Published in "Advances in Mathematics 263: 321–356, 2014"
which should be cited to refer to this work.

Chern–Moser operators and polynomial models in CR geometry

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We consider the fundamental invariant of a real hypersurface in $\mathbb{C}^N$ – its holomorphic symmetry group – and analyze its structure at a point of degenerate Levi form. Generalizing the Chern–Moser operator to hypersurfaces of finite multitype, we compute the Lie algebra of infinitesimal symmetries of the model and provide explicit description for each graded component. Compared with a hyperquadric, it may contain additional components consisting of nonlinear vector fields defined in terms of complex tangential variables.
As a consequence, we obtain exact results on jet determination for hypersurfaces with such models. The results generalize directly the fundamental result of Chern and Moser from quadratic models to polynomials of higher degree.

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[1] The first author was supported by the Project CZ.1.07/2.3.00/20.0003 of the Operational Programme Education for Competitiveness of the Ministry of Education, Youth and Sports of the Czech Republic.
[2] The second author was supported by Swiss National Science Foundation Grant 2100-063464.00/1.
[3] The third author was supported in part by the Science Foundation Ireland Grant 10/RFP/MTH2878.

1. Introduction

The holomorphic symmetry group of the unit sphere in $\mathbb{C}^2$ has been known since the seminal work of Poincaré [26]. For general signature (and dimension), computing the symmetry group of a real hyperquadric in $\mathbb{C}^N$ is the fundamental starting point for the study of CR invariants of Levi nondegenerate hypersurfaces [2,7,10,14,18,25,29–32]. Our aim in this paper is to analyze symmetry groups for polynomial models of higher degree.

Hypersurfaces with higher degree models are necessarily Levi degenerate. The study of such manifolds has been initiated by the work of J.J. Kohn in the context of boundary regularity of the $\bar\partial$ operator, and has lead to major advances in analysis and geometry, for example introducing multiplier ideal sheaves [20,21] and Y.-T. Siu's celebrated works on invariance of plurigenera [27].

Local CR geometry of Levi degenerate hypersurfaces presents completely new challenges, which are often closer to algebraic, rather than to differential geometry. In particular, if the Levi form changes rank near the given point, the differential geometric approach of Cartan, Chern and Tanaka is not available.

The Chern–Moser operator (as defined in [10]) turned out to be the most powerful algebraic tool for understanding local CR geometry at a Levi nondegenerate point. The Chern–Moser normal form construction essentially reduces to the analysis of the kernel and the image of this operator. It has been a long open question whether such techniques can be generalized also to the Levi degenerate case [1,3,11–13,15,17,31]. Let us remark that the case of CR manifolds of higher codimension has been also intensively studied (see e.g. [4,5,16,19]).

In complex dimension two, a complete normal form for hypersurfaces of finite type, based on a generalization of the Chern–Moser operator, was given by the first author in [22]. In the present paper we show that the Chern–Moser operator can be generalized in a natural way to a wide class of Levi degenerate manifolds in $\mathbb{C}^N$, namely the hypersurfaces of finite Catlin multitype. We analyze the kernel of this operator, which carries complete information about the infinitesimal automorphisms of the model hypersurface, and as a consequence gives sharp results on jet determination for the automorphisms of the hypersurface itself.

Let us recall that multitype is an essential CR invariant which Catlin defined and used to prove subelliptic estimates on pseudoconvex domains (his papers [8,9]). In particular, if a subelliptic estimate holds on a pseudoconvex model, then it is of finite multitype and holomorphically nondegenerate, and our results can be applied. On the other hand, we make no pseudoconvexity assumptions (multitype was extended to the general case in [23]), similarly as the work of Chern and Moser considers model hyperquadrics of all signatures.

Since finite multitype formalizes both the notion of model and invariantly defined weights, both essential for Chern–Moser theory, it provides a natural setting for its extension to the degenerate case. Note that hypersurfaces of finite Catlin multitype may contain complex varieties, providing a potential link between invariants of such varieties and CR invariants of the corresponding hypersurface.

Our first result deals with a hypersurface given by a homogeneous polynomial. Let $\mathbb{C}_\nu[z]$ denote the space of holomorphic homogeneous polynomials in $z = (z_1, \dots, z_n)$ of degree ν. Recall that the sharp condition generalizing Levi-nondegeneracy for the automorphism group being finite-dimensional is the *holomorphic nondegeneracy* introduced by N. Stanton. A real-analytic hypersurface M is by definition holomorphically nondegenerate if no point of M admits a holomorphic vector field in its neighborhood, whose both real and imaginary parts are tangent to M.

Theorem 1.1. *Let $P(z,\bar z)$ be a homogeneous polynomial without pluriharmonic terms of degree $d \geq 2$, such that the hypersurface*

$$M_P := \big\{\mathsf{Im}\, w = P(z,\bar z)\big\}, \quad (z,w) \in \mathbb{C}^n \times \mathbb{C}, \tag{1.1}$$

is holomorphically nondegenerate. Then the Lie algebra $\mathfrak{g}$ of all germs of infinitesimal automorphisms of M_P at 0 admits the weighted grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/d} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\tau=1}^{d-2} \mathfrak{g}_{\tau/d} \oplus \mathfrak{g}_{1-1/d} \oplus \mathfrak{g}_1, \tag{1.2}$$

and we have the following explicit description of the graded components:

(1) $\mathfrak{g}_{-1} = \{a\partial_w \colon\ a \in \mathbb{R}\}$,
(2) $\mathfrak{g}_{-1/d} = \{\sum_j a^j\partial_{z_j} + g(z)\partial_w \colon\ a^j \in \mathbb{C},\ g \in \mathbb{C}_{d-1}[z],\ 2i\sum(a^jP_{z_j} + \bar a^j P_{\bar z_j}) = g - \bar g\}$,
(3) $\mathfrak{g}_0 = \{\sum_j f^j(z)\partial_{z_j} + aw\partial_w \colon\ f^j \in \mathbb{C}_1[z],\ a \in \mathbb{R},\ \sum(f^jP_{z_j} + \bar f^j P_{\bar z_j}) = aP\}$,
(4) $\bigoplus_{\tau=1}^{d-2} \mathfrak{g}_{\tau/d} = \bigoplus_{\tau=1}^{d-2}\{\sum_j f^j(z)\partial_{z_j} \colon\ f^j \in \mathbb{C}_{\tau+1}[z],\ \sum(f^jP_{z_j} + \bar f^j P_{\bar z_j}) = 0\}$,
(5) $\mathfrak{g}_{1-1/d} = \{\sum_j (f^j(z) + a^j w)\partial_{z_j} + g(z)w\partial_w \colon\ a^j \in \mathbb{C},\ f^j \in \mathbb{C}_d[z],\ g \in \mathbb{C}_{d-1}[z],$ $\sum_j a^j\partial_{z_j} + g(z)\partial_w \in \mathfrak{g}_{-1/d},\ \sum(f^jP_{z_j} + \bar f^j P_{\bar z_j} + 2iP(a^jP_{z_j} + \bar a^j P_{\bar z_j})) = 2iP(g + \bar g)\}$,
(6) $\mathfrak{g}_1 = \{\sum_j f^j(z)w\partial_{z_j} + aw^2\partial_w \colon\ f^j \in \mathbb{C}_1[z],\ a \in \mathbb{R},\ \sum_j f^j(z)P_{z_j} = aP\}$.

Note that possible pluriharmonic terms in the expansion of P can always be easily eliminated by simple biholomorphic change of coordinates. More detailed description of the individual components is given in Sections 4, 5 and 6. In the Levi nondegenerate case the corresponding decomposition contains only five components, since $d = 2$ (see Examples 3.8 and 5.6 for manifolds which admit automorphisms of the form (4)).

Calculations show that the component $\mathfrak{g}_1$ is always at most 1-dimensional, in fact the polynomials f^j are uniquely determined by a from the equation in (6). Manifolds with nontrivial $\mathfrak{g}_1$ are characterized in Theorem 4.7.

As a consequence, we obtain a precise description of the derivatives needed to characterize an automorphism of a general hypersurface whose model is of the form (1.1). Let M be given near p by

$$\mathsf{Im}\, w = P(z,\bar z) + o\big(|z|^d, \mathsf{Re}\, w\big), \tag{1.3}$$

where P is a homogeneous polynomial without pluriharmonic terms of degree $d \geq 2$. We will denote by $(f_1, f_2, \ldots, f_n, g)$ the components of an automorphism of M (as in (2.13)).

Theorem 1.2. *The automorphisms of M at p are uniquely determined by*

(1) *the complex tangential derivatives $\frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}$ up to order $d-1$,*
(2) *the first and second order normal derivatives $\frac{\partial f_j}{\partial w}$ for $j = 1, \ldots, n$, $\frac{\partial g}{\partial w}$, $\frac{\partial^2 g}{\partial w^2}$.*

This jet determination result here is sharp, as shown by Example 3.8.

Next we consider the more general case of a weighted homogeneous model of finite Catlin multitype. Let $p \in M$ be a point of finite Catlin multitype $(m_1, \ldots, m_n)$ (see Section 2). As shown in Section 2, one can find coordinates $(z_1, \ldots, z_n, w)$ with weight of z_j equal to $\mu_j = \frac{1}{m_j}$, weight of w equal to 1, such that M is locally given by

$$\mathsf{Im}\, w = P(z, \bar{z}) + F(z, \bar{z}, \mathsf{Re}\, w), \tag{1.4}$$

where P is a weighted homogeneous polynomial of weighted degree 1 and F has Taylor expansion with terms of weighted degree > 1. To give the simplest example with unequal weights, consider the holomorphically nondegenerate hypersurface in $\mathbb{C}^3$ defined by

$$\mathsf{Im}\, w = |z_1|^2 + |z_2|^4, \tag{1.5}$$

where the weights are $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{4}$ (a finite explicit algorithm for computing multitype is given in [23]). Note that with the choice of weights $\mu_1 = \mu_2 = \frac{1}{2}$, the model becomes holomorphically degenerate.

For the rest of this section, assume that M is given by (1.4), and the associated model hypersurface

$$M_P := \left\{ \mathsf{Im}\, w = P(z, \bar{z}) \right\} \tag{1.6}$$

is holomorphically nondegenerate. Let E denote the set

$$E = \left\{ \sum_{j=1}^{n} k_j \mu_j ;\ k_j \in \mathbb{N} \cup \{-1\} \right\} \cap (0, 1). \tag{1.7}$$

Theorem 1.3. *The Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \mathsf{aut}(M_P, 0)$ of M_P admits the weighted grading given by*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\eta \in E} \mathfrak{g}_\eta \oplus \mathfrak{g}_1. \tag{1.8}$$

We have a completely analogous explicit description of the graded components as in Theorem 1.1. Note that the fourth component in (1.8) corresponds to parts (4) and (5) of Theorem 1.1 (in general it cannot be split in this way, as Example 6.4 shows).

As a consequence, we obtain the following theorem that gives a sharp characterization of the automorphisms of M.

Theorem 1.4. *The automorphisms of M at p are uniquely determined by their jets of weighted order* 2.

A precise statement giving exactly which derivatives are needed to determine an automorphism is given in Theorem 7.1. For the Levi nondegenerate case, when $\mu_j = \frac{1}{2}$, $j = 1, \dots, n$, we recover exactly the sharp statement of Chern and Moser contained in [10] (Corollary 7.2).

Let us remark that most of the results of Sections 4, 5, 6 apply in a more general case, for an arbitrary hypersurface with a weighted homogeneous model which is holomorphically nondegenerate (the weights need not coincide with the multitype weights). However, the fundamental property of the Chern–Moser operator (2.16), providing the leading linear part of the transformation law, fails in this case. Hence the infinitesimal automorphisms of the model hypersurface no longer control automorphisms of the hypersurface itself, and there exist examples for which the conclusion of Theorem 1.3 fails.

The paper is organized as follows. In Section 2, we recall the notion of Catlin multitype of a smooth hypersurface $M \subset \mathbb{C}^{n+1}$. We also study the generalized Chern–Moser operator, and show how to reduce the weighted jet determination problem for the automorphism group of M, to the study of the set of real-analytic infinitesimal CR automorphisms of M_P at p (Proposition 2.15). In Section 3, we introduce the notion of rigid vector fields and prove results regarding the determination problem for such infinitesimal automorphisms (Theorem 3.3 and Lemma 3.4). In Sections 4, 5, and 6, we study the infinitesimal automorphisms which are not rigid (Theorem 4.7, Theorem 5.5, and Theorem 6.2). In Section 7, we complete the proofs of the main results.

2. The Catlin multitype and generalized Chern–Moser operators

In this section we recall the notion of Catlin multitype and consider a generalization of the Chern–Moser operator on Levi degenerate hypersurfaces of finite multitype.

Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth hypersurface, and $p \in M$ be a point of *finite type* m in the sense of Kohn and Bloom–Graham [6]. We will consider local holomorphic coordinates (z, w) vanishing at p, where $z = (z_1, z_2, \dots, z_n)$ and $z_j = x_j + iy_j$, $w = u + iv$. The hyperplane $\{v = 0\}$ is assumed to be tangent to M at p, hence M is described near p as the graph of a uniquely determined real valued function

$$v = \psi(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n, u), \qquad d\psi(p) \neq 0. \tag{2.1}$$

Using a result of [6], we may assume that

$$\psi(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n, u) = P_m(z, \bar{z}) + o\big(|u| + |z|^m\big), \tag{2.2}$$

where $P_m(z, \bar{z})$ is a nonzero homogeneous polynomial of degree m with no pluriharmonic terms.

The definition of multitype involves *rational* weights associated to the variables $w, z_1, \dots, z_n$ in the following way.

The variables w, u and v are given weight one, reflecting our choice of variables given by (2.1). The complex tangential variables $(z_1, \dots, z_n)$ are treated according to the following definitions (for more details, see [23]).

Definition 2.1. A weight is an n-tuple of nonnegative rational numbers $\Lambda = (\lambda_1, \dots, \lambda_n)$, where $0 \le \lambda_j \le \frac{1}{2}$, and $\lambda_j \ge \lambda_{j+1}$.

Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a weight, and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ be multiindices. The weighted degree κ of a monomial

$$q(z, \bar{z}, u) = c_{\alpha\beta l} z^\alpha \bar{z}^\beta u^l, \quad l \in \mathbb{N},$$

is defined as

$$\kappa := l + \sum_{i=1}^{n} (\alpha_i + \beta_i)\lambda_i.$$

A polynomial $Q(z, \bar{z}, u)$ is Λ-homogeneous of weighted degree κ if it is a sum of monomials of weighted degree κ.

For a weight Λ, the weighted length of a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ is defined by

$$|\alpha|_\Lambda := \lambda_1\alpha_1 + \dots + \lambda_n\alpha_n.$$

Similarly, if $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_n)$ are two multiindices, the weighted length of the pair $(\alpha, \hat{\alpha})$ is

$$\big|(\alpha, \hat{\alpha})\big|_\Lambda := \lambda_1(\alpha_1 + \hat{\alpha}_1) + \dots + \lambda_n(\alpha_n + \hat{\alpha}_n).$$

The weighted order κ of a differential operator

$$D = \frac{\partial^{|\alpha| + |\hat{\alpha}| + l}}{\partial z^\alpha \partial \bar{z}^{\hat{\alpha}} \partial u^l}$$

is equal to

$$\kappa := l + \big|(\alpha, \hat{\alpha})\big|_\Lambda.$$

Definition 2.2. A weight Λ will be called distinguished for M if there exist local holomorphic coordinates (z, w) in which the defining equation of M takes form

$$v = P(z, \bar{z}) + o_\Lambda(1), \tag{2.3}$$

where $P(z, \bar{z})$ is a nonzero Λ – homogeneous polynomial of weighted degree 1 without pluriharmonic terms, and $o_\Lambda(1)$ denotes a smooth function whose derivatives of weighted order less than or equal to one vanish.

The fact that distinguished weights do exist follows from (2.2). For these coordinates (z, w), we have

$$\Lambda = \left(\frac{1}{m}, \dots, \frac{1}{m}\right).$$

In the following we shall consider the lexicographic order on the set of n-tuples defined as follows: $(\alpha_1, \dots, \alpha_n) < (\beta_1, \dots, \beta_n)$ whenever for some $1 \leq k \leq n$, $\alpha_j = \beta_j$ for $j < k$ but $\alpha_k < \beta_k$.

We recall the following definition, due to D. Catlin [8].

Definition 2.3. Let $\Lambda_M = (\mu_1, \dots, \mu_n)$ be the infimum of all possible distinguished weights Λ with respect to the lexicographic order. The multitype of M at p is defined to be the n-tuple

$$(m_1, m_2, \dots, m_n),$$

where

$$m_j = \begin{cases} \frac{1}{\mu_j} & \text{if } \mu_j \neq 0, \\ \infty & \text{if } \mu_j = 0. \end{cases}$$

Furthermore, if none of the m_j is infinity, we say that M is of *finite multitype at p*.

Since the definition of multitype includes all distinguished weights, the infimum is a *biholomorphic invariant*.

Definition 2.4. Coordinates corresponding to the multitype weight Λ_M, in which the local description of M has form (2.3), with P being Λ_M-homogeneous, are called *multitype coordinates*.

Notice that if M is of finite multitype at p, the infimum is attained, which implies that multitype coordinates do exist [8,23].

If $M \subset \mathbb{C}^2$, then M is of finite type at p if and only if M is of finite multitype at p. In this case, the type of M at p is equal to the multitype of M at p.

From now on, we assume that $p \in M$ given by (2.3) is a point of *finite multitype*

$$(m_1, m_2, \ldots, m_n),$$

where $m_j = \frac{1}{\mu_j}$, that is,

$$v = \psi(z, \bar{z}, u) = P(z, \bar{z}) + o_{\Lambda_M}(1). \tag{2.4}$$

We recall the following definition given in [23].

Definition 2.5. Let M be given by (2.4). We define a model hypersurface M_P associated to M at p by

$$M_P = \left\{(z, w) \in \mathbb{C}^{n+1} \;\middle|\; v = P(z, \bar{z})\right\}. \tag{2.5}$$

Note that multitype coordinates (z, w) are not unique. Nevertheless it is shown in [23] that all models are biholomorphically equivalent (in fact by a polynomial transformation).

The following proposition gives a useful partial normalization of P (cf. [23]).

Proposition 2.6. *Let Λ_M be as in Definition 2.3 and P as in (2.3). Then after a polynomial change of coordinates preserving the weights, we can assume that for every $1 \le k \le n$, the following hold:*

(1) *the derivatives of P satisfy*

$$P_{z_k}|_{z_{k+1}=\cdots=z_n=0} \neq 0; \tag{2.6}$$

(2) *the expansion of P contains a nontrivial monomial $cz^{\gamma^k}\bar{z}^{\widehat{\gamma}^k}$ with $\gamma_k^k \ge 1$, $\gamma_j^k = \widehat{\gamma}_j^k = 0$ for $j > k$, and no other monomial of the form*

$$e z_1^{\gamma_1^k} \ldots z_{k-1}^{\gamma_{k-1}^k} z_k^{\gamma_k^k - 1} z_{k+1}^{\alpha_{k+1}} \ldots z_n^{\alpha_n} \bar{z}^{\widehat{\gamma}^k}. \tag{2.7}$$

Proof. For reader's convenience, we first prove the statement (1) for $k = 1$. Since the type is m, we must have $\mu_1 = 1/m$. Choose $k' \ge 1$ to be the largest l such that $\mu_1 = \cdots = \mu_l$. We claim that

$$P|_{z_{k'+1}=\cdots=z_n=0} \neq 0. \tag{2.8}$$

Indeed, since the type is m, there exists a nontrivial monomial of degree m in the right-hand side of (2.3). The latter cannot be in $o_\Lambda(1)$ because all $\mu_j \le 1/m$. Hence that monomial appears in the expansion of P. Furthermore, by our choice of k', this monomial cannot contain z_j with $j > k'$, because otherwise, its weighted degree would be less than 1. This proves the claim (2.8). Then, a generic linear change of the variables $z_1, \ldots, z_{k'}$ preserves the weight Λ and achieves (2.6) for $k = 1$.

Now we prove the statement (1) for general $k = k_0$. As before, choose $k' \geq k_0$ to be the largest $l \geq k_0$ such that $\mu_{k_0} = \cdots = \mu_l$. With that k' we claim that

$$P_{z_l}|_{z_{k'+1}=\cdots=z_n=0} \neq 0 \quad \text{for some } k_0 \leq l \leq k'. \tag{2.9}$$

Indeed, otherwise $P|_{z_{k'+1}=\cdots=z_n=0}$ depends only on $z_1, \ldots, z_{k_0-1}$. Then we can decrease the weight $\mu_{k'}$ and possibly increase the weights μ_j for $j > k'$, so that the new weight Λ' becomes smaller in the lexicographic order. This contradicts the choice of Λ in Definition 2.6 and hence proves the claim (2.9). Then, again a generic linear change of the variables $z_{k_0}, \ldots, z_{k'}$ preserves the weight Λ and achieves (2.6) for $k = k_0$.

To show (2), in view of (2.6) there exists nontrivial monomial $cz^{\gamma^k}\bar{z}^{\widehat{\gamma}^k}$ with $\gamma_k^k \geq 1$, $\gamma_j^k = \widehat{\gamma}_j^k = 0$ for $j > k$, in the expansion of P. Among all such monomials, we consider ones with lexicographically maximal $\widehat{\gamma}^k$, and then among those the one (uniquely determined) with lexicographically maximal γ^k, which we denote by $cz^{\gamma^k}\bar{z}^{\widehat{\gamma}^k}$. Consider a polynomial weighted homogeneous transformation

$$z_j' = z_j \quad \text{for } j \neq k, \qquad z_k' = z_k + \sum C_\alpha z_{k+1}^{\alpha_{k+1}} \ldots z_n^{\alpha_n}. \tag{2.10}$$

Then expanding $c(z')^{\gamma^k}(\bar{z}')^{\widehat{\gamma}^k}$ we obtain monomials of the form (2.7) with $e = c\gamma_k^k C_\alpha$. It now suffices to show that no other terms can contribute to the same monomials. Indeed, all other terms $bz^\beta\bar{z}^{\widehat{\beta}}$ in the expansion of $c(z')^{\gamma^k}(\bar{z}')^{\widehat{\gamma}^k}$ will either have smaller $\widehat{\beta}$ or the same $\widehat{\beta}$ but smaller β. Moreover, by our choice of the monomial $cz^{\gamma^k}\bar{z}^{\widehat{\gamma}^k}$, all such monomials with $\gamma_k^k \geq 1$, $\gamma_j^k = \widehat{\gamma}_j^k = 0$ for $j > k$, will also have in the expansion of $P(z', \bar{z}')$ terms $bz^\beta\bar{z}^{\widehat{\beta}}$ with either smaller $\widehat{\beta}$ or the same $\widehat{\beta}$ but smaller β. Finally, if we expand another monomial $a(z')^\delta(\bar{z}')^{\widehat{\delta}}$ in $P(z', \bar{z}')$, then either (1) $\widehat{\delta}_j \geq 1$ for some $j > k$, in which case we cannot get a term (2.7), or (2) $\widehat{\delta}_j = 0$ for all $j > k$ and $\delta_j \geq 1$ for some $j > k$. In the second case, in order to have a term (2.7) in the expansion, we must have $\delta_j = \gamma_j^k$ for $j < k$ and $\delta_k \geq \gamma_k^k$, and $\widehat{\delta} = \widehat{\gamma}^k$. But then $a(z')^\delta(\bar{z}')^{\widehat{\delta}}$ would be of weight > 1 contradicting the choice of P. $\square$

We now define the notion of *weighted jets*.

Definition 2.7. Let $(z, w) \in \mathbb{C}^{n+1}$ be multitype coordinates and let $F : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function given in these coordinates. The weighted jet of F at p of weighted order κ is given by the following set

$$\left\{ \frac{\partial^{|\alpha|+|\beta|}F}{\partial z^\alpha \partial w^\beta}(p), |\alpha|_{\Lambda_M} + |\beta| \leq \kappa \right\}. \tag{2.11}$$

Definition 2.8. Let $F_1, F_2 : \mathbb{C}^{n+1} \to \mathbb{C}$ be two holomorphic functions given in some multitype coordinates. We say that F_1 and F_2 are weighted equivalent modulo κ at p if

$$\frac{\partial^{|\alpha|+|\beta|}F_1}{\partial z^\alpha \partial w^\beta}(p) = \frac{\partial^{|\alpha|+|\beta|}F_2}{\partial z^\alpha \partial w^\beta}(p), \quad |\alpha|_{\Lambda_M} + |\beta| \leq \kappa.$$

We have the following lemma.

Lemma 2.9. *The notion of weighted equivalence modulo κ at p is independent of the choice of multitype coordinates.*

Proof. This is a direct application of Theorem 4.1 of [23] combined with the Leibniz rule. Indeed, Theorem 4.1 says that any biholomorphic transformation taking multitype coordinates (z, w) into multitype coordinates (z', w') has to be of the following form

$$\begin{aligned} {z_j}' &= z_j + \sum_{|\alpha|_{\Lambda_M}=\mu_j} C_\alpha z^\alpha + o_{\Lambda_M}(\mu_j), \\ w' &= w + c \sum_{|\alpha|_{\Lambda_M}=1} D_\alpha z^\alpha + o_{\Lambda_M}(1), \end{aligned} \tag{2.12}$$

for $c \in \mathbb{R} \setminus \{0\}$, where $o_{\Lambda_M}(\mu_j)$ denotes terms in the Taylor expansion of weighted degree greater than μ_j. $\square$

We will now introduce the notion of generalized Chern–Moser operator.

Denote by $\mathsf{Aut}(M, p)$, the stability group of M, that is, those germs at p of biholomorphisms mapping M into itself and fixing p, and by $\mathsf{aut}(M, p)$, the set of germs of holomorphic vector fields in $\mathbb{C}^{n+1}$ whose real part is tangent to M.

If M admits a holomorphic vector field X in $\mathsf{aut}(M, p)$ such that $\mathsf{Im}\, X$ is also tangent (i.e. X is complex tangent), then $\mathsf{aut}(M, p)$ is of infinite dimension [28]. We recall the following definition.

Definition 2.10. A real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ is *holomorphically nondegenerate at $p \in M$* if there is no germ at p of a holomorphic vector field X tangent to M.

Denote by Θ the set of all rational numbers of the form

$$q = \sum_{j=1}^{n} k_j \mu_j + k_{n+1}$$

for some nonnegative integers $k_1, \ldots, k_{n+1}$.

We decompose the formal Taylor expansion of ψ, denoted by Ψ, into Λ_M-homogeneous polynomials of weighted degree ν, called Ψ_ν, that is,

$$\Psi = \sum_{\nu \in \Theta} \Psi_\nu.$$

Notice, using (2.4), that $\Psi_\nu = 0$, for $\nu < 1$, and $\Psi_1 = P$.

Let $h = ({z_j}', w') \in \mathsf{Aut}(M, p)$. We know by [23] that h is of the form (2.12), that we rewrite as

$$
\begin{aligned}
{z_j}' &= z_j + f^j(z,w), \\
w' &= w + g(z,w),
\end{aligned}
\tag{2.13}
$$

which takes the multitype coordinates (z,w) into the multitype coordinates (z',w').

Putting $f = (f^j, \ldots, f^n)$, we consider the mapping given by

$$
T = (f, g),
$$

and, again, decompose each power series f_j and g into Λ_M-homogeneous polynomials of weighted degree μ, called ${f^j}_\mu$ and g_μ,

$$
f_j = \sum_{\mu \in \Theta} {f^j}_\mu, \qquad g = \sum_{\mu \in \Theta} g_\mu.
$$

Let $v' = \psi(z', \bar{z}', u')$ be the defining equation of M in the coordinates (z', w'), of the form given by (2.4),

$$
\psi\big(z', \bar{z}', u'\big) = P\big(z', \bar{z}'\big) + o_{\Lambda_M}(1). \tag{2.14}
$$

Since $h \in \mathsf{Aut}(M,p)$, substituting (2.13) into $v' = \psi(z', \bar{z}', u')$ we obtain the transformation formula

$$
\begin{aligned}
&\psi\big(z + f\big(z, u + i\psi(z,\bar{z},u)\big), \overline{z + f\big(z, u + i\psi(z,\bar{z},u)\big)}, u + \mathsf{Re}\, g\big(z, u + i\psi(z,\bar{z},u)\big)\big) \\
&\quad = \psi(z,\bar{z},u) + \mathsf{Im}\, g\big(z, u + i\psi(z,\bar{z},u)\big).
\end{aligned}
\tag{2.15}
$$

Using (2.13), we only have to consider terms of weight $\mu \geq 1$ in (2.15). We get

$$
2\,\mathsf{Re} \sum_{j=1}^{n} P_{z_j}(z,\bar{z}) {f^j}_{\mu - 1 + \mu_j}\big(z, u + iP(z,\bar{z})\big) = \mathsf{Im}\, g_\mu\big(z, u + iP(z,\bar{z})\big) + \cdots, \tag{2.16}
$$

where dots denote terms depending on ${f^j}_{\nu - 1 + \mu_j}, g_\nu, \psi_\nu$, for $\nu < \mu$ (there are no dots if $\mu = 1$).

We are now in a position to introduce the analog of the Chern–Moser operator [10] for points of finite multitype.

Definition 2.11. The generalized Chern–Moser operator, denoted by L, is defined by

$$
L(f,g) = \mathsf{Re} \left\{ ig\big(z, u + iP(z,\bar{z})\big) + 2 \sum_{j=1}^{n} P_{z_j}(z,\bar{z}) f^j\big(z, u + iP(z,\bar{z})\big) \right\}. \tag{2.17}
$$

The following lemma shows the relation between the kernel of L and the infinitesimal CR automorphisms of the model hypersurface given by (2.5). (See [10] for the same result in the Levi nondegenerate case.)

Lemma 2.12. *Let L be given by (2.17) and let (f, g) be given by (2.13). Then (f, g) lies in the kernel of L if and only if the vector field*

$$Y = \sum_{j=1}^{n} f_j(z, w)\partial_{z_j} + g(z, w)\partial_w$$

lies in $\mathsf{aut}(M_P, p)$, where M_P is given by (2.5).

Proof. Applying Y to $v - P$ we obtain

$$\begin{aligned} \operatorname{Re} Y(v - P)_{|M_P} &= -\frac{1}{2} \operatorname{Re}\Big\{ ig\big(z, u + iP(z, \bar{z})\big) + 2\sum_{j=1}^{n} P_{z_j}(z, \bar{z}) f_j\big(z, u + iP(z, \bar{z})\big) \Big\} \\ &= -\frac{1}{2} L(f, g). \qquad \square \end{aligned} \tag{2.18}$$

We have the following proposition which shows how to reduce the weighted jet determination problem from $\mathsf{Aut}(M, p)$ to $\mathsf{aut}(M_P, p)$.

Proposition 2.13. *Let $h = (z + f, w + g) \in \mathsf{Aut}(M, p)$ be given by (2.13). Let*

$$(f, g) = \sum (f, g)_\mu,$$

where

$$(f, g)_\mu := \big(f^1{}_{\mu-1+\mu_1}, \dots, f^n{}_{\mu-1+\mu_n}, g_\mu\big).$$

Let μ_0 be minimal such that $(f, g)_{\mu_0} \neq 0$. Then the (nontrivial vector) field

$$Y = \sum_{j=1}^{n} f^j{}_{\mu_0-1+\mu_j} \partial_{z_j} + g_{\mu_0} \partial_w \tag{2.19}$$

lies in $\mathsf{aut}(M_P, p)$, where M_P is given by (2.5).

Proof. Using (2.16) and the definition of μ_0, we obtain that

$$L\big((f, g)_{\mu_0}\big) = 0.$$

Therefore, using Lemma 2.12, we obtain that

$$Y = \sum_{j=1}^{n} f^j_{\mu_0-1+\mu_j} \partial_{z_j} + g_{\mu_0} \partial_w$$

belongs to $\mathsf{aut}(M_P, p)$. This achieves the proof of the theorem. $\square$

Definition 2.14. We say that the vector field

$$Y = \sum_{j=1}^{n} F_j(z,w)\partial_{z_j} + G(z,w)\partial_w$$

has homogeneous weight μ (≥ -1) if F_j is a weighted homogeneous polynomial of weighted degree $\mu + \mu_j$, and G is a homogeneous polynomial of weighted degree $\mu + 1$.

The weights introduce a natural grading on $\mathsf{aut}(M_P, p)$ in the following sense. Writing $\mathsf{aut}(M_P, p)$ as

$$\mathsf{aut}(M_P, p) = \bigoplus_{\mu+1 \in \Theta} \mathfrak{g}_\mu,$$

where $\mathfrak{g}_\mu$ consists of weighted homogeneous vector fields of weight μ, we observe that each weighted homogeneous component $X_\mu \in \mathfrak{g}_\mu$ of $X \in \mathsf{aut}(M_P, p)$ lies also in $\mathsf{aut}(M_P, p)$. The reason is that $v - P$ is weighted homogeneous.

Gathering all the previous results, we obtain the following proposition.

Proposition 2.15. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth hypersurface of finite multitype $(m_1, \ldots, m_n)$ given by (2.4). Let M_P be the model hypersurface given by (2.5). Assume that there exists μ_0 such that*

$$\mathsf{aut}(M_P, p) = \bigoplus_{-1 \leq \mu < \mu_0 - 1} \mathfrak{g}_\mu. \tag{2.20}$$

Then any $h = (z + f, w + g) \in \mathsf{Aut}(M, p)$ given by (2.13) such that $(f, g)_\mu = 0$ for all $\mu < \mu_0$ is the identity map.

In the light of Proposition 2.15, we see that in order to study the weighted jet determination problem for $\mathsf{Aut}(M, p)$, it is enough to study the weighted jet determination problem for $\mathsf{aut}(M_P, p)$.

3. Rigid vector fields

In this section, we describe an important class of vector fields $X \in \mathsf{aut}(M_P, p)$, which play a crucial role in the study of $\mathsf{aut}(M_P, p)$. As before, let $M \subset \mathbb{C}^{n+1}$ be given by (2.4).

Definition 3.1. Let X be a holomorphic vector field of the form

$$X = \sum_{j=1}^{n} f^j(z,w)\partial_{z_j} + g(z,w)\partial_w. \tag{3.1}$$

We say that X is rigid if $f^1, \ldots, f^n, g$ are all independent of the variable w.

Note that the rigid vector field W, of homogeneous weight -1, given by

$$W = \partial_w \tag{3.2}$$

lies in $\mathsf{aut}(M_P, p)$. We will denote by E the weighted homogeneous vector field of weight 0 defined by

$$E = \sum_{j=1}^{n} \mu_j z_j \partial_{z_j} + w\partial_w. \tag{3.3}$$

E is the weighted Euler field. Note that by the definition of μ_j, E is a nonrigid vector field lying in $\mathsf{aut}(M_P, p)$.

Lemma 3.2. *Let $X \in \mathsf{aut}(M_P, p)$ be a rigid holomorphic vector field. Suppose that X is homogeneous of weight*

$$\nu > -\mu_n = -\min \mu_j.$$

Then $g = 0$.

Proof. Since $\nu > -\min \mu_j$, every $f^j = f^j(z)$ in (3.1) is nonconstant. Hence, writing $(\mathsf{Re}\, X)(\mathsf{Im}\, w - P(z, \bar{z})) = 0$ we see that every term involving f^j is not pluriharmonic. On the other hand, all terms involving $g = g(z)$ are pluriharmonic, and hence cannot cancel the former ones. Since $g(z)$ is also nonconstant, we immediately obtain the conclusion. □

We have the following theorem.

Theorem 3.3. *Let M_P be holomorphically nondegenerate, and let $X \in \mathsf{aut}(M_P, p)$ be a nonzero rigid vector field. Then all weighted homogeneous components of X have weight strictly less than one.*

Proof. Write

$$X = \sum_{j=1}^{n} f^j(z)\partial_{z_j} + g(z)\partial_w.$$

By assumption, we have

$$\mathsf{Re}\left(\sum_{j=1}^{n} f^j(z)\partial_{z_j} + g(z)\partial_w\right)\big(\mathsf{Im}\, w - P(z, \bar{z})\big) = 0. \tag{3.4}$$

Identifying weighted homogeneous components, we may assume, without loss of generality, that X is weighted homogeneous of weight ν. Assume $\nu \geq 1$ by contradiction. Since

$P(z,\bar{z})$ is weighted homogeneous of weight 1 and has no pluriharmonic terms, its terms have weight < 1 in z. Then extracting terms in (3.4) of weight ≥ 1 in z we obtain

$$\left(\sum_{j=1}^{n} f^j(z)\partial_{z_j} + g(z)\partial_w\right)\big(\mathsf{Im}\, w - P(z,\bar{z})\big) = 0. \tag{3.5}$$

Since M_P is holomorphically nondegenerate, it follows that $X = 0$ contradicting the assumption. $\square$

We have the following lemma.

Lemma 3.4. *Let $X \in \mathsf{aut}(M_P, p)$ be a weighted homogeneous vector field, and let $W \in \mathsf{aut}(M_P, p)$ be given by (3.2). There exists an integer $l \geq 1$, and a rigid vector field $Y \in \mathsf{aut}(M_P, p)$ such that $[[\dots[[X; W]; W]; \dots]; W] = Y$, where the string of brackets is of length l.*

Proof. Observe that the effect of taking the bracket of X with W is simply differentiation of the coefficient with respect to w. Also note that

$$\big(\mathsf{Re}[X;W]\big)\big(v - P(z,\bar{z})\big) = [\mathsf{Re}\, X, \mathsf{Re}\, W]\big(v - P(z,\bar{z})\big). \qquad \square$$

Definition 3.5. We say that $X \in \mathsf{aut}(M_P, p)$ is an l-integration of a rigid vector field $Y \in \mathsf{aut}(M_P, p)$ if the string of brackets described in the above lemma is of length l.

Remark 3.6. By the above lemma, the general case will be reduced to the rigid case by taking sufficiently many commutators with the vector field W. The problem reduces then to

(i) describing rigid vector fields;
(ii) analyzing to what extent rigid fields can be “integrated”.

As a consequence of Theorem 3.3, we can divide homogeneous rigid vector fields into three types, and introduce the following terminology.

Definition 3.7. Let $X \in \mathsf{aut}(M_P, p)$ be a rigid weighted homogeneous vector field. X is called

(1) a *shift* if the weighted degree of X is less than zero;
(2) a *rotation* if the weighted degree of X is equal to zero;
(3) a *generalized rotation* if the weighted degree of X is bigger than zero and less than one.

Note that in the Levi nondegenerate case (where $P(z,\bar z) = \langle z, z\rangle$ is a quadratic form), generalized rotations do not occur. In fact, using Lemma 3.2 and writing $(\operatorname{Re} X)\langle z, z\rangle = 0$, we conclude $X = 0$.

The same fact holds in complex dimension two. Indeed, writing $P(z,\bar z) = \sum_{k\ge k_0} c_k z^k \bar z^{m-k}$ with $c_{k_0} \neq 0$, and $X = z^l \partial_z$, the expansion of $(\operatorname{Re} X)P(z,\bar z)$ has a nonzero term with $z^{k_0}\bar z^{m-k_0+l-1}$, and hence cannot be zero.

On the other hand, in complex dimension ≥ 3, such vector fields do occur:

Example 3.8. Take

$$P(z,\bar z) := \operatorname{Re} z_1 \bar z_2^l, \qquad X := i z_2^l \partial_{z_1}, \quad l > 1,$$

where the weights are $\mu_1 = \mu_2 = \frac{1}{l+1}$. In [24], we showed that the Lie algebra of infinitesimal automorphisms has six components,

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{l+1}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1-\frac{2}{l+1}} \oplus \mathfrak{g}_{1-\frac{1}{l+1}} \oplus \mathfrak{g}_1. \tag{3.6}$$

In particular, $\mathfrak{g}_1$ is generated by the 2-integration of W

$$\left(z_1\partial_{z_1} + \frac{1}{l} z_2 \partial_{z_2}\right) w + \frac{1}{2} w^2 \partial_w.$$

Proposition 3.9. *For Λ_M and P as before, there exists a polynomial weighted homogeneous change of coordinates such that the following hold:*

(1) *any rotation is linear;*
(2) *any non-transversal shift is of the form*

$$X = \sum_{j=1}^{j_0} f^j(z)\partial_{z_j} + g(z)\partial_w \tag{3.7}$$

with $f^{j_0} = const \neq 0$.

Proof. We perform a change of coordinates as in Proposition 2.6. To show (1), observe that in view of Lemma 3.2, any rotation is of the form

$$X = \sum_{s,j} \lambda_{sj} z_s \partial_{z_j} + \sum_{\alpha,j} c_{\alpha,j} z^\alpha \partial_{z_j}, \tag{3.8}$$

where the weight of $z^\alpha \partial_{z_j}$ is zero and $\alpha_l = 0$ whenever $l \geq j$. It suffices to show that all $c_{\alpha,j} = 0$. By contradiction, assume that there exist α and j with $c_{\alpha,j} \neq 0$. Among those choose k to be the minimal possible j. Since X is a rotation, it is an automorphism of M_P, i.e. satisfies

$$(X + \bar X)P = 0. \tag{3.9}$$

Consider the nontrivial monomial $cz^{\gamma^k}\bar{z}^{\widehat{\gamma}^k}$ in the expansion of P given by Proposition 2.6. Expanding (3.9) we obtain in the left-hand side a nontrivial monomial of the form (2.7) which contradicts (3.9), showing (1).

To show (2), assume that X is a non-transversal shift of the form (3.7) with no constant $f^j(z)$. Write X in the form

$$X = \sum_{\alpha,j} c_{\alpha,j} z^{\alpha} \partial_{z_j} + g\partial_w. \tag{3.10}$$

Then choosing a nonzero monomial $c_{\alpha,j}z^{\alpha}\partial_{z_j}$ in the expansion of X with minimal possible j and arguing as before we obtain a contradiction. $\square$

4. Integrating transversal shifts

In this section, we first consider a homogeneous rigid vector field $X \in \mathsf{aut}(M_P)$, which is a shift. We call it *transversal*, if it is of weight -1, and hence

$$X = a\partial_w, \quad a \in \mathbb{R}.$$

We will show that X can be integrated at most two times, provided M_P is holomorphically nondegenerate.

We start with the following definition.

Definition 4.1. We say that the weighted homogeneous holomorphic vector field $\sum_{j=1}^n f^j \partial_{z_j}$ is a *real* reproducing field if

$$2\,\mathsf{Re} \sum_{j=1}^n f^j(z) P_{z_j}(z,\bar{z}) = P(z,\bar{z}). \tag{4.1}$$

We say that the weighted homogeneous field $\sum_{j=1}^n f^j \partial_{z_j}$ is a *complex* reproducing field if

$$2 \sum_{j=1}^n f^j(z) P_{z_j}(z,\bar{z}) = P(z,\bar{z}). \tag{4.2}$$

The following lemma is straightforward.

Lemma 4.2. *The real reproducing fields are given by $R + X$, where*

$$R = \sum_{j=1}^n \mu_j z_j \partial_{z_j} \tag{4.3}$$

and X is any rotation field.

We need the following lemma.

Lemma 4.3. *Let M_P given by (2.5) be holomorphically nondegenerate, and let X, Y, U be rigid holomorphic vector fields satisfying*

$$(\mathsf{Re}\,X)P = 0, \qquad (\mathsf{Re}\,Y)P + \big(\mathsf{Im}(X+U)\big)P^2 = 0, \qquad [U,X] = 0. \tag{4.4}$$

Assume that X is of weight ≥ 0. Then X and Y commute.

Proof. Since $\mathsf{Re}\,X$ commutes with $\mathsf{Im}\,X$, the first and the third equations in (4.4) imply

$$(\mathsf{Re}\,X)\big(\mathsf{Im}(X+U)\big)P^2 = 0. \tag{4.5}$$

Hence applying $\mathsf{Re}\,X$ to the second equation in (4.4), we obtain

$$(\mathsf{Re}\,X)(\mathsf{Re}\,Y)P = 0. \tag{4.6}$$

On the other hand, the first equation in (4.4) implies

$$(\mathsf{Re}\,Y)(\mathsf{Re}\,X)P = 0. \tag{4.7}$$

From (4.6) and (4.7) we obtain

$$\big(\mathsf{Re}[X,Y]\big)P = 0, \tag{4.8}$$

and hence $[X,Y]$ is a symmetry.

Since X has weight bigger or equal to 0, it follows from (4.4) that Y has weight bigger or equal to 1. Hence $[X,Y]$ is a symmetry of weight bigger or equal to 1. Then Theorem 3.3 implies $[X,Y]=0$ as desired. □

Let X be a field of the form

$$X = \sum_{j=1}^{n} \lambda_j z_j \partial_{z_j}, \quad \lambda_j \in \mathbb{C}. \tag{4.9}$$

It follows from the diagonal form of X that every monomial z^α is an eigenvector of X with the eigenvalue

$$w_X(z^\alpha) := \sum \lambda_j \alpha_j. \tag{4.10}$$

We have the following result.

Lemma 4.4. *Let M_P given by (2.5) be holomorphically nondegenerate, and let R be given by (4.3), $W = X + Z$ be a linear vector field in Jordan normal form with X of the form (4.9), and Z the nilpotent part, and $\lambda \in \mathbb{R}$. Suppose that X, Y and $U := Z + \lambda R$ satisfy (4.4). Then*

$$w_{X+\lambda R}(z^\alpha) \in \mathbb{R}, \quad 0 \le w_{X+\lambda R}(z^\alpha) \le \lambda, \tag{4.11}$$

for every nontrivial monomial $cz^\alpha \bar{z}^\beta$ in the expansion of P, where both inequalities are strict if $\lambda \neq 0$.

Proof. Note that clearly U and X commute since X is of weighted degree zero, and since Z commutes with X. We first show that

$$w_{X+\lambda R}(z^\alpha) \in \mathbb{R}.$$

Assume by contradiction that $\operatorname{Im} w_{X+\lambda R}(z^\alpha) \neq 0$ for some nontrivial monomial $cz^\alpha \bar{z}^\beta$ in the expansion of P. It is easily shown, using the fact that X is a rotation that

$$w_X(z^\alpha) + \overline{w_X(z^\beta)} = 0, \tag{4.12}$$

and hence, for every nontrivial monomial $z^\alpha \bar{z}^\beta$ in the expansion of P,

$$w_{X+\lambda R}(z^\alpha) - \overline{w_{X+\lambda R}(z^\beta)} = 2w_{X+\lambda R}(z^\alpha) - \lambda. \tag{4.13}$$

We choose a real linear function $l : \mathbb{R} \to \mathbb{R}$ such that

$$l\Big(\operatorname{Im}\big(w_{X+\lambda R}(z^\alpha) - \overline{w_{X+\lambda R}(z^\beta)}\big)\Big) > 0 \tag{4.14}$$

for some monomial $z^\alpha \bar{z}^\beta$ in the expansion of P. Let $z^{\alpha_0} \bar{z}^{\beta_0}$ be the minimal (in the lexicographic ordering sense) nontrivial monomial in the expansion of P maximizing the left-hand side of (4.14). Then, using the Jordan normal form, the expansion of

$$\big(\operatorname{Im}(X+U)\big)P^2 = 2P\big((\operatorname{Im} X + U)\big)P$$

contains the monomial $z^{2\alpha_0} \bar{z}^{2\beta_0}$ with

$$l\Big(\operatorname{Im}\big((w_{X+\lambda R} - w_{\overline{X+\lambda R}})(z^{2\alpha_0} \bar{z}^{2\beta_0})\big)\Big) > l\Big(\operatorname{Im}\big((w_{X+\lambda R} - w_{\overline{X+\lambda R}})(z^\alpha \bar{z}^\beta)\big)\Big) \tag{4.15}$$

for all monomials $z^\alpha \bar{z}^\beta$ in the expansion of P, the reason being that $(\operatorname{Re}(X+\lambda R))P = \lambda P$. Using Lemma 4.3 and the fact that $[R, Y] = Y$ (Y is of weighted degree one), we conclude that

$$\operatorname{Im} w_{X+\lambda R}\big((z^\gamma \partial_{z_s})(z^\alpha \bar{z}^\beta)\big) = \operatorname{Im} w_{X+\lambda R}\big((\bar{z}^\gamma \partial_{\bar{z}_s})(z^\alpha \bar{z}^\beta)\big) = \operatorname{Im} w_{X+\lambda R}(z^\alpha \bar{z}^\beta) \tag{4.16}$$

for any monomial $z^\alpha \bar{z}^\beta$ and therefore $(\mathsf{Re}\, Y)P$ contains only monomials $((\mathsf{Re}(z^\gamma \partial_{z_s}))z^\alpha \bar{z}^\beta)$ for which

$$\begin{aligned} &l\big(\mathsf{Im}(w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big((\mathsf{Re}\big(z^\gamma \partial_{z_s}\big)\big)z^\alpha \bar{z}^\beta\big)\big) \\ &\quad = l\big(\mathsf{Im}\big((w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big(z^\alpha \bar{z}^\beta\big)\big)\big). \end{aligned} \tag{4.17}$$

Summarizing, we obtain that the second equation in (4.4), (4.15) and (4.17) together contradict (4.14) and therefore

$$\mathsf{Im}\, w_{X+\lambda R}\big(z^\alpha\big) = 0. \tag{4.18}$$

But (4.18) implies that (4.13) is antisymmetric with respect to α and β. We claim that

$$w_{X+\lambda R}\big(z^\alpha\big) - \overline{w_{X+\lambda R}\big(z^\beta\big)} \le \lambda \tag{4.19}$$

which will imply by antisymmetry that

$$-\lambda \le w_{X+\lambda R}\big(z^\alpha\big) - \overline{w_{X+\lambda R}\big(z^\beta\big)} \le \lambda. \tag{4.20}$$

First, assume by contradiction that

$$w_{X+\lambda R}\big(z^\alpha\big) - \overline{w_{X+\lambda R}\big(z^\beta\big)} = \lambda + \epsilon \tag{4.21}$$

for some α and β in the expansion of P and $\epsilon > 0$ being maximal possible. We shall choose here the minimal possible α in the lexicographic order. Using again Lemma 4.3 and the fact that $[R, Y] = Y$, we obtain that

$$w_{X+\lambda R}\big(\big(z^\gamma \partial_{z_s}\big)\big(z^\alpha \bar{z}^\beta\big)\big) = w_{X+\lambda R}\big(z^\alpha \bar{z}^\beta\big) + \lambda, \tag{4.22}$$

$$w_{X+\lambda R}\big(\big(\bar{z}^\gamma \partial_{\bar{z}_s}\big)\big(z^\alpha \bar{z}^\beta\big)\big) = w_{X+\lambda R}\big(z^\alpha \bar{z}^\beta\big) \tag{4.23}$$

for any monomial $z^\alpha \bar{z}^\beta$ and therefore $(\mathsf{Re}\, Y)P$ contains only monomials $\mathsf{Re}(z^\gamma \partial_{z_s})(z^\alpha \bar{z}^\beta)$ for which

$$(w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big(\mathsf{Re}\big(z^\gamma \partial_{z_s}\big)\big(z^\alpha \bar{z}^\beta\big)\big) = (w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big(z^\alpha \bar{z}^\beta\big) + \lambda \tag{4.24}$$

or

$$(w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big(\mathsf{Re}\big(z^\gamma \partial_{z_s}\big)\big(z^\alpha \bar{z}^\beta\big)\big) = (w_{X+\lambda R} - w_{\overline{X+\lambda R}})\big(z^\alpha \bar{z}^\beta\big) - \lambda. \tag{4.25}$$

Comparing the weights in the second equation of (4.4), we see that the second term of the left-hand side of the equation contains a nontrivial monomial of weight $2\lambda + 2\epsilon$, $\epsilon > 0$, while the first term of the equation has weight at most $\lambda + \epsilon + \lambda$. Hence the contradiction. Finally, if $\lambda \neq 0$, assume by contradiction that

$$w_{X+\lambda R}(z^\alpha) - \overline{w_{X+\lambda R}(z^\beta)} = \lambda \tag{4.26}$$

for some α and β in the expansion of P. Using (4.13), we obtain that

$$w_{X+\lambda R}(z^\alpha) = \lambda, \qquad \overline{w_{X+\lambda R}(z^\beta)} = 0. \tag{4.27}$$

Let P^λ be the sum of all monomials in P for which (4.26) holds. Using the second equation of (4.4) and (4.27), we see that P^λ should satisfy

$$(2\lambda + \mathsf{Im}\, Z)\big(P^\lambda\big)^2 + Y\big(P^\lambda\big) = 0. \tag{4.28}$$

But (4.28) cannot hold since P^λ contains no harmonic terms. Indeed, take the nontrivial monomial $z^\alpha \bar{z}^\beta$ of P^λ with maximal $|\beta| > 0$ for which then (α, β) is minimal in the lexicographic order. Then $(P^\lambda)^2$ has nontrivial monomial $z^{2\alpha}\bar{z}^{2\beta}$ which cannot occur in YP^λ and in $(\mathsf{Im}\, Z)(P^\lambda)^2$. Hence we obtain the contradiction. This achieves the proof of the lemma. $\square$

Proposition 4.5. *Let M_P given by (2.5) be holomorphically nondegenerate, R be given by (4.3), W be a linear rotation, and Y be a rigid holomorphic vector field satisfying*

$$(\mathsf{Re}\, Y)P + \big(\mathsf{Im}(W + R)\big)P^2 = 0. \tag{4.29}$$

Then

$$\big(\mathsf{Im}(W + R)\big)P = 0. \tag{4.30}$$

We need the following lemma.

Lemma 4.6. *Let $W = X + Z$ be a linear vector field in Jordan normal form with X the diagonal and Z the nilpotent part. Assume that $(\mathsf{Re}\, W)P = 0$. Then $(\mathsf{Re}\, X)P = (\mathsf{Re}\, Z)P = 0$.*

Proof. Since X is diagonal, we have the spectral decomposition

$$P = \sum P_\mu, \qquad P_\mu \in \mathcal{P}_\mu := \big\langle z^\alpha \bar{z}^\beta \colon\ 2\,\mathsf{Re}\, X\big(z^\alpha \bar{z}^\beta\big) = \mu z^\alpha \bar{z}^\beta \big\rangle.$$

We consider the lexicographic order on monomials $z^\alpha \bar{z}^\beta$. We claim that $P_\mu = 0$ unless $\mu = 0$. Indeed, assume by contradiction that $P_\mu \neq 0$ for some $\mu \neq 0$ and consider the minimal nontrivial monomial $z^\alpha \bar{z}^\beta$ in the expansion of P_μ with respect to the lexicographic order. Then $\mathsf{Re}\, Z(z^\alpha \bar{z}^\beta)$ has only monomials larger than $z^\alpha \bar{z}^\beta$, therefore the coefficient of $z^\alpha \bar{z}^\beta$ in $\mathsf{Re}\, W(P)$ is equal to μ. Since $\mathsf{Re}\, W(P) = 0$, we must have $\mu = 0$ contradicting our assumption. Hence $P = P_0$ as claimed, implying $\mathsf{Re}\, X(P) = 0$ and hence $\mathsf{Re}\, W(P) = 0$, which implies $\mathsf{Re}\, Z(P) = 0$. $\square$

Proof of Proposition 4.5. After a linear change of (multitype) coordinates, we may assume that $W = X + Z$, with X the diagonal and Z the nilpotent part. By Lemma 4.6, X and Z are rotations. Now define P_μ, $0 < \mu < 1$, to be the sum of all monomials $z^\alpha z^\beta$ of P with

$$w_{X+R}\big(z^\alpha\big) = \mu.$$

By Lemma 4.4, we have

$$P = \sum P_\mu.$$

Since P is real, the monomials $z^\alpha \bar{z}^\beta$ and $z^\beta \bar{z}^\alpha$ have conjugate coefficients. Since the left-hand side of (4.13) is antisymmetric in α, β, the right-hand side satisfies

$$w_{X+R}\big(z^\beta \bar{z}^\alpha\big) = 1 - w_{X+R}\big(z^\alpha \bar{z}^\beta\big). \tag{4.31}$$

Hence if $z^\alpha \bar{z}^\beta$ enters P_μ, $z^\beta \bar{z}^\alpha$ enters $P_{1-\mu}$. Therefore using reality of P, we have

$$P_\mu = \overline{P_{1-\mu}}. \tag{4.32}$$

In course of proof we shall use the convention $P_\mu = 0$ for any $\mu \geq 1$.

Then, identifying terms of weight μ in (4.29), we obtain

$$\bar{Y} P_\mu = \big(i(\mu - 1) - \operatorname{Im} Z\big) \sum_\nu P_\nu P_{\mu-\nu}, \tag{4.33}$$

where $\mu - 1 < 0$. Denoting $T = i\bar{Y}$ we rewrite (4.33) as

$$T P_\mu = (1 - \mu - i \operatorname{Im} Z) \sum P_\nu P_{\mu-\nu}, \quad 1 - \mu > 0. \tag{4.34}$$

Without loss of generality, $P \neq 0$. Set

$$l := \min\{\mu\colon\ P_\mu \neq 0\} > 0. \tag{4.35}$$

Then (4.34) implies

$$T P_l = 0. \tag{4.36}$$

Conjugating and using (4.32) we obtain

$$\bar{T} P_{1-l} = 0. \tag{4.37}$$

Given the choice of l and using (4.32) we obtain

$$P_\mu = 0, \quad \mu < l \text{ or } \mu > 1 - l. \tag{4.38}$$

In the sequel $c_1, c_2, \ldots,$ will always denote suitable positive integers. Consider the (unique) integer $s \geq 1$ satisfying

$$1 - l \leq sl < 1. \tag{4.39}$$

Then applying s times T and using (4.34) and (4.38), we obtain

$$T^{s-1} P_{sl} = (c_1 + Q_1) P_l^s, \tag{4.40}$$

where Q_1 is a linear operator increasing the lexicographic order of monomials and commuting with T and $\bar{T}$. In the sequel we shall always denote by $Q_1, Q_2, \ldots,$ operators of this kind.

Since $P_l \neq 0$, it follows from (4.39) and (4.38) that $sl = 1 - l$, since otherwise the left-hand side vanishes and, choosing as before the minimal monomial of P_l we would reach a contradiction. Hence 1 is divisible by l.

Since T is holomorphic, it commutes with $\bar{T}$. Also T and $\bar{T}$ commute with $\mathsf{Im}\, Z$ by Lemma 4.3. Then applying $\bar{T}$ to (4.40) and using (4.37), we obtain

$$0 = (c_1 + Q_1) s P_l^{s-1} \bar{T} P_l, \tag{4.41}$$

which yields

$$\bar{T} P_l = 0 \tag{4.42}$$

using the by now frequently used argument with the minimal monomial. We shall continue using this argument without mentioning in the rest of the proof.

We next claim that

$$P_\mu = 0, \quad l < \mu < 2l. \tag{4.43}$$

Indeed, otherwise take the minimum $l < \mu < 2l$ with $P_\mu \neq 0$. Then using (4.38) we obtain

$$0 = T^s P_{sl-l+\mu} = (c_2 + Q_2) P_l^{s-1} P_\mu \tag{4.44}$$

and hence (4.43) holds as desired.

Now using (4.43) and applying T^{s-2} to P_{sl}, we obtain

$$T^{s-2} P_{sl} = (c_4 + Q_4) P_l^{s-2} P_{2l}. \tag{4.45}$$

Then applying $\bar{T}$ and using (4.42) we conclude $0 = (c_4 + Q_4) P_l^{s-2} \bar{T} P_{2l}$ implying

$$\bar{T} P_{2l} = 0. \tag{4.46}$$

Next, similarly to the claim (4.43) as before, we prove

$$P_\mu = 0, \quad 2l < \mu < 3l, \tag{4.47}$$

where we repeat the previous arguments applying T^{s-2} to $P_{sl-2l+\mu} = 0$.

Similarly, applying $T^{s'}$ for $s' = s-2, s-3, \dots,$ to $P_{sl+\varepsilon}$, $0 \le \varepsilon < l$, we conclude by induction

$$\bar{T} P_{kl} = 0 \tag{4.48}$$

for all k and $P_\mu = 0$ whenever μ is not divisible by l. In particular, we obtain $\bar{T}P_\mu = 0$ for all μ and therefore $\bar{T}P = 0$. Finally, using reality of P, we conclude $TP = 0$ and hence $\bar{Y}P = 0$ or, conjugating, $YP = 0$. This implies that $\mathsf{Im}(W+R)P = 0$ as desired. □

We may now state the main result of this section.

Theorem 4.7. *Let M_P given by (2.5) be holomorphically nondegenerate, and consider $\partial_w \in \mathsf{aut}(M_P, 0)$. Then there exists no vector field lying in $\mathsf{aut}(M_P, 0)$ that is a 3-integration of ∂_w. Moreover, if we choose coordinates as in Proposition 3.9, every 1-integration of ∂_w is of the form*

$$\sum_j l_j(z)\partial_{z_j} + w\partial_w, \tag{4.49}$$

where all l_j are linear, and every 2-integration is of the form

$$\sum_j \varphi_j(z) w \partial_{z_j} + \frac{1}{2} w^2 \partial_w \in \mathsf{aut}(M_P, 0), \tag{4.50}$$

where φ_j satisfy

$$2\sum_j \varphi_j(z) P_{z_j} = P(z, \bar{z}). \tag{4.51}$$

Proof. If ∂_w can be integrated at least once, we obtain an automorphism

$$\sum_j \varphi_j(z)\partial_{z_j} + \big(w + \varphi(z)\big)\partial_w \in \mathsf{aut}(M_P, 0), \tag{4.52}$$

where the weights of $\varphi_j(z)$ are positive. Applying twice the real part of (4.52) to $P - v$, where $w = u + iv$, and putting $w = u + iP$ we obtain

$$2\,\mathsf{Re}\sum_j \varphi_j(z) P_{z_j} - P(z, \bar{z}) - \mathsf{Im}\,\varphi(z) = 0. \tag{4.53}$$

Since the first two terms are all not pluriharmonic, we obtain $\varphi(z) = 0$. Therefore we have

$$2\,\mathsf{Re}\sum_j \varphi_j(z)P_{z_j} = P(z,\bar{z}). \tag{4.54}$$

By Lemma 4.2, we can write

$$\sum_j \varphi_j(z)\partial_{z_j} = R + W, \tag{4.55}$$

where

$$R = \sum_{j=1}^{n} \mu_j z_j \partial_{z_j} \tag{4.56}$$

is the Euler vector field and W is any rotation (see Definition 3.7).

Assuming (4.52) can be integrated, we obtain a new vector field of the form

$$\sum_j \big(\varphi_j(z)w + \psi_j(z)\big)\partial_{z_j} + \left(\frac{1}{2}w^2 + \psi(z)\right)\partial_w \in \mathsf{aut}(M_P, 0). \tag{4.57}$$

Applying twice the real part of (4.57) to $P - v$ and putting $w = u + iP$ we obtain

$$-2P(z,\bar{z})\,\mathsf{Im}\sum_j \varphi_j(z)P_{z_j} + 2\,\mathsf{Re}\sum_j P_{z_j}\psi_j(z) - \mathsf{Im}\,\psi(z) = 0. \tag{4.58}$$

Since the first two summands contain only non-pluriharmonic terms, we obtain $\psi(z) = 0$. Hence, we obtain

$$-2P(z,\bar{z})\,\mathsf{Im}\sum_j \varphi_j(z)P_{z_j} + 2\,\mathsf{Re}\sum_j P_{z_j}\psi_j(z) = 0. \tag{4.59}$$

By Proposition 3.9(1), we can assume that W is a linear rotation. Then using Proposition 4.5, we obtain in particular that

$$\mathsf{Im}\sum_j \varphi_j(z)P_{z_j} = 0. \tag{4.60}$$

Then Theorem 3.3 implies that $\psi_j = 0$, and hence, in view of (4.54), (4.59) implies

$$2\sum_j \varphi_j(z)P_{z_j} = P(z,\bar{z}). \tag{4.61}$$

Assuming again (4.57) can further be integrated, and using (4.61), we obtain a field of the form

$$Y = \sum_j \Big(\frac{1}{2} w^2 \varphi_j(z) + \chi_j(z)\Big)\partial_{z_j} + \Big(\frac{1}{6} w^3 + \chi(z)\Big)\partial_w \in \mathsf{aut}(M_P, 0). \tag{4.62}$$

Applying twice the real part of (4.62) to $P - v$, and using (4.61), we obtain (with $\chi = 0$ as above),

$$\mathsf{Re}\,\frac{1}{2}\big(u^2 - P^2 + 2iuP\big)P + \mathsf{Re}\sum_j \chi_j(z)P_{z_j} - \frac{1}{6}\big(3u^2P - P^3\big) = 0. \tag{4.63}$$

Putting $u = 0$ in (4.63), we obtain

$$-\frac{1}{3}P^3 + 2\,\mathsf{Re}\sum_j \chi_j(z)P_{z_j} = 0. \tag{4.64}$$

Multiplying (4.64) by P, we see that the expansion of the first expression has nontrivial terms of weighted bidegree $(2,2)$. On the other hand, since Y is homogeneous of weight 2, the weight of $\chi_j(z)$ is $2 + \mu_j$, and hence the right-hand side of (4.64) cannot have terms of weighted bidegree $(2,2)$. We obtain a contradiction proving that the 3-integration Y of ∂_w cannot exist, proving the theorem. $\square$

Equivalently, there exist weights $\Lambda = (\lambda_1, \ldots, \lambda_n)$ (possibly different from multitype weight), with respect to which P is diagonal, i.e. contains only monomials $z^\alpha \bar{z}^\beta$ such that $|\alpha|_\Lambda = |\beta|_\Lambda$.

5. Integrating rotations and generalized rotations

In this section, we consider rotations and generalized rotations. We show that they cannot be integrated, provided that M_P is holomorphically nondegenerate.

We write

$$P = \sum_{j=1}^{l} P_j, \tag{5.1}$$

where P_j is a sum of monomials of the form $B_{\alpha^j,\hat{\alpha}^j} z^{\alpha^j} \bar{z}^{\hat{\alpha}^j}$ of constant weighted length $|\hat{\alpha}^j|_{\Lambda_M} =: c_j$, ordered such that $c_j < c_k$ for $j < k$. We start with the following lemma.

Lemma 5.1. *Let X be a generalized rotation. Then there exists $N > 0$ such that $X^N P = 0$.*

Proof. We assume that X is a generalized rotation of weight $\nu > 0$. We then have

$$X(P_1) = 0, \qquad X(P_j) + \overline{X}(P_k) = 0, \tag{5.2}$$

where $c_j = c_k + \nu$, for every k and P_j is given by (5.1). Since X and $\overline{X}$ commute, we reach the conclusion. $\square$

We set the following definition.

Definition 5.2. Let X be a rigid holomorphic vector field and let $p \in \mathbb{C}[z, \bar{z}]$. We define

$$d_X(p) := \sup\{s + t\colon\ X^s \bar{X}^t(p) \neq 0\}. \tag{5.3}$$

Remark 5.3. Using Lemma 5.1, we see that $d_X(P) < \infty$ if X is a generalized rotation or a nilpotent linear rotation.

Lemma 5.4. *Let M_P given by (2.5) be holomorphically nondegenerate. Assume that X is either a generalized rotation or a nilpotent linear rotation, and let Y be a rigid holomorphic vector field satisfying*

$$\mathsf{Re}\, Y(P) + \mathsf{Im}\, X\big(P^2\big) = 0. \tag{5.4}$$

Then $X = 0$.

Proof. Assume by contradiction that $X(P) \neq 0$. Let D be the sum of monomials of the form $A_{\alpha,\hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}$ in P^2 for which $|\alpha|_{\Lambda_M} = |\hat{\alpha}|_{\Lambda_M}$. We claim that

$$X(D) \neq 0.$$

Indeed, writing P as in (5.1), we obtain that D can be written as

$$D = \sum_{j=1}^{l} P_j \overline{P_j}. \tag{5.5}$$

Let

$$\hat{d}_X(P) := \max_j \{d_X(P_j)\}, \tag{5.6}$$

where $d_X(P_j)$ is given by (5.3). Let ${P_j}^{\hat{d}}$ be the set of monomials of P_j of the form $B_{\alpha^j,\hat{\alpha}^j} z^{\alpha^j} \bar{z}^{\hat{\alpha}^j}$ for which there exists s such that

$$X^s \bar{X}^{\hat{d}_Z(P)-s}\big(B_{\alpha^j,\hat{\alpha}^j} z^{\alpha^j} \bar{z}^{\hat{\alpha}^j}\big) \neq 0.$$

Using Lemma 5.1 and (5.5), we obtain that

$$D = \sum_{j=1}^{l} {P_j}^{\hat{d}} \overline{{P_j}^{\hat{d}}} + R, \tag{5.7}$$

where $X^s \bar{X}^{2\hat{d}_Z(P)-s}(R) = 0$, for every s. But (5.7) implies that

$$d_X(D) = 2\hat{d}_X(P).$$

Since $X(P) \neq 0$, we have $\hat{d}_X(P) \neq 0$, and hence $d_X(D) \neq 0$. Since D is real, this implies that $X(D) \neq 0$. Since $X \in \mathsf{aut}(M_P, 0)$, the assumption (5.4) implies

$$2X\big(P^2\big) + i\,\mathsf{Re}\,Y(P) = 0. \tag{5.8}$$

Since the weighted bidegree of D is $(1,1)$, the weighted bidegree of $X(D)$ is $(1+\nu, 1)$, where $\nu \geq 0$ is the weight of X. On the other hand each term in $\mathsf{Re}\,Y(P)$ has weighted bidegree (k,l) with either $k < 1$ or $l < 1$. Hence no term from $2X(D)$ in (5.8) can get canceled by a term from the second summand. Since $X(D) \neq 0$, we obtain a contradiction with our assumption $X(P) \neq 0$. Therefore $X(P) = 0$, and hence $X = 0$ since M_P is holomorphically nondegenerate. $\square$

We may now state the main result of this section.

Theorem 5.5. *Let M_P given by (2.5) be holomorphically nondegenerate, and let $X \in \mathsf{aut}(M_P, 0)$ be either rotation or generalized rotation. There exists no vector field in $\mathsf{aut}(M_P, 0)$ that is a 1-integration of X.*

Proof. We write X as

$$X = \sum_j f_j(z)\partial_{z_j}. \tag{5.9}$$

Recall that an integration of X is any vector field $Y \in \mathsf{aut}(M_P, 0)$ satisfying $[\partial_w, Y] = X$. Then Y has to be of the form

$$Y = w\sum_j f_j(z)\partial_{z_j} + \sum_j \varphi_j(z)\partial_{z_j} + \varphi(z)\partial_w \in \mathsf{aut}(M_P, 0).$$

We then have

$$\begin{aligned} 2\,\mathsf{Re}\,Y(P - v) &= \mathsf{Re}\Big(2\sum_j P_{z_j} f_j(z)\big(u + iP(z,\bar{z})\big) + 2\sum_j P_{z_j}\varphi_j(z) + i\varphi(z)\Big) \\ &= \mathsf{Re}\Big(2\sum_j P_{z_j} f_j(z) iP(z,\bar{z}) + \sum_j 2P_{z_j}\varphi_j(z)\Big) - \mathsf{Im}\,\varphi(z) = 0, \end{aligned}$$

where we have used $\mathsf{Re}\,X(P-v) = 0$. The first two sums contain only non-pluriharmonic terms, while the last term is pluriharmonic. It implies that $\varphi(z) = 0$ and hence

$$-P(z,\bar{z})\,\mathsf{Im}\sum_j P_{z_j} f_j(z) + \mathsf{Re}\sum_j P_{z_j}\varphi_j(z) = 0. \tag{5.10}$$

In the case X is a rotation, by Proposition 3.9(1), after a polynomial weighted homogeneous change of coordinates, we may assume that X is linear. We may also assume that the matrix of X is in its Jordan normal form. Then we conclude from Lemma 4.6, that the diagonal part of X is also a rotation. Applying Lemma 4.4, we conclude that the diagonal part of X is zero. Therefore Lemma 5.4 together with (5.10) implies that $X = 0$ contradicting the assumption.

On the other hand, if X is a generalized rotation, we can directly apply Lemma 5.4 together with (5.10) to conclude that $X = 0$ contradicting the assumption. The proof is complete. $\square$

We finish this section by giving another example of a hypersurface admitting a generalized rotation.

Example 5.6. Let P be defined by

$$P(z,\bar z) = -\operatorname{Re}\big(z_1^2\overline{z_1^2 z_2^2}\big) + \big|z_1^2 z_2\big|^2, \tag{5.11}$$

and M_P by (1.6). The weights are $\mu_1 = \mu_2 = \frac{1}{6}$. M_P admits the following symmetry of weight $\frac{1}{6}$

$$Y = z_1 z_2 \partial z_1 - z_2^2 \partial z_2. \tag{5.12}$$

Indeed, we obtain

$$Y(v-P) = -2z_2 z_1^2 \overline{z_1^2 z_2^2} + 2z_2^2 z_1^2 \overline{z_1^2 z_2} \tag{5.13}$$

hence

$$\operatorname{Re} Y(v-P) = 0. \tag{5.14}$$

6. Integrating nontransversal shifts

In this section, we show that nontransversal shifts can be integrated at most one time. We start with the following lemma.

Lemma 6.1. *Let Z be a nontransversal shift on M_P of the form*

$$Z = \sum_{j=1}^{n} f^j(z)\partial_{z_j} + g(z)\partial_w. \tag{6.1}$$

Then there exist modified multitype coordinates (with pluriharmonic terms allowed) such that there exists r, $1 \le r \le n$, with

$$Z = i\partial_{z_r}, \tag{6.2}$$

and consequently,

$$P_{y_r}(z, \bar{z}) = 0. \tag{6.3}$$

Proof. We observe that there is j, $1 \le j \le n$, such that f^j is nonzero, since otherwise $Z = 0$.

We first assume that all μ_j are equal. It implies that all f^j are constant. After a possible holomorphic linear change of coordinates, we may assume that Z is given by

$$Z = \partial_{z_1} + g(z)\partial_w. \tag{6.4}$$

The following change of modified multitype coordinates leads to the desired conclusion

$$\begin{aligned} z_j^* &= z_j, \\ w^* &= w - z_1 g(z). \end{aligned} \tag{6.5}$$

Assume now that the μ_j are not all equal. Write

$$Z = \sum_{j=1}^{j_k} f^j(z)\partial_{z_j} + g(z)\partial_w, \quad f^{j_k} \neq 0, \tag{6.6}$$

where $\mu_1 \ge \dots > \mu_{j_1} = \dots = \mu_{j_k}$.

Since $Z \neq 0$ is of negative weight, f^{j_k} is nonzero and, in view of Proposition 3.9(2), can be assumed to be constant. After performing a linear change of the variables $z_{j_1}, \dots, z_{j_k}$, we may assume that $k = 1$ and $f^{j_1}(z) = 1$.

The following holomorphic change of coordinates

$$\begin{aligned} z_j^* &= z_j, \quad 1 \le j \le j_1 - 2, \\ z_{j_1-1}^* &= z_{j_1-1} - \sum_\alpha \frac{C_\alpha}{\alpha_{j_1}+1} z_{j_1}^{\alpha_{j_1}+1} \dots z_{j_n}^{\alpha_{j_n}}, \\ z_j^* &= z_j, \quad j_1 \le j \le n, \end{aligned} \tag{6.7}$$

where $f^{j_1-1}(z) = \sum C_\alpha z_{j_1}^{\alpha_{j_1}} \dots z_{j_n}^{\alpha_{j_n}}$, leads to the elimination of the term $f^{j_1-1}(z)\partial_{z_{j_1-1}}$ in (6.6).

Similarly we can eliminate any $f^j(z)$ with $\mu_j = \mu_{j_1-1}$. Furthermore, using recursively holomorphic changes of coordinates as in (6.7), we can arrange Z to become of the form

$$Z = \partial_{z_{j_1}} + g(z)\partial_w. \tag{6.8}$$

Finally, performing a change of coordinates similar to (6.5), we reach the desired conclusion. This achieves the proof of the lemma. $\square$

Assume now, according to Lemma 6.1, that M_P admits, after a possible change of modified multitype coordinates, a nontransversal shift Z, given by

$$Z = i\partial_{z_r}. \tag{6.9}$$

We may then write P as

$$P(z,\bar z) = \sum_{j=0}^{k} x_r^j P_j(z',\bar z'), \qquad P_k(z',\bar z') \neq 0, \tag{6.10}$$

where z' is the $(n-1)$-tuple of z_j's with z_r omitted. Note that if M_P is holomorphically nondegenerate, P must depend on z_r and hence $k \geq 1$.

Theorem 6.2. *Assume that M_P is holomorphically nondegenerate. Let Z be given by (6.9) and P be given by (6.10). Then there is no 2-integration of Z.*

Proof. Assuming Z can be integrated, we obtain a vector field of the form

$$wi\partial_{z_r} + \sum_{j=1}^{n} \varphi_j(z)\partial_{z_j} + \varphi(z)\partial_w \in \mathsf{aut}(M_P, 0). \tag{6.11}$$

Applying twice the real part of (6.11) to $P - v$, we obtain

$$2\,\mathsf{Re}(u - iP)iP_{z_r} + 2\,\mathsf{Re}\sum_{j=1}^{n} \varphi_j(z)P_{z_j} - \mathsf{Im}\,\varphi(z) = 0. \tag{6.12}$$

We may rewrite (6.12), using the hypothesis that $Z \in \mathsf{aut}(M_P, 0)$, as

$$-P(z,\bar z)\,\mathsf{Im}\,Z(P) + \mathsf{Re}\left(\sum_{j=1}^{n} \varphi_j(z)P_{z_j} + \frac{i}{2}\varphi(z)\right) = 0. \tag{6.13}$$

Assuming (6.11) can be integrated, and using (6.13), we obtain a vector field of the form

$$\frac{1}{2}w^2 i\partial_{z_r} + w\left(\sum_{j=1}^{n} \varphi_j(z)\partial_{z_j} + \varphi(z)\partial w\right) + \sum_{j=1}^{n} \psi_j(z)\partial_{z_j} + \psi(z)\partial_w \in \mathsf{aut}(M_P, 0). \tag{6.14}$$

Applying twice the real part (6.14) to $P - v$, we obtain

$$\begin{aligned} &\mathsf{Re}\big(u^2 - P^2 + 2iuP\big)iP_{z_r} + \mathsf{Re}(u + iP)\left(2\sum_{j=1}^{n} \varphi_j(z)P_{z_j} + i\varphi(z)\right) \\ &\quad + 2\,\mathsf{Re}\sum_{j=1}^{n} \psi_j(z)P_{z_j} - \mathsf{Im}\,\psi(z) = 0. \end{aligned} \tag{6.15}$$

Putting $u = 0$ in (6.15), we obtain

$$-P(z,\bar z)\,\mathsf{Im}\left(\sum_{j=1}^{n}\varphi_j(z)P_{z_j}+\frac{i}{2}\varphi(z)\right)+\mathsf{Re}\left(\sum_{j=1}^{n}\psi_j(z)P_{z_j}+\frac{i}{2}\psi(z)\right)=0. \tag{6.16}$$

Using the hypothesis, we may rewrite (6.13) and (6.16) as

$$-P\,\mathsf{Im}\,Z(P)+\mathsf{Re}\,X(P-v)=0, \tag{6.17}$$
$$-P\,\mathsf{Im}\,X(P-v)+\mathsf{Re}\,Y(P-v)=0, \tag{6.18}$$

where

$$X:=\sum_{j=1}^{n}\varphi_j(z)\partial_{z_j}+\varphi(z)\partial_w, \qquad Y:=\sum_{j=1}^{n}\psi_j(z)P_{z_j}+\psi(z)\partial_w. \tag{6.19}$$

Since $Z = i\partial_{z_r}$, using (6.10) we obtain

$$\left(\sum_{j=0}^{k}x_r^jP_j(z',\bar z')\right)\left(\sum_{j=0}^{k}jx_r^{j-1}P_j(z',\bar z')\right)-2\,\mathsf{Re}\,X(P-v)=0. \tag{6.20}$$

Similarly, rewriting (6.18) we have

$$\left(\sum_{j=0}^{k}x_r^jP_j(z',\bar z')\right)\mathsf{Im}\,X(P-v)+\mathsf{Re}\,Y(P-v)=0. \tag{6.21}$$

We need the following lemma.

Lemma 6.3. *Let P_k be given by (6.10) and X be as above. Then*

$$\begin{aligned}X(P-v)&=A(z',\overline{z'})z_r^k\bar z_r{}^k+\sum_{l=1}^{k-1}F_l(z',\overline{z'})z_r{}^{k+l}\bar z_r{}^{k-1-l}+F_0(z',\overline{z'})z_r{}^k\bar z_r{}^{k-1}\\&\quad+F_{-1}(z',\overline{z'})z_r{}^{k-1}\bar z_r{}^k+\cdots,\end{aligned} \tag{6.22}$$

where the dots stand for lower degree terms with respect to the variables z_r, $\overline{z_r}$, where

$$F_l(z',\overline{z'})=-c_lP_k^2,\quad l\ge 1,$$
$$F_0(z',\overline{z'})+\overline{F_{-1}(z',\overline{z'})}=-c_0P_k^2,$$

c_l are positive coefficients and $A(z',\overline{z'})$ is purely imaginary.

Proof. From (6.20) we see that

$$\operatorname{Re} X(P-v) = -\left(\sum_{s=0}^{k}\left(\frac{z_r+\overline{z_r}}{2}\right)^{s} P_s(z',\bar{z}')\right)\left(\sum_{j=0}^{k} j\left(\frac{z_r+\overline{z_r}}{2}\right)^{j-1} P_j(z',\bar{z}')\right). \tag{6.23}$$

Using binomial expansion, (6.23) yields to

$$\operatorname{Re} X(P-v) = -P_k^2 \sum_{l=0}^{2k-1} c_l {z_r}^{l}\, \overline{z_r}^{2k-1-l} + \cdots, \tag{6.24}$$

where $c_l > 0$ are positive coefficients and the dots stand for lower degree terms with respect to the variables z_r, $\overline{z_r}$. Since P is a polynomial of degree k in x_r, $X(P-v)$ cannot have terms with $z_r^s \bar{z}_r^l$ for $l > k$. The conclusion of the lemma follows now directly from (6.24). $\square$

We return to the proof of the theorem. We first assume that $A(z', \overline{z'}) \neq 0$. Consider in (6.21) the terms of order $3k$ with respect to the variables z_r, $\overline{z_r}$ coming from the first expression of the left-hand side of the equation. Expanding we obtain a nonzero term containing $z_r^{k+1}\bar{z}_r^{2k-1}$, which for $k > 1$ cannot cancel with any term coming from the second expression of the left-hand side, which is a contradiction with $A \neq 0$.

Now consider the case $k = 1$. Then the first expression in (6.21) has a term $z_r^2 \bar{z}_r P_1(z', \bar{z}') A(z', \bar{z}')$. We claim that $A \neq \text{const}$. Indeed, (6.11) implies that the weight of X is $1 - \mu_r$ and hence by (6.22), the weight of A is $2 - 3\mu_r$, which is > 0 since $\mu_r \leq 1/2$. This shows the claim. Since A is purely imaginary, it has positive order in $\bar{z}'$. Then the order of $P_1(z', \bar{z}')A(z', \bar{z}')$ is greater than that of $P_1(z', \bar{z}')$. On the other hand, any term of bidegree $(2, 1)$ in $(z_r, \bar{z}_r)$ coming from the second expression in (6.21) has order in $\bar{z}'$ equal to that of $P_1(z', \bar{z}')$. Hence the terms $z_r^2 \bar{z}_r P_1(z', \bar{z}')A(z', \bar{z}')$ cannot be compensated and therefore must be zero, which is again a contradiction with $A \neq 0$.

Hence $A(z', \overline{z'}) = 0$. Consider in (6.21) the terms of order $3k - 1$ with respect to the variables $z_r, \overline{z_r}$ coming from the first expression of the left-hand side of the equation. Using Lemma 6.3, we may rewrite (6.21) as

$$\begin{aligned} 0 = \operatorname{Re} Y(P-v) + \frac{1}{2i}\left(\sum_{j=0}^{k} x_r^j P_j(z',\bar{z}')\right)&\left(P_k^2 \sum_{l=1}^{k-1} c_l \left(z_r^{k-1-l}\bar{z}_r^{k+l} - z_r^{k+l}\bar{z}_r^{k-1-l}\right)\right. \\ &\left. - c_0 P_k^2 + 2F_{-1}(z',\bar{z}')z_r^{k-1}\bar{z}_r^{k} - 2\overline{F_{-1}(z',\bar{z}')}z_r^{k}\bar{z}_r^{k-1}\right) + \cdots, \end{aligned} \tag{6.25}$$

where the dots stand for lower degree terms with respect to the variables $z_r, \bar{z}_r$. Extracting the term containing $z_r^{2k-1}{\bar{z}_r}^{k}$ coming from the second expression of the right-hand side, we obtain that its coefficient is

$$\frac{P_k}{2i}\big(2F_{-1}(z',\overline{z'}) - 2k\overline{F_{-1}(z',\overline{z'})} - BP_k^2\big), \tag{6.26}$$

where B is a positive constant.

We claim that the weighted orders of (6.26) separately in z' and in $\bar{z}'$ are strictly larger than the weight of P_k. Indeed, the expansion of P_k^2 contains a nonzero term $c(z')^\alpha(\bar{z}')^\alpha$, where α is a multiindex such that $\sum \mu_j\alpha_j$ equals to the weight of P_k. On the other hand, it follows from (6.22) that the weighted orders of $F_{-1}(z',\bar{z}')$ separately in z' and in $\bar{z}'$ are strictly less than the total weighted order of P_k in $(z',\bar{z}')$ equal to $\sum \mu_j\alpha_j$. Hence our term $c(z')^\alpha(\bar{z}')^\alpha$ cannot appear in the expansion of the first two terms in (6.25), proving the claim.

On the other hand, the term containing $z_r^{2k-1}\overline{z_r}^k$ in the first term on the right-hand side of (6.25) has weighted order with respect to each of the variables z' and $\bar{z}'$ less than the total weighted order of P_k. This achieves the proof of the theorem. □

Example 6.4. We give an example which shows that, unlike the homogeneous case, for unequal weights a generalized rotation can occur for the same weight as an integrated nontransversal shift. Let

$$\operatorname{Im} w = \operatorname{Re}\{z_1\bar{z}_2^2 + z_3\bar{z}_4^5\}, \tag{6.27}$$

where the weights are $\mu_1 = \mu_2 = \frac{1}{3}$, $\mu_3 = \mu_4 = \frac{1}{6}$. It admits a generalized rotation (as in Example 3.8)

$$Y = iz_4^5\partial z_3 \tag{6.28}$$

and an integrated nontransversal shift

$$Z_1 = w\frac{\partial}{\partial z_1} - iz_1{z_2}^2\frac{\partial}{\partial z_1} - i\frac{1}{2}{z_2}^3\frac{\partial}{\partial z_2} + 2i{z_2}^2w\frac{\partial}{\partial w},$$

both in weight $\nu = \frac{2}{3}$.

7. Proofs of the main results

We first notice that Theorem 1.1 and Theorem 1.3 are an immediate consequence of Theorems 4.7, 5.5 and 6.2. In this section we first give a precise description of the derivatives needed to characterize an automorphism of M at p. We will denote by $(f_1, f_2, \ldots, f_n, g)$ the components of an automorphism of M, as in (2.13).

Theorem 7.1. *The automorphisms of M at p are uniquely determined by*

(1) *the complex tangential derivatives $\frac{\partial^{|\alpha|}f_j}{\partial z^\alpha}$ for $|\alpha| \le 1 - \mu_n$;*
(2) *the first and second order normal derivatives $\frac{\partial f_j}{\partial w}$ for $j = 1, \ldots, n$, $\frac{\partial g}{\partial w}$, $\frac{\partial^2 g}{\partial w^2}$.*

The proof follows immediately by combining the above results with Proposition 2.15. Theorem 1.2 and Theorem 1.4 now follow from Theorem 7.1.

Note that in the Levi nondegenerate case, when $\mu_j = \frac{1}{2}$, for all j, part (1) of Theorem 7.1 includes only the first order derivatives $\frac{\partial f_j}{\partial z_i}$. Hence we recover the sharp statement contained in the work of Chern and Moser [10].

Corollary 7.2. *If M is Levi nondegenerate at p, its automorphisms are uniquely determined by the following partial derivatives*

(1) *the first order complex tangential derivatives $\frac{\partial f_j}{\partial z_i}$ for $i, j = 1, \ldots, n$;*
(2) *the first and second order normal derivatives $\frac{\partial f_j}{\partial w}$ for $j = 1, \ldots, n$, $\frac{\partial g}{\partial w}$, $\frac{\partial^2 g}{\partial w^2}$.*

Acknowledgments

The authors thank AIM, Palo Alto, for their stay in September 2010, where this work was initiated. Also, the first two authors want to thank ETHZ for its hospitality during their stay in November 2011, where part of this work was achieved.

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