Supplementary materials for
As the world turns: short-term human spatial memory in egocentric and allocentric coordinates
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Appendix A

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Modeling the memory capacity as observed during Memory game play

N pair of identical cards are shuffled and placed face down on a table. At each step of the game (hereafter on referred to as a move), the player turns two cards face up, one after the other. If the two cards are identical, they are removed from the table. If not, the cards are replaced face down in their initial positions. The game ends when there remains a unique pair of cards. The game is scored by counting the number of pairs of cards turned over (i.e., the number of moves) necessary to find all matching pairs.

In order to build a mathematical model of the game, two types of assumptions are necessary:

Assumptions on the memorization process.

- The model assumes that the player has at his disposal L memory slots, which he can fill with memorized cards/positions.
- This memory is absolute, the player makes no errors.
- When all L memory slots are filled, the player has to delete a card from memory in order to memorize a new one.
- The cards removed from the game are forgotten.

Assumptions on the strategy of the player.

- During each move, the player turns over two cards, one after the other.
- If the player has only distinct cards (i.e., non-matching) cards in his memory, then he chooses at random (uniformly) amongst the remaining cards that are not in his memory.
- If this card corresponds to a card in his memory, he turns its double and has thus identified a pair. The pair is then removed from the game.
- If the card does not correspond to a card in his memory, he flips a second card at random.
- If the second card corresponds to the first one, he has identified a pair.
- If the second card corresponds to a card in his memory, he has two more cards in his memory, among which are a pair. If the maximum memory capacity is reached, the pair is memorized with priority.

- If the second card corresponds neither to the first one, nor to a previously memorized card, then the player places two more cards in his memory (until maximal memory capacity is reached).
- Finally, if the player has a pair of cards in his memory, he identifies and removes this pair from play.

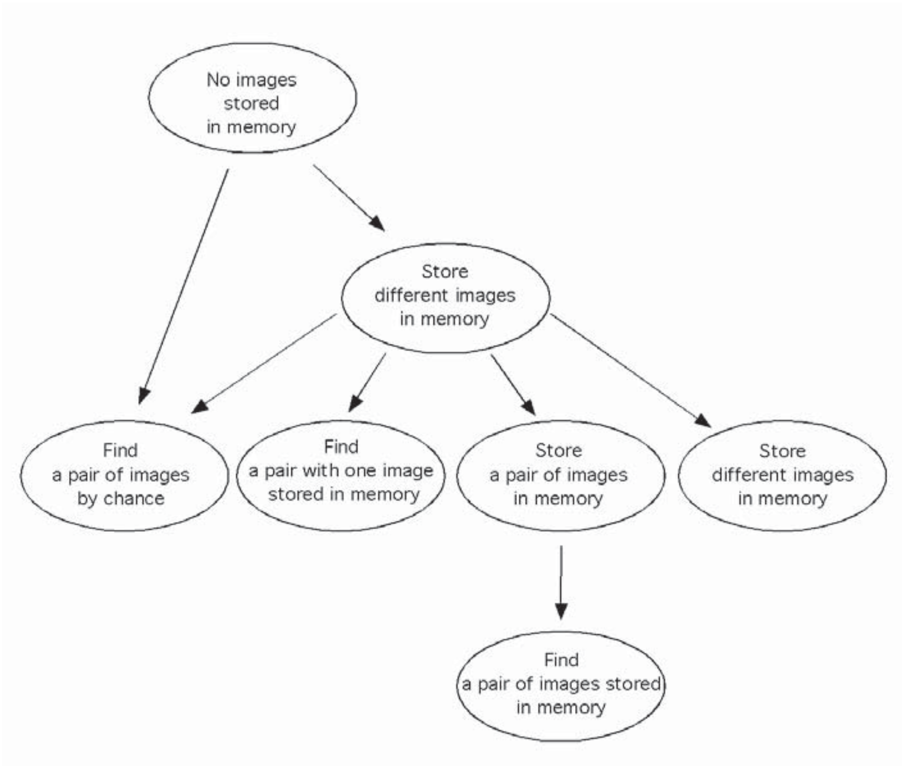

Figure 1: Figure 1 depicts the possible events which can occur during each move. The probability that a certain event occurs depends on three factors: (1) the number of remaining pairs; (2) the number of cards in memory; and (3) whether or not the player has a pair of cards in his memory or not.

Hence, the game can be viewed as a Markov chain $X_k = (n, l, \star)$ where

- n : number of pairs of cards which remain in the game
- l : number of cards memorized by the player
- $\star = \mathrm{d}$: only distinct cards in the player's memory
  $\star = \mathrm{p}$: a pair of cards in the player's memory
- k : number of trials

All possible transitions and transition probabilities are defined by the assumptions described above.

The state space of this Markov chain is given by $\Omega_1^{N,L}$ where

$$\Omega_k^{N,L} = \bigcup_{n=k}^{N} \Lambda_n^L \qquad \text{with} \qquad \Lambda_n^L = \Lambda_n^{L,\mathrm{d}} \cup \Lambda_n^{L,\mathrm{p}}$$

$$\Lambda_n^{L,\mathrm{d}} = \{n\} \times \{0, 1, \ldots, L\} \times \{\mathrm{d}\} \quad \text{and} \quad \Lambda_j^{L,\mathrm{p}} = \{j\} \times \{(2), 3, \ldots, L\} \times \{\mathrm{p}\}$$

When $L \leq 1$, we set $\Lambda_n^{L,\mathrm{p}} = \emptyset$.

We divide the state space $\Omega_1^{N,L}$ into layers Λ_n^L in order to find a simple way to describe the transitions. Each layer Λ_n^L contains all of the states where n pair of cards remain in the game. The goal is to described the transitions layer by layer.

Figure 2 shows an example of admissible transitions with the related transitions probabilities.

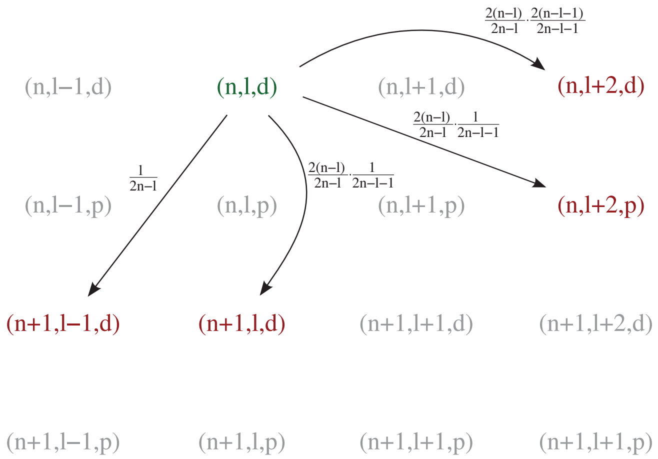

Figure 2: Example of transitions when memory limit has not been attained.

When the memory limit is attained, the transitions are different (see figure 3)

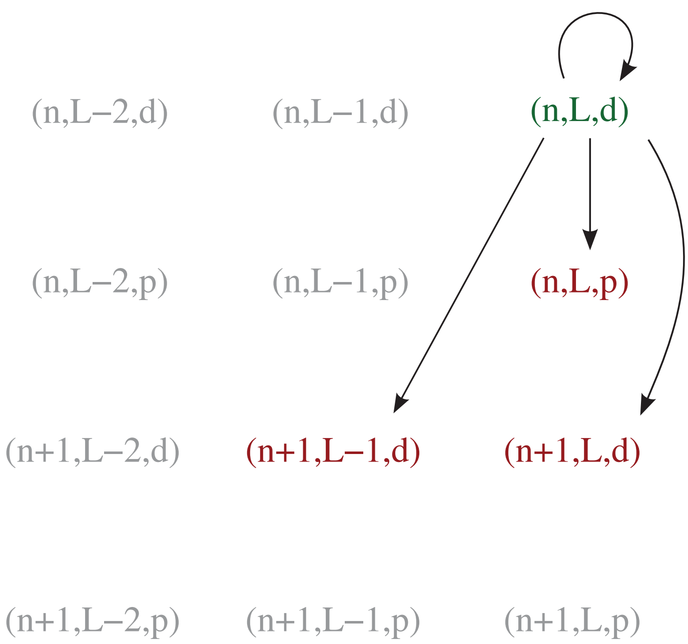

Figure 3: Example of transitions when memory limit has been attained.

Special attention must be given to the condition when the capacity of the memory exceeds the number of the remaining pairs. Indeed, the states $(n, l, \star)$ with $n < l$ are unreachable, which means that in these cases it is not possible to fill the memory because too few cards remain.

As we consider the transitions layer by layer, the transition matrix $A_{N,L}$ of this process can be divided in submatrices giving the transitions between the various layers Λ_j. If R_n^L contains the transitions $\Lambda_n^L \to \Lambda_n^L$ and P_n^L the transitions $\Lambda_n^L \to \Lambda_{n-1}^L$, we have

$$A_{N,L} = \begin{pmatrix} R_N^L & P_N^L & 0 & \dots & 0 & 0 \\ 0 & R_{N-1}^L & P_{N-1}^L & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & R_2^L & P_2^L \\ 0 & 0 & 0 & \dots & 0 & R_1^L \end{pmatrix},$$

Moreover, when $L \geq 2$, each layer Λ_n^L is divided in two sublayers $\Lambda_n^{L,\mathrm{d}}$ and $\Lambda_n^{L,\mathrm{p}}$. Hence

$$R_n^L = \begin{pmatrix} R_n^{L,\mathrm{dd}} & R_n^{L,\mathrm{dp}} \\ R_n^{L,\mathrm{pd}} & R_n^{L,\mathrm{pp}} \end{pmatrix} \qquad \text{and} \qquad P_n^L = \begin{pmatrix} P_n^{L,\mathrm{dd}} & P_n^{L,\mathrm{dp}} \\ P_n^{L,\mathrm{pd}} & P_n^{L,\mathrm{pp}} \end{pmatrix}$$

where, for example, $R_n^{L,\mathrm{dd}}$ contains the probabilities to stay in the sublayer $\Lambda_n^{L,\mathrm{d}}$ (n remaining pairs, only distinct cards in memory) and $P_n^{L,\mathrm{pd}}$ contains the probabilities to go from the sublayer $\Lambda_n^{L,\mathrm{p}}$ (n remaining pairs, a pair of cards in memory) to the sublayer $\Lambda_{n-1}^{L,\mathrm{d}}$ ($n-1$ remaining pairs, only distinct cards in memory).

Assume that $L \geq 2$. Then,

- For the submatrix $R_n^{L,\mathrm{dd}}$ we have
  - $(R_n^{L,\mathrm{dd}})_{l,l+2} = \frac{2\,(n-l)}{2\,n-l}\,\frac{2\,(n-l-1)}{2\,n-l-1}, \qquad 0 \leq l \leq \min(L-2, n-1)$,
  - $(R_n^{L,\mathrm{dd}})_{L-1,L} = \frac{2\,(n-L+1)}{2\,n-L+1}\,\frac{2\,(n-L)}{2\,n-L}, \qquad n \geq L$,
  - $(R_n^{L,\mathrm{dd}})_{L,L} = \frac{2\,(n-L)}{2\,n-L}\,\frac{2\,(n-L-1)}{2\,n-L-1}, \qquad n \geq L$,
  - $(R_n^{L,\mathrm{dd}})_{l,l} = 1, \qquad l > n$,
  - $R_{N-1}^{L,\mathrm{dd}} = \mathrm{id}$.
- For the submatrix $R_n^{L,\mathrm{dp}}$ in the case $L = 2$, we have
  - $(R_n^{L,\mathrm{dp}})_{L,L-2} = \frac{2\,(n-L)}{2\,n-L}\,\frac{L}{2\,n-L-1}, \qquad n \geq L$.
- For the submatrix $R_n^{L,\mathrm{dp}}$ in the case $L \geq 3$, we have
  - $(R_n^{L,\mathrm{dp}})_{l,l-1} = \frac{2\,(n-l)}{2\,n-l}\,\frac{l}{2\,n-l-1}, \qquad 1 \leq l \leq \min(L-2, n)$,
  - $(R_n^{L,\mathrm{dp}})_{L-1,L-3} = \frac{2\,(n-L+1)}{2\,n-L+1}\,\frac{L-1}{2\,n-L}, \qquad n \geq L-1$,
  - $(R_n^{L,\mathrm{dp}})_{L,L-3} = \frac{2\,(n-L)}{2\,n-L}\,\frac{L}{2\,n-L-1}, \qquad n \geq L$.
- $R_n^{L,\mathrm{pd}} = 0$
- $R_n^{L,\mathrm{pp}} = 0$ except $R_{N-1}^{L,\mathrm{pp}} = \mathrm{id}$
- For the submatrix $P_n^{L,\mathrm{dd}}$ we have
  - $(P_n^{L,\mathrm{dd}})_{l,l-1} = \frac{l}{2\,n-l}, \qquad 1 \leq l \leq \min(L, n)$,
  - $(P_n^{L,\mathrm{dd}})_{l,l} = \frac{2\,(n-l)}{2\,n-l}\,\frac{1}{2\,n-l-1}, \qquad 0 \leq l \leq \min(L, n)$.
- $P_n^{L,\mathrm{dp}} = 0$
- For the submatrix $P_n^{L,\mathrm{pd}}$ in the case $L = 2$, we have
  - $(P_n^{L,\mathrm{pd}})_{l-2,l-2} = 1, \qquad 2 \leq l \leq L$,

  - $P_{N-1}^{L,\mathrm{pd}} = 0$.

- For the submatrix $P_n^{L,\mathrm{pd}}$ in the case $L = 3$, we have
  - $(P_n^{L,\mathrm{pd}})_{l-3,l-2} = 1, \qquad 3 \le l \le L,$
  - $P_{N-1}^{L,\mathrm{pd}} = 0$.

- $P_n^{L,\mathrm{pp}} = 0$

When $L = 0$,

$$(A_{N,0})_{n,n} = \frac{2\,(n-1)}{2\,n-1}, \quad (A_{N,0})_{n,n+1} = \frac{1}{2\,n-1} \quad \text{and} \quad (A_{N,0})_{N,N} = 1.$$

and when $L = 1$,

$$R_n^1 = \begin{pmatrix} 0 & \frac{2\,(n-1)}{2\,n-1} \\ 0 & \frac{2\,n-3}{2\,n-1} \end{pmatrix}, \quad P_n^1 = \begin{pmatrix} \frac{1}{2\,n-1} & 0 \\ \frac{1}{2\,n-1} & \frac{1}{2\,n-1} \end{pmatrix} \quad \text{and} \quad R_N^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Duration of the game

In order to calculate the duration of the game, i.e., how many moves it will take to complete the game, we define the random variable

$$T^{N,L} = \inf\{k \mid X_k^{N,L} \in \Lambda_1^L\},$$

the law of probability of which is

$$\begin{aligned} P_L(k) = \mathbb{P}(T^{N,L} = k) &= \mathbb{P}(X_1^{N,L} \in \Omega_2^{N,L}, \dots, X_{k-1}^{N,L} \in \Omega_2^{N,L}, X_k^{N,L} \in \Lambda_1^L) \\ &= \mathbb{P}(X_{k-1}^{N,L} \in \Omega_2^{N,L}, X_k^{N,L} \in \Lambda_1^L) \\ &= \mathbb{P}(X_{k-1}^{N,L} \in \Lambda_2^L, X_k^{N,L} \in \Lambda_1^L). \end{aligned}$$

The second and third equalities follow from the relations

- $\{X_k^{N,L} \in \Omega_n^{N,L}\} \subset \{X_{k+1}^{N,L} \in \Omega_n^{N,L}\}$,
- $\{X_{k-1}^{N,L} \in \Omega_2^{N,L}\} \cap \{X_k^{N,L} \in \Lambda_1^L\} = \bigcup_{n=2}^{N} \Big(\{X_{k-1}^{N,L} \in \Lambda_n^L\} \cap \{X_k^{N,L} \in \Lambda_1^L\}\Big) = \{X_{k-1}^{N,L} \in \Lambda_2^L\} \cap \{X_k^{N,L} \in \Lambda_1^L\}$.

Moreover, we have

- for $L = 2$
- for $L \ge 3$

$$\begin{aligned} &\{X_{k-1}^{N,L} \in \Lambda_2\} \cap \{X_k^{N,L} \in \Lambda_1\} \\ &= \big(\{X_{k-1}^{N,L} = (2,0,\mathrm{d})\} \cap \{X_k^{N,L} = (1,0,\mathrm{d})\}\big) \cup \big(\{X_{k-1}^{N,L} = (2,1,\mathrm{d})\} \cap \{X_k^{N,L} = (1,0,\mathrm{d})\}\big) \\ &\quad \cup \big(\{X_{k-1}^{N,L} = (2,1,\mathrm{d})\} \cap \{X_k^{N,L} = (1,1,\mathrm{d})\}\big) \cup \big(\{X_{k-1}^{N,L} = (2,2,\mathrm{d})\} \cap \{X_k^{N,L} = (1,1,\mathrm{d})\}\big) \\ &\quad \cup \big(\{X_{k-1}^{N,L} = (2,3,\mathrm{p})\} \cap \{X_k^{N,L} = (1,1,\mathrm{d})\}\big) \end{aligned}$$

Finally, when $L \geq 3$

$$\begin{aligned}
P_L(k) &= \mathbb{P}(X_{k-1}^{N,L} = (2,0,\mathrm{d}),\, X_k^{N,L} = (1,0,\mathrm{d})) + \mathbb{P}(X_{k-1}^{N,L} = (2,1,\mathrm{d}),\, X_k^{N,L} = (1,0,\mathrm{d})) \\
&\quad + \mathbb{P}(X_{k-1}^{N,L} = (2,1,\mathrm{d}),\, X_k^{N,L} = (1,1,\mathrm{d})) + \mathbb{P}(X_{k-1}^{N,L} = (2,2,\mathrm{d}),\, X_k^{N,L} = (1,1,\mathrm{d})) \\
&\quad + \mathbb{P}(X_{k-1}^{N,L} = (2,3,\mathrm{p}),\, X_k^{N,L} = (1,1,\mathrm{d})) \\
&= \mathbb{P}(X_k^{N,L} = (1,0,\mathrm{d}) \mid X_{k-1}^{N,L} = (2,0,\mathrm{d}))\, \mathbb{P}(X_{k-1}^{N,L} = (2,0,\mathrm{d})) \\
&\quad + \mathbb{P}(X_k^{N,L} = (1,0,\mathrm{d}) \mid X_{k-1}^{N,L} = (2,1,\mathrm{d}))\, \mathbb{P}(X_{k-1}^{N,L} = (2,1,\mathrm{d})) \\
&\quad + \mathbb{P}(X_k^{N,L} = (1,1,\mathrm{d}) \mid X_{k-1}^{N,L} = (2,1,\mathrm{d}))\, \mathbb{P}(X_{k-1}^{N,L} = (2,1,\mathrm{d})) \\
&\quad + \mathbb{P}(X_k^{N,L} = (1,1,\mathrm{d}) \mid X_{k-1}^{N,L} = (2,2,\mathrm{d}))\, \mathbb{P}(X_{k-1}^{N,L} = (2,2,\mathrm{d})) \\
&\quad + \mathbb{P}(X_k^{N,L} = (1,1,\mathrm{d}) \mid X_{k-1}^{N,L} = (2,3,\mathrm{p}))\, \mathbb{P}(X_{k-1}^{N,L} = (2,3,\mathrm{p})) \\
&= \frac{1}{3}\, \mathbb{P}(X_{k-1}^{N,L} = (2,0,\mathrm{d})) + \frac{2}{3}\, \mathbb{P}(X_{k-1}^{N,L} = (2,1,\mathrm{d})) \\
&\quad + \mathbb{P}(X_{k-1}^{N,L} = (2,2,\mathrm{d})) + \mathbb{P}(X_{k-1}^{N,L} = (2,3,\mathrm{p})) \\
&= e_1^T A_{N,L}^{k-1} \big(\frac{1}{3}\, e_{R+1} + \frac{2}{3}\, e_{R+2} + e_{R+3} + e_{R+L+2} \big).
\end{aligned}$$

where $R = (2\,L-1)\,(N-2)$ and $e_n^T = (0,\ldots,0,1,0,\ldots,0)$ is a vector of length $2\,(L-1)\,N$ with the n-th entry equal to 1.

We can also calculate the generating function G_T of the random variable T. For $z < 1$, $u = e_1^T$ and $v = \frac{1}{3}\, e_{R+1} + \frac{2}{3}\, e_{R+2} + e_{R+3} + e_{R+L+2}$, we have

$$G_T(z) = \sum_{k=1}^{\infty} \mathbb{P}(T=k)\, z^k = \sum_{k=1}^{\infty} u\, A^{k-1}\, v\, z^k = z\, u \sum_{k=0}^{\infty} (z\, A)^k\, v = z\, u\, (\mathrm{Id} - z\, A)^{-1}\, v.$$

And for $S_n = \sum_{k=1}^{n} T_k$ where the random variables T_k are i.i.d. and follow the same distribution than T, we have

$$G_{S_n}(z) = \prod_{k=1}^{n} G_{T_k}(z) = (G_T(z))^n = z^n\, (u\, (\mathrm{Id} - z\, A)^{-1}\, v)^n.$$

Mean and standard deviation of the duration of the game

The mean and variance are calculate using the first and second derivatives of the generating function $G_T(z)$ evaluated in $z = 1$.

By induction, we can show that the derivatives of the generating function $G_T(z)$ for $z < 1$ are given by

$$\frac{d^k}{dz^k}\, G_T(z) = k!\, u\, A^{k-1}\, (\mathrm{Id} - z\, A)^{-k}\, (\mathrm{Id} + z\, A\, (\mathrm{Id} - z\, A)^{-1})\, v,$$

for $k \geq 1$.

Indeed for $k = 1$

$$\begin{aligned}
\frac{d}{dz}\, G_T(z) &= \frac{d}{dz}\, (z\, u (\mathrm{Id} - z\, A)^{-1} v) \\
&= \big(u\, (\mathrm{Id} - z\, A)^{-1}\, v - z\, u\, A\, (\mathrm{Id} - z\, A)^{-2}\, v \big) \\
&= u\, (\mathrm{Id} - z\, A)^{-1}\, (\mathrm{Id} - z\, A (\mathrm{Id} - z\, A)^{-1})\, v.
\end{aligned}$$

For the induction step, set $B(z) = \mathrm{Id} + z\, A\, (\mathrm{Id} - z\, A)^{-1}$. Then

$$\frac{d}{dz}\, B(z) = A\, ((\mathrm{Id} - z\, A)^{-1} + z\, A\, (\mathrm{Id} - z\, A)^{-2}) = A\, (\mathrm{Id} - z\, A)^{-1}\, B(z).$$

and therfore

$$\begin{aligned}\frac{d^{k+1}}{dz^{k+1}}\, G_T(z) &= \frac{d}{dz}\,(k!\, u\, A^{k-1}\,(\mathrm{Id} - z\, A)^{-k}\, B(z)\, v) \\ &= k!\, u\, A^{k-1}\,\Big(k\,(\mathrm{Id} - z\, A)^{-(k-1)}\, A\,(\mathrm{Id} - z\, A)^{-2}\, B(z) \\ &\qquad\qquad + (\mathrm{Id} - z\, A)^{-k}\, A\,(\mathrm{Id} - z\, A)^{-1}\, B(z)\Big)\, v \\ &= k!\, u\, A^{k}\,(k+1)\,(\mathrm{Id} - z\, A)^{-(k+1)}\, B(z)\end{aligned}$$

which ends to prove the formula for the derivatives of $G_T(z)$.

Our formulas for $G_T(z)$ and its derivatives are only valid for $z < 1$, however for the positive random variable T we have (monotone convergence)

$$\begin{aligned}\lim_{x\nearrow 1} \frac{d^k}{dz^k}\,[G_T(z)]_{z=x} &= \lim_{x\nearrow 1} \sum_{l=k}^{\infty} \frac{l!}{(l-k)!}\,\mathbb{P}(T=l)\, x^{l-k} \\ &= \sum_{k=1}^{\infty} \lim_{x\nearrow 1} \frac{l!}{(l-k)!}\,\mathbb{P}(T=l)\, x^{l-k} \\ &= \sum_{k=1}^{\infty} \frac{l!}{(l-k)!}\,\mathbb{P}(T=l),\end{aligned}$$

and therefore

$$\mathbb{E}(T) = \lim_{x\nearrow 1} \frac{d}{dz}\,[G_T(z)]_{z=x} = \lim_{z\nearrow 1} u\,(\mathrm{Id} - z\, A)^{-1}\,(\mathrm{Id} - z\, A(\mathrm{Id} - z\, A)^{-1})\, v,$$

with

$$\begin{aligned}\mathrm{Var}(T) &= \lim_{x\nearrow 1} \frac{d^2}{dz^2}\,[G_T(z)]_{z=1} + \mathbb{E}(T)\,(1 - \mathbb{E}(T)) \\ &= \lim_{z\nearrow 1} 2\, u\, A\,(\mathrm{Id} - z\, A)^{-2}\,(\mathrm{Id} + z\, A\,(\mathrm{Id} - z\, A)^{-1})\, v + \mathbb{E}(T)\,(1 - \mathbb{E}(T)).\end{aligned}$$

Since S_n is the sum of n independent replicates of T, we have simply $\mathbb{E}(S_n) = n\,\mathbb{E}(T)$ and $\mathrm{Var}(S_n) = n\,\mathrm{Var}(T)$.