

TOPOLOGY OF POSITIVELY CURVED 8-DIMENSIONAL MANIFOLDS WITH SYMMETRY

ANAND DESSAI

We show that a simply connected 8-dimensional manifold M of positive sectional curvature and symmetry rank ≥ 2 resembles a rank-one symmetric space in several ways. For example, the Euler characteristic of M is equal to the Euler characteristic of S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$. If M is rationally elliptic, then M is rationally isomorphic to a rank-one symmetric space. For torsion-free manifolds, we derive a much stronger classification. We also study the bordism type of 8-dimensional manifolds of positive sectional curvature and symmetry rank ≥ 2 . As an illustration, we apply our results to various families of 8-manifolds.

1. Introduction

We study the topology of positively curved 8-dimensional manifolds with symmetry rank ≥ 2 . A Riemannian manifold M is said to have positive curvature if the sectional curvature of all its tangent planes is positive. The symmetry rank of M is defined as the rank of its isometry group. Throughout this paper, all manifolds are assumed to be closed, that is, compact without boundary.

At present only a few manifolds are known to admit a Riemannian metric of positive curvature. Besides the examples of Eschenburg and Bazaikin (which are biquotients of dimension 6, 7 or 13), all other simply connected positively curved examples¹ are homogeneous, that is, they admit a metric of positive curvature with transitive isometry group (the latter were classified by Berger, Wallach, Aloff and Bérard Bergery). In dimension greater than 24 all known examples are symmetric of rank one.

Keywords: positive curvature, torus actions, Euler characteristic, classification of 8-manifolds.

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¹Recently, Petersen and Wilhelm [2008], Grove, Verdiani and Ziller [2008], and Dearnicott all announced the discovery of new 7-dimensional examples with positive curvature.

Classifications of various strength have been obtained for positively curved manifolds with large symmetry; see the survey [Wilking 2007, Section 4]. Among the measures of “largeness”, we shall focus on the symmetry rank.

Grove and Searle [1994] showed that the symmetry rank of a positively curved simply connected n -dimensional manifold M is $\leq [(n+1)/2]$, and that equality occurs if and only if M is diffeomorphic to a sphere or a complex projective space. Wilking [2003] proved that, if the symmetry rank of M is $\geq n/4 + 1$ and if $n \geq 10$, then either M is homeomorphic to a sphere or a quaternionic projective space, or M is homotopically equivalent to a complex projective space (it follows from [Dessai and Wilking 2004] that “homotopically equivalent” can be strengthened to “tangentially equivalent” in this classification). Building on [Wilking 2003], Fang and Rong [2005] showed that, if the symmetry rank is no less than $[(n-1)/2]$ and if $n \geq 8$, then M is homeomorphic to a sphere, a quaternionic projective space, or a complex projective space.

In dimension eight, the rank-one symmetric spaces S^8 , $\mathbb{H}P^2$ and $\mathbb{C}P^4$ are the only known simply connected positively curved examples. In this dimension, the aforementioned work of Grove and Searle and of Fang and Rong says that a positively curved simply connected manifold M is diffeomorphic to S^8 or $\mathbb{C}P^4$ if the symmetry rank of M is ≥ 4 , and homeomorphic to S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$ if the symmetry rank is ≥ 3 .

The main purpose of this paper is to give some information on the topology of positively curved 8-dimensional manifolds with symmetry rank ≥ 2 . Our first result concerns the Euler characteristic:

Theorem 1.1. *Let M be a simply connected 8-dimensional manifold. If M admits a metric of positive curvature and symmetry rank ≥ 2 , then the Euler characteristic of M is equal to the Euler characteristic of S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$; that is, $\chi(M) = 2, 3$ or 5 .*

This information on the Euler characteristic leads to a rather strong classification if one assumes in addition that the manifold is rationally elliptic. Recall that a closed simply connected manifold M is *rationally elliptic* if its rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is of finite rank. A conjecture attributed to Bott asserts that any non-negatively curved manifold is rationally elliptic; see [Grove and Halperin 1982, page 172].

Theorem 1.2. *Let M be a simply connected positively curved 8-dimensional manifold of symmetry rank ≥ 2 .*

- (1) *If M is rationally elliptic, M has the rational cohomology ring and rational homotopy type of a rank-one symmetric space, that is, of S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$.*
- (2) *If M is rationally elliptic and $H^*(M; \mathbb{Z})$ is torsion-free, M is homeomorphic to S^8 , diffeomorphic to $\mathbb{H}P^2$, or tangentially equivalent to $\mathbb{C}P^4$.*

If one drops the assumption on rational ellipticity and weakens the assumption on the symmetry rank, one can still prove a bound on the Euler characteristic:

Theorem 1.3. *Suppose M is a simply connected positively curved manifold of even dimension ≤ 8 . Assume S^1 acts smoothly on M . If some $\sigma \in S^1$ acts isometrically and nontrivially on M , then $\chi(M) \geq 2$.*

This fits well with the Hopf conjecture on the positivity of the Euler characteristic of even-dimensional positively curved manifolds.

To put our results in perspective, we briefly recall what is known about positively curved manifolds in low dimensions. Next to surfaces, manifolds of positive curvature are only classified in dimension 3; see [Hamilton 1982] and [Perelman 2002; 2003b; 2003a]. In higher dimensions, the only known obstructions to positive curvature for simply connected manifolds are given by the Betti number theorem of [Gromov 1981] and obstructions to positive scalar curvature (for example, the α -invariant of Lichnerowicz and Hitchin, and the obstructions in dimension 4 coming from Seiberg–Witten theory). In particular, the Hopf problem that asks whether $S^2 \times S^2$ admits a metric of positive curvature is still open.

The study of low-dimensional positively curved manifolds with positive symmetry rank began with Hsiang and Kleiner [1989] on 4-dimensional manifolds. Their main result says that the Euler characteristic of a simply connected positively curved 4-dimensional manifold M with positive symmetry rank is ≤ 3 . Using [Freedman 1982], they conclude that M is homeomorphic² to S^4 or $\mathbb{C}P^2$. Rong [2002] showed that a simply connected positively curved 5-dimensional manifold with symmetry rank 2 is diffeomorphic to S^5 . In dimension 6 and 7 there are examples with symmetry rank 2 and 3, respectively, that are not homotopically equivalent to a rank-one symmetric space [Aloff and Wallach 1975; Eschenburg 1982]. This indicates that, in these dimensions, a classification below the maximal symmetry rank is more complicated.

Theorems 1.1 and 1.3 imply that, for many nonnegatively curved manifolds, any metric of positive curvature must be quite nonsymmetric. For example, if M is of even dimension ≤ 8 and has Euler characteristic < 2 (for example, if M is a product of two simply connected odd-dimensional spheres, or M is a simply connected Lie group), then it follows from Theorem 1.3 that, for any positively curved metric g on M , the only isometry of (M, g) sitting in a compact connected Lie subgroup of the diffeomorphism group is the identity. As a further illustration of our results, we consider the following classes of manifolds:

Product manifolds. Let $M = N_1 \times N_2$ be a simply connected product manifold of dimension 8 (with $\dim N_i > 0$). It is straightforward to see that the Euler characteristic of M is $\neq 2, 3$ or 5 . By Theorem 1.1, M does not admit a metric of

²According to the recent preprint [Kim 2008], M is diffeomorphic to S^4 or $\mathbb{C}P^2$.

positive curvature with symmetry rank ≥ 2 . In particular, the product of two simply connected nonnegatively curved manifolds N_1 and N_2 (for example, $S^4 \times S^4$) does not admit a metric of positive curvature and symmetry rank ≥ 2 . It is interesting to compare this with [Hsiang and Kleiner 1989], which implies that $S^2 \times S^2$ does not admit a metric of positive curvature and symmetry rank ≥ 1 .

Connected sum of rank-one symmetric spaces. Cheeger [1973] has shown that $\mathbb{C}P^4 \# \pm \mathbb{C}P^4$, $\mathbb{C}P^4 \# \pm \mathbb{H}P^2$ and $\mathbb{H}P^2 \# \pm \mathbb{H}P^2$ admit a metric of nonnegative curvature. The Euler characteristic of these manifolds is 8, 6 and 4, respectively. By Theorem 1.1, none of them admits a metric of positive curvature and symmetry rank ≥ 2 .

Cohomogeneity-one manifolds. Grove and Ziller [2000] constructed invariant metrics of nonnegative curvature on cohomogeneity-one manifolds with codimension-two singular orbits. Using this construction, they exhibited metrics of nonnegative curvature on certain infinite families of simply connected manifolds that fiber over S^4 , $\mathbb{C}P^2$, $S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ [Grove and Ziller 2000; 2008]; see also the survey [Ziller 2007]. In dimension 8, the Euler characteristic of all these manifolds turns out to be $\neq 2, 3, 5$. Again, by Theorem 1.1, none of them admits a metric of positive curvature and symmetry rank ≥ 2 .

Biquotients. Another interesting class of manifolds known to admit metrics of nonnegative curvature are biquotients. A biquotient of a compact Lie group G is the quotient of a homogeneous space G/H by a free action of a subgroup K of G , where the K -action is induced from the left G -action on G/H . Note that any homogeneous space can be described as a biquotient by taking one of the factors to be trivial. If G is equipped with a biinvariant metric, the biquotient $M = K \backslash G/H$ inherits a metric of nonnegative curvature, a consequence of O'Neill's formula for Riemannian submersions. As pointed out by Eschenburg [1992], a manifold M is a biquotient if and only if M is the quotient of a compact Lie group G by a free action of a compact Lie group L , where the action of L on G is given by a homomorphism $L \rightarrow G \times G$ together with the two-sided action of $G \times G$ on G given by $(g_1, g_2)(g) := g_1 \cdot g \cdot g_2^{-1}$.

The topology of biquotients has been investigated in [Eschenburg 1992; Singhof 1993; Kapovitch 2002; Kapovitch and Ziller 2004; Totaro 2002]. Kapovitch and Ziller [2004] have classified biquotients with singly generated rational cohomology. Combining their classification with the first part of Theorem 1.2 gives this:

Corollary 1.4. *A simply connected 8-dimensional biquotient of positive curvature and symmetry rank ≥ 2 is diffeomorphic to S^8 , $\mathbb{C}P^4$, $\mathbb{H}P^2$ or $G_2/SO(4)$.*

In view of Theorems 1.1 and 1.3, the examples above contain plenty of examples of simply connected nonnegatively curved manifolds with positive Ricci curvature

for which the metric cannot be deformed to a metric of positive curvature via a symmetry-preserving process such as the Ricci flow.

The paper is structured as follows: In Section 2, we recall basic geometric and topological properties of positively curved manifolds with symmetry. In Section 3, we prove the statements on the Euler characteristic. In Section 4, we prove our classification theorem 1.2 for rationally elliptic manifolds and the corollary 1.4 for biquotients. In the final Section 5, we study the bordism type of positively curved 8-dimensional manifolds with symmetry.

2. Tools from geometry and topology

On the topological side, the proofs rely on arguments from equivariant index theory (Theorem 2.4) and the cohomological structure of fixed point sets of smooth actions on cohomology spheres and cohomology projective spaces (Theorem 2.5).

On the geometric side, the proofs rely on the work of Hsiang and Kleiner on positively curved 4-dimensional manifolds with symmetry (Theorem 2.3), the fixed point theorems of Berger, Synge and Weinstein for isometries [Kobayashi 1972, Chapter II, Corollary 5.7; Synge 1936; Weinstein 1968], and the following two properties of totally geodesic submanifolds, which we state for further reference:

Intersection theorem [Frankel 1961]. *Let M be a connected positively curved manifold of dimension n , and let N_1 and N_2 be totally geodesic submanifolds of dimension n_1 and n_2 , respectively. If $n_1 + n_2 \geq n$, then N_1 and N_2 intersect.*

Here, the dimension of a manifold is defined to be the maximal number occurring as the dimension of a connected component of it. Similarly, the codimension of a submanifold N of a connected manifold M is defined to be the minimal number occurring as the codimension of a connected component of N in M .

Building on the intersection theorem, Frankel [1966, page 71] observed that the inclusion of a connected totally geodesic submanifold N is 1-connected if the codimension of N in M is at most half of the dimension of M . Using a Morse theory argument, Wilking proved the following far-reaching generalization:

Connectivity theorem [Wilking 2003]. *Let M be a connected positively curved manifold, and let N_1 and N_2 be connected totally geodesic submanifolds of codimension k_1 and k_2 , respectively.*

- (1) *Then the inclusion $N_i \hookrightarrow M$ is $(n - 2k_i + 1)$ -connected.*
- (2) *If $k_1 + k_2 \leq n$ and $k_1 \leq k_2$, then the intersection $N_1 \cap N_2$ is a totally geodesic submanifold, and the inclusion of $N_1 \cap N_2$ in N_2 is $(n - (k_1 + k_2))$ -connected.*

The connectivity theorem leads to strong restrictions on the topology of positively curved manifolds with large symmetry. Wilking [2003] used this property to classify positively curved manifolds of dimension $n \geq 10$ (respectively, $n \geq 6000$)

with symmetry rank $\geq n/4 + 1$ (respectively, $\geq n/6 + 1$). For further reference, we point out the following rather elementary consequences:

Corollary 2.1. *Let M be a simply connected positively curved manifold of even dimension $n = 2m \geq 6$.*

- (1) *If M admits a totally geodesic connected submanifold N of codimension 2, then N is simply connected. Moreover, the integral cohomology of M and N is concentrated in even degrees and satisfies $H^{2i}(M; \mathbb{Z}) \cong H^{2j}(N; \mathbb{Z})$ for all $0 < 2i < n$ and $0 < 2j < n - 2$.*
- (2) *If M admits two different totally geodesic connected submanifolds N_1 and N_2 of codimension 2, then M is homeomorphic to S^n or homotopy equivalent to $\mathbb{C}P^m$.*

Proof. Part (1) follows directly from the connectivity theorem; see [Wilkling 2003]. For the convenience of the reader, we recall the argument. We begin with a more general discussion:

Let M be an oriented n -dimensional connected manifold, and $N \xhookrightarrow{i} M$ an oriented connected submanifold of codimension k . Let $u \in H^k(M; \mathbb{Z})$ be the Poincaré dual of the fundamental class of N in M . Then, the cup product with u is given by the composition

$$H^i(M; \mathbb{Z}) \xrightarrow{i^*} H^i(N; \mathbb{Z}) \xrightarrow{\cong} H_{n-k-i}(N; \mathbb{Z}) \xrightarrow{i_*} H_{n-k-i}(M; \mathbb{Z}) \xrightarrow{\cong} H^{i+k}(M; \mathbb{Z})$$

of maps, where the second and fourth map are the Poincaré isomorphism maps of N and M , respectively; see for example [Milnor and Stasheff 1974, page 137]. Now, assume that the inclusion $i : N \hookrightarrow M$ is $(n-k-l)$ -connected. Then it is straightforward to check that the homomorphism

$$\cup u : H^i(M; \mathbb{Z}) \rightarrow H^{i+k}(M; \mathbb{Z}), \quad x \mapsto x \cup u,$$

is surjective for $l \leq i < n - k - l$, and injective for $l < i \leq n - k - l$ (for all this, see [Wilkling 2003, Lemma 2.2]).

In the first part of the Corollary 2.1, we have $k = 2$. By the connectivity theorem, the inclusion $N \hookrightarrow M$ is $(n-3)$ -connected, that is, $l = 1$. Hence, the map $\cup u$ is surjective for $1 \leq i < n - 3$, and injective for $1 < i \leq n - 3$. Since M is simply connected and $N \hookrightarrow M$ is at least 3-connected, the first part follows.

Next, assume N_1 and N_2 are different totally geodesic connected submanifolds of codimension 2. The second part of the connectivity theorem implies that $N := N_1 \cap N_2 \hookrightarrow N_1$ is a totally geodesic submanifold, and that the inclusion $N := N_1 \cap N_2 \hookrightarrow N_1$ is at least 2-connected.

If the codimension of N in N_1 is 2, the map $H^0(N_1; \mathbb{Z}) \rightarrow H^2(N_1; \mathbb{Z})$ (given by multiplication with the Poincaré dual of N in N_1 for some fixed orientation of N) is

surjective by the connectivity theorem. Hence, $b_2(M) = b_2(N_1) \leq 1$. Using the first part of the Corollary 2.1, we conclude that M is an integral-cohomology sphere if $b_2(M) = 0$, or an integral-cohomology $\mathbb{C}P^m$ if $b_2(M) = 1$. If $b_2(M) = 0$, then M is actually homeomorphic to S^n by the work of [Smale 1961]; if $b_2(M) = 1$, then M is homotopy equivalent to $\mathbb{C}P^m$ since M is simply connected.

Next, assume the codimension of N in N_1 is 1. Using the connectivity theorem, we see that the inclusion $N \hookrightarrow N_1$ is $(n-3)$ -connected. Arguing along the lines above, it follows that M is homeomorphic to S^n . This completes the proof of the second part. \square

Remark 2.2. For 8-dimensional manifolds, one can show that, under the assumptions of Corollary 2.1(2), M is homeomorphic to S^8 or *homeomorphic* to $\mathbb{C}P^4$. This follows from the Sullivan's classification of homotopy complex projective spaces [1996]; see the argument in [Fang and Rong 2005, page 85].

Another important geometric ingredient in our proofs is the classification (due to Hsiang and Kleiner) up to homeomorphism of positively curved 4-dimensional manifolds with positive symmetry rank.

Theorem 2.3 [Hsiang and Kleiner 1989]. *If M is a positively curved simply connected 4-dimensional manifold with positive symmetry rank, the Euler characteristic of M is 2 or 3, and hence M is homeomorphic to S^4 or $\mathbb{C}P^2$ by Freedman's work.*

In particular, $S^2 \times S^2$ does not admit a metric of positive curvature and positive symmetry rank. Note that Theorem 1.1 gives analogous restrictions for the Euler characteristic in dimension 8.

Among the topological tools used in the proofs are the classical Lefschetz fixed point formula for the Euler characteristic, the rigidity of the signature on oriented manifolds with S^1 -action, and its applications to involutions [Hirzebruch 1968], as well as the \hat{A} -vanishing theorem of Atiyah and Hirzebruch [1970] for S^1 -actions on Spin-manifolds.

Theorem 2.4. *Let M be an oriented manifold with a smooth nontrivial S^1 -action, and $\sigma \in S^1$ be the element of order 2. If M^{S^1} and M^σ denote the fixed point manifolds with respect to the S^1 -action and the σ -action, respectively, then*

- (1) $\chi(M) = \chi(M^{S^1})$;
- (2) the equivariant signature $\text{sign}_{S^1}(M)$ is constant as a character of S^1 ;
- (3) $\text{sign}(M) = \text{sign}(M^{S^1})$, where the orientation of each component of the fixed point manifold M^{S^1} is chosen to be compatible with the complex structure of its normal bundle (induced by the S^1 -action) and the orientation of M ; and

- (4) *the signature of M is equal to the signature of a transversal self-intersection $M^\sigma \circ M^\sigma$.*

If, in addition, M is a Spin-manifold, then

- (5) *the \hat{A} -genus vanishes and*
 (6) *the connected components of M^σ are either all of codimension $\equiv 0 \pmod{4}$ (even case), or are all of codimension $\equiv 2 \pmod{4}$ (odd case).*

Proof. Part (1) is just a version of the classical Lefschetz fixed point theorem; see for example [Kobayashi 1972, Theorem 5.5]. We give here a simple argument: For any prime p , choose a triangulation of M that is equivariant with respect to the action of $\mathbb{Z}/p\mathbb{Z} \subset S^1$ on M . Then a counting argument shows that $\chi(M) \equiv \chi(M^{\mathbb{Z}/p\mathbb{Z}}) \pmod{p}$. For p large enough, this implies that $\chi(M) = \chi(M^{S^1})$ (note that the proof also applies to nonorientable manifolds).

Part (2) follows directly from the homotopy invariance of cohomology, or from the Lefschetz fixed point formula of Atiyah, Bott, Segal and Singer (see [Atiyah and Singer 1968, Theorem 6.12, page 582]), as explained for example in [Bott and Taubes 1989, page 142].

For part (3), we consider as a function in $\lambda \in \mathbb{C}$ the S^1 -equivariant signature $\text{sign}_{S^1}(M) \in R(S^1) \cong \mathbb{Z}[\lambda, \lambda^{-1}]$, and compute the limit $\lambda \rightarrow \infty$ using the Lefschetz fixed point formula; details can be found in [Hirzebruch et al. 1992, page 68].

Part (4) is a result of Hirzebruch; see [Hirzebruch 1968; Atiyah and Singer 1968, Proposition 6.15, page 583]. Hirzebruch shows that the signature of a transversal self-intersection $M^\sigma \circ M^\sigma$ is the equivariant signature $\text{sign}_{S^1}(M)$ evaluated at σ . The claim follows from the rigidity of the signature; see part (2).

Part (5) is the celebrated \hat{A} -vanishing theorem of Atiyah and Hirzebruch [1970]. Using the Lefschetz fixed point formula, they show that the S^1 -equivariant \hat{A} -genus extends to a holomorphic function on \mathbb{C} that vanishes at infinity. By a classical result of Liouville, this function has to vanish identically.

For a proof part (6), see [Atiyah and Bott 1968, Proposition 8.46, page 487]. \square

We also point out certain properties of smooth actions on cohomology spheres and cohomology projective spaces:

- Theorem 2.5.** (1) *If M is a $\mathbb{Z}/2\mathbb{Z}$ -cohomology sphere with a $\mathbb{Z}/2\mathbb{Z}$ -action, the $\mathbb{Z}/2\mathbb{Z}$ -fixed point manifold is again a $\mathbb{Z}/2\mathbb{Z}$ -cohomology sphere, or is empty.*
 (2) *If M is an integral-cohomology sphere with an S^1 -action, the S^1 -fixed point manifold is again an integral-cohomology sphere, or is empty.*
 (3) *If M is a $\mathbb{Z}/2\mathbb{Z}$ -cohomology complex projective space with a $\mathbb{Z}/2\mathbb{Z}$ -action such that the $\mathbb{Z}/2\mathbb{Z}$ -action extends to an S^1 -action, then each component of the $\mathbb{Z}/2\mathbb{Z}$ -fixed point manifold is again a $\mathbb{Z}/2\mathbb{Z}$ -cohomology complex projective space.*

- (4) *The action of an involution on a $\mathbb{Z}/2\mathbb{Z}$ -cohomology $\mathbb{C}P^2$ cannot have only isolated fixed points.*

Proof. The first two statements are well-known applications of Smith theory; see [Bredon 1972, Chapter III, Theorems 5.1 and 10.2]. The last two statements follow directly from the general theory on fixed point sets of actions on projective spaces; see [Bredon 1972, Chapter VII, Theorems 3.1 and 3.3]. \square

3. Euler characteristic

Here we prove the statements on the Euler characteristic given in Section 1.

Theorem 1.3. *Let M be a simply connected positively curved manifold of even dimension ≤ 8 . Assume S^1 acts smoothly on M . If some $\sigma \in S^1$ acts isometrically and nontrivially on M , then $\chi(M) \geq 2$.*

Proof. Since $\sigma \in S^1$ acts nontrivially, the dimension of M is positive; that is, $\dim M = 2, 4, 6$ or 8 . In dimension ≤ 4 , the theorem is true for purely topological reasons (Poincaré duality). So, assume the dimension of M is 6 or 8 . Note that M^σ is nonempty [Weinstein 1968] and the connected components of the fixed point manifold M^σ are totally geodesic submanifolds. Each is of even codimension, since σ preserves orientation.

If M^σ contains a connected component of codimension 2 , then, as pointed out in Corollary 2.1, the connectivity theorem implies that all odd Betti numbers of M vanish. Hence, $\chi(M) \geq 2$ by Poincaré duality.

So, assume $\text{codim } M^\sigma > 2$. Note that any connected component $F \subset M^\sigma$ is an S^1 -invariant totally geodesic submanifold of even dimension ≤ 4 . If F is not a point, then F inherits positive curvature from M . In this case, F or a twofold cover of F is simply connected by [Synge 1936]. Hence the Euler characteristic of any connected component of M^σ is positive. From the Lefschetz fixed point formula for the Euler characteristic (see Theorem 2.4(1)), we get

$$\chi(M) = \chi(M^{S^1}) = \chi((M^\sigma)^{S^1}) = \chi(M^\sigma) = \sum_{F \subset M^\sigma} \chi(F) \geq 1.$$

Here, equality holds if and only if M^σ is connected and $\chi(M^\sigma) = 1$. If so, M^σ must have the $\mathbb{Z}/2\mathbb{Z}$ -cohomology of a point, of $\mathbb{R}P^2$, or of $\mathbb{R}P^4$. Note that the connected components of $M^{S^1} = (M^\sigma)^{S^1}$ are orientable submanifolds of even dimension ≤ 4 . This implies that the case $\chi(M) = 1$ can only happen if M^{S^1} is a point; see [Bredon 1972, Chapter VII, Theorem 3.1].

However, a smooth S^1 -action on a closed orientable manifold M cannot have exactly one fixed point. To show this, consider the Lefschetz fixed point formula [Atiyah and Singer 1968] for the S^1 -equivariant signature $\text{sign}_{S^1}(M)$. The local contribution for $\text{sign}_{S^1}(M)$ at an isolated S^1 -fixed point extends to a meromorphic

function on \mathbb{C} , which has at least one pole on the unit circle; see for example [Bott and Taubes 1989, page 142]. Since $\text{sign}_{S^1}(M)$, being a character of S^1 , has no poles on the unit circle, the S^1 -action cannot have exactly one fixed point (more generally, this is true for any diffeomorphism of order p^l , with p an odd prime, as shown by [Atiyah and Bott 1968, Theorem 7.1]). Hence, $\chi(M) \geq 2$. \square

We remark that the proof simplifies drastically if M has positive symmetry rank; see for example [Püttmann and Searle 2002, Theorem 2]. Note that, by the result above, any metric of positive curvature on $S^3 \times S^3$, $S^2 \times S^3 \times S^3$, $S^3 \times S^5$ or $SU(3)$ must be very nonsymmetric.

In the remaining part of this section, we restrict to positively curved simply connected 8-dimensional manifolds with symmetry rank ≥ 2 , and prove the statement on the Euler characteristic given in Theorem 1.1.

Let T be a two-dimensional torus that acts isometrically and effectively on M , let $T_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ denote the 2-torus in T , and let $\sigma \in T$ be a nontrivial involution, that is, $\sigma \in T_2$ with $\sigma \neq \text{id}$.

By the fixed point theorem of [Weinstein 1968], the fixed point manifold M^σ is nonempty. Each connected component F of M^σ is a totally geodesic T -invariant submanifold of M . Since σ preserves orientation, F is of even codimension. By Berger's fixed point theorem (see [Kobayashi 1972, Chapter II, Corollary 5.7]), the torus T acts with fixed point on F .

Lemma 3.1. *F is orientable. If $\dim F \neq 6$, then F is homeomorphic to S^4 , $\mathbb{C}P^2$, S^2 or a point.*

Proof. If $\dim F = 6$, then F is simply connected by the connectivity theorem, and hence orientable.

Next, suppose that $\dim F = 4$ and T acts trivially on F . In this case, we can choose an S^1 subgroup of T such that M^{S^1} contains a 6-dimensional connected component Y , and F is a T -fixed point component of Y . In this case, by [Grove and Searle 1994, Theorem 1.2] M , Y and F are diffeomorphic to spheres or complex projective spaces,

Suppose that $\dim F = 4$ and T acts nontrivially on F . In this case, we can find a subgroup $S^1 \subset T$ and a connected component Y of M^{S^1} such that $F \cap Y$ has positive dimension. If $\dim Y = 6$, then M is diffeomorphic to S^8 or $\mathbb{C}P^4$; see [Grove and Searle 1994, Theorem 1.2], and F is a $\mathbb{Z}/2\mathbb{Z}$ -cohomology sphere or a $\mathbb{Z}/2\mathbb{Z}$ -cohomology complex projective space; see Theorem 2.5. Since the universal cover of F is homeomorphic to S^4 or $\mathbb{C}P^2$ (see Theorem 2.3), we conclude that F is simply connected and homeomorphic to S^4 or $\mathbb{C}P^2$. If $\dim Y = 4$, then Y is homeomorphic to S^4 or $\mathbb{C}P^2$ by Theorem 2.3. It follows from Theorem 2.5 that any connected component of $Y \cap F$ is a $\mathbb{Z}/2\mathbb{Z}$ -cohomology sphere or a $\mathbb{Z}/2\mathbb{Z}$ -cohomology complex projective space. Since $F \cap Y$ has positive dimension, we

have $\chi(F) \geq 2$. Applying Theorem 2.3 again, we conclude that F is homeomorphic to S^4 or $\mathbb{C}P^2$.

Finally, assume that $\dim F = 2$. We choose an S^1 subgroup of T that fixes F pointwise. Let Y be the connected component of M^{S^1} that contains F . If Y is 2-dimensional, then $F = Y$ is orientable of positive curvature and hence diffeomorphic to S^2 . If Y is 6-dimensional, then M is diffeomorphic to S^8 or $\mathbb{C}P^4$ [Grove and Searle 1994, Theorem 1.2], which implies that $F \cong S^2$. If Y is 4-dimensional, then we may assume that T acts nontrivially on Y (otherwise, we can replace Y by a 6-dimensional connected component for some other S^1 subgroup). By Theorem 2.3, Y is homeomorphic to S^4 or $\mathbb{C}P^2$. Using Theorem 2.5, we conclude that F is diffeomorphic to S^2 . \square

In the proof of Theorem 1.1, we will use the concept of the “type” of an involution at a fixed point component of the T_2 -action. This concept is defined as follows: For each nontrivial involution $\sigma \in T_2$ and each connected component X of M^{T_2} , we consider the dimension of the connected component F of M^σ containing X . For fixed X , this gives an unordered triple of dimensions, which we call the *type* of X . Note that F is orientable by Lemma 3.1. Since the action of T is orientation-preserving on F , any connected component X of M^{T_2} is a totally geodesic T -invariant submanifold of even dimension. By Berger’s fixed point theorem (see [Kobayashi 1972, Chapter II, Corollary 5.7]), X^T is nonempty.

We will use the following information on type, which can be easily verified by considering the isotropy representation at a T -fixed point in X .

- Lemma 3.2.** (1) *The type of X is $(6, 6, 4)$, $(6, 4, 2)$, $(6, 2, 0)$, $(4, 4, 4)$, $(4, 4, 0)$ or $(4, 2, 2)$.*
(2) *X is an isolated T_2 -fixed point if and only if X is of type $(6, 2, 0)$, $(4, 4, 0)$ or $(4, 2, 2)$.*

Example 3.3. Let M be the quaternionic plane

$$\mathbb{H}P^2 = \{[q_0, q_1, q_2] \mid q_i \in \mathbb{H}, (q_0, q_1, q_2) \neq 0\},$$

where $[q_0, q_1, q_2]$ denotes the orbit of (q_0, q_1, q_2) with respect to the diagonal action of nonzero quaternions on \mathbb{H}^3 from the right. Consider the action of $T = S^1 \times S^1 = \{(\lambda, \mu) \mid \lambda, \mu \in S^1\} \subset \mathbb{C} \times \mathbb{C}$ on M via

$$(\lambda, \mu)([q_0, q_1, q_2]) := [\lambda \cdot \sqrt{\mu} \cdot q_0, \sqrt{\mu} \cdot q_1, \sqrt{\mu} \cdot q_2].$$

Note that although the square root $\sqrt{\cdot}$ is only well-defined up to sign, the action is independent of this choice. Let σ_1 and σ_2 be the involutions in the first and second S^1 factor of T , and let $\sigma_3 := \sigma_1 \cdot \sigma_2$ denote the third nontrivial involution. Then, we

have the following fixed point manifolds:

$$\begin{aligned} M^{\sigma_1} &= \{[1, 0, 0]\} \cup \{[0, q_1, q_2] \mid q_i \in \mathbb{H}, (q_1, q_2) \neq 0\} \cong pt \cup S^4, \\ M^{\sigma_2} &= \{[q_0, q_1, q_2] \mid q_i \in \mathbb{C}, (q_0, q_1, q_2) \neq 0\} \cong \mathbb{C}P^2, \\ M^{\sigma_3} &= \{[j \cdot q_0, q_1, q_2] \mid q_i \in \mathbb{C}, (q_0, q_1, q_2) \neq 0\} \cong \mathbb{C}P^2, \\ M^{T_2} &= \{[1, 0, 0]\} \cup \{[0, q_1, q_2] \mid q_i \in \mathbb{C}, (q_1, q_2) \neq 0\} \cong pt \cup S^2 \end{aligned}$$

Hence, the type of $X = pt$ is $(4, 4, 0)$, and the type of $X = S^2$ is $(4, 4, 4)$.

The proof of Theorem 1.1 is based on the next three lemmas.

Lemma 3.4. *If $\dim M^\sigma = 6$ for some involution $\sigma \in T$, then the Euler characteristic of M is 2 or 5.*

Proof. Let $N \subset M^\sigma$ be the connected component of dimension 6. Note that all other connected components are isolated σ -fixed points, by the intersection theorem. In view of Corollary 2.1, the odd Betti numbers of M and N vanish, and the even Betti numbers satisfy $b_2(M) = b_4(M) = b_6(M) = b_2(N) = b_4(N)$. In particular, we have $\chi(M) - \chi(N) = b_2(M)$. By the Lefschetz fixed point formula for the Euler characteristic (see Theorem 2.4(1)), this difference is equal to the number of isolated σ -fixed points.

Suppose isolated σ -fixed points do occur (otherwise, $b_2(M) = 0$ and $\chi(M) = 2$). Using Lemma 3.2(2), we see that an isolated σ -fixed point is an isolated T_2 -fixed point, of type $(6, 2, 0)$ or $(4, 4, 0)$. If some isolated σ -fixed point is of type $(6, 2, 0)$, then M contains a 6-dimensional fixed point manifold different from N . In this case, M is homeomorphic to S^8 or homotopy equivalent to $\mathbb{C}P^4$; see Corollary 2.1(2). In particular, $\chi(M) = 2$ or 5.

We now consider the remaining case. Suppose all isolated σ -fixed points are of type $(4, 4, 0)$. We fix a nontrivial involution $\sigma_1 \in T$ different from σ , and denote by F_1 the 4-dimensional connected component of M^{σ_1} (note that F_1 is unique and has nonempty intersection with N , by the intersection theorem). Each isolated σ -fixed point, being of type $(4, 4, 0)$, is contained in F_1 , and hence the number d of isolated σ -fixed points is equal to $\chi(F_1) - \chi(F_1 \cap N)$. Since T acts nontrivially on F_1 , Theorem 2.3 tells us that $\chi(F_1) \leq 3$. Using Lemma 3.2, we see that the connected components of $(F_1 \cap N)^{T_2}$ are necessarily 2-dimensional of type $(6, 4, 2)$. It follows from Lemma 3.1 that at least one of the connected components of $(F_1 \cap N)^{T_2}$ is diffeomorphic to S^2 . Hence, $b_2(M) = d = \chi(F_1) - \chi(F_1 \cap N) \leq 1$, which in turn implies $\chi(M) = 2$ or 5. \square

Remark 3.5. Under the assumptions of Lemma 3.4, M is homeomorphic to S^8 or $\mathbb{C}P^4$. This follows from the proof above, together with the work of Smale [1961] on the high-dimensional Poincaré conjecture, and Sullivan's classification [1996] of homotopy complex projective spaces.

Lemma 3.6. *If $\dim M^\sigma = 2$ for some involution $\sigma \in T$, then $\chi(M) = 2$.*

Proof. We first note that the assumption on $\dim M^\sigma$ implies that the signature of M vanishes, by Theorem 2.4(4). Since $\chi(M) \equiv \text{sign}(M) \pmod{2}$, the Euler characteristic $\chi(M)$ is even.

By Lemma 3.4, we may assume that the dimension of the fixed point manifold M^τ is ≤ 4 for every nontrivial involution τ of T . Using Lemma 3.2, we see that every T_2 -fixed point component is an isolated fixed point of type $(4, 4, 0)$ or $(4, 2, 2)$.

Let σ_1 and σ_2 denote the nontrivial involutions distinct from σ . Without loss of generality, we may assume that $\dim M^{\sigma_1} = 4$. Let F_1 denote the 4-dimensional connected component of M^{σ_1} . Since T acts nontrivially on F_1 , the universal cover of F_1 is homeomorphic to S^4 or $\mathbb{C}P^2$ by Theorem 2.3. Since all T_2 -fixed points are isolated, the involution σ acts on F_1 with isolated fixed points. By Theorem 2.5, F_1 cannot be a cohomology $\mathbb{C}P^2$. Hence, $\chi(F_1) \leq 2$.

If $\dim M^{\sigma_2} < 4$, then any T_2 -fixed point component is contained in F_1 and hence $\chi(M) = \chi(F_1) = 2$ by Theorem 1.3. So assume M^{σ_2} contains a 4-dimensional connected component F_2 . Arguing as above, we see that $\chi(F_2) \leq 2$. Note that F_1 and F_2 intersect by the intersection theorem, and $F_1 \cap F_2$ consists of isolated σ -fixed points. Hence, $\chi(M) = \chi(F_1) + \chi(F_2) - \chi(F_1 \cap F_2) \leq 3$. Since $\chi(M)$ is even (as pointed out above) and no less than 2 (by Theorem 1.3), the lemma follows. \square

Lemma 3.7. *If $\dim M^\sigma = 0$ for some involution $\sigma \in T$, then $\chi(M) = 2$.*

Proof. The proof is very similar to that of Lemma 3.6. By Theorem 2.4(4), the signature of M vanishes. In particular, $\chi(M)$ is even.

Applying Lemma 3.2, we see that a connected component of M^{T_2} is of type $(6, 2, 0)$ or $(4, 4, 0)$. If some component has type $(6, 2, 0)$, then, since $\chi(M)$ is even, the Euler characteristic of M is equal to 2 by Lemma 3.4.

So, assume all components of M^{T_2} are of type $(4, 4, 0)$. Let $\sigma_1 \in T$ be a nontrivial involution distinct from σ , and let F_1 denote the unique 4-dimensional connected component of M^{σ_1} . Note that $M^{T_2} \subset F_1$. By Theorem 2.3, $\chi(F_1) \leq 3$. Since $\chi(M)$ is even, we get $\chi(M) = \chi(M^{T_2}) = \chi(F_1) \leq 2$. Now, the lemma follows from Theorem 1.3. \square

Proof of Theorem 1.1. Using the lemmas above, we may assume that $\dim M^\sigma = 4$ for every nontrivial involution $\sigma \in T$. In view of Lemma 3.2, every connected T_2 -fixed point component X is of type $(4, 4, 0)$, $(4, 2, 2)$ or $(4, 4, 4)$. In the first two cases, X is an isolated fixed point, whereas in the third case an inspection of the isotropy representation shows that X is of dimension two.

Let $\sigma_i \in T$ for $i = 1, 2, 3$ denote the nontrivial involutions, and let F_i denote the unique 4-dimensional connected component of M^{σ_i} . By the intersection theorem,

any two of the F_i intersect. Since T acts nontrivially on F_i , we have $\chi(F_i) \leq 3$ by Theorem 2.3. If $\chi(F_i) = 3$ for some i , that is, if F_i is homeomorphic to $\mathbb{C}P^2$, then F_i contains a T_2 -fixed point component of positive dimension (see Theorem 2.5), which is necessarily of type $(4, 4, 4)$. Hence, if none of the T_2 -fixed point components is of type $(4, 4, 4)$, then $\chi(F_i) \leq 2$ for all i , and

$$\chi(M) = \sum_i \chi(F_i) - \sum_{i < j} \chi(F_i \cap F_j) \leq 3 \cdot 2 - 3 = 3.$$

Since $\chi(M) \geq 2$ by Theorem 1.3, we are done in this case.

In the other case, the intersection of the F_i contains a 2-dimensional T_2 -fixed point component X of type $(4, 4, 4)$. It follows from Lemma 3.1 and Theorem 2.5 that X is diffeomorphic to $\mathbb{C}P^1$. Hence $\chi(M) \leq 3 \cdot 3 - 2 \cdot 2 = 5$ with equality holding if and only if each F_i is homeomorphic to $\mathbb{C}P^2$. Moreover, in the equality case, the T_2 -fixed point components distinct from X are all of type $(4, 2, 2)$, and for each σ_i the fixed point manifold M^{σ_i} is the union of F_i and a 2-dimensional sphere.

We claim that $\chi(M) \neq 4$. Suppose to the contrary that $\chi(M) = 4$. Then we may assume that at least one of the F_i (say, F_1) has Euler characteristic equal to 3. Now, $\chi(M) = 4$ implies that the fixed point manifold M^{σ_1} is the union of F_1 and an isolated fixed point q (in fact, arguing as for X , we see that any σ_1 -fixed point component of positive dimension distinct from F_1 would be diffeomorphic to $\mathbb{C}P^1$, implying that $\chi(M) > 4$). Note that q , being an isolated σ_1 -fixed point, must be of type $(4, 4, 0)$. Hence, q belongs to F_2 and F_3 . This implies that $\chi(F_2) = \chi(F_3) = 3$.

On the other hand, $F_1^{T_2}$ is the union of X and a point q' , distinct from q , which is of type $(4, 4, 0)$ or $(4, 2, 2)$. If q' has type $(4, 4, 0)$, then $\chi(F_2) \geq 4$ or $\chi(F_3) \geq 4$, which contradicts $\chi(F_2) = \chi(F_3) = 3$. If q' has type $(4, 2, 2)$, then $\chi(M^{\sigma_2}) \geq 5$ and $\chi(M^{\sigma_3}) \geq 5$, which contradicts $\chi(M) = 4$.

Hence, $\chi(M) \neq 4$. Since $\chi(M) \leq 5$ and $\chi(M) \geq 2$ (by Theorem 1.3), we get $\chi(M) = 2, 3, 5$. \square

Recall that, for an nonorientable even-dimensional manifold of positive curvature, a two-fold cover (the orientation cover) is simply connected [Synge 1936]. Hence, Theorem 1.1 implies this:

Corollary 3.8. *Let M be an nonorientable 8-dimensional manifold. If M admits a metric of positive curvature and symmetry rank ≥ 2 , then $\chi(M) = 1$.*

4. Rationally elliptic manifolds

In this section, we apply Theorem 1.1 to rationally elliptic manifolds. Recall that a closed simply connected n -dimensional manifold M is *rationally elliptic* if its rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is of finite rank. Rational ellipticity imposes strong

topological constraints. For example, Halperin has shown that the Euler characteristic of a rationally elliptic manifold is nonnegative, and that all odd Betti numbers vanish if the Euler characteristic is positive; see [1977, Theorem 1', page 175]. By work of [Friedlander and Halperin 1979, Corollary 1.3], the sum of degrees of generators of $\pi_{2*}(M) \otimes \mathbb{Q} := \bigoplus_i \pi_{2i}(M) \otimes \mathbb{Q}$ is $\leq n$,

On the other hand, the class of rationally elliptic manifolds contains some interesting families, for example, Lie groups, homogeneous spaces, biquotients and manifolds of cohomogeneity one. Also, all simply connected manifolds presently known to admit a metric of nonnegative curvature are rationally elliptic.

Using the information on the Euler characteristic given in Theorem 1.1, we obtain the following classification result for rationally elliptic manifolds. I thank Mikiya Masuda for explaining to me the properties of $\mathbb{Z}/2\mathbb{Z}$ -cohomology $\mathbb{C}P^4$'s with T -action that are used in the proof.

Theorem 1.2. *Suppose M is a simply connected positively curved 8-dimensional manifold of symmetry rank ≥ 2 .*

- (1) *If M is rationally elliptic, M has the rational cohomology ring and the rational homotopy type of a rank-one symmetric space, that is, of S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$.*
- (2) *If M is rationally elliptic and $H^*(M; \mathbb{Z})$ is torsion-free, then M is homeomorphic to S^8 , diffeomorphic to $\mathbb{H}P^2$, or tangentially equivalent to $\mathbb{C}P^4$.*

Proof. (1) We will show that M has the rational cohomology ring of S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$. From this, one easily deduces that M is formal. Hence, M has the same rational homotopy type as S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$.

By Theorem 1.1, the Euler characteristic of M is 2, 3 or 5. Since M is rationally elliptic, the rational cohomology ring of M is concentrated in even degrees; see [Halperin 1977, Theorem 1', page 175]. If $\chi(M) = 2$ or $\chi(M) = 3$, then M has the rational cohomology ring of S^8 or $\mathbb{H}P^2$, respectively; this follows directly from Poincaré duality. If $\chi(M) = 5$, then the rational cohomology ring of M belongs to one of the following three cases:

- (1) $b_2(M) = 0$ and $b_4(M) = 3$;
- (2) $b_2(M) = b_4(M) = 1$, and $x^2 = 0$ for a generator x of $H^2(M; \mathbb{Q})$;
- (3) $b_2(M) = b_4(M) = 1$, and $x^2 \neq 0$ for a generator x of $H^2(M; \mathbb{Q})$.

According to [Friedlander and Halperin 1979, Corollary 1.3], the sum of degrees of generators of $\pi_{2*}(M) \otimes \mathbb{Q}$ is ≤ 8 . This excludes the first two cases. In fact, in the first case the minimal model of M must have three generators of degree 4, and in the second case the minimal model of M must have generators of degree 2, 4 and 6. So, only the third case can occur; that is, $b_2(M) = b_4(M) = 1$ and $x^2 \neq 0$ for a generator x of $H^2(M; \mathbb{Q})$. By Poincaré duality, M has the rational cohomology ring of $\mathbb{C}P^4$.

(2) Now assume that M is rationally elliptic and $H^*(M; \mathbb{Z})$ is torsion-free. If $\chi(M) = 2$, then M is rationally a sphere by (1). Since $H^*(M; \mathbb{Z})$ is torsion-free, in this case M is an integral cohomology S^8 . Being simply connected, M is a homotopy sphere and hence homeomorphic to S^8 by [Smale 1961].

If $\chi(M) = 3$, then M is an integral-cohomology $\mathbb{H}P^2$ by Poincaré duality. We choose the orientation of M for which the signature of M is one. Note that M is 3-connected, and hence M is a Spin manifold. By the Atiyah–Hirzebruch vanishing theorem (see Theorem 2.4(5)), the \hat{A} -genus of M vanishes (this follows also from the result of Lichnerowicz [1963] on the vanishing of the \hat{A} -genus for Spin manifolds with positive scalar curvature). Since in dimension eight the space of Pontryagin numbers is spanned by the \hat{A} -genus and the signature, the manifolds M and $\mathbb{H}P^2$ have the same Pontryagin numbers.

From [Smale 1962, Theorem 6.3], it follows that M admits a Morse function with three critical points. The classification results of Eells and Kuiper [1962, theorem on page 216] for these manifolds imply that the diffeomorphism type of M is determined by its Pontryagin numbers, up to connected sums with homotopy spheres. In particular, M and $\mathbb{H}P^2$ are homeomorphic and diffeomorphic, up to connected sums, with a homotopy sphere. Recently, Kramer and Stolz used Kreck’s surgery theory to show that the action of the group of homotopy spheres on $\mathbb{H}P^2$ via connected sum is trivial; see [2007, Theorem A]. Hence, M and $\mathbb{H}P^2$ are diffeomorphic.

Finally, we consider the case when $\chi(M) = 5$. From (1), we know already that M is a rational cohomology $\mathbb{C}P^4$. Below, we will show that M is in fact an integral cohomology $\mathbb{C}P^4$. Assuming this for the moment, we now prove that M is tangentially equivalent to $\mathbb{C}P^4$; that is, there exists a homotopy equivalence $f : M \rightarrow \mathbb{C}P^4$ such that $f^*(T\mathbb{C}P^4)$ and TM are stably isomorphic.

We first note that M and $\mathbb{C}P^4$ are homotopy equivalent since M is an integral cohomology $\mathbb{C}P^4$ and simply connected. In the early 1970s, Petrie conjectured that a smooth S^1 -manifold N that is homotopy equivalent to $\mathbb{C}P^n$ has the same Pontryagin classes as $\mathbb{C}P^n$; that is, the total Pontryagin class $p(\mathbb{C}P^n)$ is mapped to $p(N)$ under a homotopy equivalence $N \rightarrow \mathbb{C}P^n$. Petrie’s conjecture holds for $n = 4$ [James 1985]. Hence, the homotopy equivalence $f : M \rightarrow \mathbb{C}P^4$ maps the Pontryagin classes of $\mathbb{C}P^4$ to the Pontryagin classes of M . In this situation M and $\mathbb{C}P^4$ are tangentially equivalent [Petrie 1973, page 140]. We sketch the argument: Since $H^*(M; \mathbb{Z})$ is torsion-free, the condition on the Pontryagin classes implies that the complexified vector bundles $TM \otimes \mathbb{C}$ and $f^*(T\mathbb{C}P^4) \otimes \mathbb{C}$ agree in complex K -theory. When M is a homotopy complex projective space of complex dimension $\not\equiv 1 \pmod{4}$, the complexification map $KO(M) \rightarrow K(M)$ is injective. Hence, the real vector bundles TM and $f^*(T\mathbb{C}P^4)$ are stably isomorphic (in fact, they are isomorphic, since they have up to sign the same Euler class).

To complete the proof, we need to show that M is an integral cohomology $\mathbb{C}P^4$. We choose the orientation of M for which the signature of M is 1. Since $H^*(M; \mathbb{Z})$ is torsion-free, it follows from Poincaré duality that M is a twisted $\mathbb{C}P^4$; that is, there are generators $x_{2i} \in H^{2i}(M; \mathbb{Z})$ for $i = 1, 2, 3, 4$, and an integer $m > 0$, such that x_8 is the preferred generator with respect to the chosen orientation, and

$$x_2 \cdot x_6 = x_4^2 = x_8, \quad x_2^2 = m \cdot x_4, \quad x_2 \cdot x_4 = m \cdot x_6.$$

Let T denote the 2-dimensional torus that acts isometrically and effectively on M , let $T_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ denote the 2-torus in T , and let $\sigma \in T_2$ be a non-trivial involution. By Theorem 2.4(4), the codimension of M^σ is 2 or 4. If the codimension of M^σ is 2, then M is an integral cohomology $\mathbb{C}P^4$; that is, $m = 1$. This follows directly from the proof of Corollary 2.1.

So, we are left with the case when $\dim M^\sigma = 4$ for every nontrivial involution $\sigma \in T$. Let $\sigma_i \in T$ for $i = 1, 2, 3$ denote the nontrivial involutions. From the discussion in the previous section (see the proof of Theorem 1.1), we recall the following facts: For each σ_i , the fixed point manifold M^{σ_i} is the union of a 4-dimensional connected component F_i and a 2-dimensional sphere S_i^2 . Moreover, F_i is homeomorphic to $\mathbb{C}P^2$, the three F_i intersect in a 2-dimensional T_2 -fixed point component X of type $(4, 4, 4)$, and X is diffeomorphic to $\mathbb{C}P^1$. We choose the orientation of F_i for which the signature of F_i is 1.

Therefore the normal bundle of X in M is isomorphic as a real vector bundle to three copies of the Hopf bundle. In particular, the normal bundle is not spin and the restriction of the second Stiefel–Whitney class of M to X is nonzero. This shows that the restriction homomorphism $f_i^* : H^2(M; \mathbb{Z}) \rightarrow H^2(F_i; \mathbb{Z})$ induced by the inclusion $f_i : F_i \hookrightarrow M$ maps x_2 to an odd multiple of a generator x of $H^2(F_i; \mathbb{Z}) \cong H^2(\mathbb{C}P^2; \mathbb{Z})$; that is, $f_i^*(x_2) = a \cdot x$ with a odd. It follows from the Lefschetz fixed point formula for the equivariant signature (see [Atiyah and Singer 1968]) and Theorem 2.4(4) that the Euler class $e(v_i)$ of the normal bundle v_i of $F_i \hookrightarrow M$ is equal to the preferred generator $x^2 \in H^4(F_i; \mathbb{Z})$. Hence, $f_i^*(x_4) = e(v_i) = x^2$. By restricting the identity $x_2^2 = m \cdot x_4$ to F_i , we see that $m = a^2$ is an odd square. In particular, M is a $\mathbb{Z}/2\mathbb{Z}$ -cohomology $\mathbb{C}P^4$.

Next, we recall from the proof of Theorem 1.1 that M^{T_2} is the union of X and three points p_i for $i = 1, 2, 3$, with $p_i \in F_i$. We fix a lift $\xi \in H_T^2(M; \mathbb{Z})$ of x_2 and denote by w_i the restriction of ξ to p_i . By the structure theorem for rational cohomology complex projective spaces [Hsiang 1975, Theorem VI.I, page 106], the kernel of the restriction homomorphism $H_T^*(M; \mathbb{Q}) \rightarrow H_T^*(p_i; \mathbb{Q})$ is generated by $(\xi - w_i)$.

The following argument, due to Masuda, shows that m is equal to 1. Let $f_{i!} : H_T^*(F_i; \mathbb{Z}) \rightarrow H_T^{*+4}(M; \mathbb{Z})$ denote the equivariant Gysin map (or push-forward)

induced by $f_i : F_i \hookrightarrow M$. For properties of the Gysin map, see for example [Masuda 1981, pages 132–133].

Claim 1: $f_{i!}(1) = (1/a^2) (\xi - w_j) \cdot (\xi - w_k)$, where $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Since p_j and p_k are not in F_i , the restriction of $f_{i!}(1)$ to each of these points must vanish. Hence, $f_{i!}(1)$ is divisible by $(\xi - w_j) \cdot (\xi - w_k)$. Comparing degrees, we find that $f_{i!}(1) = c \cdot (\xi - w_j) \cdot (\xi - w_k)$ for some rational constant c . By restricting this identity to ordinary cohomology, we obtain $x_4 = c \cdot x_2^2$, and hence $c = 1/m = 1/a^2$. \checkmark

Claim 2: $w_i - w_j$ is divisible by a^2 in $H^2(BT; \mathbb{Z})$.

Proof. From the first claim, we deduce that

$$f_{i!}(1) - f_{j!}(1) = \frac{1}{a^2} ((w_i - w_j) \cdot \xi - (w_i - w_j) \cdot w_k).$$

Since $(1, \xi)$ is part of a basis for the free $H^*(BT; \mathbb{Z})$ -module $H_T^*(M; \mathbb{Z})$, it follows that $(w_i - w_j)$ is divisible by a^2 . \checkmark

Recall that any two T -fixed points p_i and p_j with $i \neq j$ are contained in a T -invariant 2-dimensional sphere S_k^2 that is fixed pointwise by the involution σ_k (where $\{i, j, k\} = \{1, 2, 3\}$). Consider the T -action on S_k^2 , and let $m_{ij} \in H^2(BT; \mathbb{Z})$ denote the weight of the tangential T -representation at p_i . Note that m_{ij} is only defined up to sign and that $m_{ij} = \pm m_{ji}$ (here and in the following the notation $\alpha = \pm\beta$ is a shortcut for $\alpha = \beta$ or $\alpha = -\beta$).

Claim 3: $\pm a \cdot m_{ij} = w_i - w_j$, and m_{ij} is divisible by a .

Proof. First, note that the normal bundle of F_i in M restricted to p_i has weights $\pm m_{ij}$ and $\pm m_{ik}$, where $\{i, j, k\} = \{1, 2, 3\}$. Hence, by restricting the identity given in the first claim to the p_i , we obtain the following identities in the polynomial ring $H^*(BT; \mathbb{Z})$:

$$\pm a^2 \cdot m_{12} \cdot m_{13} = (w_1 - w_2) \cdot (w_1 - w_3),$$

$$\pm a^2 \cdot m_{23} \cdot m_{21} = (w_2 - w_3) \cdot (w_2 - w_1),$$

$$\pm a^2 \cdot m_{31} \cdot m_{32} = (w_3 - w_1) \cdot (w_3 - w_2).$$

Since m_{ij} and m_{ji} agree up to sign, $\pm a \cdot m_{ij} = w_i - w_j$. Using Claim 2, m_{ij} is divisible by a . \checkmark

Suppose $X \subset M^T$. We can choose a subgroup S^1 of T such that F_1 is fixed pointwise by S^1 . By Claim 3, the subgroup $\mathbb{Z}/a\mathbb{Z}$ of S^1 acts trivially on M . Since the T -action is effective, we get $a = \pm 1$.

If $X \not\subset M^T$, then M^T consists of five isolated points $\{p, p', p_1, p_2, p_3\}$, where $p, p' \in X$ and $p_i \in F_i$. Recall that F_i is homeomorphic to $\mathbb{C}P^2$. In particular, there is a unique T -invariant 2-dimensional sphere in F_i that contains p and p_i .

Consider the T -action on this sphere and let $m_i \in H^2(BT; \mathbb{Z})$ denote the weight of the tangential T -representation at p_i . Similarly, let $m'_i \in H^2(BT; \mathbb{Z})$ denote the weight of the tangential T -representation that corresponds to p' and p_i . Note that m_i and m'_i are only defined up to sign.

Let w and w' denote the restriction of ξ to p and p' , respectively. Since any torus action on a homotopy $\mathbb{C}P^2$ is of linear type and x_2 restricted to F_i is equal to a times a generator of $H^2(F_i; \mathbb{Z})$, we get

$$(1) \quad \pm a \cdot m_i = w_i - w \quad \text{and} \quad \pm a \cdot m'_i = w_i - w'.$$

Now consider the circle subgroup $S \hookrightarrow T$ defined by $w = w'$. Since T acts linearly on F_i , the fixed point set F_i^S is the union of X and p_i .

For $u \in H^*(BT; \mathbb{Z})$, let \bar{u} denote the restriction of $u \in H^*(BT; \mathbb{Z})$ to $H^*(BS; \mathbb{Z})$. Since $\bar{w} = \bar{w}'$, it follows from equations (1) that \bar{m}_i and \bar{m}'_i agree up to sign. Since T acts effectively on M and the weights m_{ij} are divisible by a (see Claim 3), \bar{m}_i and \bar{m}'_i are both coprime to a .

Suppose $m = a^2$ is not equal to 1. Consider the action of $\mathbb{Z}/a\mathbb{Z} \subset S$. Since \bar{m}_{ij} is divisible by a , and \bar{m}_i and \bar{m}'_i are both coprime to a , the connected $\mathbb{Z}/a\mathbb{Z}$ -fixed point component Z that contains p_1 contains both p_2 and p_3 but does not contain X . Hence, the S -equivariant Gysin map $f_! : H_S^*(Z; \mathbb{Z}) \rightarrow H_S^{*+4}(M; \mathbb{Z})$ induced by the inclusion $f : Z \hookrightarrow M$ vanishes after restricting to X . Applying the structure theorem for rational cohomology complex projective spaces [Hsiang 1975, Theorem VI.I, page 106] to M and Z , we find that $f_!(1)$ is divisible by $(\bar{\xi} - \bar{w})^2$. Comparing degrees, we see that there is a rational constant C such that $f_!(1) = C \cdot (\bar{\xi} - \bar{w})^2$. By restricting this identity to the T -fixed point p_i , we obtain $\pm \bar{m}_i \cdot \bar{m}'_i = C \cdot (\bar{w}_i - \bar{w})^2$. Using (1), we get $C = \pm 1/a^2$. Hence $(1/a^2) \cdot (\bar{\xi} - \bar{w})^2 \in H_S^4(M; \mathbb{Z})$. Recall from Claim 1 that $(1/a^2) \cdot (\bar{\xi} - \bar{w}_1) \cdot (\bar{\xi} - \bar{w}_2)$ is also in $H_S^4(M; \mathbb{Z})$. Taking the difference of these two elements, we obtain

$$\frac{1}{a^2} ((\bar{w}_2 - \bar{w}_1 + 2 \cdot (\bar{w}_1 - \bar{w})) \cdot \bar{\xi} + (\bar{w}^2 - \bar{w}_1 \cdot \bar{w}_2)) \in H_S^4(M; \mathbb{Z}).$$

Since $(1, \bar{\xi})$ is part of a basis of the free $H^*(BS; \mathbb{Z})$ -module $H_S^*(M; \mathbb{Z})$, it follows that $(\bar{w}_2 - \bar{w}_1 + 2 \cdot (\bar{w}_1 - \bar{w}))$ is divisible by a^2 . Now $(\bar{w}_2 - \bar{w}_1)$ is divisible by a^2 by Claim 3, and a is odd. Hence $(\bar{w}_1 - \bar{w})$ is divisible by a^2 . Using (1), we deduce that a divides \bar{m}_i . This contradicts $a^2 \neq 1$ since \bar{m}_i is coprime to a . Hence $m = a^2$ is equal to 1. \square

We close this section with an application to biquotients. Recall that any biquotient of a compact connected Lie group G is rationally elliptic and comes with a metric of nonnegative curvature induced from a biinvariant metric on G .

Corollary 4.1. *A simply connected 8-dimensional biquotient of positive curvature and symmetry rank ≥ 2 is diffeomorphic to S^8 , $\mathbb{C}P^4$, $\mathbb{H}P^2$ or $G_2/\text{SO}(4)$.*

Proof. According to Theorem 1.2, a simply connected positively curved 8-dimensional biquotient with symmetry rank ≥ 2 is rationally singly generated. Rationally singly generated biquotients were classified by Kapovitch and Ziller in [2004, Theorem A]. In dimension 8, these are the homogeneous spaces given. \square

Remark 4.2. From the classification of homogeneous positively curved manifolds, it follows that $G_2/\mathrm{SO}(4)$ does not admit a homogeneous metric of positive curvature. We do not know whether $G_2/\mathrm{SO}(4)$ admits a positively curved metric with symmetry rank two.

5. Bordism type

In this section, we consider the bordism type of closed simply connected Riemannian 8-manifolds with positive curvature. We determine the Spin-bordism type and comment on the oriented bordism type for manifolds with symmetry rank ≥ 2 .

In dimension eight, the Spin-bordism group Ω_8^{Spin} is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and the Spin-bordism type is detected by Pontryagin numbers; see [Milnor 1963, page 201]. Since in this dimension the Pontryagin numbers are uniquely determined by the \hat{A} -genus and the signature, it suffices to compute these numerical invariants.

Proposition 5.1. *Let M be an 8-dimensional Spin-manifold. If M admits a metric of positive curvature and symmetry rank ≥ 2 , then $\chi(M) = 2$ or 3 , and M is Spin-bordant to S^8 or $\pm\mathbb{H}P^2$.*

Proof. By the Atiyah–Hirzebruch vanishing theorem (see Theorem 2.4(5)) or alternatively by the theorem of Lichnerowicz [1963], the \hat{A} -genus of M vanishes. Hence, it suffices to show that the signature of M is equal to the signature of S^8 or $\pm\mathbb{H}P^2$; that is, we want to show that $|\mathrm{sign}(M)| \leq 1$. If some isometry in T acts with a fixed point component of codimension 2, this follows directly from Corollary 2.1 and Theorem 1.1 (in fact M is bordant to S^8 since an integral-cohomology $\mathbb{C}P^4$ is never spin). So assume that for any $\tau \in T$ the fixed point manifold

(*) M^τ has no fixed point component of codimension 2.

Recall from Theorem 1.1 that $\chi(M) = 2, 3$ or 5 . If $\chi(M) = 2$, $\mathrm{sign}(M) = 0$. To see this consider a subgroup $S^1 \subset T$ of positive fixed point dimension (that is, $\dim M^{S^1} > 0$). By condition (*), any connected component of M^{S^1} is of dimension no more than 4. It follows from Theorem 2.4(1) that M^{S^1} is S^2 or an integral cohomology S^4 . Since the signature of M is the sum of the signatures of the connected components of M^{S^1} , the signature of M vanishes; see Theorem 2.4(3).

If $\chi(M) = 3$, then $|\mathrm{sign}(M)| = 1$. The reasoning is similar to the one above. Choose an S^1 subgroup of T such that the fixed point manifold M^{S^1} has a connected

component F of dimension 2 or 4. Any such F is simply connected by [Synge 1936] and satisfies $|\text{sign}(F)| \leq \chi(F) - 2$. Since $\chi(M) = \chi(M^{S^1}) = 3$ and the signature of M is the sum of the signatures of the connected components of M^{S^1} (taken with the appropriate orientation), we get $|\text{sign}(M)| = 1$.

Finally, we claim that the case $\chi(M) = 5$ cannot occur. First note that in this case the signature of M is odd since $\text{sign}(M) \equiv \chi(M) \pmod{2}$. Let σ_i for $i \in \{1, 2, 3\}$ denote the three nontrivial involutions in T . It follows from condition (*) and Theorem 2.4(4) that M^{σ_i} contains a 4-dimensional connected component F_i (which is unique by the intersection theorem). By Lemma 3.1, F_i is homeomorphic to S^4 or $\mathbb{C}P^2$. Since M is spin, the action of σ_i must be even; see Theorem 2.4(6). Hence M^{σ_i} is the union of F_i and isolated σ_i -fixed points. Using Lemma 3.2, we see that each connected component of M^{T_2} has type $(4, 4, 4)$ or $(4, 4, 0)$. In particular, any T_2 -fixed point component is contained in some F_i .

To derive a contradiction, we will compute the Euler characteristic. Consider the case when for one of the F_i , say F_1 , the Euler characteristic is equal to 3 and hence F_i is homeomorphic to $\mathbb{C}P^2$. Since σ_2 acts nontrivially on F_1 , we get $F_1^{\sigma_2} = F_1^{\sigma_3} = S^2 \cup \{pt\}$, where S^2 and pt are connected components of M^{T_2} of type $(4, 4, 4)$ and $(4, 4, 0)$, respectively. Hence, one of the other F_i , say F_2 , contains $S^2 \cup \{pt\}$. This leads to the contradiction

$$5 = \chi(M) = \chi(M^{T_2}) = \chi(F_2) + \chi(F_3) - \chi(F_2 \cap F_3) \leq 3 + 3 - 2 = 4.$$

So, assume $\chi(F_i) = 2$ for all i . Note that M^{T_2} cannot contain a connected component of type $(4, 4, 4)$ since otherwise $\chi(M) = 2$ by a computation similar to the one above. Hence, each connected component of M^{T_2} is of type $(4, 4, 0)$. In particular, the F_i intersect pairwise in different points, which gives the contradiction

$$5 = \chi(M) = \sum_i \chi(F_i) - \sum_{i < j} \chi(F_i \cap F_j) + \chi(F_1 \cap F_2 \cap F_3) \leq 6 - 3 = 3.$$

In conclusion, we have shown that $\chi(M) \neq 5$, which completes the proof of the theorem. \square

A more natural and apparently more difficult problem is to understand the *oriented* bordism type of an 8-dimensional positively curved manifold M with symmetry rank ≥ 2 .

Wall [1960] has shown that the only torsion in the oriented bordism ring Ω_*^{SO} is 2-torsion, and that the oriented bordism type of a manifold is determined by Pontryagin and Stiefel–Whitney numbers. In dimension eight, Ω_8^{SO} is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Hence, in this dimension the oriented bordism type of an oriented manifold is determined by its Pontryagin numbers.

In contrast to the case of Spin-manifolds, the \hat{A} -genus does not have to vanish on positively curved 8-dimensional oriented manifolds with symmetry (consider for

example $\mathbb{C}P^4$). This makes the problem of determining the oriented bordism type more difficult. One way to attack this problem is to prove the stronger statement that for some orientation of M and some $S^1 \subset T$, the S^1 -action has locally the same S^1 -geometry as a suitable chosen S^1 -action on one of the symmetric spaces S^8 , $\mathbb{H}P^2$ or $\mathbb{C}P^4$ (that two S^1 -manifolds have the same local S^1 -geometry just means that there exists an equivariant orientation-preserving diffeomorphism between the normal bundles of the S^1 -fixed point manifolds). Once this has been accomplished, one can glue the complements of the normal bundles together to get a new manifold W , with fixed point free S^1 -action, that is bordant to the difference of M and the symmetric space in question. As observed by Bott [1967], all Pontryagin numbers of a manifold with fixed point-free S^1 -action vanish and hence W is rationally zero-bordant. Since the oriented bordism ring has no torsion in degree 8, the manifold M is bordant to the symmetric space in question.

This line of attack can be applied successfully at least if $\chi(M) \neq 5$. Details will appear elsewhere.

It is interesting to compare the results above with [Dessai and Tuschmann 2007], in which it is shown that there exists an infinite sequence of closed simply connected Riemannian 8-manifolds with *nonnegative* curvature and mutually distinct oriented bordism type.

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ANAND DESSAI
 DÉPARTEMENT DE MATHÉMATIQUES
 CHEMIN DU MUSÉE 23
 FACULTÉ DES SCIENCES
 UNIVERSITÉ DE FRIBOURG, PÉROLLES
 1700 FRIBOURG
 SWITZERLAND
 anand.dessai@unifr.ch
 http://homeweb1.unifr.ch/dessaia/pub/