Cofinite hyperbolic Coxeter groups, minimal growth rate and Pisot numbers

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Abstract. By a result of R. Meyerhoff, it is known that among all cusped hyperbolic 3-orbifolds the quotient of $\mathbb{H}^3$ by the tetrahedral Coxeter group (3,3,6) has minimal volume. We prove that the group (3,3,6) has smallest growth rate among all non-cocompact cofinite hyperbolic Coxeter groups, and that it is as such unique. This result extends to three dimensions some work of W. Floyd who showed that the Coxeter triangle group (3,∞) has minimal growth rate among all non-cocompact cofinite planar hyperbolic Coxeter groups. In contrast to Floyd's result, the growth rate of the tetrahedral group (3,3,6) is not a Pisot number.

Keywords: Hyperbolic Coxeter group, cusp, growth rate, Pisot number

1. Introduction

Let $\mathbb{H}^n$ denote the standard hyperbolic n-space. A Coxeter polytope $P \subset \mathbb{H}^n$ is a convex polytope all of whose dihedral angles are of the form π/k for an integer $k \geq 2$. We always assume that P is of finite volume so that it is bounded by finitely many hyperplanes H_i, $i \in I$. The reflections s_i with respect to H_i, $i \in I$, generate a discrete group G of hyperbolic isometries which is a Coxeter group $G = (G, S)$ with presentation $< S \mid R >$ where

$$S = \{\, s_i \mid i \in I \,\} \quad , \quad R = \{\, s_i^2 = 1 \,,\, (s_i s_j)^{k_{ij}} = 1 \mid i, j \in I \,,\, i \neq j \,\} \quad . \tag{1.1}$$

In (1.1), the exponents k_{ij} are integers ≥ 2, symmetric with respect to i, j, and related to the dihedral angles formed by H_i, H_j when intersecting in $\mathbb{H}^n$. We often represent G (and P) by its Coxeter graph Σ or its Coxeter symbol if the presentation (1.1) for G is simple enough (see §2.2).

In the focus of this work are non-compact Coxeter polyhedra $P \subset \mathbb{H}^3$ of finite volume which form a vast, infinite set (see §2.2). The Coxeter tetrahedron with graph

$$\Sigma_* \quad : \quad \bullet\!-\!\!\!-\!\!\!-\!\bullet\!-\!\!\!-\!\!\!-\!\bullet\overset{6}{-\!\!\!-\!\!\!-}\!\bullet \tag{1.2}$$

and Coxeter symbol $(3,3,6)$ is of particular importance. It is a building block for an ideal regular tetrahedron and has one vertex at infinity. It yields the 1-cusped quotient space

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$\mathbb{H}^3/(3,3,6)$ which has minimal volume among *all* cusped hyperbolic 3-orbifolds as proven by Meyerhoff [M].

We shall study another quantity related to hyperbolic Coxeter groups $G = (G, S)$, namely the growth rate τ_G associated to the growth series (see §3)

$$f_S(x) = 1 + |S|\,x + \sum_{n\geq 2} a_n x^n \quad , \quad x \in \mathbb{C} \quad , \tag{1.3}$$

where a_n denotes the number of words $w \in G$ of S-length equal to n. More precisely, the growth series f_S of G has radius of convergence $R < 1$ and is the series expansion of a rational function $p(x)/q(x)$ with coprime elements $p(x), q(x) \in \mathbb{Z}[x]$. With this said, the growth rate τ_G is defined to be the reciprocal of R. It follows that $\tau_G > 1$ is a root of maximal absolute value of $q(x)$ and an algebraic integer. As such τ_G is an interesting object in the realm of Salem numbers, Pisot numbers and Perron numbers (see §3.2). We shall prove the following main result of this work (see §4).

Theorem. *Among all hyperbolic Coxeter groups with non-compact fundamental polyhedron of finite volume in* $\mathbb{H}^3$*, the tetrahedral group* $(3,3,6)$ *has minimal growth rate, and as such the group is unique.*

The Theorem completes the picture of growth rate minimality for cofinite hyperbolic Coxeter groups in three dimensions. Indeed, in [KKol], we showed that the growth rate of the group $(3,5,3)$ is minimal among all growth rates (being Salem numbers) of Coxeter groups acting cocompactly on $\mathbb{H}^3$.

Let us briefly discuss the proof of the Theorem. We exploit Steinberg's formula

$$\frac{1}{f_S(x^{-1})} = \sum_{\substack{G_T < G \\ |G_T| < \infty}} \frac{(-1)^{|T|}}{f_T(x)} \tag{1.4}$$

expressing $f_S(x^{-1})$ in terms of the growth functions $f_T(x)$ of the *finite* subgroups G_T of G. Each part $f_T(x)$ in (1.4) is, by Solomon's formula, a product of certain polynomials related to the Coxeter exponents of T (see §3). Our first observation is that these exponents satisfy a certain monotonicity property (see §4.1). Although the function $f_G(x) := f_S(x)$ is - in presence of ideal vertices of P - not anti-reciprocal anymore (see §3.1), we are able to prove

$$\frac{1}{f_G(x)} < \frac{1}{f_{(3,3,6)}(x)} \quad \text{for all} \quad x \in (0, 1/\tau_{(3,3,6)}] \tag{1.5}$$

for Coxeter groups G different from $(3,3,6)$ in the following way. By Andreev's Theorem (see §2.2), the ideal vertices of P are either 3-valent or 4-valent so that the set of vertices of P can be partitioned according to $\Omega_0 = \Omega_f \cup \Omega^3_\infty \cup \Omega^4_\infty$ where Ω_f is the set of finite (3-valent) vertices of P. If $\Omega^4_\infty \neq \emptyset$, a result of Kolpakov [Kol] shows that τ_G is the limit of an increasing sequence of growth rates τ_{G_n} for Coxeter groups G_n having less 4-valent ideal vertices than G. This implies that we can restrict to the case $\Omega^4_\infty = \emptyset$ and consider *simple* Coxeter polyhedra, allowing us to identify $1/f_G(x)$ in a new and very efficient way (see (4.9), (4.11) and (4.12)), and at the end, to verify (1.5).

Finally, the proof of the Theorem can be easily adapted to the two-dimensional case and provides an elementary verification of the result of Floyd [F] that the Coxeter triangle group $(3,\infty)$, which is closely related to the modular group $SL(2,\mathbb{Z})$, has minimal growth rate among all hyperbolic Coxeter groups with non-compact fundamental polygon of finite area in $\mathbb{H}^2$. Notice that our proof (see §5.2) does not rely upon the theory of Pisot polynomials, the identification of $\tau_{(3,\infty)}$ with the Pisot number $\alpha_S \simeq 1.324718$ of $x^3 - x - 1$, and the result of Smyth [Sm] that α_S is the smallest Pisot number (see §3.2). In this context, observe that the growth rate $\tau_{(3,3,6)}$ is not a Pisot number but a Perron number, at least (see Remark, §5.2, §5.1 and [KoU2]).

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2. Cofinite hyperbolic Coxeter groups

2.1. Convex hyperbolic polytopes. Denote by $\mathbb{X}^n$, $n \geq 2$, either the standard hyperbolic n-space $\mathbb{H}^n$, the unit sphere $\mathbb{S}^n$, or the Euclidean space $\mathbb{E}^n$. Interpret $\mathbb{X}^n \neq \mathbb{E}^n$ in its vector space model, that is, $\mathbb{X}^n$ is a subset of a real vector space $\mathbb{Y}^{n+1}$ equipped with a bilinear form $< \cdot, \cdot >$ inducing the metric structure on $\mathbb{X}^n$. In particular, we view hyperbolic space $\mathbb{H}^n$ as embedded in the Lorentz-Minkowski space $\mathbb{Y}^{n+1} = \mathbb{E}^{n,1}$ of signature $(n,1)$ so that points of the boundary $\partial\mathbb{H}^n$ are vectors of vanishing norm. A convex polytope $P \subset \mathbb{X}^n$ is defined to be the intersection of finitely many half-spaces bounded by hyperplanes H_i , $i \in I$, in $\mathbb{X}^n$, where each H_i can be written as orthogonal complement of a vector $e_i \in \mathbb{Y}^{n+1}$ of positive norm, directed outwards with respect to P, say. In the sequel, we consider convex polytopes of finite volume, only. In the hyperbolic context, the finite volume condition is equivalent to the property that P is the convex hull of finitely many points or *vertices* in $\mathbb{H}^n \cup \partial\mathbb{H}^n$. A vertex $v \in P$ lying in $\mathbb{H}^n$ is called a *finite* vertex, and a vertex $v_\infty \in \overline{P}$ lying on $\partial\mathbb{H}^n$ is called an *ideal* vertex of P. If all vertices of P are finite, then P is compact. If all vertices of P are ideal, we call P an *ideal* hyperbolic polytope. Consider the Gram matrix $\mathrm{Gram}(P)$ associated to the vectors e_i , $i \in I$, whose non-diagonal entries are metrically related to the dihedral angles and distances between the hyperplanes H_i bounding P. In particular, a non-diagonal entry g_{ij} of $\mathrm{Gram}(P)$, which is of absolute value smaller than one, can be interpreted according to

$$g_{ij} =< e_i, e_j >= -\cos\angle(H_i, H_j) \quad ,$$

that is, the hyperplanes H_i, H_j intersect under the dihedral angle $\angle(H_i, H_j)$ in the face $F_{ij} = H_i \cap H_j \cap P$ of P. In [V1, chapter I] and [V2, part I, chapter 6], Vinberg developped explicit criteria for the existence of an acute-angled polytope $P \subset \mathbb{H}^n$ in terms of the Gram matrix, as well as criteria for compactness, finite volume and vertices to be finite or ideal. As an example, a hyperbolic tetrahedron $S \subset \mathbb{H}^3$ with dihedral angles not bigger than $\pi/2$ is characterised by a 4×4 matrix $G = (g_{ij})$ with $g_{ii} = 1$ such that the signature of G equals $(3, 1)$. A vertex $v \in S$ is finite if the principal submatrix G_v, formed by the three hyperplanes passing through v, is positive definite, while an ideal vertex $v_\infty \in \overline{S}$ is characterised by a principal submatrix G_{v_∞} which is positive semi-definite.

2.2. Coxeter polytopes and Coxeter groups in $\mathbb{X}^n$. A *Coxeter polytope* $P \subset \mathbb{X}^n$ is a convex polytope such that all its dihedral angles are submultiples of π, that is, they are of the form π/k with an integer $k \geq 2$. We call two hyperplanes in $\mathbb{H}^n$ *parallel* resp. *ultra-parallel* if they meet on the boundary $\partial\mathbb{H}^n$ resp. if they admit a common perpendicular in $\mathbb{H}^n$ realising their distance. For a given Coxeter polytope $P \subset \mathbb{X}^n$, consider the group G generated by the reflections s_i in the hyperplanes H_i bounding P. G is called a *geometric Coxeter group*. It is known that $G \subset \mathrm{Isom}(\mathbb{X}^n)$ is a discrete group with fundamental domain P. If P is compact (or of finite volume), the group G is called *cocompact* (or *cofinite*). Notice that a compact acute-angled polytope P in $\mathbb{X}^n$ is *simple*, that is, each k-dimensional face of P is contained in precisely $n-k$ bounding hyperplanes of P (cf. [V1, §3]). In particular, a vertex of a simple Coxeter polytope is contained in exactly $n-1$ bounding hyperplanes of P. Denote by f_k , $k = 0, \ldots, n-1$, the number of k-dimensional faces of P. By the Euler-Schläfli identity, one has

$$\sum_{k=0}^{n-1} (-1)^k f_k = 1 - (-1)^n \quad . \tag{2.1}$$

If P is simple, then obviously $n f_0 = 2 f_1$. We shall be mainly interested in non-cocompact but cofinite hyperbolic Coxeter groups in $\mathrm{Isom}(\mathbb{H}^3)$. In this particular case, Andreev's existence theorem for acute-angled polyhedra in hyperbolic 3-space (cf. [V2, part I, p. 112], for example) implies that vertices of the associated Coxeter polyhedron P are k-valent (intersections of exactly k edges of P) for $k = 3$ or $k = 4$, only. In particular, a compact or a simple Coxeter polyhedron $P \subset \mathbb{H}^3$ satisfies, by (2.1), $f_0 - f_1 + f_2 = 2$ and $3f_0 = 2f_1$ so that the number $f_0 = 2\,(f_2 - 2) \geq 4$ of vertices is even, and the number f_1 of edges and dihedral angles of P satisfies $f_1 = 3\,(f_2 - 2)$.

Consider a geometric Coxeter group with fundamental polytope $P \subset \mathbb{X}^n$. Denote by $S = \{s_i \mid i \in I\}$ the set of generators of G. Together with the set R of relations

$$s_i^2 = 1 \quad , \quad (s_i s_j)^{k_{ij}} = 1 \quad \text{if} \quad \angle(H_i, H_j) = \frac{\pi}{k_{ij}} \quad , \tag{2.2}$$

we obtain the presentation $< S \mid R >$ for G. The stabiliser of any vertex of P is generated by the reflections in the hyperplanes passing through it and gives rise to a subgroup of G which itself is a geometric Coxeter group G_T for some $T \subset S$.

For simple presentations we prefer a description of G (and of its subgroups) by means of *Coxeter diagrams.* More precisely, the Coxeter diagram $\Sigma = \Sigma(G) = \Sigma(P)$ of a geometric Coxeter group G (and its fundamental Coxeter polytope P) consists of nodes ν_i , $i \in I$, corresponding to the reflections s_i (and its mirrors H_i), which are pairwise connected by a weighted edge $\varepsilon_{ij} = \nu_i\nu_j$ if the hyperplanes H_i, H_j are not orthogonal. For hyperplanes forming the dihedral angle $\angle(H_i, H_j) = \pi/3$, the edge ε_{ij} is drawn without weight, while for hyperplanes intersecting with dihedral angle π/k , $k \geq 4$, the edge is marked by k. The nodes in Σ corresponding to parallel hyperplanes are connected by an edge with weight ∞, while ultra-parallel hyperbolic hyperplanes give rise to nodes joined by a dotted edge (omitting the weight given by their hyperbolic distance). The *order* and the *rank* of the diagram $\Sigma = \Sigma(P)$ are defined by the cardinality of S and by the rank of the Gram matrix of P. Furthermore, Σ is called *elliptic*, *parabolic* or *hyperbolic* if the Coxeter group (and the Coxeter polytope P) is spherical, euclidean or hyperbolic. Hence, by Vinberg's criterion (see §2.1), a finite (resp. ideal) vertex of a Coxeter polytope $P \subset \mathbb{H}^n$ gives rise to an elliptic Coxeter diagram of rank $n-1$ (resp. parabolic Coxeter diagram of rank $n-2$) which can be identified with a certain subdiagram of Σ (see [V2, part II, chapter 5]). All connected elliptic and parabolic Coxeter diagrams, and therefore all irreducible spherical and euclidean Coxeter groups, were classified by Coxeter [Co] in 1934. In Table 1, we reproduce his results for order two and three, only. Furthermore, for the elliptic diagrams, we add the associated exponents (cf. [CoMo, 9.7]).

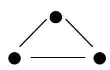
Elliptic case			*Parabolic case*	
Diagram	Notation	Exponents	Diagram	Notation
$\bullet\overset{k}{\text{——}}\bullet$	I_k	$1, k-1$	$\bullet\overset{\infty}{\text{——}}\bullet$	$\widetilde{A}_1$
$\bullet\text{——}\bullet\text{——}\bullet$	A_3	$1, 2, 3$		$\widetilde{A}_2$
$\bullet\text{——}\bullet\overset{4}{\text{——}}\bullet$	B_3	$1, 3, 5$	$\bullet\overset{4}{\text{——}}\bullet\overset{4}{\text{——}}\bullet$	$\widetilde{B}_2$
$\bullet\text{——}\bullet\overset{5}{\text{——}}\bullet$	H_3	$1, 5, 9$	$\bullet\text{——}\bullet\overset{6}{\text{——}}\bullet$	$\widetilde{I}$

Table 1. *Connected elliptic and parabolic Coxeter diagrams of orders 2 and 3*

For the description of geometric Coxeter groups given by linear diagrams

$$\bullet\overset{k_1}{\text{———}}\bullet-\cdots-\bullet\overset{k_n}{\text{———}}\bullet \quad , \tag{2.3}$$

we use the *Coxeter symbol* $(k_1, \ldots, k_n)$. Diagrams of type (2.3) describe so-called *Coxeter orthoschemes* and play an important role in the theory of regular polytopes and tesselations (cf. [V2, part II, chapter 5, §3]). We are particularly interested in the hyperbolic Coxeter orthoscheme $(3,3,6)$ in $\mathbb{H}^3$, having precisely one ideal vertex related to the group $\widetilde{I} = (3,6)$, and which forms a building block for an ideal regular tetrahedron. Let us provide some further instructive examples.

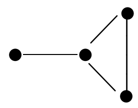
Example 1. The Coxeter diagram

$$\Sigma_2 \quad : \quad \bullet\!\!-\!\!-\!\!-\!\!\bullet\overset{\infty}{-\!\!-\!\!-\!\!-}\!\!\bullet \tag{2.4}$$

is intimately related to the modular group $\mathrm{SL}(2,\mathbb{Z})$. By a result of C. L. Siegel (see [F], for example), it provides the unique non-compact hyperbolic 2-orbifold $Q_2 = \mathbb{H}^2/\Sigma_2$ of minimal volume.

Example 2. The Coxeter graph

Σ_3^2 :



(2.5)

describes a hyperbolic tetrahedron with precisely one ideal and three finite vertices. As indicated by the two-fold graph symmetry in (2.5), the tetrahedron has an internal symmetry plane along which it can be dissected into two isometric copies of the Coxeter orthoscheme $\Sigma_3 := (3,3,6)$. Hence, the tetrahedron Σ_3^2 is the *double* with twice the volume of Σ_3. Let us point out that the quotient space $\mathbb{H}^3/(3,3,6)$ is distinguished by the fact that it has minimal volume among all cusped hyperbolic 3-orbifolds. This is a result of Meyerhoff [M].

Example 3. Let $p,q,r \geq 2$ be integers such that $1/p + 1/q < 1/2 \leq 1/q + 1/r$, and consider the Coxeter diagram of order five

$$\Sigma_{p,q,r} \quad : \quad \bullet\overset{p}{-\!\!-\!\!-\!\!-}\bullet\overset{q}{-\!\!-\!\!-\!\!-}\bullet\overset{r}{-\!\!-\!\!-\!\!-}\bullet\cdots\bullet \quad . \tag{2.6}$$

By Vinberg's criterion, one sees that the graph $\Sigma_{p,q,r}$ describes a straight triangular hyperbolic Coxeter prism of finite volume, that is, a prism with one triangular face F_1 being orthogonal to the three quadrilateral faces. The prism has five finite vertices and, according to the (in-)equality $1/q + 1/r \geq 1/2$, one further (finite or) ideal vertex. The infinite sequence (2.6) is related to infinite-volume hyperbolic Coxeter orthoschemes (p,q,r) having one ultra-ideal vertex v_* (with positive norm and with triangular vertex figure (p,q)). By cutting off from (p,q,r) the part of infinite volume by means of the associated polar plane $H_* = \{x \in \mathbb{H}^3 \mid < x, v_* >= 0\}$, we get a finite-volume triangular straight Coxeter prism in $\mathbb{H}^3$, also called a simply truncated Coxeter 3-orthoscheme. In the limiting case

$$\Sigma_{\infty,q,r} \quad : \quad \bullet\overset{\infty}{-\!\!-\!\!-\!\!-}\bullet\overset{q}{-\!\!-\!\!-\!\!-}\bullet\overset{r}{-\!\!-\!\!-\!\!-}\bullet\overset{\infty}{-\!\!-\!\!-\!\!-}\bullet \quad , \quad \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2} \quad , \tag{2.7}$$

the polar plane H_* is parallel to the second triangular face F_2. The polyhedron (2.7) has the combinatorial type of a pyramid over a product of two segments and has exactly one 4-valent vertex. Hence, the polyhedron $\Sigma_{\infty,q,r}$ is not simple.

In contrast to the spherical and euclidean cases, the classification of cofinite hyperbolic Coxeter groups and Coxeter polytopes is not available and far out of reach. For some families of given simple combinatorial type, there are complete classification results. For example, non-compact Coxeter simplices were classified by Koszul [Kos], and straight Coxeter prisms of finite volume were classified by Kaplinskaja [Ka]. Non-compact Coxeter simplices of finite volume exist up to dimension nine. For their volumes, we refer to [JKRT, p. 347-348]. The 23 examples in dimension three are listed in Table 2.

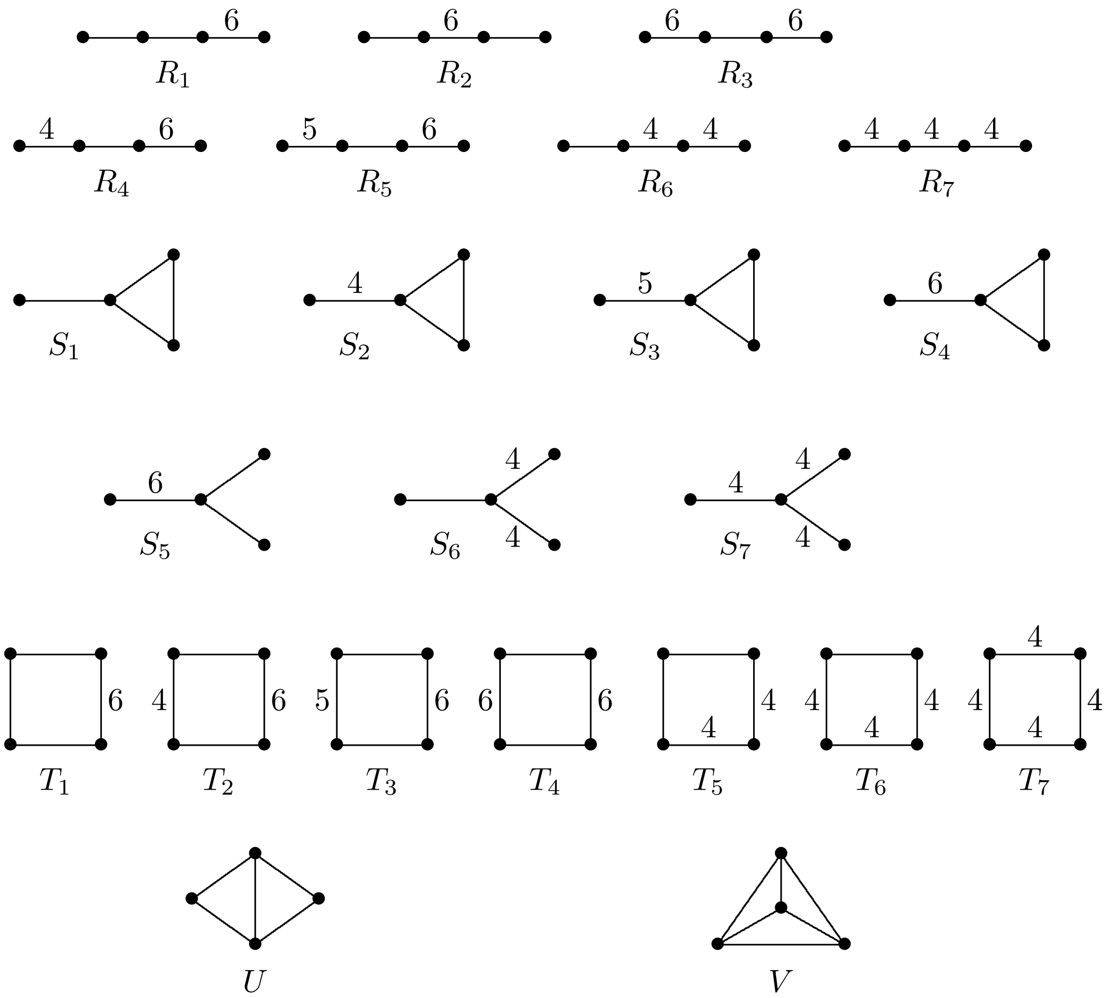


Table 2. *The 23 non-compact hyperbolic Coxeter tetrahedra of finite volume*

3. Growth rates of cofinite hyperbolic Coxeter groups

3.1. Growth functions and growth rates. Let G be a geometric Coxeter group with set S of natural generators, and denote by $P \subset \mathbb{X}^n$ a Coxeter fundamental domain for G.

The (spherical) growth series of (G,S)

$$f_S(x)=1+|S|\,x+\sum_{k\geq 2}a_kx^k\quad , \tag{3.1}$$

where a_k is the number of words in G of S-length k, is by Steinberg's result [St] the series expansion of a rational function, that is,

$$f_S(x)=\frac{p(x)}{q(x)}\quad , \quad \text{where} \quad p,q\in\mathbb{Z}[x] \tag{3.2}$$

are coprime polynomials. In order to investigate growth functions, Steinberg's formula [St]

$$\frac{1}{f_S(x^{-1})}=\sum_{\substack{G_T<G\\ |G_T|<\infty}}\frac{(-1)^{|T|}}{f_T(x)} \tag{3.3}$$

is very important since it allows us to compute the growth function of a *given* group in terms of the growth functions of its *finite* subgroups G_T. Each such G_T is notabene a spherical Coxeter group acting as stabiliser of a certain face of P and yields a polynomial term $f_T(x)$ in (3.2). In the case of a reducible group $(G_T,T)=(G_{T_1}\times G_{T_2},T_1\cup T_2)$, the polynomial $f_T(x)$ equals the product of the growth polynomials of (G_{T_1},T_1) and (G_{T_2},T_2). By a result of Solomon [So], the function $f_T(x)$ can be derived easily in terms of its *Coxeter exponents* $m_1=1,m_2,\ldots,m_t$ (cf. [CoMo]) according to

$$f_T(x)=\prod_{i=1}^{t}\,[m_i+1]\quad , \tag{3.4}$$

where each factor in (3.4) is a polynomial of type $[k]=1+x+\cdots+x^{k-1}$. Later, we shall write $[k,l]=[k]\cdot[l]$ and so on.

Notation	Symbol	m_1,m_2,m_3	$f_S(x)$
A_1	$-$	1	$[2]$
D_2^k , $k\geq 2$	(k)	$1,k-1$	$[2,k]$
A_3	$(3,3)$	$1,2,3$	$[2,3,4]$
B_3	$(4,3)$	$1,3,5$	$[2,4,6]$
H_3	$(5,3)$	$1,5,9$	$[2,6,10]$

Table 3. *Growth of irreducible finite Coxeter groups of rank at most three*

Since

$$[k](x)=x^{k-1}\cdot[k](x^{-1}) \tag{3.5}$$

for each positive integer k, the polynomial $f = f_T$ (of degree d, say) in (3.3) is not only monic but also palindromic, that is, $f(x) = x^d f(x^{-1})$. This property and formulas (3.2), (3.3) imply that $p(0) = 1$, which together with $f_S(0) = 1$ (see (3.1)) yields $q(0) = 1$.

In the cofinite hyperbolic case, the radius of convergence R of the infinite series $f_S(x)$ is smaller than 1 (see [Ha]), and its inverse

$$\tau := \limsup_{k\to\infty} \sqrt[k]{a_k} > 1 \tag{3.6}$$

is called the *growth rate* of (G, S) (and of P). By (3.2), R is equal to the smallest real positive root of $q(x)$, and by (3.3) and (3.4), the growth rate $\tau = 1/R > 1$ is an algebraic integer.

If G acts cocompactly on $\mathbb{H}^3$, then the growth function $f(x) := f_S(x)$ is anti-reciprocal, that is, $f(x^{-1}) = -f(x)$ (cf. [ChD]). Since $p(x)$ is palindromic by (3.3) and (3.4), the denominator $q(x)$ (of degree d, say) is anti-palindromic, that is, $q(x) = -x^d q(x^{-1})$. Therefore, τ and $1/\tau$ are Galois conjugates.

In the non-cocompact case, the anti-reciprocity property does not hold anymore. As an illustration, consider the non-cocompact cofinite Coxeter groups $G_1 = R_1 = (3, 3, 6)$, $G_2 = V$, which is related to an ideal regular tetrahedron, and $G_3 = \Sigma_{\infty,3,3}$ (see Table 2 and (2.7)). By (3.3) and Table 3, and by using some well-known factorisation properties of $[k]$ such as $[2k] = (x^k + 1)\,[k]$, their growth functions f_1, f_2 and f_3 can be computed as follows (cf. also [KoU1, Proposition 1]).

$$\begin{aligned} f_1(x) &= \frac{[2,2,2,3]\,(x^2+1)\,(x^2-x+1)}{(x-1)\,(x^7+x^6+x^5+x^4-1)} \quad , \\ f_2(x) &= \frac{[2,3]}{(x-1)(3\,x^2+x-1)} \quad , \\ f_3(x) &= \frac{[2,2,2,3]\,(x^2+1)}{(x-1)(x^5+2\,x^4+2\,x^3+x^2-1)} \quad . \end{aligned} \tag{3.7}$$

The fact that $x = 1$ is a pole of f_i follows from the vanishing of the Euler characteristic $\chi(G_i)$. For the numerators p_i and denominators q_i of f_i $(i = 1, 2, 3)$ in (3.7), we see that $\deg p_1 \neq \deg q_1$, q_2 is not monic, and we calculate

$$\begin{aligned} x^9 q_1(x^{-1}) &= x^9 - x^8 - x^5 + x = x\,(x-1)\,(x^7 - x^3 - x^2 - x - 1) \quad , \\ x^3 q_2(x^{-1}) &= x^3 - 2\,x^2 - 2\,x + 3 = (x-1)\,(x^2 - x - 3) \quad , \\ x^6 q_3(x^{-1}) &= x^6 - x^5 - x^4 - x^3 + x + 1 = (x-1)\,(x^5 - x^3 - 2\,x^2 - 2\,x - 1) \quad . \end{aligned}$$

As a consequence, the functions f_i are not anti-reciprocal. The growth rates are given by

$$\tau((3,3,6)) \simeq 1.296466 \quad , \quad \tau(V) \simeq 2.302776 \quad , \quad \tau(\Sigma_{\infty,3,3}) \simeq 1.734691 \quad . \tag{3.8}$$

Finally, one can check numerically that the Galois conjugates of $\tau((3, 3, 6))$ lie inside and outside of the unit circle but are all of absolute value strictly smaller than $\tau((3, 3, 6))$. The Galois conjugates of $\tau(\Sigma_{\infty,3,3})$ lie all inside the unit circle.

3.2. Pisot numbers and Perron numbers. A very interesting arithmetic aspect in the study of growth rates of cofinite hyperbolic Coxeter groups is that certain classes of real algebraic integers show up. An algebraic integer $\sigma > 1$ is a *Salem number* if its inverse $1/\sigma$ is a Galois conjugate of σ and all other Galois conjugates lie on the unit circle. It is known by results of Cannon, Wagreich and Parry (see [Pa], for example) that the growth rate of a Coxeter group acting cocompactly on $\mathbb{H}^2$ or $\mathbb{H}^3$ is a Salem number. In [KKol], it was shown that the cocompact hyperbolic Coxeter group of minimal growth rate in three dimensions is the tetrahedral group $(3,5,3)$, while E. Hironaka proved in [Hi] that the triangle group $(3,7)$ is the cocompact hyperbolic Coxeter group of minimal growth rate in two dimensions. Both Coxeter groups are closely related to the (unique) compact hyperbolic orbifolds of minimal volume in dimensions two and three (see [KKol], for example).

An algebraic integer $\alpha > 1$ is a *Pisot-Vijayaraghavan number*, or a *Pisot number* for short, if all its Galois conjugates are less than 1 in absolute value. In contrast to Salem numbers, the smallest Pisot number is known. More precisely, Smyth [Sm] proved that this one is given by the algebraic integer $\alpha_S \simeq 1.324718$ with minimal polynomial $x^3 - x - 1$. Floyd [F] proved that the growth rate of any non-cocompact cofinite *planar* hyperbolic Coxeter group is a Pisot number and then, based on Smyth's result, that the smallest growth rate equals α_S and is realised by the triangle group $(3,\infty)$.

Finally, an algebraic integer $\beta > 1$ is called a *Perron number* if all its Galois conjugates are less than β in absolute value. Of course, any Pisot or Salem number is a Perron number.

In [KoU2], Komori and Umemoto show among other things that the growth rates of the non-cocompact groups $G_i, i = 1,2,3$, in §3.1 and of the Coxeter tetrahedra in Table 2 are all Perron numbers. However, neither $\tau((3,3,6))$ nor $\tau(V)$ are Pisot number. While this is evident for $\tau(V)$, having minimal polynomial $x^2 - x - 3$, the verification for $\tau((3,3,6))$, with minimal polynomial $x^7 - x^3 - x^2 - x - 1$, follows by comparing $\tau((3,3,6)) < \alpha_S$ and by using Smyth's minimality result mentionned above (see (3.8)). In contrast to this, a numerical check shows that $\tau(\Sigma_{\infty,3,3})$ is a Pisot number with minimal polynomial $x^5 - x^3 - 2\,x^2 - 2\,x - 1$. This can be shown rigorously as follows (see Example 4 below).

In [Kol], a geometric characterisation of Pisot numbers has been proven which explains to some extent the above discrepancies. Consider a Coxeter polyhedron $P \subset \mathbb{H}^3$ of finite volume. An edge e of P is a *ridge of type* $< 2,2,n,2,2 >$ if e is bounded with 3-valent vertices v, w such that the dihedral angles at the incident edges equal $\pi/2$ while the dihedral angle at the edge e equals π/n. If a Coxeter polyhedron P_∞ has a 4-valent ideal vertex, then Vinberg [Vi2, p. 238] indicated the following degeneration feature which was proved in detail by Kolpakov [Kol, Proposition 2].

Proposition 1. *Let $P_\infty \subset \mathbb{H}^3$ be a Coxeter polyhedron of finite volume with at least one 4-valent ideal vertex v_∞. Then there exists a sequence of finite-volume Coxeter polyhedra $P_n \subset \mathbb{H}^3$ having the same combinatorial type and dihedral angles as P_∞ except for a ridge e of type $< 2,2,n,2,2 >$ with n sufficiently large, giving rise to the vertex v_∞ under contraction of e as $n \to \infty$.*

Based on this point of view, Kolpakov [Kol, Proposition 3 and Theorem 5] proved the following results, which generalise Floyd's work [F] from the planar to the spatial case.

Proposition 2. *Let $P_\infty \subset \mathbb{H}^3$ be a Coxeter polyhedron of finite volume with at least one 4-valent ideal vertex obtained from a sequence of finite-volume Coxeter polyhedra P_n by contraction of a ridge of type $< 2,2,n,2,2 >$ as $n \to \infty$. Then, the growth rates $\tau(P_n)$ tend from below to the growth rate $\tau(P_\infty)$.*

Proposition 3. *Let $P_n \subset \mathbb{H}^3$ be a compact Coxeter polyhedron with a ridge e of type $< 2,2,n,2,2 >$ for sufficiently large n. Denote by P_∞ the polyhedron arising by contraction of the ridge e. Let τ_n and τ_∞ be the growth rates of P_n and P_∞, respectively. Then $\tau_n < \tau_\infty$ for all n, and $\tau_n \to \tau_\infty$ as $n \to \infty$. Furthermore, τ_∞ is a Pisot number.*

Example 4. Consider the sequence $\Sigma_{p,3,3}$, $p \geq 7$, of cocompact hyperbolic Coxeter groups (see (2.6)). Each member $\Sigma_{p,3,3}$ is a simple straight triangular Coxeter prism, that is, a compact simply truncated Coxeter orthoscheme. The dihedral angle π/p sits at an edge e, which connects the top and bottom faces F_1 and F_2, and which is of type $< 2,2,p,2,2 >$. For $p \to \infty$, the sequence $\Sigma_{p,3,3}$ degenerates under contraction of e to the polyhedron $\Sigma_{\infty,3,3}$ with precisely one 4-valent ideal vertex v_∞, whose stabiliser in the group

$$\Sigma_{\infty,3,3} \quad : \quad \bullet\overset{\infty}{\text{——}}\bullet\text{——}\bullet\text{——}\bullet\overset{\infty}{\text{——}}\bullet \tag{3.9}$$

is the quadrilateral affine Coxeter group $\widetilde{A}_1 \times \widetilde{A}_1$. Now, by Proposition 3, the growth rates $\tau(\Sigma_{p,3,3})$, $p \geq 7$, which are all Salem numbers, tend from below to the Pisot number $\tau(\Sigma_{\infty,3,3}) \simeq 1.734691$.

4. The main result

Let G be a cofinite hyperbolic Coxeter group acting on hyperbolic 3-space with non-compact fundamental Coxeter polyhedron $P \subset \mathbb{H}^3$. Denote by $\Omega_0 = \Omega_f \cup \Omega_\infty$ the set of vertices of P where Ω_f and $\Omega_\infty \neq \emptyset$ denote the subsets of finite vertices and ideal vertices of P. In the case of the Coxeter orthoscheme $(3,3,6)$, the set Ω_0 consists of four 3-valent vertices, and $|\Omega_\infty| = 1$.

Theorem. *Among all hyperbolic Coxeter groups with non-compact fundamental polyhedron of finite volume in $\mathbb{H}^3$, the tetrahedral group $(3,3,6)$ has minimal growth rate, and as such the group is unique.*

Before we provide a proof of the Theorem, let us recapitulate the growth data of $(3,3,6)$ (see §3). The growth function of $(3,3,6)$ is given by (see (3.7))

$$f_{(3,3,6)}(x) = \frac{[2,2,2,3]\,(x^2+1)\,(x^2-x+1)}{(x-1)\,(x^7+x^6+x^5+x^4-1)} \quad , \tag{4.1}$$

and the growth rate $\tau_{(3,3,6)} \simeq 1.296466$ has minimal polynomial $x^7 - x^3 - x^2 - x - 1$ (see (3.8)). As already mentionned in §3.2, it is known that $\tau_{(3,3,6)}$ is a Perron number, but it is not a Pisot number since its value is strictly smaller than the least Pisot number $\alpha_S \simeq 1.324718$.

4.1. Proof of the Theorem. Consider a hyperbolic Coxeter group $G = (G, S)$ with non-compact fundamental Coxeter polyhedron $P \subset \mathbb{H}^3$ of finite volume. The polyhedron P is the convex hull of finitely many points in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, whose number is denoted by f_0 as usually (see §2.2). Together with the number f_1 of edges and the number f_2 of polygonal faces of P, we have that $f_0 - f_1 + f_2 = 2$. We shall focus on the non-empty vertex subset $\Omega_\infty \subset \Omega_0$ of P and the valencies of its elements. For $i = 3, 4$, we denote by Ω^i_∞ the set of i-valent ideal vertices of P and notice that $\Omega_\infty = \Omega^3_\infty \cup \Omega^4_\infty$.

Step 1. Suppose that $\Omega^4_\infty \neq \emptyset$ and let $v_\infty \in \Omega^4_\infty$. By Proposition 1, P is the result of a contraction process by means of finite-volume Coxeter polyhedra $P_n \subset \mathbb{H}^3$ having the same combinatorial type and dihedral angles as P except for a ridge of type $< 2, 2, n, 2, 2 >$ with n sufficiently large, giving rise to the vertex v_∞. By Proposition 2, the growth rates $\tau(P_n)$ tend from below to $\tau(P)$.
If $|\Omega_\infty| = 1$, then the polyhedra P_n are all compact, and by Proposition 3, $\tau(P)$ is a Pisot number. Since $\tau_{(3,3,6)}$ is smaller than any Pisot number, the conclusion follows.
If $|\Omega_\infty| \geq |\Omega^4_\infty| \geq 2$, we perform the contraction process successively for each further vertex in Ω^4_∞, by using Proposition 1, so that Proposition 2 allows us to conclude that non-compact Coxeter polyhedra in $\mathbb{H}^3$ of smallest growth rates are characterised by $\Omega^4_\infty = \emptyset$.

Step 2. Let $\Omega^4_\infty = \emptyset$, that is, $\Omega_0 = \Omega_f \cup \Omega^3_\infty$. In particular, all vertices of P are 3-valent and

$$f_0 = 2\,(f_2 - 2) \geq 4 \quad , \quad f_1 = \frac{3}{2}\, f_0 \quad . \tag{4.2}$$

Denote by π/n_i for integers $n_i \geq 2$, $i = 1, \ldots, f_1$, the dihedral angles of P. By Steinberg's formula (see (3.3)) and Table 3,

$$\frac{1}{f_S(x^{-1})} = \sum_{\substack{G_T < G \\ |G_T| < \infty}} \frac{(-1)^{|T|}}{f_T(x)} = 1 - \frac{|S|}{[2]} + \sum_{i=1}^{f_1} \frac{1}{[2, n_i]} - \sum_{v \in \Omega_f} \frac{1}{f_v(x)} \quad , \tag{4.3}$$

where f_v is the growth polynomial of the finite Coxeter group G_v of rank three which is the stabiliser of the vertex $v \in \Omega_f$ in G. In Table 3 are listed all possible irreducible components for realisations of G_v. By Solomon's formula (3.4), the growth polynomial f_v equals $[2, m_2 + 1, m_3 + 1]$ where m_2, m_3 depend on G_v according to Table 3. We point out the following simple, but crucial fact which we term "exponent monotonicity". Let $G_1 \neq A_1$ be a group in Table 3. By increasing one entry in the Coxeter symbol of G_1 and passing from G_1 to another group G_2 in Table 3, the exponents different from $m_1 = 1$ all increase as well. For example, the passage from B_3 to H_3 (increase the first entry 4 of the

Coxeter symbol $(4,3)$ to 5) transforms the non-trivial exponents according to $m_2 : 3 \mapsto 5$ and $m_3 : 5 \mapsto 9$.

Now, since $|S| = f_2 = f_0/2 + 2$, and since each of the f_1 edges has precisely two vertices, (4.3) can be rewritten as

$$\begin{aligned}\frac{1}{f_S(x^{-1})} &= 1 - \frac{f_2}{[2]} + \frac{1}{2}\sum_{w\in\Omega_0}\sum_{i=1}^{3}\frac{1}{[2,n_i^w]} - \sum_{v\in\Omega_f}\frac{1}{[2,m_2+1,m_3+1]} \\ &= \frac{1}{[2]}\left\{x-1+\frac{1}{2}\sum_{w\in\Omega_0}\Big(\sum_{i=1}^{3}\frac{1}{[n_i^w]}-1\Big)-\sum_{v\in\Omega_f}\frac{1}{[m_2+1,m_3+1]}\right\} \quad ,\end{aligned} \tag{4.4}$$

where we denote by π/n_i^w , $i = 1,2,3$, the dihedral angles at the three edges giving rise to the vertex $w \in \Omega_0$. By definition of $[n]$, we have

$$x^n - 1 = (x-1)\,[n] \quad , \tag{4.5}$$

which, together with the working hypothesis $\Omega_0 = \Omega_\infty^3 \cup \Omega_f$, allows us to write

$$\begin{aligned}\frac{1}{f_S(x^{-1})} =& \frac{x-1}{[2]} + \frac{x-1}{2\,[2]}\sum_{w\in\Omega_\infty^3}\Big(\sum_{i=1}^{3}\frac{1}{x^{n_i^w}-1}-\frac{1}{x-1}\Big)+ \\ &+\frac{x-1}{2\,[2]}\sum_{v\in\Omega_f}\left(\sum_{i=1}^{3}\frac{1}{x^{n_i^v}-1}-\frac{1}{x-1}-\frac{2\,(x-1)}{(x^{m_2+1}-1)\,(x^{m_3+1}-1)}\right) \quad .\end{aligned} \tag{4.6}$$

By analysing the different types of finite subgroups G_v according to Table 3, Parry identified the terms in the sum running over the vertices $v \in \Omega_f$ in the following coherent way (see [Pa, (2.13) and (2.14)]).

$$\begin{aligned}&\frac{x-1}{2\,[2]}\sum_{v\in\Omega_f}\left(\sum_{i=1}^{3}\frac{1}{x^{n_i^v}-1}-\frac{1}{x-1}-\frac{2\,(x-1)}{(x^{m_2+1}-1)\,(x^{m_3+1}-1)}\right) \\ &= -\frac{1}{2}\,x(x-1)\sum_{v\in\Omega_f}\frac{(x^{m_1}-1)(x^{m_2}-1)(x^{m_3}-1)}{(x^{m_1+1}-1)(x^{m_2+1}-1)(x^{m_3+1}-1)} \quad ,\end{aligned} \tag{4.7}$$

where we used $m_1 = 1$. As for the sum running over the infinite vertices $w \in \Omega_\infty^3$ in (4.6), Table 1 shows that each term belongs to a euclidean subgroup G_w of type $\widetilde{A}_2$, $\widetilde{B}_2$ or $\widetilde{I}$ (cf. also §2.1). An easy calculation in each of these cases reveals that

$$\sum_{w\in\Omega_\infty^3}\Big(\sum_{i=1}^{3}\frac{1}{x^{n_i^w}-1}-\frac{1}{x-1}\Big) = \frac{-(\,[n_w-1]+1\,)}{[n_w]} \quad , \tag{4.8}$$

where

$$n_w := \max\ (n_1^w, n_2^w, n_3^w) = \begin{cases} 3 & \text{if} \quad G_w = \widetilde{A}_2\,, \\ 4 & \text{if} \quad G_w = \widetilde{B}_2\,, \\ 6 & \text{if} \quad G_w = \widetilde{I}\,. \end{cases} \tag{4.9}$$

By (4.5) and (4.7)–(4.9), the identity (4.6) can be rewritten according to

$$\frac{1}{f_S(x^{-1})} = \frac{x-1}{x+1}\left\{1 - \frac{1}{2}\left(\sum_{w\in\Omega_\infty^3} \frac{[n_w-1]+1}{[n_w]} + x \sum_{v\in\Omega_f} \frac{[m_2,m_3]}{[m_2+1,m_3+1]}\right)\right\} \quad . \tag{4.10}$$

Now, the inversion $x \mapsto x^{-1}$ for $x \neq 0$ combined with (3.5), that is,

$$\frac{1}{[n](x^{-1})} = \frac{x^{n-1}}{[n](x)} = \frac{x^{n-1}}{[n]} \quad ,$$

transforms (4.10) into the expression

$$\begin{aligned} \frac{1}{f_S(x)} &= \frac{1-x}{1+x}\left\{1 - \frac{x}{2}\left(\sum_{w\in\Omega_\infty^3} \frac{[n_w-1]+x^{n_w-2}}{[n_w]} + \sum_{v\in\Omega_f} \frac{[m_2,m_3]}{[m_2+1,m_3+1]}\right)\right\} \\ &=: \frac{1-x}{[2]}\left\{1 - \frac{x}{2}\,H(x)\right\} \quad , \end{aligned} \tag{4.11}$$

where

$$H(x) := \sum_{w\in\Omega_\infty^3} \frac{[n_w-1]+x^{n_w-2}}{[n_w]} + \sum_{v\in\Omega_f} \frac{[m_2,m_3]}{[m_2+1,m_3+1]} \quad . \tag{4.12}$$

In order to prove the Theorem it suffices to show that, for each (G,S) different from $(3,3,6)$, and for all $x \in (0, 1/\tau_{(3,3,6)}]$,

$$\frac{1}{f_S(x)} < \frac{1}{f_{(3,3,6)}(x)} \quad , \tag{4.13}$$

which, by (4.9), (4.11) and (4.12), is equivalent to (see also (4.1))

$$\begin{aligned} H(x) > H_{(3,3,6)}(x) &= \frac{[5]+x^4}{[6]} + \frac{[5]}{[2,6]} + \frac{[2]}{[2,3]} + \frac{[2,3]}{[3,4]} \\ &= 2\,\frac{1+[2]\,(x^6+2\,x^5+2\,x^4+3\,x^3+2\,x^2+2\,x+1)}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} \quad . \end{aligned} \tag{4.14}$$

To this end, we consider the function in (4.12) and write

$$H(x) = \sum_{w\in\Omega_\infty^3} g_{n_w}(x) + \sum_{v\in\Omega_f} h_{m_2+1}(x)\,h_{m_3+1}(x) \quad , \tag{4.15}$$

where we put

$$h_k(x) = \frac{[k-1]}{[k]} \quad \text{and} \quad g_k(x) = h_k(x) + \frac{x^{k-2}}{[k]} = \frac{[k-1] + x^{k-2}}{[k]} \quad . \tag{4.16}$$

Observe that the functions $h_k(x)$ are strictly monotonely decreasing on $[0,1]$. Furthermore, the functions h_k and g_k satisfy the following properties.

Lemma 1. *For all integers $k \geq 2$ and for all $x \in (0,1)$,*

(a) $0 < h_k(x) < h_{k+1}(x) < 1\,,$

(b) $g_k(x) > g_{k+1}(x) > 1\,,$

(c) $g_6(x) > \dfrac{1}{[2]^2}\,.$

Proof. Claim (a) follows from [KKol, Lemma]. For the proof of (b), we observe first that the definition of $[k]$ and g_k according to (4.16) imply that $g_k(x) > 1$ for all $x \in (0,1)$ and all integers $k \geq 2$. Secondly, for the difference function $d_k(x) = g_k(x) - g_{k+1}(x)$, we compute

$$d_k(x) = \frac{[k+1]\,([k-1] + x^{k-2}) - [k]\,([k] + x^{k-1})}{[k, k+1]} \quad . \tag{4.17}$$

Since (see [KP, (2.4)])

$$[n_1, n_2] = \frac{[n_1 + n_2] - [n_1] - [n_2]}{x-1} \quad , \tag{4.18}$$

for all integers $n_1, n_2 \geq 2$, the numerator of d_k yields, by (4.17) and (4.18), the estimate

$$\frac{2\,[k] - [k+1] - [k-1]}{x-1} + x^{k-2}\,(\,[k+1] - x\,[k]\,) = \frac{x^{k-1}(1-x)}{x-1} + x^{k-2} = x^{k-2}(1-x) > 0 \quad .$$

Finally, the inequality (c) is equivalent to the (obvious) positivity of the expression

$$[2]^2(\,[5] + x^4) - [6] = x\,[2]\,(2\,x^4 + 2\,x^3 + 2\,x^2 + x + 2) \quad ,$$

so that, by definition (4.16), claim (c) follows. □

In order to prove (4.14), the strategy is to distinguish between the two cases $f_2 = 4$ and $f_2 \geq 5$ and to find a respective "sandwich function" $\widetilde{H}(x)$ satisfying $H(x) \geq \widetilde{H}(x) > H_{(3,3,6)}(x)$ for all $x \in (0, 1/\tau_{(3,3,6)}]$ which is easier to handle for our purpose. It will turn out that the delicate case is $f_2 = 4$ (and in particular the groups R_2 and R_6) requiring a certain amount of case-by-case analysis in view of Table 2.

Case 1. Suppose that $f_2 \geq 5$. By (4.2), $f_0 \geq 6$, and $|\Omega^3_\infty| \geq 1$. By Lemma 1,

$$g_l(x) > g_6(x) > 1 > h_k(x) > h_k(x)\,h_2(x) \geq h_2^2(x) > 0 \tag{4.19}$$

for integers $k \geq 2$, $l = 2, \ldots, 5$, and for all $x \in (0,1)$. In view of (4.14) and (4.15), this motivates the definition of

$$\widetilde{H}(x) := \frac{[5]+x^4}{[6]} + \frac{5}{[2]^2} \quad , \tag{4.20}$$

which leads to the estimate

$$H(x) = \sum_{w \in \Omega_\infty^3} g_{n_w}(x) + \sum_{v \in \Omega_f} h_{m_2+1}(x)\, h_{m_3+1}(x) \geq \widetilde{H}(x)$$

on the interval $(0,1)$. Since

$$\frac{5}{[2]^2} - \Big(\frac{[5]}{[2,6]} + \frac{[2]}{[2,3]} + \frac{[2,3}{[3,4]} \Big) = \frac{(x-1)^2\,(2\,x^4+3\,x^2+2)}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} > 0 \quad ,$$

it follows from (4.20) that $\widetilde{H}(x) > H_{(3,3,6)}(x)$ for all $x \in (0, 1/\tau_{(3,3,6)}]$, Hence, the conclusion (4.14) follows.

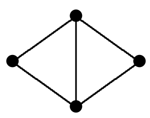
Case 2. Suppose that $f_2 = 4$, that is, P is one of the 23 non-compact Coxeter tetrahedra which are listed in Table 2. Although the combinatorial type of P is most elementary, the proof of $\tau(P) \geq \tau((3,3,6))$ with equality only if P is isometric to $(3,3,6)$ is more delicate.

(i) Suppose first that P is an *ideal* Coxeter tetrahedron, that is, $P = T_4, T_7$ or V (cf. Table 2). Then, by (4.12), (4.14) and by Lemma 1,

$$\begin{aligned} H(x) = \sum_{w \in \Omega_\infty^3} \frac{[n_w - 1] + x^{n_w-2}}{[n_w]} &\geq 4\,\frac{[5]+x^4}{[6]} > \frac{[5]+x^4}{[6]} + 3 > \\ > \frac{[5]+x^4}{[6]} + \frac{[5]}{[2,6]} + \frac{[2]}{[2,3]} &+ \frac{[2,3]}{[3,4]} = H_{(3,3,6)}(x) \quad , \end{aligned} \tag{4.21}$$

which holds for all $x \in (0,1)$ and finishes the verification in this particular case.

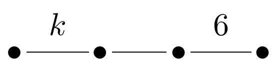
(ii) Let us treat another simple case, namely $P = U$, given by the Coxeter diagram

U : 

which, by Lemma 1, yields

$$H_U(x) = 2\,\frac{[2]+x}{[3]} + 2\,\frac{[2,3]}{[3,4]} > \frac{[5]+x^4}{[6]} + \frac{[5]}{[2,6]} + \frac{1}{[3]} + \frac{[2,3]}{[3,4]} = H_{(3,3,6)} \quad .$$

(iii) Consider the Coxeter groups (see Table 2)

Σ_k : 

, $k = 3,4,5,6$,

where $\Sigma_3 = (3,3,6) = R_1$, $\Sigma_4 = R_4$, $\Sigma_5 = R_5$ and $\Sigma_6 = R_3$. We want to show that, for $k = 4,5,6$,

$$H_{\Sigma_k}(x) > H_{(3,3,6)}(x) \quad \text{for all} \quad x \in (0,1) \quad , \tag{4.22}$$

by exploiting the exponent monotonicity for m_2, m_3. Indeed, by (4.12) and by Lemma 1, we obtain, for $k = 4,5$,

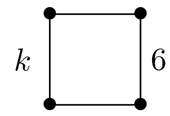
$$\begin{aligned} H_{\Sigma_k}(x) - H_{(3,3,6)}(x) &= \frac{[5]+x^4}{[6]} + \sum_{v \in \Omega_f} \frac{[m_2,m_3]}{[m_2+1,m_3+1]} - H_{(3,3,6)}(x) \\ &= \frac{[k-1]}{[2,k]} - \frac{[2]}{[2,3]} + \frac{[m_2,m_3]}{[m_2+1,m_3+1]} - \frac{[2,3]}{[3,4]} > 0 \quad , \end{aligned} \tag{4.23}$$

where the exponents m_2, m_3 are associated to the subgroup $(k,3)$ of Σ_k. In a similar way, we can conclude that on $(0,1)$

$$H_{\Sigma_6}(x) = 2\,\frac{[5]+x^4}{[6]} + \sum_{v \in \Omega_f} \frac{[m_2,m_3]}{[m_2+1,m_3+1]} > H_{(3,3,6)}(x) \quad . \tag{4.24}$$

(iv) Consider the Coxeter groups (see Table 2)

T_k , $k = 3,4,5$:



which have all $|\Omega^3_\infty| = 2$. We proceed as in (iii) and see that the functions

$$H_{T_k}(x) = 2\,\frac{[5]+x^4}{[6]} + 2\,\frac{[m_2,m_3]}{[m_2+1,m_3+1]} \quad ,$$

where the exponents m_2, m_3 are again associated to the subgroup $(k,3)$ of T_k, are strictly bigger than $H_{(3,3,6)}(x)$ for $x \in (0,1)$.

(v) Let us pass to the Coxeter groups with precisely one subgroup of type $\widetilde{A}_2$

S_1 S_2 S_3 S_4

which yield - for $k = 1,2,3$ - the functions

$$H_{S_k}(x) = \frac{[2]+x}{[3]} + \frac{1}{[2]^2} + 2\,\frac{[m_2,m_3]}{[m_2+1,m_3+1]} \geq \frac{[2]+x}{[3]} + \frac{1}{[2]^2} + 2\,\frac{[2,3]}{[3,4]} \quad , \tag{4.25}$$

where we again exploited Lemma 1 and the monotonicity properties of m_2, m_3. It follows from (4.14) and (4.25) that $H_{S_k}(x) > H_{(3,3,6)}(x)$ on $(0,1)$ if

$$\frac{1}{[2]^2} + \frac{[2,3]}{[3,4]} > \frac{[5]}{[2,6]} + \frac{1}{[3]} \quad , \tag{4.26}$$

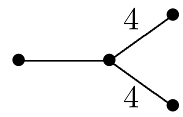
which is equivalent to the positivity of the associated difference function $\Delta(x)$ as given by

$$\Delta(x) = \frac{x^2(x-1)^2}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} \quad . \tag{4.27}$$

As for the group S_4 with $|\Omega_f| = 1$, it follows easily from (4.27) that

$$H_{S_4}(x) = 2\,\frac{[5]+x^4}{[6]} + \frac{[2]+x}{[3]} + \frac{1}{[2]^2} > H_{(3,3,6)}(x) \quad . \tag{4.28}$$

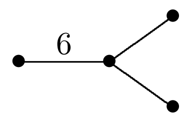
(vi) Next, we treat the Coxeter groups S_6, S_7, T_5 and T_6 which have among its euclidean subgroups only those of type $\widetilde{B}_2$, and have, if $|\Omega^3_\infty| < 3$, two *irreducible* spherical subgroups of rank three. By computing the respective functions $H(x)$ according to (4.12) and by using the inequality (4.26), one can easily see that $H(x) > H_{(3,3,6)}(x)$ on $(0,1)$. Let us illustrate this for the Coxeter group

S_6 :


with $|\Omega^3_\infty| = 1$ and whose function $H(x)$ compares with $H_{(3,3,6)}(x)$ according to (see Lemma 1, (4.14) and (4.26))

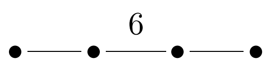
$$H(x) = \frac{[3]+x^2}{[4]} + 2\,\frac{[5,9]}{[6,10]} + \frac{1}{[2]^2} > \frac{[5]+x^4}{[6]} + 2\,\frac{[2,3]}{[3,4]} + \frac{1}{[2]^2} > H_{(3,3,6)}(x) \quad . \tag{4.29}$$

(vii) In a similar way as in (4.29), and based on Lemma 1 and (4.26) as well, we can conclude that the Coxeter group

S_5 :


satisfies $H(x) > H_{(3,3,6)}(x)$ on $(0,1)$.

(viii) We finish the proof by considering the remaining Coxeter tetrahedra

R_2 :


R_6 :


, R_7 :


.

First, by Lemma 1, it is evident that, on $(0,1)$,

$$H_{R_7}(x) = 2\,\frac{[3]+x^2}{[4]} + 2\,\frac{[3]}{[2,4]} > \frac{[3]+x^2}{[4]} + \frac{[5,9]}{[6,10]} + \frac{[3]}{[2,4]} + \frac{1}{[3]} = H_{R_6}(x) \quad . \tag{4.30}$$

Hence, it remains to prove (4.14) for the two simply asymptotic orthoschemes R_2 and R_6. To this end, we compute the difference functions $\Delta_i(x) := H_{R_i}(x) - H_{(3,3,6)}(x)$ for $i = 2, 6$. By using (4.14) and $[2k] = [k]\,(1 + x^k)$, as usually, we easily deduce the expressions

$$\Delta_2(x) = 2\,\frac{x^3\,[5]}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} = x\,[5]\,\Delta_6(x) \quad , \tag{4.31}$$

which are clearly positive on $(0, 1)$. ∎

5. Final remarks

5.1. About the function $H_{(3,3,6)}$ and a related Salem polynomial. Consider the extremal tetrahedral group $(3, 3, 6)$, providing the minimal volume cusped hyperbolic 3-orbifold and the non-cocompact Coxeter group with minimal growth rate $\tau_* \simeq 1.296466$. As pointed out, the minimal polynomial $x^7 - x^3 - x^2 - x - 1$ of τ_* is neither a Salem nor a Pisot polynomial. However, τ_* is a Perron number. Consider the auxiliary function (see (4.14))

$$\begin{aligned} H_{(3,3,6)}(x) =& 2\,\frac{1 + [2]\,(x^6 + 2\,x^5 + 2\,x^4 + 3\,x^3 + 2\,x^2 + 2\,x + 1)}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} \\ =:& 2\,\frac{1 + [2]\,p(x)}{[2,2,3]\,(x^2+1)\,(x^2-x+1)} \quad , \end{aligned} \tag{5.1}$$

which is related inversely to the growth function $f_{(3,3,6)}(x)$ according to (see (4.11))

$$\frac{1}{f_{(3,3,6)}(x)} = \frac{1-x}{1+x}\left\{1 - \frac{x}{2}\,H_{(3,3,6)}(x)\right\} \quad .$$

Lemma 2. *The polynomial $p(x)$ in (5.1) given by*

$$p(x) = x^6 + 2\,x^5 + 2\,x^4 + 3\,x^3 + 2\,x^2 + 2\,x + 1 \tag{5.2}$$

is a Salem polynomial.

Proof. Obviously, $p(\pm 1) \neq 0$. Furthermore, $p(x)$ is a palindromic monic polynomial of (even) degree six over the integers. By an adaption of a result of Kempner (see [ZZ, Proposition 1]), we conclude that $p(x)$ is a Salem polynomial in the following way. Consider the polynomial

$$q(x) = (x-i)^6\,p\!\left(\frac{x+i}{x-i}\right) = 13\,x^6 - 37\,x^4 + 15\,x^2 + 1 \in \mathbb{Z}[x] \quad , \tag{5.3}$$

which is of degree six and even. There is the following equivalence between the roots of p and q.

(a) p has $2i$ roots on the unit circle if and only if q has i positive real roots.

(b) p has $2j$ real roots if and only if q has j positive imaginary roots.

Let us show that q has two positive real roots and one positive imaginary root as follows (see [R, Classical Formulas], for example). By substituting $y = x^2$ into (5.3), we see that the cubic equation $13\,y^3 - 37\,y^2 + 15\,y + 1 = 0$ has positive discriminant and therefore three distinct real roots. These roots are given by explicit formulas, and their inspection shows that exactly one root is negative. Since $y = x^2$, the equation $13\,x^6 - 37\,x^4 + 15\,x^2 + 1 = 0$ has two positive real roots and one positive imaginary root. Therefore, $p(x)$ is a Salem polynomial.

□

5.2. The two-dimensional case. The method which we developed in order to prove that the Coxeter group $(3,3,6)$ has minimal growth rate among all non-cocompact cofinite Coxeter groups in $\mathbb{H}^3$ can be applied to the two-dimensional case as well. This approach gives an alternative proof of the following result of Floyd [F, p. 482], which is more elementary, without reference to Pisot polynomials and the respective minimality result of Smyth (see §3.2).

Proposition. *Among all hyperbolic Coxeter groups with non-compact fundamental polygon of finite volume in $\mathbb{H}^2$, the triangle group $(3,\infty)$ has minimal growth rate.*

Proof. Denote by $P \subset \mathbb{H}^2$ a fundamental polygon of a hyperbolic Coxeter group (G,S) such that P is non-compact but of finite volume. Let $\Omega_0 = \Omega_f \cup \Omega_\infty$ be the set of vertices partitionned into the set Ω_f of finite vertices and the set Ω_∞ of ideal vertices of P. Let $f_0 = |\Omega_0|$ and $f_0^\infty = |\Omega_\infty|$. It is a particular feature in two dimensions that the number of vertices f_0 equals the number of edges $f_1 = |S|$ and that $f_0 - f_0^\infty$ equals the number of (positive) angles of P which are of the form $\frac{\pi}{k}$ for integers $k \geq 2$. Denote by $2\,k_v$ the order of the stabiliser D_2^k of the vertex $v \in \Omega_f$. By the formula (3.3) and (3.4) of Steinberg and Solomon, and by Table 3, we can derive the following expression for the growth function $f := f_S$ of $G = (G,S)$.

$$\begin{aligned}\frac{1}{f(x^{-1})} &= 1 - \frac{f_0}{[2]} + \sum_{v\in\Omega_f} \frac{1}{[2,k_v]} = 1 - \frac{1}{[2]}\,\{\, f_0^\infty + \sum_{v\in\Omega_f} \big(1 - \frac{1}{[k_v]}\big)\} \\ &= 1 - \frac{1}{1+x}\,\{\, f_0^\infty + x \sum_{v\in\Omega_f} \frac{[k_v - 1]}{[k_v]}\,\} \quad .\end{aligned} \tag{5.4}$$

The passage $x \mapsto x^{-1}$ and the property (3.5) yield

$$\frac{1}{f(x)} = 1 - \frac{x}{[2]}\,\{\, f_0^\infty + \sum_{v\in\Omega_f} \frac{[k_v - 1]}{[k_v]}\,\} =: 1 - \frac{x}{[2]}\,H(x) \quad , \tag{5.5}$$

where we put

$$H(x) := f_0^\infty + \sum_{v\in\Omega_f} \frac{[k_v - 1]}{[k_v]} \quad , \tag{5.6}$$

so that

$$H_{(3,\infty)}(x) = 1 + \frac{1}{[2]} + \frac{[2]}{[3]} \quad . \tag{5.7}$$

As in the three-dimensional case, we shall prove the Proposition by showing that each Coxeter group G different from the triangle group $(3,\infty)$ satisfies (see (5.5))

$$\frac{1}{f(x)} < \frac{1}{f_{(3,\infty)}(x)} = \frac{1 - x^2 - x^3}{[2,2,3]} \quad \text{for} \quad x \in (0, 1/\tau_{(0,\infty)}] \quad , \tag{5.8}$$

which, by (4.16), (5.6) and (5.7), is equivalent to the verification of

$$H(x) = f_0^\infty + \sum_{v\in\Omega_f} h_{k_v}(x) > 1 + \frac{1}{[2]} + \frac{[2]}{[3]} = H_{(3,\infty)}(x) \tag{5.9}$$

on the interval $(0, 1/\tau_{(0,\infty)}]$. We have $f_0^\infty \geq 1$, and by Lemma 1, each term $h_k(x) := h_{k_v}(x)$ satisfies $0 < h_k(x) < h_{k+1} < 1$ on $(0,1)$. These properties and the realisation condition for hyperbolic Coxeter polygons with f_0 vertices and $f_0 - f_0^\infty$ positive angles π/k_v , $v \in \Omega_f$, that is,

$$\sum_{v\in\Omega_f} \frac{1}{k_v} < f_0 - 2 \quad , \tag{5.8}$$

shows that a minimiser of the functions $H(x)$ on $(0,1)$ must have $f_0^\infty = 1$ as well as $f_0 = 3$ with $k_{v_1} = 2$ and $k_{v_2} = 3$. These conditions are fulfilled only by the triangle group $(3,\infty)$. ■

Remark. By the Proposition, §3.2 and §5.1, the growth rate τ of any hyperbolic Coxeter group G with non-compact fundamental polygon of finite volume in $\mathbb{H}^2$ satisfies $\tau \geq \tau_{(3,\infty)} = \alpha_S \simeq 1.324718 > 1.296466 \simeq \tau_* = \tau_{(3,3,6)}$, with equality $\tau = \tau_{(3,\infty)}$ if and only if $G = (3,\infty)$.

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