

Non-equilibrium dynamics of Gaudin models

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Abstract – In classical mechanics the theory of non-linear dynamics provides a detailed framework for the distinction between near-integrable and chaotic systems. Quite in opposition, in quantum many-body theory no generic microscopic principle at the origin of complex dynamics is known. Here we show that the non-equilibrium dynamics of homogeneous Gaudin models can be fully described by underlying classical Hamiltonian equations of motion. The original Gaudin system remains fully quantum and thus cannot exhibit chaos, but the underlying classical description can be analyzed using the powerful tools of the classical theory of motion. We specifically apply this strategy to the Tavis-Cummings model for quantum photons interacting with an ensemble of two-level systems. We show that scattering in the classical phase space can drive the quantum model close to thermal equilibrium. Interestingly, this happens in the fully quantum regime, where physical observables do not show any dynamic chaotic behavior.

Many aspects of the transition from regular dynamics of an integrable system to erratic behavior of a complex system are understood in classical mechanics. On the one hand, there is the Kolmogorov-Arnold-Moser (KAM) theorem [1], that proves the stability of weakly perturbed integrable systems. On the other hand, a variety of mechanisms leading to chaos and eventually to the ergodic exploration of phase space have been found (see, *e.g.*, [2]). For quantum systems, the main paradigms for the description of quantum chaotic phenomena are quasiclassical [3] and random matrix [3,4] theories. Moreover, in a number of specific model studies thermalization processes have been observed (*e.g.*, [5]). However, there remains an important conceptual gap between regular and complex behaviors. In this letter we investigate a non-trivial integrable quantum system without going to the quasiclassical limit and gain microscopic insight into the emergence of irregularity when breaking integrability by driving an internal parameter.

A good starting point to approach regular dynamics of non-trivial quantum systems are Bethe ansatz (BA) integrable models, which possess a complete set of integrals of motion. The exact solutions of time-independent

BA many-body solvable systems played a crucial role in the understanding of various fundamental phenomena and concepts in physics. Famous examples are the solutions for the Ising model, the Heisenberg spin chain, the one-dimensional Hubbard model or the Lieb-Liniger gas [6,7]. Also certain aspects of quantum chromodynamics can be described by the integrable quantum spin chain with complex spin [8]. However, the non-equilibrium dynamics of these models are rich [9–18] and much more difficult to be calculated within the BA than the static properties. Formulating a theory of integrability breaking for time-dependent problems is thus not only a conceptual, but also a technical challenge.

In this letter we restrict ourselves to a certain class of Gaudin-type models for which we can derive a description in terms of a classical many-body interacting system. Deviation from integrability for the quantum system can then be understood in terms of the classical system, for which powerful tools such as the KAM theorem are available. To be specific, we consider the Tavis-Cummings model which was introduced in the context of interaction of light and matter in quantum optics [19]. It can be seen as a Dicke model [20] in the rotating-wave approximation. The model

is a representative for the first non-trivial members of BA integrable systems [21]. An application is for instance the description of the Bose-Einstein condensate in an optical cavity [22]. It is important to note that our auxiliary classical representation is not connected to the quasiclassical limit of the related Dicke model [23], but exact for the full quantum model. We will derive this classical representation and analyze its dynamics under periodic driving of the detuning.

Our results can be straightforwardly applied to arbitrary homogeneous Gaudin models, such as the Lipkin-Meshkov-Glick model for phase transitions in nuclei [24]. The challenges one faces when extending our approach to inhomogeneous models, for instance the Richardson or the central spin model, will be discussed in the conclusions.

The quantum Hamiltonian of the Tavis-Cummings model reads

$$\hat{H}_D = \Delta \hat{S}^z + g(\hat{b}^\dagger \hat{S}^- + \hat{b} \hat{S}^+), \quad (1)$$

where $\hat{S}^\mu = \sum_{j=1}^{2S} \frac{\hat{\sigma}_j^\mu}{2}$ is a collective spin operator with $\sum_{\mu=1}^3 (\hat{S}^\mu)^2 = S(S+1)$, the single-mode bosonic field (\hat{b} and \hat{b}^\dagger , photon annihilation and creation operators) is detuned by Δ . The total number of excitations $M = \hat{b}^\dagger \hat{b} + \hat{S}^z + S$ is a conserved quantity. Therefore, the relative strength of the detuning Δ/g is the only free parameter in the system in a given sector with well defined quantum numbers M and S .

The Tavis-Cummings model belongs to the class Gaudin-type models [25] and one can introduce the Bethe wave function

$$|\{\lambda_\alpha\}\rangle = \prod_{\alpha=1}^M \hat{B}(\lambda_\alpha)|0\rangle. \quad (2)$$

The rapidities (or spectral parameters) λ_α , $\alpha = 1, \dots, M$, are complex numbers, while the excitation creation operators $\hat{B}(\lambda)$ and the vacuum $|0\rangle$ are defined as

$$\hat{B}(\lambda) = \hat{b}^\dagger - g \frac{\hat{S}^+}{\lambda}, \quad \text{and} \quad \hat{b}|0\rangle = \hat{S}^-|0\rangle = 0. \quad (3)$$

The action of the Hamiltonian on the Bethe wave function is given by

$$\begin{aligned} \hat{H}|\{\lambda_\alpha\}\rangle = & \left[E_{S,M}(\{\lambda_\alpha\}) + \sum_{\alpha=1}^M f_\alpha(\{\lambda_\alpha\}) \right] \prod_{\alpha=1}^M \hat{B}(\lambda_\alpha)|0\rangle \\ & + g \sum_{\alpha=1}^M \frac{f_\alpha(\{\lambda_\alpha\})}{\lambda_\alpha} \prod_{\beta \neq \alpha} \hat{B}(\lambda_\beta) \hat{b}^\dagger |0\rangle. \end{aligned} \quad (4)$$

where $f_\alpha(\{\lambda_\alpha\})$ is defined as

$$f_\alpha(\{\lambda_\alpha\}) = -\frac{2g^2 S}{\lambda_\alpha} + \lambda_\alpha - \Delta + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^M \frac{2g^2}{\lambda_\alpha - \lambda_\beta}. \quad (5)$$

Equation (4) is known as the off-shell Bethe equation [26]. If the rapidities satisfy the Bethe equations $f_\alpha(\{\lambda_\alpha\}) = 0$, then the Bethe wave function is an eigenstate with the eigenenergy

$$E_{S,M}(\{\lambda_\alpha\}) = \Delta(M - S) - \sum_{\alpha=1}^M \lambda_\alpha. \quad (6)$$

Indeed, it is possible to construct a basis of Bethe states, and this is how the system is solved when the Hamiltonian is time-independent. However, for the Tavis-Cummings model, one can explicitly include the off-shell term in order to describe the dynamics of the wave function under a time-dependent detuning $\Delta(t)$. The Bethe wave function including the off-shell part completely describes the solution of the time-dependent Schrödinger equation with rapidities $\{\lambda_\alpha\}$ moving in time,

$$|\Psi(t)\rangle = \exp[-ie(t)] \prod_{\alpha=1}^M \hat{B}(\lambda_\alpha(t))|0\rangle, \quad (7)$$

with a phase $e(t) = \sum_{\alpha} \int_0^t [E_{S,M}(\lambda_\alpha(t)) + f_\alpha(\lambda_\alpha)] - S\Delta(t)$ and where the rapidities are subject to the following set of equations:

$$i \frac{\dot{\lambda}_\alpha(t)}{\lambda_\alpha(t)} = f_\alpha(\lambda_\alpha(t)). \quad (8)$$

It can be verified that in the stationary case $\dot{\lambda}_\alpha(t) = 0$ the time-dependent wave function (7) reduces to the static one (2) with a phase given by the eigenenergy (6).

The appeal of the representation (8) is its equivalence to an integrable classical many-body problem. After the change of variables

$$\lambda_\alpha(t) = 2x_\alpha^2, \quad (9)$$

the dynamical Bethe equation (8) reads

$$\begin{aligned} \dot{x}_\alpha = & i \frac{g^2 S}{2x_\alpha} + \frac{i\Delta(t)}{2} x_\alpha - ix_\alpha^3 \\ & - \frac{ig^2}{4} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^M \left[\frac{1}{x_\alpha + x_\beta} + \frac{1}{x_\alpha - x_\beta} \right]. \end{aligned} \quad (10)$$

Therefore, it becomes apparent that the x_α move according to a classical Hamiltonian $H_I = \sum_{\alpha=1}^M \frac{p_\alpha^2}{2} + V_\alpha(\{x_\alpha\})$, with potential

$$\begin{aligned} V_\alpha(\{x_\alpha\}) = & \frac{g^4}{16} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^M \left(\frac{1}{(x_\alpha - x_\beta)^2} + \frac{1}{(x_\alpha + x_\beta)^2} \right) \\ & + \frac{1}{2} x_\alpha^6 - \frac{\Delta(t)}{2} x_\alpha^4 + \frac{\gamma(t)}{2} x_\alpha^2 + \frac{g^4 S^2}{8} \frac{1}{x_\alpha^2}, \end{aligned} \quad (11)$$

where

$$\gamma(t) = (M - 1 - S)g^2 - 2g^2 S + \frac{\Delta^2}{4} - i \frac{\dot{\Delta}(t)}{2}. \quad (12)$$

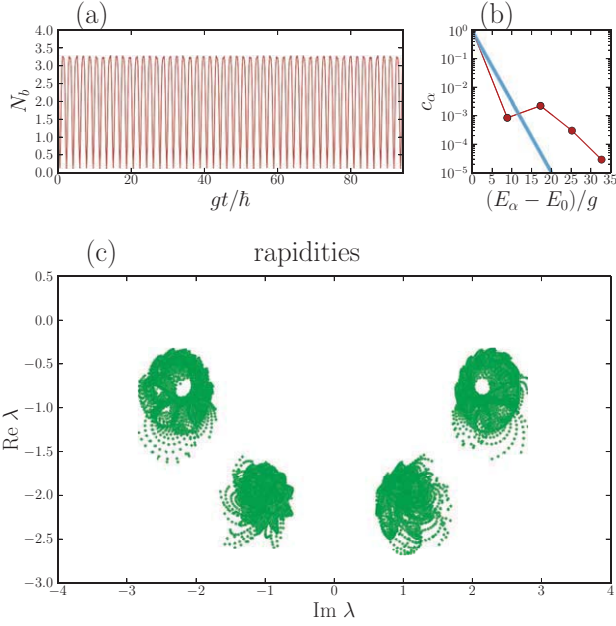


Fig. 1: (Colour on-line) Dynamics of the Tavis-Cummings model driven non-resonantly with the amplitude $\Delta_0/g = 5$ and a frequency $\omega = 3.57g/\hbar$, $S = 6$ and $M = 4$. (a) The boson occupation number N_b monitored over some interval of time, (b) the weights of eigenstates (13) c_α and (c) the stroboscopic maps of all rapidities λ_m , $m = 1, \dots, M$ after 4000 cycles.

This model is a complexified version of the BC-type Izozentsev model [27] and belongs to the family of generalizations of the Calogero model. It is integrable on the classical level for time-independent parameters. We can therefore interpret the full quantum dynamics and breaking of integrability in terms of the classical equations of motion.

Here we break the integrability by the time-dependent driving of the detuning. Namely, we consider the following setup: at $t = 0$ the system is prepared in its ground state at $\Delta = \Delta_0$. Then we evaluate numerically its time evolution under the periodic detuning $\Delta(t) = \Delta_0 \cos(\omega t)$. We solve the time-dependent Schrödinger equation by using a Runge-Kutta integration scheme. The rapidities can be obtained from the coefficients of the wave functions by finding the roots of symmetric polynomials. For illustration of the principle, we choose a small number of excitations, $M = 4$, $S = 6$ and a strong amplitude of the detuning $\Delta_0/g = 5$, such that the bosonic modes are highly occupied initially, $N_b = \langle \hat{b}^\dagger \hat{b} \rangle \approx 3.2$, and the population of excited spins is small. The high driving amplitude causes strong dynamical redistribution of excitations between bosonic and spin degrees of freedom. If the driving frequency is non-resonant, dynamics remain almost adiabatic and observables are expected to exhibit periodic oscillations along the instantaneous ground-state values.

In fig. 1(a) the following example example is shown: At frequency $\omega = 3.57g/\hbar$, there are regular oscillations of

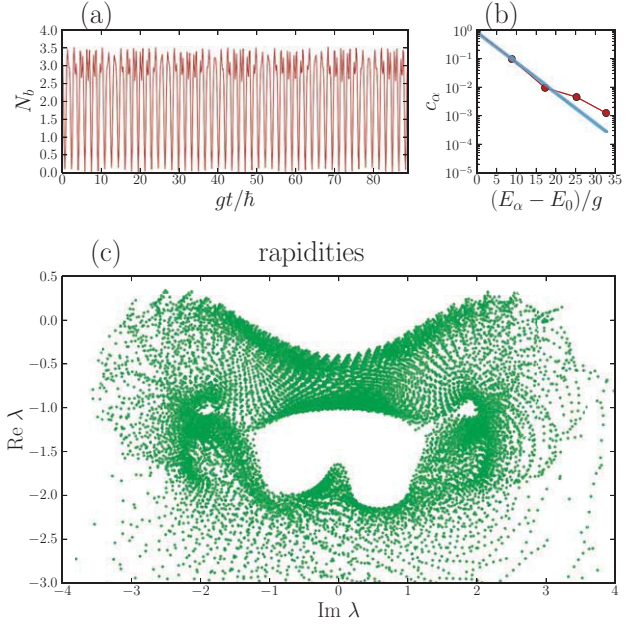


Fig. 2: (Colour on-line) The Tavis-Cummings model driven near-resonantly with amplitude $\Delta_0/g = 5$ and frequency $\omega = 3.68g/\hbar$. For explanations see caption of fig. 1.

the boson populations $N_b(t)$ between 3.2 and 0.2. The rapidities, which correspond to the position variables of the classical model (11), are monitored stroboscopically after each cycle (*i.e.* at time $t_p = 2\pi p/\omega$, $p = 0, \dots, P$, where $P = 4000$ in the present case) by collapsing them onto a single complex plane. Figure 1(c) shows that in this non-resonant case the rapidities cluster on circles located around the ground state positions. These circles indicate the existence of stable KAM-tori in the 16-dimensional phase space of the classical system and according to our correspondence between dynamics of the quantum and the auxiliary classical system, we can classify such behavior as nearly integrable. In order to characterize the statistical properties of this system we measure the distribution of states averaged over all cycles

$$c_\alpha = \frac{1}{P} \sum_p |\langle \psi(t_p) | \alpha \rangle|^2, \quad (13)$$

where $|\psi(t_p)\rangle$ is a state of the driven system at $t = t_p$ and $|\alpha\rangle$ are the eigenstates of the Hamiltonian $H(t_p)$ after p cycles. In fig. 1(b), we found their distribution to decay rapidly, which is expected in this nearly adiabatically driven system. We compare this distribution to a Boltzmann distribution, $c_\alpha = e^{-\beta E_\alpha}/Z$, with the same average energy. It turns out that the Boltzmann distribution cannot describe the weights. Figure 1(b) also shows that very few energy is pumped into the system.

In fig. 2 we consider a slight increase of the frequency with respect to non-resonant case to $\omega = 3.68g/\hbar$. The boson occupation, which starts to exhibit an additional beating frequency (fig. 2(a)), suggests that a resonance

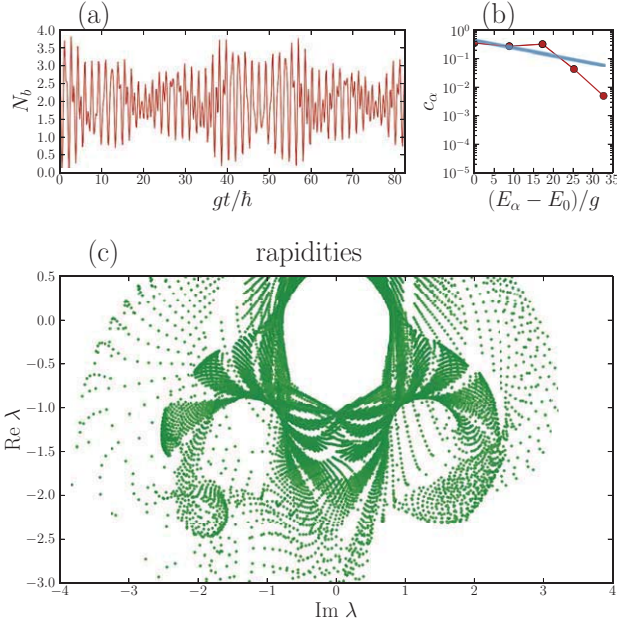


Fig. 3: (Colour on-line) The Tavis-Cummings model driven with amplitude $\Delta_0/g = 5$ and frequency $\omega = 3.75g/\hbar$. For explanations see caption of fig. 1.

is approached in the quantum model. Interestingly, this comes along with a scattering of the rapidities on the collapsed 2-dimensional stroboscopic maps (fig. 2(c)). From the point of view of the auxiliary classical system, these dynamics rather strongly deviate from the integrable limit. It has to be noted that despite the relatively dense exploration of the phase space, we could not find an indication of truly chaotic behavior. Nevertheless, and despite the small number of degrees of freedom in the system, this leads to a state distribution remarkably close to Boltzmann distribution (fig. 2(b)).

Further increasing the driving frequency to $\omega = 3.75g/\hbar$ as in fig. 3 leads to strongly beating dynamics of boson occupancies. This resonance of the quantum model leads to a new structured pattern in the stroboscopic map of the classical variables. This hints that there are new emerging quasiperiodic orbits, which reside on a topological structure different from the one of the near-adiabatic case. The state distribution in fig. 3(c) shows that the weights deviate considerably from the Boltzmann distribution. Unlike in the non-resonant case (fig. 1(c)), a large amount of energy is absorbed by the system.

The cycle structure, well-localized rapidities in the non-resonant cases, and a special pattern in a resonantly driven cases, repeats when further increasing driving frequencies or by modifying other parameters. A special case is the two-particle problem, $M = 1$, where a ring-pattern is transformed into a line and back to a ring upon changing the driving frequency.

In summary, we derived a correspondence between a time-dependent quantum model and an auxiliary classical

system. The strength of this approach is illustrated by an example of a driven Tavis-Cummings model with a frequency tuned from a non-resonant to a resonant value. The emerging dynamics can be interpreted in terms of the classical underlying system, whose trajectories show very different pattern in their stroboscopic maps. At the point where one pattern is deformed into the other, irregularity in the classical dynamics is most pronounced and time-averages of quantum observables approached thermal equilibrium.

The Tavis-Cummings model belongs to the special class of homogeneous Gaudin models. The fact that the rapidities can be used to describe the full quantum dynamics is due to the completeness of the off-shell BA for the homogeneous model. Therefore, extending the approach to the inhomogeneous Tavis-Cummings model [28] or Richardson models [11] is not straightforward. For these models, it is impossible to describe an arbitrary state in terms of a single off-shell Bethe state with time-dependent rapidities, and a linear superposition of these states would be necessary to capture the dynamics of these systems. We believe that the approach based on the separation of variables [29] could give an important insight into a further development of our approach.

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