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Enhancing synchronization by directionality in complex networks

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We propose a method called the residual edge-betweenness gradient (REBG) to enhance the synchronizability of networks by assigning the link direction while keeping the topology and link weights unchanged. Direction assignment has been shown to improve the synchronizability of undirected networks in general, but we find that in some cases incommunicable components emerge and networks fail to synchronize. We show that the REBG method improves the residual degree gradient (RDG) method by effectively avoiding the synchronization failure. Further experiments show that the REBG method enhances the synchronizability in networks with a community structure compared with the RDG method.

Synchronization is an important phenomenon in various fields including biology, physics, engineering, and even sociology [1]. In particular, synchronization in complex networks has been intensively studied in the past decade [2–8]. One important objective of these studies is to enhance the synchronizability [6–8], that is, the ability to coordinate oscillators in synchronization. Various methods have been proposed, taking into account degree centrality [6], betweenness [7], and node age [8] to enhance the synchronizability. Nishikawa and Motter proposed assigning zero weight to particular links, which leads to an oriented spanning tree with normalized input strength and no directed loop, and proved that synchronizability is highest in this tree [9]. Recently, the shortest oriented spanning tree was shown to be optimal for both synchronizability and convergence time [10]. However, all these methods are based on the link weight, and the influence of the link direction on synchronization has not been intensively studied.

How to improve synchronization in a directed network is still an unsolved problem. Though previous works suggest that hierarchical structures and the absence of a feedback loop can enhance synchronizability, the underlying mechanism is not clear. In contrast, the directionality plays a significant role in the dynamic of networks [11–14]. With the understanding of relations between link direction and synchronization, a lot of applications can be made. For example, simply regulating the direction of the phase signal of alternating current can facilitate phase match in power grids without additional construction cost to alter the topology. Thus Ref. [15] proposed the residual degree gradient (RDG) method to enhance the synchronizability of networks by assigning only the link direction without changing the topology and link weights.

We find, however, that the RDG method results in incommunicable components for some cases, which leads to incomplete synchronization in graphs. Incommunicable components of a network correspond to the components on which information cannot be transmitted from one to the other in both direction. In these circumstances, the networks can never reach a completely synchronized state. In this paper, we propose the so-called residual edge-betweenness gradient (REBG) method to resolve the problem of incommunicable components. The effectiveness of the algorithm lies in the use of edge betweenness, which embeds global information, compared to the node degree, which reflects only local information. We find that the REBG method effectively enhances synchronizability and avoids incommunicable components.

To begin our analysis, we note that undirected networks can never reach complete synchronization when there are isolated nodes or components, since communications between the isolated components are absent. We call this phenomenon *synchronization failure*, which is in general more likely in directed networks, as isolated components are not necessary. To show this, we first denote the *cut vertex* as the only node through which two or more components communicate with each other. In directed networks, the failure happens whenever cut vertexes have only incoming links. In this case, there is no communication between those components, and the cut vertex becomes an information sink. Hence, complete synchronization cannot be achieved among the incommunicable components. Actually, the synchronization failure is reflected by the eigenvalue properties of the Laplacian matrix, defined as $L_{ij} = \delta_{ij} \sum_h A_{ih} - A_{ij}$ [16], where $A_{ij} = 1$ when there exists a *directed* link from j to i, and $A_{ij} = 0$ otherwise. The second-smallest eigenvalue λ_2 is called the algebraic connectivity of a graph since $\lambda_2 = 0$ when incommunicable components emerge [17]. Therefore, λ_2 can be used to detect the synchronization failure.

To examine synchronization failure, we first describe briefly the RDG method in Ref. [15], which enhances the synchronizability of undirected networks by simply assigning the link direction. Links that have not yet been assigned a direction are referred to as *residual edges*, and the number of connected residual edges is referred as the *residual degree* of a node. In each step, the node with the smallest residual degree is selected and a maximum of $\lceil \langle k \rangle /2 \rceil$ residual edges is set to be incoming [18], where $\langle k \rangle$ is the average degree in the original network. Once a node has been selected, it will not be chosen again. Nodes that have not yet been selected are called *residual nodes*. The RDG assignments are finished when there is no residual node left in the network. In addition,

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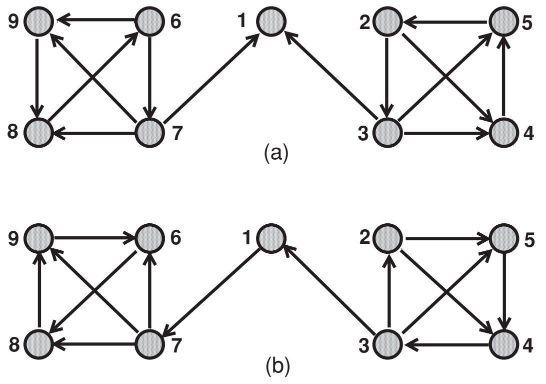

FIG. 1. (a) A simple example of an RDG network with synchronization failure. (b) The REBG network from the same original network, with $\theta = 1$ and without synchronization failure.

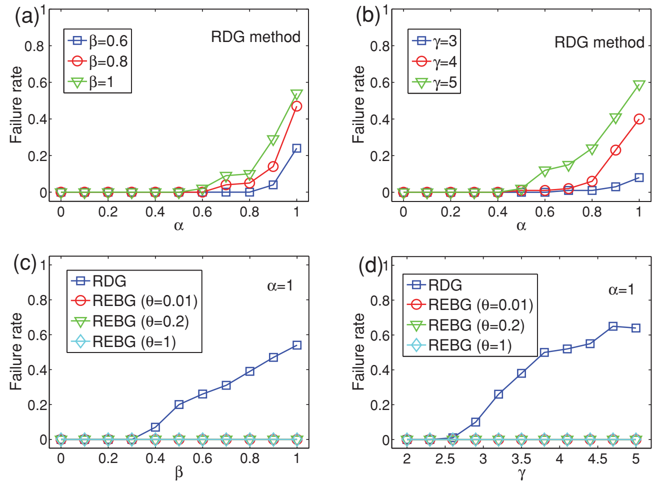

FIG. 2. (Color online) Synchronization failure rate as a function of α by the RDG method in (a) random exponential networks [$P(k) \sim e^{-\beta k}$, $N = 500$, $k_{\min} = 2$, $\beta = 1$] and (b) random scale-free networks [$P(k) \sim k^{-\gamma}$, $N = 500$, $k_{\min} = 2$, $\gamma = 5$]. Given $\alpha = 1$, the failure rate as a function of β in (c) random exponential networks and the failure rate as a function of γ in (d) random scale-free networks when the RDG and the REBG ($\theta = 0.01$, $\theta = 0.2$, and $\theta = 1$) methods are used.

there is a *directionality* α to control the final fraction of links with direction assignment. When $\alpha = 0$, all the links remain undirected. When $\alpha = 1$, all the links are assigned a direction.

The RDG method may lead to directed graphs with synchronization failure, that is, cut vertexes with only incoming links. A simple example is given in Fig. 1(a). According to the rule of RDG, node 1 ($k = 2$) is selected first and the two remaining communities are left incommunicable. As we have discussed, this RDG network cannot reach complete synchronized state.

As exponential and power-law degree distributions are widely observed in empirical data [19,20], we study the RDG method in *random exponential networks* and *random scale-free networks* [20], which correspond to random networks with exponential and power-law degree distributions, respectively. Their degree distributions are given by $P(k) \sim e^{-\beta k}$ and $P(k) \sim k^{-\gamma}$, respectively. For each β and γ, we tested the RDG method on 100 network realizations. To examine incommunicable components, we made sure that there are no original isolated nodes. We find that the resultant RDG networks show synchronization failure, and the failure rate, that is, the fraction of realizations with $0\ \lambda_2$, is reported in Fig. 2. As shown in Figs. 2(a) and 2(b), when α increases (i.e., more links are assigned direction), the failure rate increases for both networks. These results imply that the synchronization failure really happens in both networks, given the RDG is employed. Further, when $\alpha = 1$, where there is the largest synchronization improvement [15], we find that the failure rate is high when β and γ are large. We also note that the failure rate decreases with $k_{\min}$ but increases with N. Synchronization failure is relatively frequent, especially in sparse networks, and therefore a new method is required to assign the link direction avoiding incommunicable components.

We thus introduce the REBG method to avoid synchronization failure. Instead of the node degree, we take the edge betweenness into account. First, we define s_i for node i as

$$s_i = \sum_{j=1}^{N} a_{ij} l_{ij}^{\theta}, \tag{1}$$

where $a_{ij} = 1$ when there an *undirected* link exists between i and j and $a_{ij} = 0$ otherwise. The edge betweenness l_{ij}, that is, the betweenness of the link between i and j, is defined as $l_{ij} = \sum_{m>n} \frac{C_{mn}^{ij}}{C_{mn}}$, where C_{mn} is the number of shortest paths from m to n, and C_{mn}^{ij} is the number of shortest paths from m to n that pass through a link ij. l_{ij} is evaluated on the original undirected networks and is subjected to a power θ with $0 \leqslant \theta \leqslant 1$ in the evaluation of s_i. To assign the link direction, we select the node with the smallest residual s_i in each step and assign an incoming direction for a maximum of $\lceil \langle k \rangle / 2 \rceil$ residual links. As more directed links are assigned, the residual s_i has to be updated at every step. If there are multiple nodes of the smallest s_i, we choose the node with the smallest initial s_i. If, again, their initial s_i values are identical, we randomly choose one node. The REBG method stops when there is no residual node left and all links are directed. We remark that when $\theta = 0$, $s_i = k_i$, and the REBG reduces to the RDG method. When $\theta > 0$, it contains the global information delivered from the edge betweenness.

As shown in Fig. 1(b), the REBG method effectively avoids synchronization failure. In this simple network, whenever node 1 is chosen before nodes 3 and 7, synchronization failure occurs. We note that node 1 is the cut vertex connecting the two components, and its edges are of high betweenness. To determine the details of how the algorithm avoids the failure, we evaluate the initial s_i before any direction assignment, as given by $s_1 = 20^\theta + 20^\theta$, $s_x = 1^\theta + 1^\theta + 6^\theta$ for $x = 2,4,5,6,8,9$ and $s_y = 6^\theta + 6^\theta + 6^\theta + 20^\theta$ for $y = 3,7$. By tracing all the possibilities of the subsequent direction assignment, we find that synchronization failure does not occur provided that node 1 is not selected at the first assignment. In this case, $s_1 > s_x$, which implies $\theta > 0.18$. We denote this value θ_c, which marks the value of θ from which failure ceases. We note that θ_c depends on the network topology.

We then examine the failure rate in random exponential networks and random scale-free networks. As shown in Figs. 2(c) and 2(d), the failure rate vanishes for both networks when $\theta = 0.01$, 0.2, and 1, which suggests that θ_c is nonzero and infinitesimally positive. Since cut vertexes have the same

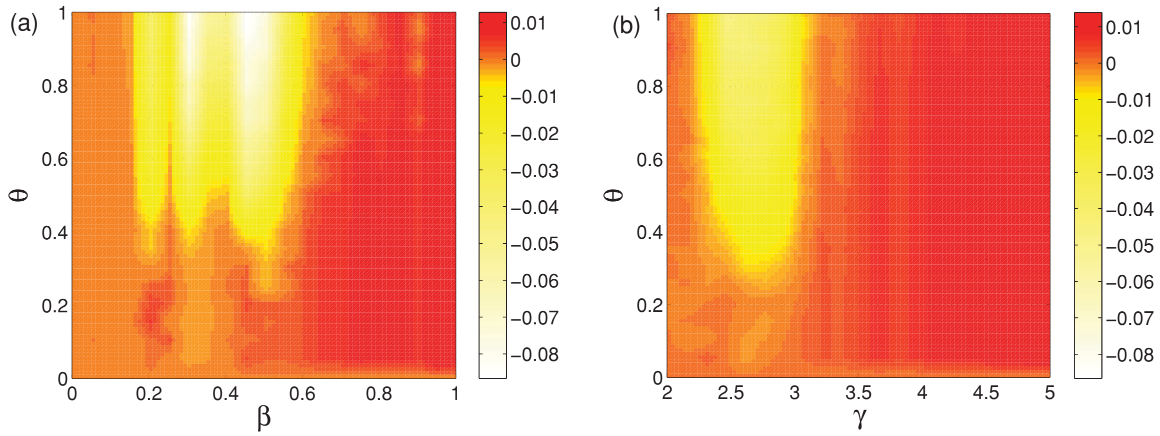

FIG. 3. (Color online) Synchronizability difference $D = \sigma_{\mathrm{RDG}} - \sigma_{\mathrm{REBG}}$ in (a) random exponential networks and (b) random scale-free networks with $N = 500$ and $k_{\min} = 2$.

degree of but a larger edge betweenness compared to other nodes in these networks, failure ceases when edge betweenness has nonvanishing contributions in direction assignment to identify cut vertexes. This implies that for these two types of network, θ in Eq. (1) should be positive to stop synchronization failure.

We also compare the resultant synchronizability between the REBG and the RDG methods. We represent network synchronizability by the normalized spread of eigenvalues in the complex plane, which is given by

$$\sigma^2 = \frac{1}{d^2(N-1)} \sum_{i=2}^{N} |\lambda_i - \bar{\lambda}|^2, \tag{2}$$

where $\bar{\lambda} = \frac{1}{N-1}\sum_{i=2}^{N}\lambda_i$ and $d = \frac{1}{N}\sum_i \sum_{j\neq i} A_{ij}$ [5]. Generally, the smaller the value of σ, the stronger the synchronizability of networks. Both random exponential networks and random scale-free networks are examined. For better illustration, we report the difference $D = \sigma_{\mathrm{RDG}} - \sigma_{\mathrm{REBG}}$ between the eigenvalue spread of the REBG and RDG networks as a function of θ and β in Fig. 3(a) and θ and γ in Fig. 3(b). The positive D shows that the REBG method results in higher synchronizability compared to the RDG method. This enhancement mainly results from the absence of failure in REBG, which avoids zero eigenvalues and reduces the eigenvalue spread.

Moreover, we observe how often cyclic loops emerge in resulting networks. We show in Fig. 4 that for the regime when β and γ are large, the resulting networks are free of directed loops. From the lines of $\theta = 0$ and $\theta = 1$, we see that the number of networks with directed loops is higher when $\theta = 1$, which hinders synchronization and leads to unfavorable results with the REBG. These results also explain the negative D observed at intermediate values of β and γ, since loops would increase the imaginary part of the eigenvalue and lead to a larger spread. We, further, show that the REBG method with $\theta = 0.2$ is similar to the RDG in terms of the fraction of loopy realizations, suggesting that $\theta = 0.2$ is an effective value for direction assignment and, at the same time, avoids synchronization failure. Hence, one can use the REBG method with small θ to enhance synchronizability effectively.

We further compare the RDG and the REBG methods in graphs where failure does not occur. We find that the REBG method would outperform the RDG method in networks with obvious modular structures. Here, we present the result by studying the two methods in a Girvan-Newman benchmark (GN benchmark) network that consists of 128 nodes divided into four4 communities [21]. In the GN benchmark network, $k_{\mathrm{inter}} + k_{\mathrm{outer}} = 16$, where k_{inter} is the average node degree in each community and k_{outer} is the average node degree between different communities. We show in Fig. 5 the results of $k_{\mathrm{outer}} = 1$, in which the GN benchmark networks are highly clustered.

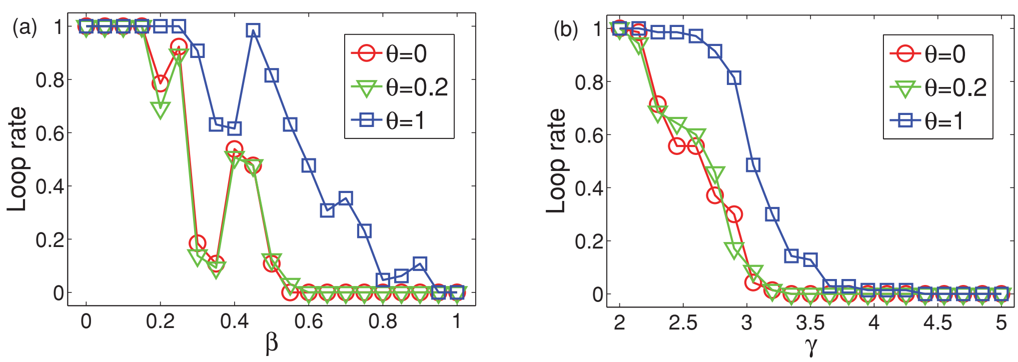

FIG. 4. (Color online) Fraction of realizations with a directed loop as obtained by the REBG and the RDG methods in (a) random exponential networks and (b) random scale-free networks. The abrupt jumps in random exponential networks come from the discontinuity in the assignment constraint $\lceil\langle k\rangle/2\rceil$.

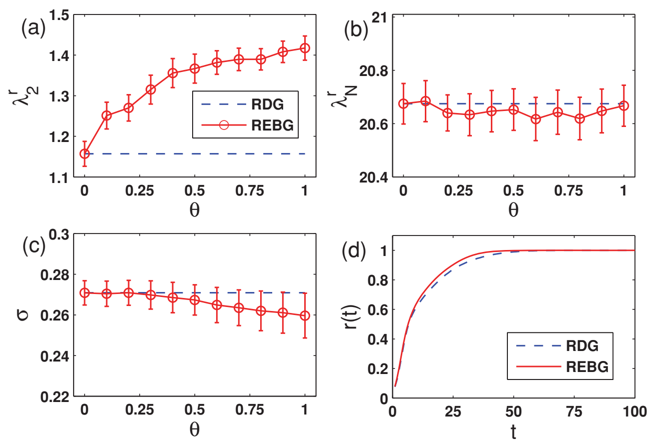

FIG. 5. (Color online) Value of (a) λ_2^r, (b) λ_N^r, and (c) σ of the REBG and the RDG networks from the GN benchmark ($k_{\mathrm{outer}} = 1$). (d) Order parameter $r(t)$ of the Kuramoto model ($\varepsilon = 5$) on the RDG and the REBG networks ($\theta = 1$). Results were obtained by averaging 100 independent realizations.

The σ in Fig. 5(c) indicates that synchronizability is further enhanced in the REBG method. Moreover, we check the ratio of the second smallest and the largest real part of the eigenvalue (λ_2^r/λ_N^r) as an alternative indicator [8,15]. A higher ratio value indicates a generally better synchronizability. The results in Figs. 5(a) and 5(b) show that λ_2^r increases with θ while λ_N^r stays almost the same when θ varies.

Although current indicators can approximately characterize the synchronizability of directed networks, a simple universal measure is still absent [5,15]. It is necessary to use a model of oscillators and test its actual ability for synchronization on directed networks. We therefore employ the Kuramoto model to study synchronization on the resultant directed networks [22]. The phase of the oscillator on node i of the networks is described by $\dot{\phi}_i = \omega_i + \varepsilon \sum_j^N A_{ij} \sin(\phi_j - \phi_i)$, where A is, again, the directed adjacency matrix, and the collective phase synchronization can be investigated by the order parameter defined as $r(t) = \langle |\sum_{j=1}^N e^{i\phi_j(t)}/N|\rangle$. $r(t) \approx 1$ and $r(t) \approx 0$ describe the limits in which all oscillators are, respectively, phase locked and moving incoherently. In Fig. 5(d), it can be seen from $r(t)$ that oscillators in the REBG networks converge more rapidly to complete synchronization than those in the RDG networks. These results show that the REBG method leads to a greater improvement in synchronizability than the RDG method in highly clustered networks, despite the absence of failure. This is because the RDG method does not distinguish links between communities that are of high betweenness [21], while the REBG method deals with them in the final steps of direction assignment to establish a more effective information flow between communities.

In summary, we have introduced the REBG method for direction assignment, which overcomes the problem of emergence of incommunicable components in the RDG method. The effectiveness of our method lies in the use of edge betweenness, which reflects global information compared to the node degree in the RDG. Further tests in highly clustered networks show that the REBG leads to a greater synchronizability improvement despite the absence of synchronization failure. In general, incommunicable components cause problems in various systems such as power grids, wireless communication networks, neural networks, and social interactions [1]. The proposed REBG method is effective in enhancing, avoiding synchronization failures, and may lead to wide applications.

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