Scissors Congruence, the Golden Ratio and Volumes in Hyperbolic 5-Space

Ruth Kellerhals

Abstract By different scissors congruence techniques a number of dissection identities are presented between certain quasi-Coxeter polytopes, whose parameters are related to the golden section, and an ideal regular simplex in hyperbolic 5-space. As a consequence, several hyperbolic polyhedral 5-volumes can be computed explicitly in terms of Apéry's constant $\zeta(3)$ and the trilogarithmic value $Л_3(\frac{\pi}{5})$.

Keywords Scissors congruence · Hyperbolic 5-space · Quasi-Coxeter polytopes · Trilogarithms · Zeta values

1 Introduction

Polytopes in hyperbolic space $\mathbb{H}^n$ are important objects in the study of hyperbolic space forms and arise in various contexts ranging from geometry and topology to mathematical physics. They serve as combinatorial models and allow us sometimes to compute geometric-topological data such as Betti numbers, Euler characteristic, systole and volume. These quantities are related by (in-)equalities that allow us to concentrate on the most important quantity, the volume of a hyperbolic polytope.

In contrast to lower dimensions, polyhedral volume in hyperbolic 5-space is only partially understood but forms the higher dimensional doorway in view of Schläfli's volume differential formula (see e.g. [9, 12], [16, p. 118]). In [10], we provide, for a certain two parameter-family of doubly asymptotic orthoschemes a comparatively simple formula in terms of their dihedral angles and the so-called trilobachevsky function $Л_3(\omega)$. As a consequence, the covolume of the hyperbolic Coxeter group

$$\Gamma \quad : \quad \circ \overset{3}{\rule[0.5ex]{2em}{0.4pt}} \circ \overset{4}{\rule[0.5ex]{2em}{0.4pt}} \circ \overset{3}{\rule[0.5ex]{2em}{0.4pt}} \circ \overset{3}{\rule[0.5ex]{2em}{0.4pt}} \circ \overset{3}{\rule[0.5ex]{2em}{0.4pt}} \bullet$$

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could be computed and equals $7\zeta(3)/46080$, allowing to identify the (arithmetic) quotient orbifold $Q = \mathbb{H}^5/\Gamma$ as the (unique) cusped hyperbolic 5-space form of minimal volume (see [7]; the bullet indicates the occurrence of one cusp in Q). In [11], we completely solved the volume problem for hyperbolic 5-space by deriving a volume formula for the generators of the scissors congruence group $\mathcal{P}(\overline{\mathbb{H}^5})$ in terms of (many) trilogarithm expressions. In practice, however, this formula requires the dissection/extension of a given polytope $P \subset \overline{\mathbb{H}^5}$ by doubly asymptotic orthoschemes before its evaluation which per se is already a difficult task. In particular, such a procedure does not allow the characterisation of volumes of arithmetically defined or other distinguished hyperbolic 5-space forms in terms of their (number theoretical) basic data.

In this work, we describe several scissors congruence procedures in detail, providing new relations in the group $\mathcal{P}(\overline{\mathbb{H}^5})$ between non-compact orthoschemes and orthoprisms. In the case of certain quasi-Coxeter polytopes (see Sect. 2.4, (11) and (12)), whose dihedral angles are related to the golden ratio, we are able to represent their volumes as explicit $\mathbb{Q}$-linear combinations of three fundamental constants, Apéry's constant $\zeta(3)$, the trilobachevsky value $Ⅱ_3(\frac{\pi}{5})$ and the volume μ_5 of an ideal regular hyperbolic 5-simplex (see Sect. 4, Theorems I and II). This will be achieved by scissors congruence techniques and without analytic manipulations!

This approach may be of relevance by studying quantitative and arithmetic aspects of further extremal volume problems in hyperbolic 5-space. In this context, let us point out the particular role of an ideal regular hyperbolic 5-simplex $S_{\text{reg}}^\infty(2\lambda)$ with dihedral angle 2λ satisfying $\cos(2\lambda) = 1/4$. In fact, the volume of an arbitrary hyperbolic 5-simplex is maximal if and only if it is ideal and regular. A first analytical proof, for arbitrary dimension n, is due to U. Haagerup and H.J. Munkholm [6], a second and geometrical proof based on Steiner symmetrisation is due to N. Peyerimhoff [13]. This extremality property is a key, for example, in Gromov's proof [5] of Mostow rigidity.

Finally, let us mention that L. Schläfli considered the three-dimensional spherical analogues of the quasi-Coxeter polytopes in (11) and (12). By applying his reduction principle, he was able to determine their volumes and used them for the volume computation of a regular spherical simplex $S_{\text{reg}}(2\lambda) \subset S^3$ as rational linear combination in terms of $\lambda\pi$ and π^2 (cf. [15, Sect. 30], and Sect. 4.3, Remark 5).

2 Cycles of Orthoschemes

2.1 Orthoschemes and Orthoprisms

Let $\mathbb{X}^n = \mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n$ be a standard space of constant curvature $K \in \{+1, 0, -1\}$. Put $\mathbb{H}^n$ in the Lorentz–Minkowski vector space $\mathbb{R}^{n,1}$ of signature $(n, 1)$ or identify it with the Klein–Beltrami model in real projective space $\mathbb{R}P^n$. An n-orthoscheme $R \subset \mathbb{X}^n$ is an n-simplex bounded by hyperplanes $H_0, \ldots, H_n$ in $\mathbb{X}^n$ which are such that $H_i \perp H_k$ for $|i - k| > 1$. Orthoschemes can be seen as higher dimensional analogues of right-angled triangles and were introduced by L. Schläfli [15]. They are characterised by many nice metrical and orthogonality relations. Apart from right dihedral angles, an

Fig. 1 An orthoscheme
$R = p_0 \cdots p_5$ in $\mathbb{X}^5$
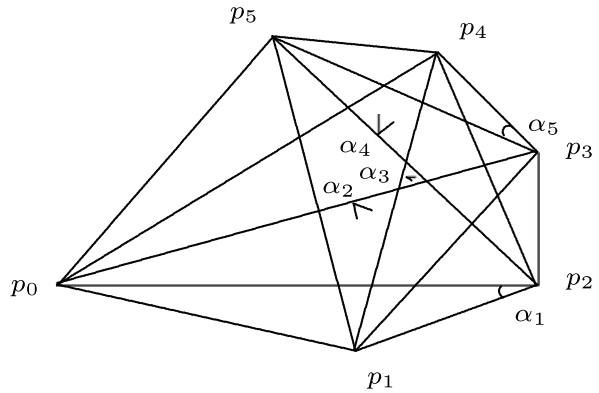
orthoscheme R has dihedral angles $\alpha_i := \angle(H_{i-1}, H_i)$, $1 \leq i \leq n$, which—in the hyperbolic case—are acute and form a complete system of parameters. We shall only consider orthoschemes with non-obtuse dihedral angles. For $0 \leq i \leq n$, consider the vertex p_i opposite to the hyperplane H_i in R and write $R = p_0 \cdots p_n$ (cf. Fig. 1).

The edges $p_0 p_1, \ldots, p_{n-1} p_n$ of R form an orthogonal polygonal path π in $\mathbb{X}^n$ with the special property that, for $\mathbb{X}^n = \mathbb{H}^n$, at most the initial vertex p_0 and/or the final vertex p_n of the path π may be *ideal* points, i.e. points belonging to the boundary $\partial \mathbb{H}^n$ at infinity. In the latter case, we call R *simply* or *doubly asymptotic*. By considering $\mathbb{H}^n$ in $\mathbb{R}P^n$, we may extend this process and allow the initial vertex p_0 and/or the final vertex p_n of π to be *ultra-ideal* points, that is, points lying outside the defining quadric $Q_{n,1} = \partial \mathbb{H}^n$ of signature $(n, 1)$, as long as *all* edges of π emanating from p_0 and/or p_n intersect non-trivially the interior $\mathbb{H}^n$. We continue to call R a hyperbolic n-orthoscheme. Notice, however, that the convex body $R \cap \mathbb{H}^n$ is of infinite volume. By cutting with the polar (or projectively dual) hyperplane P_0 (resp. P_n) associated with p_0 (resp. p_n) in $\mathbb{R}P^n$, the orthoscheme R will be truncated yielding a polyhedron R^t of finite volume in $\mathbb{H}^n$ with boundary formed by $H_0, \ldots, H_n$ and by the hyperplane P_0 and/or P_n (for more details, compare [8, 9]). In the sequel, we are mainly interested in *simply truncated n-orthoschemes* or *orthoprisms R^t*, that is, in hyperbolic n-prisms with an $(n-1)$-orthoscheme basis which arise by polar truncation of R in $\mathbb{R}P^n$ having only one ultra-ideal vertex, say p_0. By construction, the polar hyperplane P_0 has a common perpendicular with the hyperplane H_0 in $\mathbb{H}^n$ which coincides with the hyperbolic line defined by p_0 and p_1. We always assume that the distance between P_0 and H_0 is positive. Furthermore, since P_0 intersects orthogonally all hyperplanes through p_0 of R, the orthoprism R^t has—apart from n additional right dihedral angles—the same dihedral angles and parameters $\alpha_1, \ldots, \alpha_n$ as R, at least for $n \geq 3$. In fact, a 2-orthoprism is a Lambert quadrilateral in $\mathbb{H}^2$ with a single angular parameter $\alpha_1 \in [0, \frac{\pi}{2}[$.

2.2 Graphs of Orthoschemes and Orthoprisms

Let R be an n-orthoscheme in $\mathbb{X}^n$ with vertices $p_0, \ldots, p_n$ lying opposite to the bounding hyperplanes $H_0, \ldots, H_n$, and denote by $\alpha_i = \angle(H_{i-1}, H_i)$ the dihedral angle formed by H_{i-1}, H_i. If p_0 is ultra-ideal, denote by R^t the orthoprism given by the truncation of R by means of the polar hyperplane P_0.

By discarding the hyperplane H_0 from the boundary of R, the configuration $H_1, \ldots, H_n$ yields a simplicial cone R_{p_0} with apex p_0. If p_0 is an ordinary point in $\mathbb{X}^n$ (resp. an ideal point of $\mathbb{H}^n$), then the apex geometry of R_{p_0} in $\mathbb{X}^n$ is given by a spherical (resp. Euclidean) $(n-1)$-orthoscheme with dihedral angles $\alpha_2, \ldots, \alpha_n$. If p_0 is an ultra-ideal point in $\mathbb{R}P^n$, the apex geometry in $\mathbb{H}^n$ of the cone R_{p_0} with apex p_0 in $\mathbb{R}P^n$ is given by a hyperbolic $(n-1)$-orthoscheme with dihedral angles $\alpha_2, \ldots, \alpha_n$. The latter is identical with the prism basis of R^t.

Let us associate with an n-orthoscheme R (and R^t) a graph $\Sigma = \Sigma(R)$, characterising R up to congruence, by assigning to each hyperplane H_i a node ν_i in Σ. Two nodes are joined by an edge if the corresponding hyperplanes are not orthogonal. We attach the weight α_i to the edge joining the nodes of the consecutive hyperplanes if they intersect under the dihedral angle α_i. A *subgraph* of a graph $\Sigma = \Sigma(R)$ arises by discarding nodes together with their edges. Observe that Σ consists of (at most $n+1$) connected components of linear graphs. In the hyperbolic finite volume case, it is easy to see that Σ is always connected.

To every graph $\Sigma(R)$ as above corresponds the (symmetric) Gram matrix $G(R) = (g_{ik})$ of R which is of order $n+1$, with $g_{ii} = 1$, $g_{i,i+1} = -\cos\alpha_{i+1} \leq 0$ and $g_{ik} = 0$ otherwise ($0 \leq i < k \leq n$). On the other hand, if $G = (g_{ik})$ is an indecomposable symmetric matrix of order $n+1$ with $g_{ii} = 1$, $g_{i,i+1} \leq 0$ and $g_{ik} = 0$ otherwise, then G can be realised as Gram matrix $G(R)$ for a n-orthoscheme R in $\mathbb{X}^n$ as follows (cf. [16, Sect. 2] and [14, Sects. 7.2, 7.3]). Denote by G_l the principal submatrix of order n of G arising by removing the $(l+1)$-th row and the $(l+1)$-th column.

(1) If G is positive definite, then $G = G(R)$ for a spherical n-orthoscheme R defined up to isometry. Both $G(R)$ and the graph Σ are called *elliptic*.

(2) If G is positive semidefinite, then $G = G(R)$ for a Euclidean n-orthoscheme R defined up to similarity. Both, $G(R)$ and the graph Σ are called *parabolic*.

(3) If G is of signature $(n, 1)$, then $G = G(R)$ for a hyperbolic n-orthoscheme R (of finite or infinite volume) defined up to isometry. The matrix $G(R)$ and the graph $\Sigma(R)$ are called *hyperbolic*. We distinguish further the following subcases.

   (a) If all principal submatrices of G are elliptic, then R is a compact orthoscheme. Its graph $\Sigma = \Sigma(R)$ will be drawn according to

$$\Sigma \ : \ \circ \overset{\alpha_1}{\rule{1.5cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_n}{\rule{1.5cm}{0.4pt}} \circ$$

   (b) If, apart from elliptic principal submatrices of G, only G_0 (and G_n) is parabolic, then R is a simply (doubly) asymptotic orthoscheme. In particular, if R is simply (doubly) asymptotic of finite volume with G_0 (and G_n) parabolic, we blacken the node ν_0 (and ν_n) and write

$$\Sigma \ : \ \bullet \overset{\alpha_1}{\rule{1.5cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_n}{\rule{1.5cm}{0.4pt}} \circ \quad .$$

   (c) If, among all principal submatrices of G, precisely G_0 is hyperbolic, then R is an orthoscheme with ultra-ideal vertex p_0 and associated orthoprism R^t of finite volume. The *graph* $\Sigma = \Sigma(R^t)$ *of* R^t is given by the graph $\Sigma(R)$ by putting a diamond in place of the node ν_0 (indicating that P_0 is added to the

hyperplane configuration $H_0, \ldots, H_n$ bounding R). A compact n-orthoprism R^t is therefore described by

$$\Sigma \; : \; \diamond \overset{\alpha_1}{\rule{1.5cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_n}{\rule{1.5cm}{0.4pt}} \circ \quad .$$

2.3 The Coxeter and Pseudo-Coxeter Cases

Consider a convex n-polyhedron P in $\mathbb{X}^n$. If all its dihedral angles are of the form π/k with $k \geq 2$, the polyhedron P is a *Coxeter polyhedron*. In the theory of regular polyhedra and regular honeycombs, the family of Coxeter orthoschemes plays an important role. In fact, Coxeter orthoschemes provide fundamental domains for the associated symmetry groups. They were classified by Coxeter, Lannér and Koszul-Chein (cf. [16] for lists and references). The family of simply and doubly truncated Coxeter orthoschemes in $\mathbb{H}^n$ was studied and classified by H.-C. Im Hof (cf. [8]). It is noteworthy that hyperbolic Coxeter orthoschemes, truncated or not, exist only up to dimension 9. In the compact case, they exist only up to dimension 5.

A convex n-polyhedron $P \subset \mathbb{X}^n$ is called a *pseudo-Coxeter polyhedron* if all its dihedral angles are commensurable with π. They arise naturally in the study of regular star-polyhedra and star-honeycombs (cf. [2, p. 161 ff]). The notion of *graph* of a pseudo-Coxeter orthoscheme is defined in an analogous way by a graph with edge weigths $\frac{q}{p}$ corresponding to (non-right) dihedral angles of the form $p\pi/q$ for integers $p \geq 1$, $q \geq 3$. Of interest will be, among others, the pseudo-Coxeter orthoschemes in $\mathbb{H}^5$ given by

$$
\begin{aligned}
&\bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet \\
&\bullet \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \quad , \\
&\bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \diamond \quad .
\end{aligned}
\tag{1}
$$

In the sequel, by considering dissections of certain Coxeter and quasi-Coxeter polyhedra, new polyhedra will arise with dihedral angles of mixed type, that is, some angles are commensurable with π, some are not. In order to keep the notations as concise as possible, we describe these polyhedra by graphs with rational weights in the first case and with letters such as α, λ, ω representing angular parameters in the second case (see the examples (5), (6) below).

Remark Observe that $\cos^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{5} + \cos^2 \frac{2\pi}{5} = 1$ which is responsible for the parabolicity of the subgraphs in (1). In general, the parabolicity of a graph

$$\circ \overset{\alpha_1}{\rule{1.2cm}{0.4pt}} \circ \overset{\alpha_2}{\rule{1.2cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1.2cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1.2cm}{0.4pt}} \circ$$

is equivalent to the condition that (cf. [2, p. 264])

$$\frac{\cos^2 \alpha_2}{\sin^2 \alpha_1} + \frac{\cos^2 \alpha_3}{\sin^2 \alpha_4} = 1. \tag{2}$$

2.4 Orthoscheme Cycles

Consider a doubly asymptotic 5-orthoscheme R with graph

$$\Sigma \; : \; \bullet \overset{\alpha_1}{\rule{1.2cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_5}{\rule{1.2cm}{0.4pt}} \bullet$$

and define the angle $\alpha_0 \in {]0, \frac{\pi}{2}[}$ such that the graph

$$\circ \overset{\alpha_0}{\rule{1.2cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_3}{\rule{1.2cm}{0.4pt}} \circ$$

is parabolic according to (2). Then, by [10, Sect. 1.4, Lemma], the graphs

$$\Sigma_i \; : \; \bullet \overset{\alpha_i}{\rule{1.2cm}{0.4pt}} \circ - \cdots - \circ \overset{\alpha_{3+i}}{\rule{1.2cm}{0.4pt}} \circ \overset{\alpha_{4+i}}{\rule{1.2cm}{0.4pt}} \bullet \quad ,$$

wherein indices $i \geq 1$ are taken modulo 6, form a Σ-*cycle of length* 6 consisting of six different doubly asymptotic 5-orthoschemes where two neighbours share a vertex at infinity. According to the weights, a cycle may consist of tuples of isometric members and "collapse" to a cycle consisting of a smaller number of non-isometric members. This phenomenon reduces the length of the cycle, accordingly. Here are the most prominent examples for the subsequent investigations:

$$U_i \; : \; \bullet \overset{\frac{5}{2}}{\rule{0.9cm}{0.4pt}} \circ \overset{5}{\rule{0.9cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.9cm}{0.4pt}} \circ \overset{5}{\rule{0.9cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.9cm}{0.4pt}} \bullet \quad , \quad i \bmod 2 \; ; \tag{3}$$

$$V_i \; : \; \bullet \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{5}{\rule{0.9cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.9cm}{0.4pt}} \circ \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{5}{\rule{0.9cm}{0.4pt}} \bullet \quad , \quad i \bmod 3 \; ; \tag{4}$$

$$W_i \; : \; \bullet \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.9cm}{0.4pt}} \circ \overset{\omega_1}{\rule{0.9cm}{0.4pt}} \circ \overset{\omega_2}{\rule{0.9cm}{0.4pt}} \bullet \quad , \quad i \bmod 6 \; ; \tag{5}$$

$$Z_i \; : \; \bullet \overset{\lambda}{\rule{0.9cm}{0.4pt}} \circ \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{3}{\rule{0.9cm}{0.4pt}} \circ \overset{\lambda}{\rule{0.9cm}{0.4pt}} \bullet \quad , \quad i \bmod 4 \; , \tag{6}$$

where the weights in (5) and (6) are given by the conditions

$$0 < \lambda, \omega_1, \omega_2 < \frac{\pi}{2} \quad \text{with}$$

$$\cos(2\lambda) = \frac{1}{4}, \qquad \omega_1 = \frac{\pi}{3} - \lambda, \qquad \omega_2 = \frac{\pi}{3} + \omega_1. \tag{7}$$

Denote by τ the *golden ratio* given by $\tau = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$ satisfying the quadratic equation $\tau^2 = \tau + 1$. Then, we deduce the following relations:

$$\sin\omega_1 = \sqrt{\frac{3}{8}(\tau - 1)}, \qquad \sin\omega_2 = \sqrt{\frac{3}{8}}\,\tau. \tag{8}$$

Observe that the V-cycle (4) is a realisation of the cycle of length 3 given by

$$\Sigma_i \; : \; \bullet \overset{\alpha}{\rule{0.9cm}{0.4pt}} \circ \overset{\beta}{\rule{0.9cm}{0.4pt}} \circ \overset{\gamma}{\rule{0.9cm}{0.4pt}} \circ \overset{\alpha}{\rule{0.9cm}{0.4pt}} \circ \overset{\beta}{\rule{0.9cm}{0.4pt}} \bullet \quad , \quad i \bmod 3 \; , \tag{9}$$

where $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ satisfy the equation (cf. also (1))

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{10}$$

The Z-cycle (6) of length 4 is not of type (9), (10). However, its members belong to different dissections of an ideal regular 5-simplex $S_{\mathrm{reg}}^{\infty}(2\lambda)$ with dihedral angle 2λ satisfying (7) (see Sect. 3.2, Remark 2).

Closely related to the doubly asymptotic 5-orthoschemes above is the following group of simply asymptotic 5-orthoschemes:

$$
\begin{array}{rl}
A\ : & \bullet\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ \quad, \\[6pt]
B\ : & \bullet\!\overset{3}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ \quad, \\[6pt]
C\ : & \bullet\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ \quad, \\[6pt]
D\ : & \bullet\!\overset{3}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ \quad,
\end{array} \tag{11}
$$

as well as the following group of (simply) asymptotic 5-orthoprisms:

$$
\begin{array}{rl}
\overline{A}\ : & \bullet\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\diamond \quad, \\[6pt]
\overline{B}\ : & \bullet\!\overset{3}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\diamond \quad, \\[6pt]
\overline{C}\ : & \bullet\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\diamond \quad, \\[6pt]
\overline{D}\ : & \bullet\!\overset{3}{-\!-}\!\circ\!\overset{3}{-\!-}\!\circ\!\overset{5}{-\!-}\!\circ\!\overset{\frac{5}{2}}{-\!-}\!\circ\!\overset{3}{-\!-}\!\diamond \quad.
\end{array} \tag{12}
$$

We shall be able to determine all volumes for (11) and (12) as rational linear combinations of the characteristic units $\zeta(3)$, $\mathrm{JI}_3(\frac{\pi}{5})$. Then, it will be possible to relate them to some members of (5) and to some members of the Z-cycle and therefore to the volume $\mu_5 := \mathrm{vol}_5(S_{\mathrm{reg}}(2\lambda))$ in hyperbolic 5-space (cf. Sect. 4).

3 Scissors Congruences in Hyperbolic 5-Space

In the hyperbolic space, asymptotic orthoschemes play a particularly important role in the context of scissors congruence.

Let Y^n be either $\mathbb{H}^n$ or $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$. Recall that the scissors congruence group $\mathcal{P}(Y^n)$ is the abelian group generated by $[P]$ for each polyhedron P in Y^n equipped with the relations (i) $[P \sqcup Q] = [P] + [Q]$ ($\sqcup$ denotes disjoint interior union) and (ii) $[P] = [Q]$ for P isometric to Q. For $n \geq 2$, the group $\mathcal{P}(\mathbb{H}^n)$ is generated by the classes of simply asymptotic orthoschemes, while, for $n > 1$ *odd*, the group $\mathcal{P}(\overline{\mathbb{H}^n})$ is generated by the classes of doubly asymptotic orthoschemes (for references, see for example [3]). The proof of these results is based on several cutting and pasting procedures for orthoschemes (cf. Sects. 3.2 and Sect. 3.3 below). By applying the homomorphism given by the volume functional, these two scissors congruences, combined with some more relations including the barycentric dissection for regular simplices, will enable us to derive the volumes of all polytopes in (11) and (12) and

Fig. 2 Bisecting the angle of parallelism
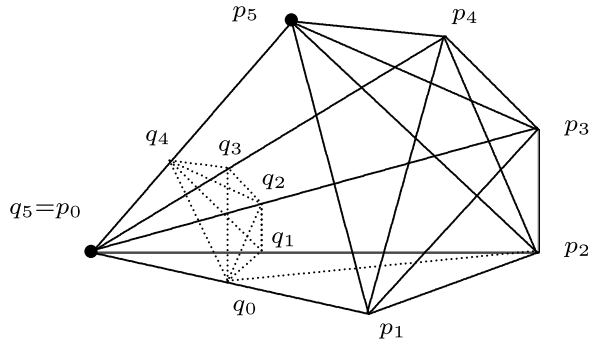
to relate the volume of some members of the W-cycle to the volume μ_5 of an ideal regular 5-simplex. This will be achieved by elegant decomposition relations, avoiding hard analytic computations. In the sequel, we present the scissors congruences in detail for asymptotic orthoschemes and orthoprisms of dimension $n = 5$, only.

3.1 Bisecting the Angle of Parallelism

Let $R = p_0 \cdots p_5$ denote a doubly asymptotic 5-orthoscheme with vertices $p_0, \ldots, p_5$ and with graph

$$\Sigma(R) \quad : \quad \bullet \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \overset{\alpha_4}{\text{———}} \circ \overset{\alpha_5}{\text{———}} \bullet \quad .$$

The dihedral angles α_1 and α_5 appear as angles of parallelism in simply asymptotic triangles. For instance, α_1 is the angle of parallelism in the triangle $p_0 p_1 p_2$ (cf. Fig. 1) whence

$$\sin \alpha_1 = \frac{1}{\cosh p_1 p_2}. \tag{13}$$

Write $\alpha_1 =: 2\alpha$. In the sequel, we decompose R into 6 simply asymptotic orthoschemes two of which are isometric to one another with angle of parallelism equal to α. Draw the angle bisector l of α_1 in the triangle $p_0 p_1 p_2$ and denote by $q_0 := l \cap p_0 p_1$ the intersection of l with the line defined by p_0, p_1. Consider the hyperplane H through q_0 which is orthogonal to the line $p_0 p_5$ and define $q_i := H \cap p_0 p_{i+1}$ for $1 \leq i \leq 4$ (cf. Fig. 2).

Write $q_5 := p_0 \in \partial \mathbb{H}^5$ and form the convex hull

$$R_k := q_0 \cdots q_{k-1} p_k p_{k+1} \cdots p_5, \quad 0 \leq k \leq 6. \tag{14}$$

By construction, each R_k, $0 \leq k \leq 6$, is a 5-orthoscheme with $R = R_0$ and $R_6 =: Q = q_0 \cdots q_5$. For $k \neq 0$, the orthoscheme R_k is simply asymptotic. Furthermore, R_1 is isometric to R_2, and R_5 is isometric to R_6 (cf. Fig. 1 and use angle parallelism). The construction (14) yields the scissors congruence relation

$$[R] = \sum_{k=1}^{6} [R_k] = 2[R_2] + [R_3] + [R_4] + 2[R_5] \tag{15}$$

called the *bisection of the angle of parallelism*. In the following, we determine the metric properties of the dissecting orthoschemes. Denote the graphs associated with $R_2, \ldots, R_5$ by

$$
\begin{array}{ll}
\Sigma(R_2) & : \quad \circ \overset{\alpha}{\rule{1cm}{0.4pt}} \circ \overset{u}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \\[6pt]
\Sigma(R_3) & : \quad \circ \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\overline{u}}{\rule{1cm}{0.4pt}} \circ \overset{v}{\rule{1cm}{0.4pt}} \circ \overset{x}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \\[6pt]
\Sigma(R_4) & : \quad \circ \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{w}{\rule{1cm}{0.4pt}} \circ \overset{y}{\rule{1cm}{0.4pt}} \circ \overset{\overline{z}}{\rule{1cm}{0.4pt}} \bullet \\[6pt]
\Sigma(R_5) & : \quad \circ \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \overset{z}{\rule{1cm}{0.4pt}} \bullet
\end{array}
\tag{16}
$$

where we put $\overline{\omega} := \pi - 2\omega$. By (2), the parabolicity of the subgraph of R_2 associated with the common vertex p_5 yields

$$
\cos^2 u = \left(1 - \frac{\cos^2 \alpha_3}{\sin^2 \alpha_4} \right) \sin^2 \alpha. \tag{17}
$$

In order to determine the remaining parameters in (16), we consider the 3-face $q_0 q_1 q_2 q_3$ in Q (cf. Fig. 2). The vertex figure associated with q_3 is a right-angled spherical triangle with angles α_2, α_3 so that its edge length δ opposite to α_3 appears as the planar angle opposite to the edge $q_1 q_2$ in the face $q_1 q_2 q_3$. By (13) and with some non-Euclidean trigonometry, one easily computes δ and the lengths of the orthogonal edge path in $q_0 q_1 q_2 q_3$ according to

$$
\cosh q_0 q_1 = \frac{2 \cos^2 \alpha}{\sqrt{4 \cos^2 \alpha - 1}}, \qquad \cos \delta = \frac{\cos \alpha_3}{\sin \alpha_2}, \tag{18}
$$

$$
\sinh q_1 q_2 = \tanh q_0 q_1 \cdot \cot \alpha_2, \qquad \sinh q_2 q_3 = \tanh q_1 q_2 \cdot \cot \delta.
$$

In a similar way, by considering the vertex figures associated with q_2 in the 3-faces $q_2 q_3 q_4 p_5$ of R_5, $q_2 q_3 p_4 p_5$ of R_4 and $q_2 p_3 p_4 p_5$ of R_3 respectively, we obtain the relations

$$
\tanh q_2 q_3 = \cot z \cot \alpha_5 = \cot y \cot \overline{z}; \qquad \sin x = \frac{\cos \alpha_5 \sin y}{\cos \overline{z}}. \tag{19}
$$

The remaining parameters v, w are determined by a parabolicity condition (2).

Example 1 For the pseudo-Coxeter orthoscheme R with graph given by (cf. (3))

$$
U_1 \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet ,
$$

one derives the relations

$$
[R_2] = [R_3] = \left[\circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet \right] = [A],
$$

$$
[R_4] = [R_5] = \left[\circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet \right] = [B].
$$

Hence, by (15), it follows that

$$
[U_1] = 3[A] + 3[B]. \tag{20}
$$

Example 2 For the pseudo-Coxeter orthoscheme R with graph (cf. (4))

$$V_2 \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \bullet \quad ,$$

one derives the relations

$$[R_2] = [R_3] = [R_4] = [\,\circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \bullet\,] = [C] \quad ,$$

$$[R_5] = [\,\circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \bullet\,] = [A] \quad .$$

Again, by (15), we obtain that

$$[V_2] = 2[A] + 4[C]. \tag{21}$$

Example 3 For the cycle neighbour R of V_2 in (4) given by

$$V_3 \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \bullet \quad ,$$

one derives the relations

$$[R_2] = [\,\circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \bullet\,] = [B] \quad ,$$

$$[R_3] = [R_4] = [R_5] = [\,\circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{5}{\rule{1.2em}{0.5pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \circ \overset{3}{\rule{1.2em}{0.5pt}} \bullet\,] = [D] \quad .$$

Here, we obtain that

$$[V_3] = 2[B] + 4[D]. \tag{22}$$

Remark 1 It is not hard to generalise the dissection (15) by dissecting the angle of parallelism $\alpha_1 = \beta_1 + \beta_2$ for arbitrary $0 < \beta_1, \beta_2 < \alpha_1$.

3.2 Reversing to Infinity

Let $R = p_0 \cdots p_5$ be a simply asymptotic 5-orthoscheme with vertices $p_0, \ldots, p_4 \in \mathbb{H}^5$ and with $p_5 \in \partial\mathbb{H}^5$ encoded by the graph

$$\Sigma(R) \quad : \quad \circ \overset{\alpha_1}{\rule{1.2em}{0.5pt}} \circ \overset{\alpha_2}{\rule{1.2em}{0.5pt}} \circ \overset{\alpha_3}{\rule{1.2em}{0.5pt}} \circ \overset{\alpha_4}{\rule{1.2em}{0.5pt}} \circ \overset{\alpha_5}{\rule{1.2em}{0.5pt}} \bullet \quad .$$

Prolong the (finite) oriented geodesic segment $p_0 p_1$ from p_0 to p_1 in the reversed sense to a geodesic half-line l_1 with limit point $q_0 \in \partial\mathbb{H}^5$, say. In the same way, prolong all oriented edges $p_0 p_i$ in the reversed sense in order to obtain geodesic half-lines l_i for $2 \le i \le 5$ (cf. Fig. 3). Consider the hyperplane H through q_0 which is orthogonal to l_5 and define the points $q_i := H \cap l_{i+1}$ for $i = 1, \ldots, 4$, and let $q_5 := p_0$. Form the convex hull

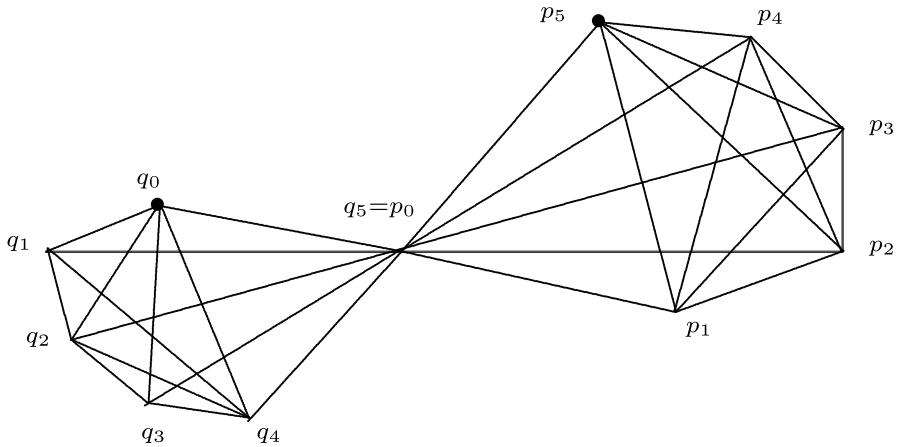$$R_k := q_0 \cdots q_{k-1} p_k \cdots p_5, \quad 0 \le k \le 6.$$

Fig. 3 Reversing to infinity

It follows easily (cf. also [3, Theorem (2.6), (ii)]) that $R_0 = R$ and $R_6 =: Q$ are simply asymptotic and that R_k, $1 \le k \le 5$, are doubly asymptotic orthoschemes satisfying the relation

$$[R] + [Q] = \sum_{k=1}^{5} (-1)^{k-1} [R_k].\tag{23}$$

Since Q is simply asymptotic with vertex $q_0 \in \partial\mathbb{H}^5$ and with spherical vertex figure at $q_5 = p_0$ isometric to that one of R at p_0, we deduce that its graph equals

$$\Sigma(Q) \quad : \quad \bullet \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_6}{\rule{1cm}{0.4pt}} \circ$$

where $\alpha_6 \in {}]0, \frac{\pi}{2}[$ is given by the parabolicity condition (2), that is,

$$\frac{\cos^2 \alpha_4}{\sin^2 \alpha_3} + \frac{\cos^2 \alpha_5}{\sin^2 \alpha_6} = 1.\tag{24}$$

For the graphs of the orthoschemes R_k, $1 \le k \le 5$, write (cf. also Fig. 1)

$$
\begin{aligned}
\Sigma(R_1) &: \quad \bullet \overset{u_1}{\rule{0.8cm}{0.4pt}} \circ \overset{x_1}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{0.8cm}{0.4pt}} \bullet \quad &&\text{with} \quad u_1 = \alpha_1 + u\\
\Sigma(R_2) &: \quad \bullet \overset{u}{\rule{0.8cm}{0.4pt}} \circ \overset{x_2}{\rule{0.8cm}{0.4pt}} \circ \overset{y_1}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{0.8cm}{0.4pt}} \bullet \\
\Sigma(R_3) &: \quad \bullet \overset{\alpha_2}{\rule{0.8cm}{0.4pt}} \circ \overset{x_3}{\rule{0.8cm}{0.4pt}} \circ \overset{y_2}{\rule{0.8cm}{0.4pt}} \circ \overset{z_1}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{0.8cm}{0.4pt}} \bullet \\
\Sigma(R_4) &: \quad \bullet \overset{\alpha_2}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{0.8cm}{0.4pt}} \circ \overset{y_3}{\rule{0.8cm}{0.4pt}} \circ \overset{z_2}{\rule{0.8cm}{0.4pt}} \circ \overset{v}{\rule{0.8cm}{0.4pt}} \bullet \\
\Sigma(R_5) &: \quad \bullet \overset{\alpha_2}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{0.8cm}{0.4pt}} \circ \overset{z_3}{\rule{0.8cm}{0.4pt}} \circ \overset{v_6}{\rule{0.8cm}{0.4pt}} \bullet \quad &&\text{with} \quad v_6 = v + \alpha_6 .
\end{aligned}\tag{25}
$$

The dihedral angles x_1, u_1, z_3, v_6 (and therefore u, v) can be determined by the parabolicity condition (2). Then, the dihedral angles x_2, z_2 follow from the conditions

$$\tan u \tan x_2 = \tan \alpha_4 \tan \alpha_5, \qquad \tan \alpha_2 \tan \alpha_3 = \tan z_2 \tan v,\tag{26}$$

Fig. 4 Vertex figures at q_0

which reflect the bi-asymptoticity of the 5-orthoschemes R_2 and R_4 (cf. [10, Sect. 1.4, Lemma]). Again, the parameters y_1, y_3 follow from (2). As for x_3, z_1, observe that

$$x_1 + x_3 = x_2, \qquad z_1 + z_3 = z_2. \tag{27}$$

Indeed, for the first part of (27), consider the 3-faces $q_0 p_1 p_2 p_3$ of R_1, $q_0 q_1 p_2 p_3$ of R_2 and $q_0 q_1 q_2 q_3$ of R_3, respectively. The (Euclidean) vertex figures associated with q_0 yield a picture (with obvious labellings) as drawn in Fig. 4. Now, the first relation in (27) is the angle sum $x_2 = x_1 + x_3$ at p_3'. The second relation of (27) is verified accordingly by considering the vertex p_5. Finally, the dihedral angle y_2 follows from the parabolicity condition (2).

Example 1 For the pseudo-Coxeter orthoscheme R with graph (cf. (11))

$$B = \Sigma(R) \quad : \quad \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \bullet \quad ,$$

one obtains (cf. Sect. 2.3, Remark, and (3)–(5), (7))

$$
\begin{aligned}
A = \Sigma(Q) &\quad : \quad \bullet \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \circ \xrightarrow{5} \circ \\
V_3 = \Sigma(R_1) &\quad : \quad \bullet \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \bullet \\
W_2 = \Sigma(R_2) &\quad : \quad \bullet \xrightarrow{5} \circ \xrightarrow{\omega_2} \circ \xrightarrow{\omega_1} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \bullet \\
W_3 = \Sigma(R_3) &\quad : \quad \bullet \xrightarrow{\frac{5}{2}} \circ \xrightarrow{\omega_1} \circ \xrightarrow{\omega_2} \circ \xrightarrow{5} \circ \xrightarrow{3} \bullet \\
V_2 = \Sigma(R_4) &\quad : \quad \bullet \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \circ \xrightarrow{3} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \bullet \\
U_1 = \Sigma(R_5) &\quad : \quad \bullet \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \bullet \quad .
\end{aligned}
\tag{28}
$$

Hence, by (23), we deduce that

$$[A] + [B] = [W_3] - [W_2] + [U_1] + [V_3] - [V_2]. \tag{29}$$

Notice that we obtain the identical relation (29) by starting with an orthoscheme R given by the graph $\Sigma(R) = A$.

Example 2 For the pseudo-Coxeter orthoscheme R with graph (cf. Sect. 3.1)

$$D = \Sigma(R) \quad : \quad \circ \xrightarrow{3} \circ \xrightarrow{5} \circ \xrightarrow{\frac{5}{2}} \circ \xrightarrow{3} \circ \xrightarrow{3} \bullet \quad ,$$

one obtains (cf. Sect. 2.3, Remark, and (3)–(5), (7))

$$
\begin{array}{lll}
C = \Sigma(Q) & : & \bullet\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\circ \\[4pt]
W_1 = \Sigma(R_1) & : & \bullet\overset{\omega_2}{\rule{1.2em}{0.4pt}}\circ\overset{\omega_1}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet \\[4pt]
W_4 = \Sigma(R_2) & : & \bullet\overset{\omega_1}{\rule{1.2em}{0.4pt}}\circ\overset{\omega_2}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet \\[4pt]
V_1 = \Sigma(R_3) & : & \bullet\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet \\[4pt]
U_2 = \Sigma(R_4) & : & \bullet\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\bullet \\[4pt]
V_2 = \Sigma(R_5) & : & \bullet\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{5}{\rule{1.2em}{0.4pt}}\circ\overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}}\bullet \; .
\end{array}
\tag{30}
$$

Hence, by (23), we deduce that

$$
[C] + [D] = [W_1] - [W_4] - [U_2] + [V_1] + [V_2]. \tag{31}
$$

Observe that the relations obtained so far do not allow as yet to express the elements $[A]$, $[B]$, $[C]$, $[D]$ only in terms of members of the U- and the V-cycle (3) and (4) whose volumes are well known (cf. Sect. 4).

Remark 2 Consider the orthoscheme R with graph

$$
Z = \Sigma(R) \quad : \quad \circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet
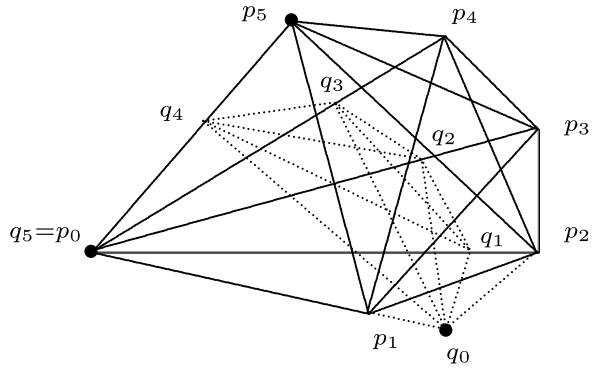$$

which is related to the orthoscheme cycle (6) associated with an ideal regular hyperbolic 5-simplex $S := S^\infty_{\mathrm{reg}}(2\lambda)$ of dihedral angle 2λ given by (7). By reversing to infinity according to (23)–(27), we deduce easily the scissors relation

$$
2\,[Z] = 2\,[Z_2] - 2\,[Z_3] + [Z_4] =
$$
$$
= 2\,[\,\bullet\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] - 2\,[\,\bullet\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] +
$$
$$
+ [\,\bullet\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] \;\; ,
$$

which is compatible with the well-known barycentric subdivison for $S := S^\infty_{\mathrm{reg}}(2\lambda)$. More precisely, let c be the centre of gravity and q one of the vertices at infinity of S. By drawing successively perpendiculars starting from c or from q, or by applying the dissection procedure above and in Sect. 3.3 below (cf. [3, (7.4), p. 147], for example), one obtains the scissors relations

$$
\begin{aligned}
[S] &= 720\,[\,\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] \\[4pt]
&= 120\,[\,\bullet\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] = 120\,[Z_2] \\[4pt]
&= 48\,[\,\bullet\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] = 48\,[Z_3] \\[4pt]
&= 36\,[\,\bullet\overset{3}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{2\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{\lambda}{\rule{1.2em}{0.4pt}}\circ\overset{3}{\rule{1.2em}{0.4pt}}\bullet\,] = 36\,[Z_4] \quad .
\end{aligned}
$$

Fig. 5 Extending to infinity

In particular, we deduce that

$$\mu_5 := \mathrm{vol}_5(S^\infty_{\mathrm{reg}}(2\lambda)) = \mathrm{vol}_5(S) = 720\,\mathrm{vol}_5(Z)$$

$$= 120\,\mathrm{vol}_5(Z_2) = 48\,\mathrm{vol}_5(Z_3) = 36\,\mathrm{vol}_5(Z_4). \quad (32)$$

3.3 Extending to Infinity

Let us start with the easiest case and consider a doubly asymptotic 5-orthoscheme $R = p_0 \cdots p_5$ with vertices $p_0, \ldots, p_5$ and with graph

$$\Sigma(R) \quad : \quad \bullet \overset{\alpha_1}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \quad .$$

The bi-asymptoticity of R can be expressed according to (cf. (26))

$$\tan\alpha_1 \tan\alpha_2 = \tan\alpha_4 \tan\alpha_5. \quad (33)$$

Now, prolong the geodesic half-line $p_0 p_1$ defined by p_0 and p_1 beyond p_1 in order to obtain a geodesic line l. Denote by $q_0 \in \partial\mathbb{H}^5$ the limiting point associated with l. Let H be the hyperplane through q_0 orthogonal to the line defined by p_0 and p_5, and let $q_i := H \cap p_0 p_{i+1}$ for $i = 1, \ldots, 4$. Set $q_5 := p_0$ and consider the convex hull

$$R_k := q_0 \cdots q_{k-1} p_k \cdots p_5, \quad 0 \le k \le 6.$$

Obviously, $R = R_0$, and write $Q := R_6$. Then, by [3, Theorem (2.6) and Sect. 6], each R_k is a doubly asymptotic 5-orthoscheme which takes part of the scissors congruence relation

$$[R] = -\sum_{k=1}^{p} [R_k] + \sum_{k=p+1}^{6} [R_k], \quad (34)$$

where the index $p \in [1, 5]$ depends on the measures of the dihedral angles $\alpha_1, \ldots, \alpha_5$ of R. For example, if the dihedral angle $\alpha_1 < \pi/4$ (resp. $\alpha_1 > \pi/4$), then the foot point q_1 of the perpendicular from q_0 to the half-line $p_0 p_2$ lies *inside* of R (resp. *outside* of R), a fact which we denote by $q_1 \in R$ (resp. $q_1 \notin R$); see also Fig. 5. By

the identity (33) and the parabolicity condition (2), it is easy to see that R is isometric to R_1 and that Q is isometric to R_5 (cf. dihedral angle distribution in Fig. 1). The graph of Q is given by

$$\Sigma(Q) \quad : \quad \bullet \!\!\overset{\alpha_2}{-\!\!-\!\!-}\!\! \circ \overset{\alpha_3}{-\!\!-\!\!-} \circ \overset{\alpha_4}{-\!\!-\!\!-} \circ \overset{\alpha_5}{-\!\!-\!\!-} \circ \overset{\alpha_6}{-\!\!-\!\!-}\!\! \bullet \quad ,$$

where $\alpha_6 \in \left]0, \frac{\pi}{2}\right[$ is given by (24). Hence, if $\alpha_1 < \pi/4$ and $q_1 \in R$, then $p = 1$, and (34) turns into (cf. [10, (18)])

$$2[R] = 2[Q] + [R_2] + [R_3] + [R_4]. \tag{35}$$

Now, denote the graphs of R_2, R_3, R_4 as follows (cf. also Fig. 1):

$$
\begin{aligned}
\Sigma(R_2) \quad &: \quad \bullet \overset{\varphi_1}{-\!\!-} \circ \overset{x_1}{-\!\!-} \circ \overset{y_1}{-\!\!-} \circ \overset{\alpha_4}{-\!\!-} \circ \overset{\alpha_5}{-\!\!-} \bullet \\
\Sigma(R_3) \quad &: \quad \bullet \overset{\alpha_2}{-\!\!-} \circ \overset{x_2}{-\!\!-} \circ \overset{y_2}{-\!\!-} \circ \overset{z_1}{-\!\!-} \circ \overset{\alpha_5}{-\!\!-} \bullet \\
\Sigma(R_4) \quad &: \quad \bullet \overset{\alpha_2}{-\!\!-} \circ \overset{\alpha_3}{-\!\!-} \circ \overset{y_3}{-\!\!-} \circ \overset{z_2}{-\!\!-} \circ \overset{\varphi_6}{-\!\!-} \bullet \quad ,
\end{aligned}
\tag{36}
$$

where the parameter $\varphi_k \in \left]0, \frac{\pi}{2}\right[$, $k = 1, 6$, depends—by parallelism considerations—on α_k as follows:

$$
\varphi_k := \begin{cases} 2\alpha_k, & \text{if } \alpha_k < \frac{\pi}{4}; \\ \pi - 2\alpha_k, & \text{if } \alpha_k > \frac{\pi}{4}. \end{cases}
\tag{37}
$$

By the parabolicity conditions (2) and (33), the dihedral angles x_1, y_1, z_2, y_3 are easy to determine. As for $x_2 \in \left]0, \frac{\pi}{2}\right[$ and finally p (cf. (34)), consider the different positions of q_1 and q_2 relative to R inside the 3-faces $q_0 q_1 p_2 p_3$ and $q_0 q_1 q_2 p_3$ and the position of q_3 relative to R. It follows at once that

$$
x_2 = \begin{cases} \alpha_2 - x_1, & \text{if } q_1 \in R \text{ and } p = 1; & \text{(38a)} \\ \alpha_2 + x_1, & \text{if } q_1 \notin R \text{ and } q_2 \in R \text{ and } p = 2; & \text{(38b)} \\ \pi - (\alpha_2 + x_1), & \text{if } q_1, q_2 \notin R \text{ and } q_3 \in R \text{ and } p = 3; & \text{(38c)} \\ \pi - (\alpha_2 + x_1), & \text{if } q_1, q_2, q_3 \notin R \text{ and } p = 4. & \text{(38d)} \end{cases}
$$

We illustrate these dissections by providing several useful examples connecting members of the W-cycle with members of the V-cycle and—most importantly—the orthoscheme Z_4 (cf. (4)–(7), also Sect. 3.4 below).

Example 1 For the pseudo-Coxeter orthoscheme R with graph (cf. (5))

$$W_2 = \Sigma(R) \quad : \quad \bullet \overset{5}{-\!\!-} \circ \overset{\omega_2}{-\!\!-} \circ \overset{\omega_1}{-\!\!-} \circ \overset{\frac{5}{2}}{-\!\!-} \circ \overset{3}{-\!\!-} \bullet \quad ,$$

one obtains

$$W_1 = \Sigma(Q) \quad : \quad \bullet \overset{\omega_2}{-\!\!-} \circ \overset{\omega_1}{-\!\!-} \circ \overset{\frac{5}{2}}{-\!\!-} \circ \overset{3}{-\!\!-} \circ \overset{3}{-\!\!-} \bullet \quad ,$$

so that $q_1, q_3 \in R$ by (37). Furthermore, by (9) and (38a), one derives that $p = 1$ and

$$V_3 = \Sigma(R_2) \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet$$

$$W_1 = \Sigma(R_3) \quad : \quad \bullet \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet$$

$$W_1 = \Sigma(R_4) \quad : \quad \bullet \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet \quad .$$

Hence, by (35), we deduce that

$$2[W_2] = 4[W_1] + [V_3]. \tag{39}$$

Example 2 For the pseudo-Coxeter orthoscheme R with graph

$$W_3 = \Sigma(R) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

one gets $q_1 \notin R$ and

$$W_2 = \Sigma(Q) \quad : \quad \bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

so that $q_3 \in R$ by (37). By (38b), we obtain $p = 2$ and

$$W_3 = \Sigma(R_2) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet$$

$$V_2 = \Sigma(R_3) \quad : \quad \bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet$$

$$W_2 = \Sigma(R_4) \quad : \quad \bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \bullet \quad .$$

Hence, by (34), we deduce that

$$3[W_3] = 3[W_2] + [V_2]. \tag{40}$$

Example 3 For the pseudo-Coxeter orthoscheme R with graph

$$W_4 = \Sigma(R) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

one gets $q_1 \notin R$ and

$$W_3 = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

so that $q_3 \in R$ by (37). By (38b), we obtain $p = 3$ and

$$W_4 = \Sigma(R_2) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \bullet$$

$$W_4 = \Sigma(R_3) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \bullet$$

$$V_1 = \Sigma(R_4) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet \quad .$$

Hence, by (34), we obtain

$$4[W_4] = 2[W_3] + [V_1]. \tag{41}$$

Example 4 For the pseudo-Coxeter orthoscheme R with graph

$$V_3 = \Sigma(R) \quad : \quad \bullet\!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{5}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\bullet \quad ,$$

one gets $q_1 \notin R$ and

$$V_1 = \Sigma(Q) \quad : \quad \bullet\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{5}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{5}{\rule{1cm}{0.4pt}}\!\!\bullet \quad ,$$

so that $q_3 \notin R$ by (37). By (38d), we obtain $p = 4$ and, by (6) and (7),

$$W_2 = \Sigma(R_2) \quad : \quad \bullet\!\!\overset{5}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\omega_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\omega_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\bullet$$

$$Z_4 = \Sigma(R_3) \quad : \quad \bullet\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\lambda}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{2\lambda}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\lambda}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\bullet$$

$$W_3 = \Sigma(R_4) \quad : \quad \bullet\!\!\overset{3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{5}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\omega_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\omega_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\!\!\bullet \quad .$$

By (34), we deduce that

$$2[V_1] = 2[V_3] + [W_2] + [W_3] + [Z_4]. \tag{42}$$

Remark 3 The extension to infinity as described by (34) can be easily generalised to the case when the vertex p_0 of R is finite (respectively ultra-ideal). The orthoschemes R_k, $1 \le k \le 5$, will be 2-asymptotic, and the orthoscheme $R_6 = Q$ will be simply asymptotic (respectively truncated with respect to p_0). More precisely, we obtain a dissection according to (34) where the graphs of $R_1, \ldots, R_5$ are as follows:

$$\Sigma(R_1) \quad : \quad \bullet\!\!\overset{\varphi_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\varphi_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_4}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_5}{\rule{1cm}{0.4pt}}\!\!\bullet$$

$$\Sigma(R_2) \quad : \quad \bullet\!\!\overset{\psi_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{x_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{y_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_4}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_5}{\rule{1cm}{0.4pt}}\!\!\bullet$$

$$\Sigma(R_3) \quad : \quad \bullet\!\!\overset{\alpha_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{x_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{y_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{z_1}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_5}{\rule{1cm}{0.4pt}}\!\!\bullet \tag{36'}$$

$$\Sigma(R_4) \quad : \quad \bullet\!\!\overset{\alpha_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{y_3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{z_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\psi_2}{\rule{1cm}{0.4pt}}\!\!\bullet \quad ,$$

$$\Sigma(R_5) \quad : \quad \bullet\!\!\overset{\alpha_2}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\alpha_4}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\varphi_3}{\rule{1cm}{0.4pt}}\!\!\circ\!\!\overset{\varphi_4}{\rule{1cm}{0.4pt}}\!\!\bullet \quad ,$$

where the parameters $\varphi_k \in \,]0, \frac{\pi}{2}[$ as well as x_1, y_1, y_2, z_1, z_2 are determined by the parabolicity conditions (2) and (33), while

$$\psi_1 = \begin{cases} \alpha_1 + \varphi_1, & \text{if } \alpha_1 + \varphi_1 < \frac{\pi}{2}, \\ \pi - (\alpha_1 + \varphi_1), & \text{if } \alpha_1 + \varphi_1 > \frac{\pi}{2}; \end{cases}$$

$$\psi_2 = \begin{cases} \alpha_6 + \varphi_4, & \text{if } \alpha_6 + \varphi_4 < \frac{\pi}{2}, \\ \pi - (\alpha_6 + \varphi_4), & \text{if } \alpha_6 + \varphi_4 > \frac{\pi}{2} \end{cases}$$

and

$$x_2 = \begin{cases} \varphi_2 - x_1, & \text{if } q_1 \in R \text{ and } p = 1; & (38a') \\ \varphi_2 + x_1, & \text{if } q_1 \notin R \text{ and } q_2 \in R \text{ and } p = 2; & (38b') \\ \pi - (\varphi_2 + x_1), & \text{if } q_1, q_2 \notin R \text{ and } q_3 \in R \text{ and } p = 3; & (38c') \\ \pi - (\varphi_2 + x_1), & \text{if } q_1, q_2, q_3 \notin R \text{ and } p = 4. & (38d') \end{cases}$$

We describe this case by giving some useful examples.

Example 5 For the pseudo-Coxeter orthoscheme R with graph

$$A = \Sigma(R) \quad : \quad \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \bullet \quad ,$$

one gets $q_1 \notin R$ and

$$B = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \quad ,$$

so that $q_3 \in R$. Furthermore, one sees that $q_2 \in R$ implying $p = 2$. It follows that

$$U_1 = \Sigma(R_1) = \Sigma(R_2) \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \bullet$$

$$V_3 = \Sigma(R_3) = \Sigma(R_4) = \Sigma(R_5) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \bullet \quad .$$

By (34), we deduce that

$$[A] = [B] + 3[V_3] - 2[U_1]. \tag{43}$$

Example 6 For the pseudo-Coxeter orthoscheme R with graph

$$C = \Sigma(R) \quad : \quad \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \bullet \quad ,$$

one gets $q_1 \notin R$ and

$$D = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \quad ,$$

so that $q_3 \in R$. Furthermore, one sees that $q_2 \in R$ implying $p = 2$. Using (6) and (7), one derives that

$$V_2 = \Sigma(R_1) = \Sigma(R_2) \quad : \quad \bullet \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \bullet$$

$$W_2 = \Sigma(R_3) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{\omega_1}{\rule{1.2em}{0.4pt}} \circ \overset{\omega_2}{\rule{1.2em}{0.4pt}} \circ \overset{5}{\rule{1.2em}{0.4pt}} \bullet$$

$$Z_3 = \Sigma(R_4) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\lambda}{\rule{1.2em}{0.4pt}} \circ \overset{2\lambda}{\rule{1.2em}{0.4pt}} \circ \overset{\lambda}{\rule{1.2em}{0.4pt}} \bullet$$

$$W_1 = \Sigma(R_5) \quad : \quad \bullet \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{3}{\rule{1.2em}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1.2em}{0.4pt}} \circ \overset{\omega_1}{\rule{1.2em}{0.4pt}} \circ \overset{\omega_2}{\rule{1.2em}{0.4pt}} \bullet \quad .$$

By (34), we deduce that

$$[C] = [D] + [W_1] + [W_2] + [Z_3] - 2[V_2]. \tag{44}$$

The next two examples serve exclusively to illustrate the differences between the seemingly related ordinary orthoschemes belonging to (11) and satisfying (43), (44) and the truncated counterparts belonging to (12).

Example 7 For the pseudo-Coxeter orthoprism R with graph

$$\overline{A} = \Sigma(R) \quad : \quad \diamond \!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \circ \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \bullet \quad ,$$

one gets

$$\overline{B} = \Sigma(Q) \quad : \quad \bullet \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \diamond \quad ,$$

and $p = 3$. Here, one derives

$$[\overline{A}] = [\overline{B}] + [W_3] - [W_2] + [V_1] - [V_2] - [U_2]. \tag{45}$$

Example 8 For the pseudo-Coxeter orthoprism R with graph

$$\overline{C} = \Sigma(R) \quad : \quad \diamond \!\!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \circ \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \bullet \quad ,$$

one gets

$$\overline{D} = \Sigma(Q) \quad : \quad \bullet \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\frac{5}{2}}{\rule{1cm}{0.4pt}}\! \circ \!\overset{3}{\rule{1cm}{0.4pt}}\! \diamond \quad ,$$

so that $q_3 \notin R$ implying $p = 4$. Here, one derives

$$[\overline{C}] = [\overline{D}] + [W_4] - [W_1] - [U_1] - [V_2] - [V_3]. \tag{46}$$

Remark 4 Another version of extending to infinity is to consider an orthoscheme R with vertex at infinity p_0 and with ultra-ideal vertex p_5 which, by truncation, leads to an asymptotic orthoprism with graph

$$\Sigma(R) \quad : \quad \bullet \!\overset{\alpha_1}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_2}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_4}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_5}{\rule{1cm}{0.4pt}}\! \diamond \quad .$$

The corresponding case with finite vertex p_5 will not be considered here since the reversion process of Sect. 3.1 applied to the group (11) provides identical dissections. By extending to infinity the half-line $p_0 p_1$, the limit point $q_0 \in \partial \mathbb{H}^5$ together with the vertices $p_1, \ldots, p_5$ yields a polytope R_1 isometric to R. The hyperplane H through q_0 and orthogonal to the line determined by p_0, p_5 provides as in (34) a dissection of R by means of truncated orthoschemes $R_k = q_0 \cdots q_{k-1} p_k \cdots p_5$ with $q_i = H \cap p_0 p_{i+1}$, with the exception that the member $Q = R_6$ is a doubly asymptotic orthoscheme with graph

$$\Sigma(Q) \quad : \quad \bullet \!\overset{\alpha_2}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_3}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_4}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_5}{\rule{1cm}{0.4pt}}\! \circ \!\overset{\alpha_6}{\rule{1cm}{0.4pt}}\! \bullet \quad ,$$

where α_6 is given by (24). Hence,

$$2[R] + \sum_{k=2}^{p} [R_k] = \sum_{k=p+1}^{5} [R_k] + [Q], \tag{47}$$

where the sum of the left-hand side is void if $p = 1$, and where the explicit description of $R_2, \ldots, R_5$ is based on standard non-Euclidean computations (see Sects. 3.1 and 3.2) yielding

$$
\begin{array}{llll}
\Sigma(R_2) & : & \bullet \overset{\varphi_1}{\rule{1.2em}{0.4pt}} \circ \overset{x_1}{\rule{1.2em}{0.4pt}} \circ \overset{y_1}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_4}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_5}{\rule{1.2em}{0.4pt}} \diamond \\[4pt]
\Sigma(R_3) & : & \bullet \overset{\alpha_2}{\rule{1.2em}{0.4pt}} \circ \overset{x_2}{\rule{1.2em}{0.4pt}} \circ \overset{y_2}{\rule{1.2em}{0.4pt}} \circ \overset{z_1}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_5}{\rule{1.2em}{0.4pt}} \diamond \\[4pt]
\Sigma(R_4) & : & \bullet \overset{\alpha_2}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_3}{\rule{1.2em}{0.4pt}} \circ \overset{y_3}{\rule{1.2em}{0.4pt}} \circ \overset{z_2}{\rule{1.2em}{0.4pt}} \circ \overset{u_1}{\rule{1.2em}{0.4pt}} \diamond \\[4pt]
\Sigma(R_5) & : & \bullet \overset{\alpha_2}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_3}{\rule{1.2em}{0.4pt}} \circ \overset{\alpha_4}{\rule{1.2em}{0.4pt}} \circ \overset{z_3}{\rule{1.2em}{0.4pt}} \circ \overset{u_2}{\rule{1.2em}{0.4pt}} \diamond \quad .
\end{array}
\tag{48}
$$

The angle φ_1 satisfies the condition (37), and the angle x_2 is subject to the conditions (38a)–(38d). The relation

$$\tan x_1 = \tan \alpha_1 \tan \alpha_2 \cot \varphi_1$$

follows from comparing the hypotenuse of the vertex figures associated with p_3 in the 3-faces $q_0 q_1 p_2 p_3$ and $q_0 p_1 p_2 p_3$, and the parameter y_1 is determined by the parabolicity condition (2). As for the remaining parameters, observe that

$$
y_2 = \begin{cases} \pi - (y_1 + y_3), & \text{if } p = 1, 4, \\ y_1 - y_3, & \text{if } p = 2, \\ y_3 - y_1, & \text{if } p = 3, \end{cases}
\quad \text{and} \quad
z_2 = \begin{cases} \pi - (z_1 + z_3), & \text{if } p = 1, 2, \\ z_1 - z_3, & \text{if } p = 3, \\ z_3 - z_1, & \text{if } p = 4. \end{cases}
$$

Indeed, the relation between the dihedral angles y_1, y_2, y_3 which are attached to the edge $q_1 p_4$ in the 3-faces $q_1 p_2 p_3 p_4$, $q_1 q_2 p_3 p_4$, $q_1 q_2 q_3 p_4$ follows by their relative position in the cone C with apex p_0 over the triangle $p_2 p_3 p_4$. The identity between z_1, z_2, z_3 follows by considering the edge $q_2 p_5$ in C in a similar fashion. By these identities and the parabolicity condition (2), it suffices to determine z_1 and u_1. First, observe that the edge length $p_2 p_3$ (resp. $q_2 q_3$) in the simply asymptotic 3-face $p_0 p_1 p_2 p_3$ of R (resp. $q_0 q_1 q_2 q_3$ of Q) is given by

$$\tanh p_2 p_3 = \cot \alpha_1 \cot \alpha_2 \quad (\text{resp. } \tanh q_2 q_3 = \cot \alpha_2 \cot \alpha_3). \tag{49}$$

Next, consider the vertex figures at p_3, q_2 (resp. p_2) in the 3-faces $q_0 q_1 q_2 p_3$, $q_2 p_3 p_4 p_5$ (resp. $p_2 p_3 p_4 p_5$) of R_3. For the planar angles $\delta := \angle(q_0 p_3 q_2)$, $\sigma := \angle(p_3 q_2 p_4)$ and $\rho := \angle(p_3 p_2 p_4)$ one easily deduces the identities

$$
\cos \delta = \cot \alpha_2 \cot x_2, \qquad \sin \delta = \frac{1}{\sinh q_2 p_3},
$$

$$
\cos \sigma = \frac{\cos z_1}{\sin \alpha_5},
\tag{50}
$$

$$
\cos \rho = \frac{\cos \alpha_4}{\sin \alpha_5}, \qquad \tan \rho = \frac{\tanh p_3 p_4}{\sinh p_2 p_3},
$$

which together with (49) allow us to determine z_1. It remains to find u_1 which we perform in a similar way. Consider the planar angles $\zeta := \angle(p_4 q_2 p_5)$ and $\xi := \angle(q_3 q_2 p_4)$ in the 3-face $q_2 q_3 p_4 p_5$ of R_4. We obtain the trigonometrical identities

$$\cos\zeta = \frac{\cos\alpha_5}{\sin z_1}, \qquad \sin u_1 = \frac{\tan\zeta}{\sin\xi}, \qquad \frac{\tanh q_2 q_3}{\cos\xi} = \tanh q_2 p_4 = \frac{\tanh q_2 p_3}{\cos\sigma},$$

which by means of (49) and (50) lead to u_1. Here are some illustrating and important examples.

Example 9 For the pseudo-Coxeter orthoprism R with graph

$$\overline{B} = \Sigma(R) \quad : \quad \bullet \overset{3}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \diamond \quad,$$

one gets $p = 2$ and

$$U_2 = \Sigma(Q) \quad : \quad \bullet \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \bullet \quad,$$

as well as

$$\overline{B} = \Sigma(R_2) \quad : \quad \bullet \overset{3}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \diamond$$

$$\overline{A} = \Sigma(R_3) = \Sigma(R_4) = \Sigma(R_5) \quad : \quad \bullet \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \diamond \quad.$$

Therefore,

$$3[\overline{B}] = 3[\overline{A}] + [U_2]. \tag{51}$$

Example 10 For the pseudo-Coxeter orthoprism R with graph

$$\overline{A} = \Sigma(R) \quad : \quad \bullet \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \diamond \quad,$$

one gets $p = 1$ and

$$V_2 = \Sigma(Q) \quad : \quad \bullet \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \bullet \quad,$$

and

$$\overline{C} = \Sigma(R_2) = \Sigma(R_3) = \Sigma(R_4) = \Sigma(R_5) \quad : \quad \bullet \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \diamond \quad.$$

Hence,

$$2[\overline{A}] = 2[\overline{C}] + [V_2]. \tag{52}$$

Example 11 For the pseudo-Coxeter orthoprism R with graph

$$\overline{D} = \Sigma(R) \quad : \quad \bullet \overset{3}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \circ \overset{5}{\underline{\quad}} \circ \overset{\frac{5}{2}}{\underline{\quad}} \circ \overset{3}{\underline{\quad}} \diamond \quad,$$

one gets $p = 3$ and

$$V_1 = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

as well as

$$\overline{D} = \Sigma(R_2) = \Sigma(R_3) \quad : \quad \bullet \overset{3}{\rule{0.8cm}{0.4pt}} \circ \overset{3}{\rule{0.8cm}{0.4pt}} \circ \overset{5}{\rule{0.8cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.8cm}{0.4pt}} \circ \overset{3}{\rule{0.8cm}{0.4pt}} \diamond$$

$$\overline{B} = \Sigma(R_4) = \Sigma(R_5) \quad : \quad \bullet \overset{3}{\rule{0.8cm}{0.4pt}} \circ \overset{5}{\rule{0.8cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.8cm}{0.4pt}} \circ \overset{5}{\rule{0.8cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{0.8cm}{0.4pt}} \diamond \quad .$$

Therefore,

$$4[\overline{D}] = 2[\overline{B}] + [V_1]. \tag{53}$$

It is interesting to observe the similarity of the scissors relations (52)–(54) to the ones in (20)–(22)!

3.4 Extension and Polar Reflection

Let $R = p_0 \cdots p_5 \subset \mathbb{H}^5$ denote an asymptotic orthoscheme with ideal vertex p_5 and ultra-infinite vertex p_0 which is truncated by the polar hyperplane P_0 associated with p_0. Denote the graph of R by

$$\Sigma(R) \quad : \quad \diamond \overset{\alpha_1}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_2}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{0.8cm}{0.4pt}} \bullet \quad .$$

The hyperplane P_0 intersects the line determined by p_0 and p_1 at an ordinary point π, say. The two rays (of opposite orientation) πp_0 and πp_1 intersect the boundary $\partial \mathbb{H}^5$ in the ideal points $\bar{p}_0$ and q_0, say, which lie symmetrically with respect to π and P_0. The following scissors procedure has first been described by Debrunner in the case of $\mathbb{H}^3$ (cf. [3, p. 145]). Consider as usual the hyperplane H through q_0 and orthogonal to $p_0 p_5$ providing the intersection points $q_1, \ldots, q_4$ on the edges $p_0 p_2, \ldots, p_0 p_5$. In the same way, consider the hyperplane $\overline{H}$ through $\bar{p}_0$ and orthogonal to $p_0 p_5$ providing the intersection points $\bar{p}_1, \ldots, \bar{p}_4$ on the edges $p_0 p_2, \ldots, p_0 p_5$. Since the polar hyperplane P_0 cuts orthogonally all hyperplanes passing through p_0, it follows from the construction that the prism $Q =: q_0 \cdots q_4 P_0$ given by the convex hull of the set $\{q_0, \ldots, q_4, P_0 \cap R\}$ is isometric to the prism $\overline{R} =: \bar{p}_0 \cdots \bar{p}_4 P_0$ given by the convex hull of the set $\{\bar{p}_0, \ldots, \bar{p}_4, P_0 \cap R\}$. The prism $\overline{R}$ has the orthoscheme $\bar{p}_0 \cdots \bar{p}_4$ as top face and shares the base $P_0 \cap R$ with Q whose top face is given by the orthoscheme $q_0 \cdots q_4$ (cf. Fig. 6).

For their graph, we obtain

$$\Sigma(Q) = \Sigma(\overline{R}) \quad : \quad \bullet \overset{\alpha_2}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{0.8cm}{0.4pt}} \circ \overset{\alpha_6}{\rule{0.8cm}{0.4pt}} \diamond \quad ,$$

where α_6 is given by the parabolicity identity (24). The following dissection is a natural consequence of our considerations. Consider the doubly asymptotic orthoschemes $\overline{R}_k := \bar{p}_0 \cdots \bar{p}_{k-1} p_k \cdots p_5$, $1 \leq k \leq 5$. Then,

$$[R] + [Q] = [R] + [\overline{R}] = \sum_{k=1}^{5} [\overline{R}_k], \tag{54}$$
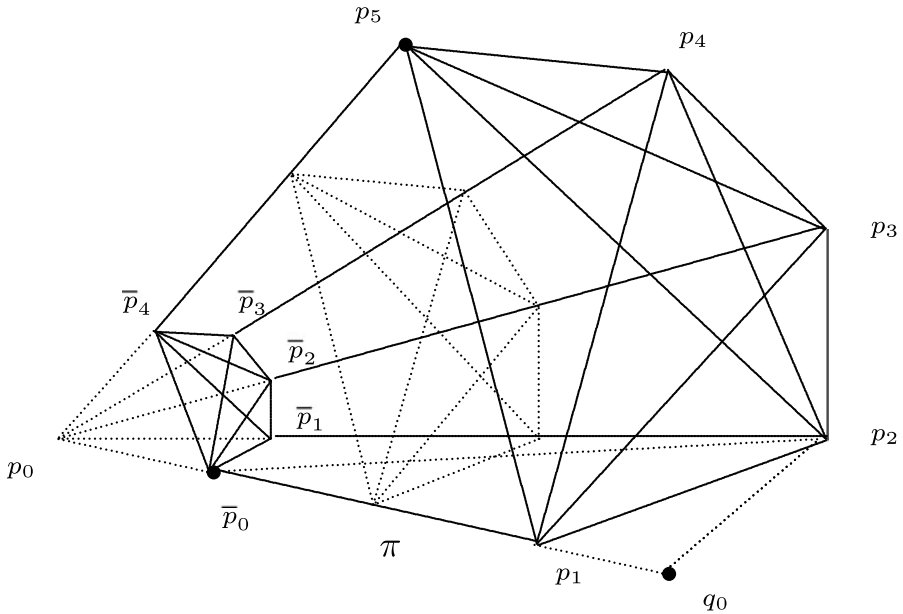
Fig. 6 Extension and polar reflection w.r.t. the polar hyperplane P_0 of p_0

where the graphs of $\overline{R}_k$, $1 \leq k \leq 5$, are given by

$$\begin{array}{lll}
\Sigma(\overline{R}_1) & : & \bullet \overset{u_1}{\rule{1cm}{0.4pt}} \circ \overset{x_1}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \\
\Sigma(\overline{R}_2) & : & \bullet \overset{u_2}{\rule{1cm}{0.4pt}} \circ \overset{x_2}{\rule{1cm}{0.4pt}} \circ \overset{y_1}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \\
\Sigma(\overline{R}_3) & : & \bullet \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{x_3}{\rule{1cm}{0.4pt}} \circ \overset{y_2}{\rule{1cm}{0.4pt}} \circ \overset{z_1}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_5}{\rule{1cm}{0.4pt}} \bullet \\
\Sigma(\overline{R}_4) & : & \bullet \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{y_3}{\rule{1cm}{0.4pt}} \circ \overset{z_2}{\rule{1cm}{0.4pt}} \circ \overset{v_1}{\rule{1cm}{0.4pt}} \bullet \\
\Sigma(\overline{R}_5) & : & \bullet \overset{\alpha_2}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_3}{\rule{1cm}{0.4pt}} \circ \overset{\alpha_4}{\rule{1cm}{0.4pt}} \circ \overset{z_3}{\rule{1cm}{0.4pt}} \circ \overset{v_2}{\rule{1cm}{0.4pt}} \bullet
\end{array} \qquad (55)$$

subject to the relations

$$\begin{aligned}
u_1 + u_2 &= \alpha_1, \qquad v_2 - v_1 = \alpha_5, \\
x_1 + x_2 + x_3 &= \pi.
\end{aligned} \qquad (56)$$

In fact, the first two identities of (56) follow from the induced decompositions in the triangles $p_0 p_1 p_2$ and $p_0 p_3 p_4$, both truncated by P_0. The third identity follows from the fact that x_1, x_2 and x_3 are attached to the edge $\bar{p}_0 p_3$ which is common to the 3-faces $\bar{p}_0 p_1 p_2 p_3$, $\bar{p}_0 \bar{p}_1 p_2 p_3$ and $\bar{p}_0 \bar{p}_1 \bar{p}_2 p_3$ (cf. also Remark 4). These relations together with the parabolicity condition (2) allow to determine all parameters in (55). We illustrate the dissection (54) with some examples.

Example 12 For the pseudo-Coxeter orthoprism R with graph

$$\overline{A} = \Sigma(R) \quad : \quad \diamond \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

one gets

$$\overline{B} = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \diamond \quad ,$$

and

$$U_2 = \Sigma(\overline{R}_1) = \Sigma(\overline{R}_2) \quad : \quad \bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet$$

$$V_1 = \Sigma(\overline{R}_3) = \Sigma(\overline{R}_4) = \Sigma(\overline{R}_5) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \bullet \quad .$$

Therefore,

$$[\overline{A}] + [\overline{B}] = 2[U_2] + 3[V_1]. \tag{57}$$

Example 13 For the pseudo-Coxeter orthoprism R with diagram

$$\overline{C} = \Sigma(R) \quad : \quad \diamond \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet \quad ,$$

one gets

$$\overline{D} = \Sigma(Q) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \diamond \quad ,$$

as well as

$$V_2 = \Sigma(\overline{R}_1) = \Sigma(\overline{R}_2) \quad : \quad \bullet \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet$$

$$W_3 = \Sigma(\overline{R}_3) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \circ \overset{\frac{5}{2}}{\rule{1cm}{0.4pt}} \bullet$$

$$Z_3 = \Sigma(\overline{R}_4) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{\lambda}{\rule{1cm}{0.4pt}} \circ \overset{2\lambda}{\rule{1cm}{0.4pt}} \circ \overset{\lambda}{\rule{1cm}{0.4pt}} \bullet$$

$$W_4 = \Sigma(\overline{R}_5) \quad : \quad \bullet \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{3}{\rule{1cm}{0.4pt}} \circ \overset{5}{\rule{1cm}{0.4pt}} \circ \overset{\omega_2}{\rule{1cm}{0.4pt}} \circ \overset{\omega_1}{\rule{1cm}{0.4pt}} \bullet \quad .$$

Therefore,

$$[\overline{C}] + [\overline{D}] = 2[V_2] + [W_3] + [W_4] + [Z_3]. \tag{58}$$

The identities (57) and (58) should be compared with (43) and (44) of their non-truncated counterparts. The similarities are striking but the geometric background is mysterious.

4 Volume Computations

4.1 A Simple Volume Formula

In [11, Theorem 3], we derived an explicit formula for the volume of a doubly asymptotic hyperbolic 5-orthoscheme. Since the scissors congruence group $\mathcal{P}(\overline{\mathbb{H}^5})$ of extended hyperbolic 5-space $\overline{\mathbb{H}^5} = \mathbb{H}^5 \cup \partial\mathbb{H}^5$ is generated by doubly asymptotic orthoschemes, this result solves in principle the volume problem in hyperbolic 5-space. However, applying the formula to concrete cases such as (6) yields very complicated

combinations of trilogarithmic values related to the weigths of (6). There is very little hope to simplify the expression by using the few known functional equations for the trilogarithm function involved. However, the volume formula simplifies drastically when restricting to doubly asymptotic orthoschemes $R = R(\alpha, \beta, \gamma)$ with graph of type (9), i.e.

$$\Sigma(R) \quad : \quad \bullet \overset{\alpha}{\rule{1.5em}{0.4pt}} \circ \overset{\beta}{\rule{1.5em}{0.4pt}} \circ \overset{\gamma}{\rule{1.5em}{0.4pt}} \circ \overset{\alpha}{\rule{1.5em}{0.4pt}} \circ \overset{\beta}{\rule{1.5em}{0.4pt}} \bullet \quad , \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad .$$

For this case, we proved that (cf. [10, (4)])

$$\mathrm{vol}_5(R) = \frac{1}{4}\left\{ Л_3(\alpha) + Л_3(\beta) - \frac{1}{2}Л_3\left(\frac{\pi}{2} - \gamma\right) \right\}$$

$$- \frac{1}{16}\left\{ Л_3\left(\frac{\pi}{2} + \alpha + \beta\right) + Л_3\left(\frac{\pi}{2} - \alpha + \beta\right) \right\} + \frac{3}{64}\zeta(3), \qquad (59)$$

where the *trilobachevsky function* $Л_3(\omega)$ is related to the classical trilogarithm function

$$\mathrm{Li}_3(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^3}, \quad z \in \mathbb{C},$$

by the identity (cf. [10, Sect. 2]; for $Л_2(\omega)$, see (69))

$$Л_3(\omega) = \frac{1}{4}\Re\left(\mathrm{Li}_3(e^{2i\omega})\right) = \frac{1}{4}\sum_{r=1}^{\infty}\frac{\cos(2r\omega)}{r^3} = \frac{1}{4}\zeta(3) - \int_0^{\omega} Л_2(t)\, dt. \qquad (60)$$

The function $Л_3(\omega)$ is even, π-periodic and satisfies the distribution law

$$\frac{1}{m^2}Л_3(m\omega) = \sum_{r=0}^{m-1} Л_3\left(\omega + \frac{r\pi}{m}\right). \qquad (61)$$

In particular, we obtain the values

$$Л_3(0) = \frac{1}{4}\zeta(3), \qquad Л_3\left(\frac{\pi}{2}\right) = -\frac{3}{16}\zeta(3),$$

$$Л_3\left(\frac{\pi}{5}\right) + Л_3\left(\frac{2\pi}{5}\right) = -\frac{3}{25}\zeta(3). \qquad (62)$$

Notice that formula (59) is a specialised 5-dimensional analogue of Lobachevsky's formula (see [9] and (68), for example) for a hyperbolic 3-orthoscheme $R = R(\alpha, \beta, \gamma)$ with graph

$$\Sigma(R) \quad : \quad \circ \overset{\alpha}{\rule{1.5em}{0.4pt}} \circ \overset{\beta}{\rule{1.5em}{0.4pt}} \circ \overset{\gamma}{\rule{1.5em}{0.4pt}} \circ \quad \text{existing for} \quad \cos\beta > \sin\alpha\sin\gamma \quad .$$

In five dimensions, as an application of (59)–(62), one can determine easily the volumes of all members of the V-cycle (4), as first done in [10, Sect. 3.2]. In the sequel, we use the following notation.

Notation For an orthoscheme F with graph $\Sigma(F)$, put $f = \mathrm{vol}_5(\Sigma(F)) := \mathrm{vol}_5(F)$. Therefore,

$$v_1 = \mathrm{vol}_5(V_1) = \frac{1}{144}\left\{ Л_3\left(\frac{\pi}{5}\right) + \frac{\zeta(3)}{5} \right\} \approx 0.001998,$$

$$v_2 = \mathrm{vol}_5(V_2) = \frac{\zeta(3)}{1200} \approx 0.001002, \tag{63}$$

$$v_3 = \mathrm{vol}_5(V_3) = \frac{1}{144}\left\{ -Л_3\left(\frac{\pi}{5}\right) + \frac{2\zeta(3)}{25} \right\} \approx 0.000339.$$

Furthermore, by using the dissection by extending to infinity as described in Sect. 3.3, we computed in [10, Sect. 3.2] the volumes of the two members of the U-cycle (3), and we obtained

$$u_1 = \mathrm{vol}_5(U_1) = \frac{1}{96} Л_3\left(\frac{\pi}{5}\right) \approx 0.000493,$$

$$u_2 = \mathrm{vol}_5(U_2) = \frac{1}{96} Л_3\left(\frac{\pi}{5}\right) + \frac{\zeta(3)}{800} \approx 0.001996. \tag{64}$$

4.2 The Volumes of A, B, C, D and $\overline{A}, \overline{B}, \overline{C}, \overline{D}$

The simplices A, B, C, D of the group (11) are simply asymptotic orthoschemes while the polytopes $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ of the group (12) are simply asymptotic orthoprisms (or 1-truncated orthoschemes). For the graph Σ of each of these polytopes P, we note

$$\Sigma \ : \ \bullet \!\!\xrightarrow{\alpha_1}\!\! \circ \!\!\xrightarrow{\alpha_2}\!\! \circ \!\!\xrightarrow{\alpha_3}\!\! \circ \!\!\xrightarrow{\alpha_4}\!\! \circ \!\!\xrightarrow{\alpha_5}\!\! \square \quad , \tag{65}$$

where the vertex $\square$ represents either an ordinary vertex $\circ$ or an ultra-ideal vertex $\diamond$. Observe that a polytope of type (65) is described by four independent (acute) angle parameters uniquely up to isometry. Indeed, the parameters $\alpha_2, \ldots, \alpha_5$ satisfy the parabolicity relation (2). By the Schläfli volume differential (see e.g. [16, p. 118]), the volume of P is—up to constants—equal to the simple integral of the differential

$$\mathrm{vol}_3(\circ \!\!\xrightarrow{r}\!\! \circ \!\!\xrightarrow{\alpha_4}\!\! \circ \!\!\xrightarrow{\alpha_5}\!\! \square)\, d\alpha_1 + \mathrm{vol}_3(\bullet \!\!\xrightarrow{s}\!\! \circ \!\!\xrightarrow{\alpha_5'}\!\! \circ \!\!\xrightarrow{\alpha_5}\!\! \square)\, d\alpha_2 +$$

$$\mathrm{vol}_3(\bullet \!\!\xrightarrow{t}\!\! \circ \!\!\xrightarrow{u'}\!\! \circ \!\!\xrightarrow{u}\!\! \square)\, d\alpha_3 + \mathrm{vol}_3(\bullet \!\!\xrightarrow{\alpha_1}\!\! \circ \!\!\xrightarrow{v'}\!\! \circ \!\!\xrightarrow{v}\!\! \square)\, d\alpha_4 +$$

$$+ \mathrm{vol}_3(\bullet \!\!\xrightarrow{\alpha_1}\!\! \circ \!\!\xrightarrow{\alpha_2}\!\! \circ \!\!\xrightarrow{\alpha_2'}\!\! \square)\, d\alpha_5 \tag{66}$$

where $x' := \pi/2 - x$,

$$\tan r \tan \alpha_1 = \tan s \cot \alpha_2 = \tan t \tan \alpha_3 = \tan \theta_1 = \frac{\sqrt{\sin^2 \alpha_1 \sin^2 \alpha_3 - \cos^2 \alpha_2}}{\cos \alpha_1 \cos \alpha_3},$$

$$\tan v \cot \alpha_4 = \tan u \tan \alpha_3 = \tan \theta_2 = \frac{\sqrt{\sin^2 \alpha_3 \sin^2 \alpha_5 - \cos^2 \alpha_4}}{\cos \alpha_3 \cos \alpha_5}, \tag{67}$$

and where the integrands are given by the analytic expression (cf. [9, Theorem II])

$$\text{vol}_3(\circ\!\!\overset{x}{-\!\!-}\!\!\circ\!\!\overset{y}{-\!\!-}\!\!\circ\!\!\overset{z}{-\!\!-}\Box) = \frac{1}{4}\left\{ \text{Л}_2(x+\theta) - \text{Л}_2(x-\theta) + \text{Л}_2\left(\frac{\pi}{2}+y-\theta\right) + \right.$$
$$\left. + \text{Л}_2\left(\frac{\pi}{2}-y-\theta\right) + \text{Л}_2(z+\theta) - \text{Л}_2(z-\theta) + 2\text{Л}_2\left(\frac{\pi}{2}-\theta\right) \right\}, \tag{68}$$

with

$$\tan\theta = \frac{\sqrt{\cos^2 y - \sin^2 x \sin^2 z}}{\cos x \cos z},$$

and the classical Lobachevsky function

$$\text{Л}_2(\omega) = \frac{1}{2}\Im\left(\text{Li}_2(e^{2i\omega})\right) = \frac{1}{2}\sum_{r=1}^{\infty}\frac{\sin(2r\omega)}{r^2} = -\int_0^{\omega}\log|2\sin t|\,dt. \tag{69}$$

In particular, in the limiting case, one obtains

$$\text{vol}_3(\bullet\!\!\overset{x}{-\!\!-}\!\!\circ\!\!\overset{y}{-\!\!-}\!\!\circ\!\!\overset{y'}{-\!\!-}\Box) = \frac{1}{4}\left\{\text{Л}_2\left(\frac{\pi}{2}+x-y\right) - \text{Л}_2\left(\frac{\pi}{2}+x+y\right) + 2\text{Л}_2(y)\right\}.$$

Formulas (66)–(69) furnish evidence that the computation of $\text{vol}_5(P)$ is very challenging and that the evaluation for particular examples does not provide simple expressions in the angle parameters. However, by means of the scissors relations (20)–(57) and the elegant expressions (63), (64), we are able to determine the volumes of all members of the groups (11) and (12), and this without painful (and possibly unsuccessful) polylogarithmic calculations as just mentioned.

Theorem I

(a) *The volumes of the simply asymptotic hyperbolic* 5-*orthoschemes* A, B, C, D *as given by* (11) *are equal to*

$$a = \frac{\zeta(3)}{1200} - \frac{11}{576}\text{Л}_3\left(\frac{\pi}{5}\right), \qquad b = -\frac{\zeta(3)}{1200} + \frac{13}{576}\text{Л}_3\left(\frac{\pi}{5}\right),$$
$$c = -\frac{\zeta(3)}{4800} + \frac{11}{1152}\text{Л}_3\left(\frac{\pi}{5}\right), \qquad d = \frac{\zeta(3)}{1800} - \frac{5}{384}\text{Л}_3\left(\frac{\pi}{5}\right). \tag{70}$$

(b) *The volumes of the simply asymptotic hyperbolic* 5-*orthoprisms* $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ *as given by* (12) *are equal to*

$$\bar{a} = \frac{\zeta(3)}{320} + \frac{11}{576}\text{Л}_3\left(\frac{\pi}{5}\right), \qquad \bar{b} = \frac{17\zeta(3)}{4800} + \frac{13}{576}\text{Л}_3\left(\frac{\pi}{5}\right),$$
$$\bar{c} = \frac{13\zeta(3)}{9600} + \frac{11}{1152}\text{Л}_3\left(\frac{\pi}{5}\right), \qquad \bar{d} = \frac{61\zeta(3)}{28800} + \frac{5}{384}\text{Л}_3\left(\frac{\pi}{5}\right). \tag{71}$$

Proof In order to verify the values in (a), combine the scissors equations (20) and (43) and derive $b = \frac{7}{6}u_1 - \frac{3}{2}v_3$ and $a = -\frac{5}{6}u_1 + \frac{3}{2}v_3$. Equation (21) allows us to conclude that $c = \frac{5}{12}u_1 + \frac{1}{4}v_2 - \frac{3}{4}v_3$. Finally, by (22), we obtain $d = -\frac{7}{12}u_1 + v_3$. Now, plug (63) and (64) into the equations which yields (70).

For the proof of the identities (b), we proceed similarly and combine (51) and (57) in order to derive $\bar{b} = \frac{7}{6}u_2 + \frac{3}{2}v_1$ and $\bar{a} = \frac{5}{6}u_2 + \frac{3}{2}v_1$. Equation (52) then yields $\bar{c} = \frac{5}{12}u_2 + \frac{3}{4}v_1 - \frac{1}{4}v_2$. Together with (53), this implies that $\bar{d} = \frac{7}{12}u_1 + v_1$. Again, by using the identities (63) and (64), the proof can be completed. □

4.3 The W-Cycle and the Ideal Regular 5-Simplex

For some members of the W-cycle we can express their volume as rational linear combinations of *three* universal constants, namely by adjoining to Apéry's constant $\zeta(3)$ and the value $\text{Л}_3(\frac{\pi}{5}) \simeq 0.0473419788$ the volume $\mu_5 = \text{vol}_5(S_{\text{reg}}^\infty(2\lambda)) = 720z \simeq 0.0575647377$ of an ideal regular 5-simplex (see (32)). We point out that the field $\mathbb{Q}(\sqrt{5})$, containing the golden ratio τ, is the ground field associated with the (unique) arithmetic hyperbolic 5-orbifold of *minimal* volume (cf. [4] and [1]) while μ_5 is the *maximal* volume among all hyperbolic 5-simplex volumes (cf. [6] and [13]). Of course, all three constants are certain trilogarithmic expressions. More precisely, μ_5 can be expressed as

$$\mu_5 = 45 \int_\lambda^{\lambda_0} \Big\{ \text{Л}_2\big(\beta(t) + \theta(t)\big) - \text{Л}_2\big(\beta(t) - \theta(t)\big)$$

$$+ \text{Л}_2\Big(\frac{\pi}{6} - \theta(t)\Big) - \text{Л}_2\Big(\frac{\pi}{6} + \theta(t)\Big) + \text{Л}_2\Big(\frac{\pi}{3} + \theta(t)\Big)$$

$$- \text{Л}_2\Big(\frac{\pi}{3} - \theta(t)\Big) + 2\text{Л}_2\Big(\frac{\pi}{2} - \theta(t)\Big) \Big\} \, dt$$

where $\lambda_0 = \frac{1}{2}\arccos\frac{1}{5}$, $\beta(x) = \arccos\frac{\sin x}{\sqrt{4\sin^2 x - 1}}$ and $\theta(x) = \arctan\frac{\sqrt{2 - 5\sin^2 x}}{\sin x}$.

In terms of $\zeta(3)$, $\text{Л}_3(\frac{\pi}{5})$ and μ_5, we obtain the following result.

Theorem II *The volumes of the doubly asymptotic hyperbolic 5-orthoschemes W_1, W_2, W_3, W_4 of the W-cycle as given by (5) are equal to*

$$w_1 = \frac{1}{144}\Big\{ \frac{5}{4}\text{Л}_3\Big(\frac{\pi}{5}\Big) + \frac{3\zeta(3)}{100} - \mu_5 \Big\}, \qquad w_2 = \frac{1}{72}\Big\{ \text{Л}_3\Big(\frac{\pi}{5}\Big) + \frac{\zeta(3)}{20} - \mu_5 \Big\},$$

$$w_3 = \frac{1}{72}\Big\{ \text{Л}_3\Big(\frac{\pi}{5}\Big) + \frac{7\zeta(3)}{100} - \mu_5 \Big\}, \qquad w_4 = \frac{1}{144}\Big\{ \frac{5}{4}\text{Л}_3\Big(\frac{\pi}{5}\Big) + \frac{3\zeta(3)}{25} - \mu_5 \Big\}.$$

$$(72)$$

Proof By means of the scissors identities (32), (40) and (42), we obtain the equation $w_2 = v_1 - \frac{1}{6}v_2 - v_3 - \frac{1}{72}\mu_5$. Then, (39) yields $w_1 = \frac{1}{2}v_1 - \frac{1}{12}v_2 - \frac{3}{4}v_3 - \frac{1}{144}\mu_5$. By using again (40), we obtain $w_3 = v_1 + \frac{1}{6}v_2 - v_3 - \frac{1}{72}\mu_5$. Finally, by (41), we deduce

that $w_4 = \frac{3}{4}v_1 + \frac{1}{12}v_2 - \frac{1}{2}v_3 - \frac{1}{144}\mu_5$. The results then follow with the help of (63) and (64). □

Remark 5 In [15, pp. 269–270], L. Schläfli considered two so-called periods of spherical 3-orthoschemes $R(\alpha, \beta, \gamma)$ with angle parameters $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \lambda, 2\lambda, \lambda$ (respectively $\frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{5}$ $\omega_1 = \frac{\pi}{3} - \lambda, \omega_2 = \frac{2\pi}{3} - \lambda$) and deduced the values

$$\mathrm{vol}_3\left(R\left(\frac{\pi}{3}, \frac{\pi}{3}, \lambda\right)\right) = \frac{3\pi}{4}\lambda - \frac{\pi^2}{60},$$

$$\mathrm{vol}_3\left(R\left(\frac{\pi}{3}, \lambda, 2\lambda\right)\right) = 3\pi\lambda - \frac{\pi^2}{15},$$

$$\mathrm{vol}_3\left(R(\lambda, 2\lambda, \lambda)\right) = \frac{\pi}{2}\lambda - \frac{\pi^2}{10},$$

$$\mathrm{vol}_3\left(R\left(\frac{\pi}{3}, \frac{2\pi}{5}, \omega_1\right)\right) = -\frac{3\pi}{4}\lambda + \frac{43\pi^2}{2400}, \qquad (73)$$

$$\mathrm{vol}_3\left(R\left(\frac{2\pi}{5}, \omega_1, \omega_2\right)\right) = -\frac{\pi}{4}\lambda - \frac{391\pi^2}{7200},$$

$$\mathrm{vol}_3\left(R\left(\omega_1, \omega_2, \frac{\pi}{5}\right)\right) = -\frac{\pi}{4}\lambda - \frac{401\pi^2}{7200},$$

$$\mathrm{vol}_3\left(R\left(\omega_2, \frac{\pi}{5}, \frac{2\pi}{5}\right)\right) = -\frac{3\pi}{4}\lambda - \frac{53\pi^2}{2400}.$$

Observe that $\zeta(2) = \pi^2/6$ and that the totally orthogonal regular 3-simplex has volume $\pi^2/8$ which tops $\lambda\pi/2$. The results in (73) are based—among other things—on the reduction principle for even dimensional (spherical) simplices. This important tool is not applicable when computing the volumes of the W-cycle!

References

1. Belolipetsky, M., Emery, V.: On volumes of arithmetic quotients of PO(*n*, 1), *n* odd. Preprint. arXiv:1001.4670. To appear in Proc. Lond. Math. Soc.
2. Coxeter, H.S.M.: Regular Polytopes. Dover, New York (1973)
3. Debrunner, H.E.: Dissecting orthoschemes into orthoschemes. Geom. Dedic. **33**, 123–152 (1990)
4. Emery, V.: Du volume des quotients arithmétiques de l'espace hyperbolique. Ph.D. thesis no. 1648, Université de Fribourg (2009)
5. Gromov, M.: Hyperbolic manifolds according to Thurston and Jørgensen. Séminaire Bourbaki, vol. 22 (1979–1980). Exposé no. 546
6. Haagerup, U., Munkholm, H.J.: Simplices of maximal volume in hyperbolic *n*-space. Acta Math. **147**, 1–12 (1981)
7. Hild, T.: The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten. J. Algebra **313**, 208–222 (2007)
8. Im Hof, H.-C.: Napier cycles and hyperbolic Coxeter groups. Bull. Soc. Math. Belg., Sér. A **42**, 523–545 (1990)

9. Kellerhals, R.: On the volume of hyperbolic polyhedra. Math. Ann. **285**, 541–569 (1989)
10. Kellerhals, R.: On the volumes of hyperbolic 5-orthoschemes and the trilogarithm. Comment. Math. Helv. **67**, 648–663 (1992)
11. Kellerhals, R.: Volumes in hyperbolic 5-space. Geom. Funct. Anal. **5**, 640–667 (1995)
12. Milnor, J.: How to compute volume in hyperbolic space. In: Geometry, Collected Papers, vol. 1. Publish or Perish, Berkeley (1994)
13. Peyerimhoff, N.: Simplices of maximal volume or minimal total edge length in hyperbolic space. J. Lond. Math. Soc., II Ser. **66**, 753–768 (2002)
14. Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds. Springer, New York (1994)
15. Schläfli, L.: Theorie der vielfachen Kontinuität. In: Gesammelte Mathematische Abhandlungen, vol. I. Birkhäuser, Basel (1950)
16. Vinberg, E.B.: Geometry II: Spaces of Constant Curvature. Encyclopaedia of Mathematical Sciences, vol. 29. Springer, Berlin (1993)