathcal{T}_{ls}(R)$ and, in fact,

$$d_{ls}(X, Y_\infty) \leq \frac{N}{2C}. \tag{8}$$

Suppose that there is a sequence of $X_k \in \mathcal{T}_{qc}(R)$ such that $d_{ls}(Y_\infty, X_k) \to 0$ as $k \to \infty$. Denote the difference of the twist coordinates of each X_k from X (with respect to the pants decomposition $\mathcal{P}$) by the sequence $(\tau_k(C_i))$. Since the pants decomposition is upper-bounded and since $X_k \in \mathcal{T}_{qc}(R)$, we have $\sup_i |\tau_k(C_i)| < \infty$, since otherwise, Corollary 3.3 would imply $d_{qc}(X, X_k) = \infty$.

Note that for each X_k, its difference of twist coordinates with Y_∞ is equal to

1. $T_n - \tau_k(\alpha_n)$ for each α_n;

2. $-\tau_k(C_i)$ for each $C_i \in \mathcal{P}\backslash\{\alpha_n\}$.

For each α_n, choose a simple closed curve β_n as in the construction before Theorem 4.6; then, by Theorem 4.6 we have

$$d_{ls}(X_k, Y_\infty) \geq \sup_n \frac{1}{2} \log \frac{(N+2)|\log \epsilon_n| - C - \tau_k(\alpha_n)}{2|\log \epsilon_n| + C}.$$

The right hand side of the above inequality has a positive lower bound that is independent of k (because $\tau_k(\alpha_n)$ is bounded). As a result, the sequence X_k cannot approximate Y_∞ in the length-spectrum metric. □

Denote the closure of $\mathcal{T}_{qc}(R)$ in $\mathcal{T}_{ls}(R)$ (both spaces equipped with the length-spectrum metric) by $\overline{\mathcal{T}_{qc}(R)}$. By Theorem 5.6, $\mathcal{T}_{ls}(R)\backslash\overline{\mathcal{T}_{qc}(R)}$ is not empty.

Proposition 5.7 *Under the above assumptions, for any $X \in \mathcal{T}_{qc}(R)$, there is a sequence of points $Z_k \in \mathcal{T}_{ls}(R)\backslash\overline{\mathcal{T}_{qc}(R)}$ such that $d_{ls}(X, Z_k) \to 0$.*

Proof For any $X \in \mathcal{T}_{qc}(R)$, we let Z_k be the same as Y_∞, which we constructed in the proof of Theorem 5.6, by setting $N = \frac{1}{k}$ for each k.

As we have shown before, $Z_k \in \mathcal{T}_{ls}(R)\backslash\overline{\mathcal{T}_{qc}(R)}$. Moreover, by Inequality (8), we have

$$d_{ls}(X, Z_k) \leq \frac{1}{2Ck},$$

which tends to 0 as k tends to ∞. □

Using the previous proposition, we can prove now the following result:

Theorem 5.8 *Under the above assumptions, the space $(\mathcal{T}_{qc}(R), d_{ls})$ is nowhere dense in $(\mathcal{T}_{ls}(R), d_{ls})$.*

Proof It is equivalent to prove that $\overline{\mathcal{T}_{qc}(R)}$ has no interior point. Consider an arbitrary $Y \in \overline{\mathcal{T}_{qc}(R)}$ and an arbitrary $\epsilon > 0$. If $Y \in \partial\overline{\mathcal{T}_{qc}(R)}$, then we let X be a point in $\mathcal{T}_{qc}(R)$ such that $d_{ls}(X, Y) < \frac{\epsilon}{2}$. If $Y \in \mathcal{T}_{qc}(R)$, we just set $X = Y$. By Proposition 5.7, there is a point $Z \in \mathcal{T}_{ls}(R)\backslash\overline{\mathcal{T}_{qc}(R)}$ such that $d_{ls}(X, Z) < \frac{\epsilon}{2}$. By the triangle inequality, $d_{ls}(Y, Z) < \epsilon$. □

6 Connectedness of the length-spectrum Teichmüller space

In this section, we will prove that if $(R, \mathcal{P})$ is upper-bounded but does not satisfy Shiga's property, then $(\mathcal{T}_{ls}(R), d_{ls})$ is path-connected.

Assume that $\sup_i \ell_R(C_i) \leq M$. We associate to R the Fenchel–Nielsen coordinates

$$((\ell_R(C_i), \tau_R(C_i)))_{i=1,2,\ldots},$$

and we choose the twist coordinates in such a way that $|\tau_R(C_i)| < \ell_R(C_i)$.

Consider now an arbitrary $X \in \mathcal{T}_{ls}(R)$, with Fenchel–Nielsen coordinates

$$((\ell_X(C_i), \tau_X(C_i)))_{i=1,2,\dots}.$$

We write $X = (\ell_X(C_i), \tau_X(C_i))$ for simplicity.

Lemma 6.1 *Suppose that $d_{ls}(R, X) < 2K$. We can find a hyperbolic metric $Y \in \mathcal{T}_{qc}(R)$, with Fenchel–Nielsen coordinates $Y = (\ell_Y(C_i), \tau_Y(C_i))$ such that $\ell_Y(C_i) = \ell_X(C_i)$ and $|\tau_Y(C_i) - \tau_R(C_i)| < 2e^K \ell_R(C_i) \le 2e^K M$.*

Proof Using a theorem of Bishop [5], for any pair of geodesic pair of pants P and Q there is a quasiconformal map from P to Q which satisfies the following:

1. the quasiconformal dilatation of the map is less than $1+C d_{ls}(P, Q)$, where $C > 0$ depends on an upper bound of $d_{ls}(P, Q)$ and the boundary length of P;
2. the map is affine on each of the boundary components.

Cut R into pairs of pants along $\mathcal{P}$. We use Bishop's construction [5] to deform each hyperbolic pair of pants with boundary lengths $\ell_R(\cdot)$ into a pair of pants with boundary lengths $\ell_X(\cdot)$. Then we patch together the new hyperbolic pairs of pants to get a new hyperbolic metric Y homeomorphic to R, in the following way. Suppose two pairs of pants P_1, P_2 are joined at $\alpha \in \mathcal{P}$ and deformed into Q_1, Q_2. Then we patch together Q_1 and Q_2 by identifying the common image α by an affine map. In this way, the quasiconformal map between pairs of pants provided by Bishop can be extended to a global quasiconformal map between R and Y, with dilatation bounded by

$$1 + C \sup_i \left| \log \frac{\ell_X(C_i)}{\ell_R(C_i)} \right|,$$

where C is a positive constant depending on K, M.

As a result,

$$2d_{qc}(R, Y) \le \log \left(1 + C \sup_i \left| \log \frac{\ell_X(C_i)}{\ell_R(C_i)} \right| \right) \le C \sup_i \left| \log \frac{\ell_X(C_i)}{\ell_R(C_i)} \right|.$$

Let $Y = (\ell_Y(C_i), \tau_Y(C_i))$. Then, it follows from our construction that $|\tau_Y(C_i)| \le e^{K\ell_R(C_i)}$ and $|\tau_Y(C_i) - \tau_R(C_i)| \le 2e^{K\ell_R(C_i)}$. □

Now we construct a continuous path in $\mathcal{T}_{ls}(R)$ from Y to X by varying the twist coordinates. First we need the following result.

Theorem 6.2 *The hyperbolic structure $X = (\ell_X(C_i), \tau_X(C_i))$ lies in $\mathcal{T}_{ls}(R)$ if and only if there exists a constant $N > 0$ such that for each i,*

$$\left| \log \frac{\ell_X(C_i)}{\ell_Y(C_i)} \right| < N$$

and

$$|\tau_X(C_i) - \tau_R(C_i)| < N \max\{|\log \ell_R(C_i)|, 1\}.$$

Proof By Lemma 6.1, we only need to consider the case where $\ell_X(C_i) = \ell_R(C_i)$ for each i. In this case, X is a multi-twist deformation of R along $\mathcal{P}$. If for each i,

$$|\tau_X(C_i) - \tau_R(C_i)| < N \max\{|\log \ell_R(C_i)|, 1\},$$

then we can use the proof of Theorem 5.6 to show that $d_{ls}(R, X) < \infty$.

Conversely, suppose that $d_{ls}(R, X) < \infty$ and that there is a subsequence $\{\alpha_n\}$ of $\mathcal{P}$ such that

$$|\tau_X(\alpha_n) - \tau_R(\alpha_n)| > n \max\{|\log \ell_R(\alpha_n)|, 1\}.$$

If there is a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, with length $\ell_R(\alpha_n)$ tending to zero, then, by Theorem 4.6, we have $d_{ls}(R, X) = \infty$, which contradicts our assumption. If $\{\ell_R(\alpha_n)\}$ is bounded below (and bounded above by assumption), then it also follows from Proposition 3.3 of [4] that $d_{ls}(R, X) = \infty$. As a result, there is a sufficiently large constant N such that for each i,

$$|\tau_X(C_i) - \tau_R(C_i)| < N \max\{|\log \ell_R(C_i)|, 1\}.$$

$\square$

Now we can prove the following:

Theorem 6.3 *If $(R, \mathcal{P})$ is upper-bounded but does not satisfy Shiga's property, then $(\mathcal{T}_{ls}(R), d_{ls})$ is path-connected.*

Proof Given any $X = (\ell_X(C_i), \tau_X(C_i))$ in $\mathcal{T}_{ls}(R)$ with $d_{ls}(R, X) < 2K$, we first construct the point $Y = (\ell_Y(C_i), \tau_Y(C_i))$ as we did in Lemma 6.1. Then we have $\ell_Y(C_i) = \ell_X(C_i)$ and $|\tau_Y(C_i) - \tau_R(C_i)| < 2KM$ for each C_i. By Theorem 6.2, we also have

$$|\tau_X(C_i) - \tau_Y(C_i)| < N \max\{|\log \ell_R(C_i)|, 1\},$$

if we choose N sufficiently large (depending on K, M).

Now we use the multi-twist deformation to construct a path in $\mathcal{T}_{ls}(R)$ connecting Y and X, by letting $Y_t = (\ell_X(C_i), (1-t)\tau_Y(C_i) + t\tau_X(C_i)), 0 \leq t \leq 1$. Theorem 6.2 shows that Y_t lies in $\mathcal{T}_{ls}(R)$.

Moreover, we can use the proof of Proposition 5.4 to prove that

$$d_{ls}(Y_s, Y_t) \leq \frac{(N + 2e^K M)|s - t|}{2C}.$$

If we take the Teichmüller geodesic path from R to Y and then take the path obtained by the multi-twist deformations from Y to X, then we get a continuous (in fact, Lipschitz) path in $\mathcal{T}_{ls}(R)$ connecting R and X. $\square$

Šarić proved in [15] that if $(R, \mathcal{P})$ is upper-bounded but does not satisfy Shiga's condition, then $(\mathcal{T}_{ls}(R), d_{ls})$ is contractible. This answers a question that was asked in the first version of the present paper.

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