

Non-discrete hybrids in $SU(2, 1)$

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Abstract We show that a natural hybridization construction of lattices in $SU(n, 1)$ to produce (non-arithmetic) lattices in $SU(n + 1, 1)$ fails when $n = 1$ for most triangle groups in $SU(1, 1)$.

Keywords Non-arithmetic lattices · Complex hyperbolic geometry · Fuchsian triangle groups · Hybridation of lattices

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1 Introduction

By celebrated work of Margulis, non-arithmetic lattices in real simple Lie groups can only exist in real rank one, that is when the associated symmetric space is a hyperbolic space. In real hyperbolic space, where the Lie group is $PO(n, 1)$, the first examples of non-arithmetic lattices beyond $n = 2$ were constructed by Makarov ($n = 3$, 1965) and Vinberg ($n = 3, 4, 5$, 1967). Later Gromov and Piatetski-Shapiro produced in [5] a construction yielding non-arithmetic lattices in $PO(n, 1)$ for all $n \geq 2$ (see below for more details), in fact producing in each dimension infinitely many non-commensurable lattices, both cocompact and non-cocompact.

In quaternionic hyperbolic spaces (and the Cayley octave plane), work of Corlette and Gromov-Schoen implies as in the higher rank case that all lattices are arithmetic.

The case of complex hyperbolic spaces (where the associated Lie group is $PU(n, 1)$) is much less understood. Non-arithmetic lattices in $PU(2, 1)$ were first constructed by Mostow in 1980 in [11], and subsequently Deligne-Mostow and Mostow constructed further examples as monodromy groups of certain hypergeometric functions in [1] and [12] (in fact, the lattices from [1] for $n = 2$ were known to Picard who did not consider their arithmetic nature). Taken

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together, these constructions yield a list of 14 non-arithmetic lattices in $\text{PU}(2, 1)$ (with only 7 to 9 up to commensurability) and one example in $\text{PU}(3, 1)$ (see also [18] and [14]). Recently, new examples in $\text{PU}(2, 1)$ were produced in a series of papers by Parker, the author and Deraux [2, 3, 13, 15, 16], but the list of known examples is still finite. The major open question in this area remains the existence of non-arithmetic lattices in $\text{PU}(n, 1)$ for $n \geq 4$. Hunt proposes an analog of the Gromov–Piatetski–Shapiro construction for complex hyperbolic lattices, and announces in [6] that this produces non-arithmetic lattices in $\text{PU}(n, 1)$ for all $n \geq 3$. The purpose of this short note is to show that the construction fails for $n = 2$, and to hopefully give some insight into the difficulties in higher dimensions as well.

The Gromov–Piatetski–Shapiro construction, which they call interbreeding of 2 arithmetic lattices (often referred to as hybridation), produces a lattice $\Gamma < \text{PO}(n, 1)$ from 2 lattices Γ_1 and Γ_2 in $\text{PO}(n, 1)$ which have a common sublattice $\Gamma_{12} < \text{PO}(n - 1, 1)$. Geometrically, this provides 2 hyperbolic n -manifolds $V_1 = \Gamma_1 \backslash H_{\mathbb{R}}^n$ and $V_2 = \Gamma_2 \backslash H_{\mathbb{R}}^n$ with a hyperbolic $(n - 1)$ -manifold V_{12} which is isometrically embedded in V_1 and V_2 as a totally geodesic hypersurface. This allows one to produce the hybrid manifold V by gluing $V_1 - V_{12}$ and $V_2 - V_{12}$ along V_{12} . The resulting manifold is also hyperbolic because the gluing took place along a totally geodesic hypersurface, and its fundamental group Γ is therefore a lattice in $\text{PO}(n, 1)$. The point is then that if Γ_1 and Γ_2 are both arithmetic but non-commensurable, their hybrid Γ is non-arithmetic.

It is not straightforward to adapt this construction to construct lattices in $\text{PU}(n, 1)$, the main difficulty being that there do not exist in complex hyperbolic space any totally geodesic hypersurfaces (i.e. of real codimension 1). In fact, it has been a famous open question since the work of Gromov–Piatetski–Shapiro to find some analogous construction in $\text{PU}(n, 1)$. Hunt proposes in [6] the following idea. Start with 2 arithmetic lattices Γ_1 and Γ_2 in $\text{PU}(n, 1)$, and suppose that one can embed them in $\text{PU}(n + 1, 1)$ in such a way that (a) each stabilizes a totally geodesic $H_{\mathbb{C}}^n \subset H_{\mathbb{C}}^{n+1}$ (b) these 2 complex hypersurfaces are orthogonal, and (c) the intersection of the embedded Γ_i is a lattice in the corresponding $\text{PU}(n - 1, 1)$. The resulting hybrid Γ is then defined as the subgroup of $\text{PU}(n + 1, 1)$ generated by the images of Γ_1 and Γ_2 .

Hunt’s main results are (1) producing lattices Γ_i satisfying (a)–(b)–(c) ((c) is not obvious when $n \geq 2$), and (2) proving that the resulting hybrid Γ is indeed a lattice in $\text{PU}(n + 1, 1)$. Intuitively, this should use the fact that condition (c) is highly restrictive when $n \geq 2$. Indeed, when $n = 1$ it only means that the embedded Fuchsian groups have finite intersection in the stabilizer of a point (the intersection of the orthogonal complex lines in $H_{\mathbb{C}}^2$). The present note illustrates quantitatively that this is not enough to ensure that the hybrid Γ is discrete. Namely, we consider 2 Fuchsian triangle groups $\Gamma_1 = (l_1, m_1, n_1)$ and $\Gamma_2 = (l_2, m_2, n_2)$, embed them in the stabilizers of 2 orthogonal complex lines in $H_{\mathbb{C}}^2$, and prove that in infinitely many cases the resulting Γ is non-discrete (see Theorems 5.2 and 5.4 for a more precise statement). The tool which we use to prove non-discreteness is the complex hyperbolic Jørgensen inequality developed by Jiang et al. [7] and Xie and Jiang [19], which can be thought of as a quantitative version of the Margulis lemma.

Our negative result gives rise to the following question: can one embed two given Fuchsian groups in such a way that the resulting hybrid is a lattice? We give at the end of Sect. 4 a family of examples where the hybrid is indeed discrete, in the case where $\Gamma_1 = \Gamma_2 = \text{SU}(1, 1, \mathcal{O}_d)$, but it is by construction contained in an arithmetic lattice in $\text{SU}(2, 1)$. Also our argument only applies when one of the rotations has order ≥ 6 , so it is still possible that the hybrid of two Fuchsian triangle groups with large rotation angles is a lattice (this only concerns a small number of groups). See also the final remarks about “self-hybrids” of a Fuchsian triangle group with itself.

2 Fuchsian triangle groups

For $l, m, n \in \mathbb{N} \cup \{\infty\}$, we denote by (l, m, n) the triangle group with presentation $\langle A, B \mid A^l = B^m = (AB)^n = 1 \rangle$. We will assume that $1/l + 1/m + 1/n < 1$, so that (l, m, n) is a Fuchsian group (the orientation-preserving subgroup of the group generated by the reflections in the sides of a hyperbolic triangle with angles $(\pi/l, \pi/m, \pi/n)$). We use the following form for the generators $A, B \in \text{SL}(2, \mathbb{R})$ given by Knapp in [9] (up to change of sign), denoting, for $k \in \mathbb{N} \cup \{\infty\}$, $c_k = \cos(\pi/k)$ and $s_k = \sin(\pi/k)$. Note that these generators rotate clockwise (so by an angle $-2\pi/k$).

$$\begin{aligned}
 A(l) &= \begin{pmatrix} c_l & -s_l \\ s_l & c_l \end{pmatrix} \\
 B(l, m, n) &= \begin{pmatrix} c_m & -\frac{c_n+c_m c_l}{s_l} - \sqrt{\left(\frac{c_n+c_m c_l}{s_l}\right)^2 - s_m^2} \\ \frac{c_n+c_m c_l}{s_l} - \sqrt{\left(\frac{c_n+c_m c_l}{s_l}\right)^2 - s_m^2} & c_m \end{pmatrix} \quad (2.1)
 \end{aligned}$$

It will be more convenient for our purposes (embedding $\text{SU}(1, 1)$ in $\text{SU}(2, 1)$) to work with matrices in $\text{SU}(1, 1)$, the subgroup of $\text{SL}(2, \mathbb{C})$ preserving the Hermitian form of signature $(1, 1)$ given by the diagonal matrix $\text{Diag}(1, -1)$. We will use the following isomorphism between $\text{SL}(2, \mathbb{R})$ and $\text{SU}(1, 1)$, which is induced by the map $z \mapsto \frac{z-i}{z+i}$ (sending the upper-half plane to the unit disk in \mathbb{C}):

$$\begin{aligned}
 \delta : \text{SL}(2, \mathbb{R}) &\longrightarrow \text{SU}(1, 1) \\
 \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{1}{2} \begin{pmatrix} a+d+i(b-c) & a-d-i(b+c) \\ a-d+i(b+c) & a+d-i(b-c) \end{pmatrix} \quad (2.2)
 \end{aligned}$$

3 Complex hyperbolic space and isometries

The standard reference for complex hyperbolic geometry is [4]. For the reader’s convenience we include a brief summary of key definitions and facts. We will consider only the case of dimension $n = 2$ in this note, but the general setup is identical for higher dimensions so we state it for all $n \geq 1$. Consider $\mathbb{C}^{n,1}$, the vector space \mathbb{C}^{n+1} endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. Let $V^- = \{Z \in \mathbb{C}^{n,1} \mid \langle Z, Z \rangle < 0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}P^n$ denote projectivization. Define $H_{\mathbb{C}}^n$ to be $\pi(V^-) \subset \mathbb{C}P^n$, endowed with the distance ρ (Bergman metric) given by:

$$\cosh^2 \frac{1}{2} \rho(\pi(X), \pi(Y)) = \frac{|\langle X, Y \rangle|^2}{\langle X, X \rangle \langle Y, Y \rangle} \quad (3.1)$$

From this formula it is clear that $\text{PU}(n, 1)$ acts by isometries on $H_{\mathbb{C}}^n$ (where $\text{U}(n, 1)$ is the subgroup of $\text{GL}(n+1, \mathbb{C})$ preserving $\langle \cdot, \cdot \rangle$, and $\text{PU}(n, 1)$ is its image in $\text{PGL}(n+1, \mathbb{C})$).

Fact: $\text{Isom}^+(H_{\mathbb{C}}^n) = \text{PU}(n, 1)$, and $\text{Isom}(H_{\mathbb{C}}^n) = \text{PU}(n, 1) \times \mathbb{Z}/2$ (complex conjugation).

Classification: $g \in \text{PU}(n, 1)$ is of one of the following types:

- *elliptic:* g has a fixed point in $H_{\mathbb{C}}^n$
- *parabolic:* g has (no fixed point in $H_{\mathbb{C}}^n$ and) exactly one fixed point in $\partial H_{\mathbb{C}}^n$
- *loxodromic:* g has (no fixed point in $H_{\mathbb{C}}^n$ and) exactly two fixed points in $\partial H_{\mathbb{C}}^n$

Definitions: A *complex k-plane* is a projective k -dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $H_{\mathbb{C}}^k \subset H_{\mathbb{C}}^n$). Complex 1-planes are usually called *complex lines*. A *complex reflection* is an elliptic isometry

$g \in \text{PU}(n, 1)$ whose fixed-point set is a complex $(n - 1)$ -plane. An elliptic isometry g is called *regular* if any of its matrix representatives $A \in U(n, 1)$ has distinct eigenvalues. The eigenvalues of a matrix $A \in U(n, 1)$ representing an elliptic isometry g have modulus one. Exactly one of these eigenvalues has eigenvectors in V^- (projecting to a fixed point of g in $H_{\mathbb{C}}^n$), and such an eigenvalue will be called *of negative type*. Regular elliptic isometries have an isolated fixed point in $H_{\mathbb{C}}^n$.

If $L = \pi(\tilde{L})$ is a complex $(n - 1)$ -plane, any $v \in \mathbb{C}^{n+1} - \{0\}$ orthogonal to \tilde{L} is called a *polar vector* of L . Such a vector satisfies $\langle v, v \rangle > 0$, and we will usually normalize v so that $\langle v, v \rangle = 1$. The discreteness test which we will use in the second part of Sect. 5 (the complex hyperbolic Jørgensen’s inequality established in [7]) uses the following classic fact (p. 100 of [4]):

Proposition 3.1 *Let L_1 and L_2 be 2 complex $(n - 1)$ -planes with polar vectors p_1 and p_2 , suitably normalized so that $\langle p_1, p_1 \rangle = \langle p_2, p_2 \rangle = 1$. Then one of the following holds:*

- (1) L_1 and L_2 intersect in $H_{\mathbb{C}}^n \iff |\langle p_1, p_2 \rangle| < 1$. In that case the angle ψ between L_1 and L_2 satisfies $\cos \psi = |\langle p_1, p_2 \rangle|$.
- (2) L_1 and L_2 intersect in $\partial H_{\mathbb{C}}^n \iff |\langle p_1, p_2 \rangle| = 1$.
- (3) L_1 and L_2 are ultraparallel $\iff |\langle p_1, p_2 \rangle| > 1$. In that case the distance ρ between L_1 and L_2 satisfies $\cosh \frac{\rho}{2} = |\langle p_1, p_2 \rangle|$.

4 Embedding the Fuchsian groups in $\text{SU}(2, 1)$

We will use the ball model of $H_{\mathbb{C}}^2$, using the Hermitian form of signature $(2, 1)$ on \mathbb{C}^3 given by the diagonal matrix $\text{Diag}(1, 1, -1)$. (In affine coordinates $(z_1, z_2, 1)$, $H_{\mathbb{C}}^2$ is then the unit ball $\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$.)

Two orthogonal complex lines L_1 and L_2 (the coordinate axes in the above affine chart) are given by $L_1 = \pi(\text{Span}(e_1, e_3))$ and $L_2 = \pi(\text{Span}(e_2, e_3))$ (where (e_1, e_2, e_3) denotes the canonical basis of \mathbb{C}^3). These intersect at the origin $O = \pi(e_3)$.

We will embed $\text{SU}(1, 1)$ in the stabilizer of each of these complex lines in the obvious block matrix form, namely:

$$\begin{aligned} \rho_1 : \text{SU}(1, 1) &\longrightarrow \text{SU}(2, 1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \end{aligned} \tag{4.1}$$

$$\begin{aligned} \rho_2 : \text{SU}(1, 1) &\longrightarrow \text{SU}(2, 1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \end{aligned} \tag{4.2}$$

Note that the triangle group generators A and B given in (2.1) have eigenvalues of the form $e^{\pm i\theta}$, so that $\rho_i(\delta(A))$ and $\rho_i(\delta(B))$ are regular elliptic with eigenvalues 1 and $e^{\pm i\theta}$. Moreover such an A fixes the point i in the upper half-plane, so that $\delta(A)$ fixes the center of the unit disk and $\rho_i(\delta(A))$ fixes the center of the unit ball.

Definition: Following Hunt [6], we will define the *hybrid* Γ of two Fuchsian groups $\Gamma_1, \Gamma_2 < \text{SU}(1, 1)$ as the subgroup of $\text{SU}(2, 1)$ generated by $\rho_1(\Gamma_1)$ and $\rho_2(\Gamma_2)$: $\Gamma := \langle \rho_1(\Gamma_1), \rho_2(\Gamma_2) \rangle$.

A family of discrete examples: If $\Gamma_1 = \Gamma_2 = \text{SU}(1, 1, \mathcal{O}_d)$ (where \mathcal{O}_d denotes the ring of integers in $\mathbb{Q}(\sqrt{d})$ for a negative squarefree integer d), then the hybrid $\langle \rho_1(\Gamma_1), \rho_2(\Gamma_2) \rangle$ has its matrix entries in the discrete ring \mathcal{O}_d so is a discrete subgroup of $\text{SU}(2, 1)$. We will see however that for some cocompact arithmetic lattices $\Gamma_1 < \text{SU}(1, 1)$, the self-hybrid $\langle \rho_1(\Gamma_1), \rho_2(\Gamma_1) \rangle$ is non-discrete.

Other possible embeddings: In fact, the stabilizer in $\text{SU}(2, 1)$ of a complex line is conjugate to $\text{S}(\text{U}(1, 1) \times \text{U}(1))$, so that one has some flexibility on the $\text{U}(1)$ factor (geometrically, on how an isometry preserving a complex line acts on complex lines orthogonal to it). For instance it would make sense geometrically to map the generators of the Fuchsian triangle group to complex reflections (acting trivially on complex lines orthogonal to the corresponding L_i). Explicitly we map the generators of the Fuchsian triangle groups to complex reflections in the following way. The block matrices are the same as for the embeddings ρ_1, ρ_2 , we simply change the $\text{U}(1)$ factor by repeating the eigenvalue of negative type. Also the image is here in $\text{U}(2, 1)$; one could rescale to have determinant 1 but that is irrelevant here. We only write ρ'_1 (whose image stabilizes L_1), ρ'_2 is entirely analogous.

$$\begin{aligned} \rho'_1 : A(l) &\longmapsto \begin{pmatrix} e^{-i\pi/l} & 0 & 0 \\ 0 & e^{i\pi/l} & 0 \\ 0 & 0 & e^{i\pi/l} \end{pmatrix} \\ B(l, m, n) &\longmapsto \begin{pmatrix} c_m - i \frac{c_n + c_m c_l}{s_l} & 0 & i \sqrt{\left(\frac{c_n + c_m c_l}{s_l}\right)^2 - s_m^2} \\ 0 & e^{i\pi/m} & 0 \\ -i \sqrt{\left(\frac{c_n + c_m c_l}{s_l}\right)^2 - s_m^2} & 0 & c_m + i \frac{c_n + c_m c_l}{s_l} \end{pmatrix} \end{aligned} \tag{4.3}$$

Definition: We will define the *twisted hybrid* Γ' of two Fuchsian groups $\Gamma_1, \Gamma_2 < \text{SU}(1, 1)$ as the subgroup of $\text{SU}(2, 1)$ generated by $\rho'_1(\Gamma_1)$ and $\rho'_2(\Gamma_2)$: $\Gamma' := \langle \rho'_1(\Gamma_1), \rho'_2(\Gamma_2) \rangle$.

5 Jørgensen’s inequality and non-discreteness

The classical Jørgensen inequality [8] gives a necessary condition for discreteness of a non-elementary 2-generator subgroup of $\text{PSL}(2, \mathbb{C})$. It was generalized to the group $\text{PU}(2, 1)$ in [7] in the case where one of the generators is loxodromic or a complex reflection, and in [19] in the case where one of the generators is regular elliptic. In this note we will use both criteria.

5.1 Hybrids generated by regular elliptic isometries

The relevant version of Jørgensen’s inequality is here:

Theorem 5.1 [19] *Let $A \in \text{PU}(2, 1)$ be a regular elliptic isometry with eigenvalues u, v, w , where w is of negative type, and let q_A be the fixed point of A in $\mathbb{H}^n_{\mathbb{C}}$. If $B \in \text{PU}(2, 1)$ satisfies $B(q_A) \neq q_A$ and:*

$$\cosh \frac{1}{2} \rho(q_A, B(q_A)) \text{Max}(|u - w|, |v - w|) < 1$$

then the subgroup generated by A and B is elementary or non-discrete.

Note that the eigenvalues of $A \in \text{PU}(2, 1)$ are only defined up to multiplication by a common unit complex number, but that their pairwise distances are well-defined. Note also that the above inequality can only hold if $\text{Max}(|u - w|, |v - w|) < 1$, which in our case will mean that the rotation angle of A is at most $2\pi/7$ (i.e. $l_1 \geq 7$).

We now focus on the case of the hybrid of 2 Fuchsian triangle groups $\Gamma_1 = (l_1, m_1, n_1)$ and $\Gamma_2 = (l_2, m_2, n_2)$. To be completely explicit, we consider the subgroup $\Gamma(l_1, m_1, n_1; l_2, m_2, n_2)$ of $\text{SU}(2, 1)$ generated by $A_1 = \rho_1\delta(A(l_1))$, $B_1 = \rho_1\delta(B(l_1, m_1, n_1))$, $A_2 = \rho_2\delta(A(l_2))$ and $B_2 = \rho_2\delta(B(l_2, m_2, n_2))$. We apply the above Jørgensen inequality with $A = A_1$ and $B = B_2$. The quantity appearing in Jørgensen’s inequality is then given by the following expression, denoting as above, for $k \in \mathbb{N} \cup \{\infty\}$, $c_k = \cos(\pi/k)$ and $s_k = \sin(\pi/k)$:

Lemma 5.1 *With $A_1 = \rho_1\delta(A(l_1))$ and $B_2 = \rho_2\delta(B(l_2, m_2, n_2))$ as above, we have:*

$$\cosh \frac{1}{2}\rho(q_{A_1}, B_2(q_{A_1}))\text{Max}(|u - w|, |v - w|) = 2s_{l_1}\sqrt{c_{m_2}^2 + \left(\frac{c_{n_2} + c_{m_2}c_{l_2}}{s_{l_2}}\right)^2} \tag{5.1}$$

Proof. Start by noting that $A(l_1)$ has eigenvalues $e^{\pm i\pi/l_1}$, so that A_1 has eigenvalues $e^{\pm i\pi/l_1}$ and 1, with $e^{i\pi/l_1}$ of negative type. Therefore: $\text{Max}(|u - w|, |v - w|) = 2s_{l_1}$.

Now as we have said the fixed point q_{A_1} is the origin $\pi(e_3)$, so that by (3.1):

$$\cosh \frac{1}{2}\rho(q_{A_1}, B_2(q_{A_1})) = |\langle e_3, B_2(e_3) \rangle| = |B_2[3, 3]|$$

where $X[i, j]$ denotes the (i, j) -entry of the matrix X . Note that by (2.1), (4.2) and (2.2):

$$B_2[3, 3] = -c_{m_2} + i \frac{c_{n_2} + c_{m_2}c_{l_2}}{s_{l_2}}.$$

□

Using this expression we have the following result; the quantitative version is obtained by evaluating the right-hand side of (5.1). (This can be done for instance with a little Maple routine). The basic observation is that this expression decreases (to 0) as l_1 increases, and increases with l_2, m_2, n_2 . The values we have chosen to list contain most of the arithmetic Fuchsian triangle groups (listed on p. 418 of [10] following [17]). This is why we pay special attention to the values 7,8,9,10,12,14,15,16,18,24,30. One could however extend the list indefinitely, for example by using the weaker inequality $2s_{l_1}\sqrt{1 + 4/s_{l_2}^2} < 1$. For our purposes we choose $l_1 \geq m_1 \geq n_1$ and $l_2 \leq m_2 \leq n_2$ (we only really need to take l_1 as large as possible).

Theorem 5.2 *Infinitely many of the triangle group hybrids $\Gamma(l_1, m_1, n_1; l_2, m_2, n_2)$ are non-discrete. More precisely, for all values of l_1 and (l_2, m_2, n_2) listed in Table 1, $\Gamma(l_1, m_1, n_1; l_2, m_2, n_2)$ is non-discrete. In fact, for these values any subgroup of Γ containing A_1 and B_2 is non-discrete.*

It is interesting to note that for $(l_2, m_2, n_2) = (l_1, 3, 2)$ one has equality in Jørgensen’s inequality, indicating that the corresponding self-hybrids $\Gamma(l_1, 3, 2; l_1, 3, 2)$ could be discrete, with small covolume if they are lattices. However if one allows reordering then some “asymmetric “self-hybrids” such as $\Gamma(l_1, 3, 2; 2, 3, l_1)$ are non-discrete.

Table 1 Some values of l_1 and (l_2, m_2, n_2) for which $\Gamma(l_1, m_1, n_1; l_2, m_2, n_2)$ is non-discrete

l_1	(l_2, m_2, n_2)
≥ 7	$(2, 3, k), k \geq 7; (2, 4, k), 5 \leq k \leq 7; (2, 5, 5)$
≥ 8	$(2, 5, k), k \geq 5; (2, 6, k), 6 \leq k \leq 15; (2, 7, k), 7 \leq k \leq 9; (3, 3, 4)$
≥ 9	$(2, 6, k), k \geq 6; (2, 7, k), k \geq 7; (2, 8, k), k \geq 8; (2, 9, k), k \geq 9; (2, 10, k), k \geq 10;$ $(2, 12, k), k \geq 12; (2, 15, k), k \geq 15; (3, 3, k), 4 \leq k \leq 8; (3, 4, 4)$
≥ 10	$(3, 3, k), k \geq 4; (3, 4, k), 4 \leq k \leq 7$
≥ 12	$(3, 4, k), k \geq 4; (3, 5, k), k \geq 5; (3, 6, k), k \geq 6; (3, 7, k), k \geq 7;$ $(3, 8, k), k \geq 8; (3, 9, k), 9 \leq k \leq 24; (3, 10, k), 10 \leq k \leq 15; (4, 4, 4)$
≥ 14	$l_2 = 2; (3, 9, k), k \geq 9; (3, 10, k), k \geq 10; (3, 12, k), k \geq 12; (4, 4, k), k \geq 4; (4, 5, 5)$
≥ 15	$(4, 6, k), 6 \leq k \leq 13$
≥ 16	$l_2 \leq 3; (4, 6, k), k \geq 6$
≥ 18	$(4, 16, k), k \geq 16; (5, 5, k), 5 \leq k \leq 12$
≥ 24	$l_2 \leq 5; (6, 6, k), k \geq 6; (6, 12, k), k \geq 12$
≥ 30	$l_2 \leq 7; (8, 8, 8)$

5.2 Twisted hybrids generated by complex reflections

One can as in (4.3) map the generators of the Fuchsian triangle groups to complex reflections. The relevant version of Jørgensen’s inequality is now the following (Theorem 5.2, case (3) of [7]):

Theorem 5.3 [7] *Let $A \in \text{PU}(2, 1)$ be a complex reflection through angle θ , and let L_A be the fixed complex line of A . Let $B \in \text{PU}(2, 1)$ such that $B(L_A)$ and L_A are ultraparallel and denote $\rho = \rho(L_A, B(L_A))$. If:*

$$2 \left| \sin \frac{\theta}{2} \right| \cosh \frac{\rho}{2} < 1$$

then the subgroup generated by A and B is elementary or non-discrete.

Let $A_1 = \rho'_1(A(l_1)), B_1 = \rho'_1(B(l_1, m_1, n_1)), A_2 = \rho'_2(A(l_2))$ and $B_2 = \rho'_2(B(l_2, m_2, n_2))$. We denote $\Gamma'(l_1, m_1, n_1; l_2, m_2, n_2) = \langle A_1, B_1, A_2, B_2 \rangle$. The quantity appearing in Jørgensen’s inequality is then given by the following expression, denoting as above, for $k \in \mathbb{N} \cup \{\infty\}$, $c_k = \cos(\pi/k)$ and $s_k = \sin(\pi/k)$:

Lemma 5.2 *With $A = B_2$ and $B = B_1$, we have:*

$$2 \left| \sin \frac{\theta}{2} \right| \cosh \frac{\rho}{2} = \left(4c_{m_1}^2 s_{m_2}^2 + \left(s_{m_1} \left(s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right) + \frac{c_{n_1} + c_{m_1} c_{l_1}}{s_{l_1}} \left(s_{m_2} - \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right) \right)^2 \right)^{1/2} \tag{5.2}$$

Proof. Start by noting that $B(l_2, m_2, n_2)$ rotates through angle $\theta = 2\pi/m_2$, its eigenvalues being $e^{\pm i\pi/m_2}$ with $e^{i\pi/m_2}$ of negative type. Any eigenvector of $A = B_2$ corresponding to the eigenvalue $e^{-i\pi/m_2}$ of positive type is a normal vector n_A for the fixed line L_A . A straightforward computation gives:

$$n_A = \left[\begin{array}{c} 0 \\ s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \\ \sqrt{\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2} \end{array} \right] \quad \text{and}$$

$$B(n_A) = \left[\begin{array}{c} i \sqrt{\frac{(c_{n_1} + c_{m_1} c_{l_1})^2}{s_{l_1}^2} - s_{m_1}^2} \sqrt{\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2} \\ e^{i\pi/m_1} \left(s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right) \\ \left(c_{m_1} + i \frac{c_{n_1} + c_{m_1} c_{l_1}}{s_{l_1}} \right) \sqrt{\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2} \end{array} \right]$$

By Proposition 3.1 one has: $\cosh \frac{1}{2} \rho(L_A, B(L_A)) = \frac{|(n_A, B(n_A))|}{|(n_A, n_A)|}$, with:

$$\begin{aligned} \langle n_A, B(n_A) \rangle &= e^{i\pi/m_1} \left(s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right)^2 - \left(c_{m_1} + i \frac{c_{n_1} + c_{m_1} c_{l_1}}{s_{l_1}} \right) \\ &\quad \times \left(\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2 \right) \\ \langle n_A, n_A \rangle &= \left(s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right)^2 - \left(\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2 \right) \\ &= 2s_{m_2}^2 + 2s_{m_2} \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \end{aligned}$$

Therefore:

$$2 \left| \sin \frac{\theta}{2} \right| \cosh \frac{\rho}{2} = \frac{\left| e^{i\pi/m_1} \left(s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}} \right)^2 - \left(c_{m_1} + i \frac{c_{n_1} + c_{m_1} c_{l_1}}{s_{l_1}} \right) \left(\frac{(c_{n_2} + c_{m_2} c_{l_2})^2}{s_{l_2}^2} - s_{m_2}^2 \right) \right|}{s_{m_2} + \frac{c_{n_2} + c_{m_2} c_{l_2}}{s_{l_2}}} \quad \square$$

Using this expression we have the following result, obtained by evaluating the right-hand side of (5.2). (This can be done for instance with a little Maple routine). In fact in most cases we only check that this expression is less than 1, but not the extra hypothesis that $B(L_A)$ and L_A are ultraparallel, by appealing to parts (1) and (2) of Theorem 5.2 of [7]. Indeed these state that if $B(L_A)$ and L_A intersect (respectively, are asymptotic) and if A has order ≥ 6 (respectively 7) then the group is again non-discrete.

The dependence on the parameters $l_1, m_1, n_1, l_2, m_2, n_2$ is now less clear, however we still obtain non-discreteness for infinite families of right-angled triangles by observing that, for $l_2 = 2$ and l_1, m_1, n_1, n_2 fixed, the right-hand side of (5.2) decreases as m_2 increases.

Again we pay special attention to arithmetic triangle groups; the order reflects that of commensurability classes of arithmetic triangle groups as listed in [10]. Note that in this case almost all self-hybrids $\Gamma'(l_1, m_1, n_1; l_1, m_1, n_1)$ are non-discrete.

Theorem 5.4 *Infinitely many of the twisted triangle group hybrids $\Gamma'(l_1, m_1, n_1; l_2, m_2, n_2)$ are non-discrete. More precisely, for all values of (l_1, m_1, n_1) and (l_2, m_2, n_2) listed in Tables 2 and 3, $\Gamma'(l_1, m_1, n_1; l_2, m_2, n_2)$ is non-discrete. In fact, for these values any subgroup of Γ containing B_1 and B_2 is non-discrete.*

Table 2 Non-discrete twisted hybrids of right-angled triangles $\Gamma'(2, m_1, n_1; 2, m_2, n_2)$

(l_1, m_1, n_1)	(l_2, m_2, n_2)
$(2, 6, 4), (2, 8, 3), (2, 12, 3), (2, 12, 4), (2, 5, 4), (2, 6, 5),$ $(2, 10, 3), (3, 6, 4), (2, 7, 3), (2, 9, 3), (2, 18, 4), (2, 16, 3),$ $(2, 20, 5), (2, 24, 3), (2, 30, 3), (2, 8, 5), (2, 11, 3)$ $(2, 30, 5)$	$(2, k, 3), k \geq 7; (2, k, 4), k \geq 6;$ $(2, k, 5), k \geq 6; (2, k, 6), k \geq 6$ $(2, k, 3), k \geq 7; (2, k, 4), k \geq 6;$ $(2, k, 5), k \geq 7; (2, k, 6), k \geq 7$

Table 3 Other values of (l_1, m_1, n_1) and (l_2, m_2, n_2) for which $\Gamma'(l_1, m_1, n_1; l_2, m_2, n_2)$ is non-discrete

(l_1, m_1, n_1)	(l_2, m_2, n_2)
$(3, 4, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), k \geq 6; (3, k, k), k \geq 6; (3, 6, 18), (3, 8, 24), (3, 10, 30),$ $(4, k, 4), k \geq 5; (4, 5, 5), (k, k, k), 5 \leq k \leq 8; (5, 10, 5), (5, 15, 5)$
$(3, 5, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), k \geq 6; (3, k, k), k \geq 6; (3, 6, 18), (3, 8, 24), (3, 10, 30),$ $(4, k, 4), k \geq 5; (4, 5, 5), (5, 5, 5), (5, 10, 5), (5, 15, 5)$
$(3, 6, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), k \geq 6; (3, k, k), k \geq 6; (3, 6, 18), (3, 8, 24), (3, 10, 30),$ $(4, k, 4), 6 \leq k \leq 29$
$(3, 7, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), k \geq 6; (3, k, k), 6 \leq k \leq 29; (3, 6, 18), (3, 8, 24), (3, 10, 30),$ $(4, k, 4), 7 \leq k \leq 10$
$(3, 8, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), 6 \leq k \leq 313; (3, k, k), 6 \leq k \leq 15; (3, 8, 24), (3, 10, 30)$
$(3, 9, 3)$	$(3, k, 3), k \geq 6; (3, k, 4), 6 \leq k \leq 50; (3, 8, 8)$
$(3, 12, 3)$	$(3, k, 3), 6 \leq k \leq 368; (3, k, 4), 6 \leq k \leq 14$
$(3, 15, 3)$	$(3, k, 3), 7 \leq k \leq 49$
$(3, 4, 4)$	$(3, k, 4), k \geq 5; (3, k, k), k \geq 5; (3, 6, 18), (3, 8, 24), (3, 10, 30), (4, k, 4), 5 \leq k \leq 170;$ $(4, 5, 5), (5, 5, 5), (5, 10, 5)$
$(3, 6, 4)$	$(3, k, 4), 6 \leq k \leq 38; (3, k, k), 6 \leq k \leq 11; (3, 6, 18), (3, 8, 24), (3, 10, 30)$
$(3, 5, 5)$	$(3, k, k), 5 \leq k \leq 14; (3, 6, 18), (3, 8, 24), (3, 10, 30), (4, 5, 4), (4, 6, 4)$

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