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Light propagation in media with highly nonlinear response: an analytical study

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Abstract

The problem of light propagation in highly nonlinear media is studied with the help of a recently introduced systematic approach to the analytical solution of equations of nonlinear optics (2008 Phys. Rev. A 78 021806(R)). Numerous particular cases of media exhibiting hight order nonlinear refractive indexes are considered. We obtain analytical expressions for determining the self-focusing position and new exact expression for calculating the filament intensity. The constructed solutions allowed to revise a so-called self-focusing scaling law, i.e. the functional dependence of the self-focusing position on the initial light peak intensity. It was demonstrated that this dependence is governed by the form of the nonlinear refractive index and not by the laser beam shape at the boundary.

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1. Introduction

Accurate theoretical predictions of the laser intensity distribution in materials are fundamental importance for a large number of applications in science and technology. Among them one could mention, for example, the effect of filamentation in the atmosphere which is used for pollution and weather monitoring, attosecond pulse generation, THz irradiation generation, material microprocessing, etc. (see e.g. reviews [18, 11] and Refs. therin).

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The problem of light propagation in nonlinear media has been under detailed theoretical investigation since the early 1960s. Due to the moderate light intensity available at that time, the material response could be modeled by a linear dependence of the refractive index on the laser beam intensity, $n(I) = n_0 + n_2 I$ [21, 29, 2, 35].

First, for this form of the refractive index exact analytical solutions under the geometrical optics approximation were obtained by Akhmanov *et al* (see Ref. [1]). They demonstrated that for both cases hyperbolic-secant initial beam in (1+1) dimensions and parabolic beam in (1+2) dimensions, the solutions exhibited the beam collapse at a certain distance. This classical result is presented in reviews Refs. [2, 37] as well.

Later, Zakharov and Shabat in Ref. [45] studied the problem in (1+1) dimensions avoiding the geometrical optics approximation and demonstrated exitance of stable soliton solutions provided by the collapse arrest by of the hight order derivatives term. They discussed transition to the semi-classical (geometrical optics) approximation and appearance of the beam collapse under this limit.

This problem for Kerr nonlinearity was also studied analytically by Kovalev for various initial beam profiles in (1+1) dimensions under the geometrical optics approximation in Refs. [33], and for an arbitrary beam profile in (1+2) dimensions in Refs. [32, 31]. For both cases (1+1) and (1+2) dimensions analytical expressions for the beam self-focusing position were obtained, and in Refs. [32, 31] an influence of the diffraction was discussed.

Nowadays, modern experimental pulse laser facilities allow one to achieve electric field intensity comparable with the intensity of the atomic field [15, 20, 18, 11]. For a theoretical description of such experiments, the linear approximation to the function $n(I)$ becomes insufficient, and more complicated forms of the refraction index must be considered [27, 12, 14]. As a result, the problem becomes rather complicated to be treated analytically. Therefore, this is not surprisingly that the theoretical results obtained so far were mainly based on extensive numerical simulations. In particular, self-guiding versus collapse in air [7, 8], filament formation in fused silica [38, 41, 19], splitting of a filament into several ones [10], their stability and features of their interactions [42, 44] were investigated. In general, such numerical simulations are very time consuming and sometimes require outstanding computational facilities [16]. Therefore, analytical expressions capable to accurately predict a global beam behavior (exitance of collapse or its arrest, the collapse position etc.) in case of a complicated nonlinearity are very desirable.

Now the most widely used analytical method for this goal is known as the variational approach [4] with its different corrections (see e.g. [27, 28, 17, 8, 5]). Its basic assumption is that the beam profile (usually assumed as a Gaussian) remains unchanged upon propagation. This assumption, however, contradicts to results of analytical studies [33, 32, 24] numerical simulations [8] and experiments [3].

In the present paper we search for approximate analytical solutions of the problem of light propagation in nonlinear media, avoiding any assumptions on the form of the beam during propagation and any restrictions on the functional dependence of the refraction index on the light intensity. Previously, in Refs. [39, 40] we have already demonstrated that for numerous experimental conditions the problem can be studied under the geometrical optics approximation. In Ref. [39] approximate solutions for arbitrary refractive index were constructed. A more detailed study allowed to improve the accuracy of the result in Ref. [40] and to get exact expressions for the nonlinear self-focusing positions and the light intensity in a single filament. In that paper, a new general approach to solutions construction was formulated and the major attention through the applications was payed to the case of the cubic-quintic form of the refractive index.

In the present paper, different additional forms of $n(I)$ are studied systematically. Firstly, we present in detail our approach to the construction of the analytical solutions for an arbitrary nonlinear refractive index avoiding artificial assumptions on the fixed beam shape. Then, we consider different examples and present solutions for particular forms of $n(I)$. A special attention is given to the case of power, saturating and two term nonlinearities. For each form of $n(I)$ explicit of implicit analytical expressions for the beam self-focusing position $z_{\rm sf}$ are obtained. It is demonstrated that the functional dependence of the self-focusing position on the initial light peak intensity governed by the form of the nonlinear refractive index and not by the laser beam shape at the boundary.

The paper is organized as follows: in section II we start from the generalized nonlinear Schrödinger equation and derive its semiclassical approximation, the eikonal equations. We formulate a general systematic approach to the construction of the approximate analytical solutions for eikonal equations in (1+1) dimensions for arbitrary refractive index. Further, in section III, the suggested approach is applied to several particular forms of nonlinear media response. We demonstrate that i) for Kerr refractive index, the constructed solution coincides with the result of Refs. [33, 37] obtained on the basis of the renormalization group symmetry analysis; ii) the obtained solutions are exact on the beam axis and, in such a way, provide exact expressions for the positions of the Gaussian beam self-focusing of light canal. In this section we derive a new general expression relating the refractive index to the singe filament intensity. In section IV approximate solutions for Kerr nonlinearty and arbitrary initial beam profile are presented and investigated. In section V approximate solutions for the eikonal equation in (1+2) dimensions and arbitrary refractive index are constructed. As a particular case, Kerr and two-term refractive index are considered and comparison with results of numerical simulations are discussed. In the last section our results are summarized.

2. Model equations

The basic mathematical model for the laser beam propagation in media is the nonlinear Schrödinger equation [15]:

$$i\partial_z \mathcal{E} + \frac{1}{2k_0}\nabla_\perp^2 \mathcal{E} + k_0 n(|\mathcal{E}|^2)\mathcal{E} = 0. \tag{1}$$

where $\mathcal{E}$ is the slowly-varying envelope of the electric field, z is the propagation length, k_0 is the wave number $k_0 = n_0\omega_0/c$, ω_0 is the carrier frequency of the laser irradiation and c is the velocity of light. In case of (1+1) dimension, Eq. (1) describes the cw laser beam propagation in a planar geometry, or propagation of the laser pulse in a fiber where radial intensity distribution is not changed. In the first case the second spatial variable represents the space, and the Laplacian $\nabla_\perp^2$ describes wave diffraction. In the second case the Laplasian describes the dispersion in the plan transverse to the propagation axis z. $n = n(|\mathcal{E}|^2)$ is the index of nonlinear refraction which, in general, includes all nonlinear processes: nonlinear polarization; multiphoton, tunnel ionization etc. As a result, $n(I)$ can become a quite complicated function of the intensity.

Let us consider propagation of a planar laser beam in space. Case of the pulse evolution along the fiber can be studied formally in a similar way. We represent electric field $\mathcal{E}$ in the eikonal form: $\mathcal{E} = \sqrt{I}\exp(ik_0 S)$. Then, starting from (1), after some algebraic manipulations we obtain [32]

$$\partial_z S = -\frac{1}{2}(\partial_x S)^2 + n(I) + \frac{1}{2k_0^2}\left(\frac{x^{1-\nu}}{\sqrt{I}}\partial_x(x^{\nu-1}\partial_x\sqrt{I})\right), \tag{2}$$

$$\partial_z I = -\partial_x(I\partial_x S) - (\nu - 1)\frac{I\partial_x S}{x}, \tag{3}$$

where ν correspond to the dimensionality of the problem.

Let us differentiate the first equation with respect to x and introduce a new variable $v \equiv \partial_x S$. Then the refractive index term gives $\partial_x n(I) \equiv \varphi\partial_x I$, where

$$\varphi(I) \equiv \partial_I n(I). \tag{4}$$

For convenience, we introduce dimensionless variables $\tilde{I} \equiv I/I_0$, $\tilde{x} \equiv x/w_{\text{in}}$, $\tilde{z} \equiv z/w_{\text{in}}$, $\alpha \equiv n_2 I_0$ where n_2 is a part of refractive index corresponding to the Kerr nonlinearity. Further, all other functions in the refractive index are rescaled into this value. $\tilde{\varphi}(\tilde{I})$ is a function of the dimensionless intensity. We use dimensionless variables throughout the rest of the paper, omitting the tilde for the sake of simplicity.

Thus, Eqs. (2-3) turn into equations

$$\partial_z v + v\partial_x v - \alpha\varphi\partial_x I - \theta\partial_x\left(\frac{x^{1-\nu}}{\sqrt{I}}\partial_x(x^{\nu-1}\partial_x\sqrt{I})\right) = 0, \tag{5}$$

$$\partial_z I + v\partial_x I + I\partial_x v + (\nu - 1)\frac{vI}{x} = 0. \tag{6}$$

For the diffraction of the plane wave, $\theta = (2k_0^2 w_0^2)^{-1}$, for dispersion $\theta \sim (2\omega_0^2 t_p^2)^{-1}$, where t_p is the initial pulse duration.

Typical values of the parameters in modern experiments are: $\lambda = 800$ nm, initial beam radius $w_0 = 1$ mm. Using these values, we get $\theta \simeq 8 \times 10^{-8}$. This means that the hight-order derivatives term is negligible and that the geometrical optics approximation should give a reasonable description of the problem of intense laser beam propagation in nonlinear media for several experimental conditions. If this term describes the dispersions, a validity of the semi-classical approximation is discussed in detail in Ref. [43].

Thus, we have the following boundary value problem in (1+1) dimensions

$$\partial_z v + v\partial_x v - \alpha\varphi\partial_x I = 0, \tag{7}$$

$$\partial_z I + v\partial_x I + I\partial_x v = 0, \tag{8}$$

$$v(0, x) = 0, \qquad I(0, x) = I_0 \exp(-x^2/w_{\text{in}}^2),$$

these equations describe propagation of an initially collimated Gaussian beam, with waist $w_{\text{in}} = w_0/\sqrt{2}$, in an arbitrary nonlinear medium. Solutions of

Eqs. (7-8) and their derivatives can exhibit singularities for particular values of z. Analyzing these points we obtain the nonlinear self-focusing positions $z_{\rm sf}$ of the laser beam.

We have to notice that, as it is well-known, the original equation (1) does not provide a beam collapse in (1+1) dimension and for Kerr-type refractive index. However, as a consequence of the geometrical optics approximation, a singularity appears due to neglecting of the higher-order derivative term in Eq. (5). From the physical point of view this behavior finds the following explanation (see e.g. [24], [25]): for highly intense laser light (when the geometrical optics approximation is valid) one does not observe a pure beam collapse when the light stops propagating, but the beam compression into a filament at a certain point. The point of the beam collapse obtained from the geometrical optics approximation is usually identified with a point of the single filament formation [24, 25].

If the beam is not collimated at the boundary but is focused by a lens with the focal length f, then the total focusing distance $z_{\rm sf,f}$ is defined as

$$\frac{1}{z_{\rm sf,f}} = \frac{1}{z_{\rm sf}} + \frac{1}{f} \tag{9}$$

We notice that equations (7-8) are linear with respect to the first order derivatives. Therefore, it is convenient to use the hodograph transformation [33] in order to transform it into a linear system of first order partial differential equations (see Appendix):

$$\partial_w \tau - \frac{I}{\varphi(I)} \partial_I \chi = 0, \tag{10}$$

$$\partial_w \chi + \alpha \partial_I \tau = 0, \tag{11}$$

where $\tau = Iz$, $\chi = x - vz$, $w = v/\alpha$. The boundary conditions are transformed as follows:

$$w = 0, \qquad \tau = 0, \qquad \chi = \sqrt{\ln(1/I)}. \tag{12}$$

We notice that, e.g. for $I_0 = 10^{10}$ W/cm^2 and the light propagation in air ($n_2 = 3.2 \times 10^{-19}$ cm^2/W) $\alpha \simeq 10^{-4}$. Therefore, α can be considered as a small parameter. The similar estimates for dispersion confirm this assumption.

Due to smallness of α we can construct an approximate solution to the system (10-11) step-by-step distinguishing two stages in the dynamics of the system [34, 26]. At the first step, we put $\alpha \simeq 0$, this allows us to assume that in Eq. (11) $\partial_w \chi = 0$ and, consequently, χ approximately does not depend on w. Differentiate boundary conditions Eq. (12) with respect to I, we get $\partial_I \chi = -(2\chi I)^{-1}$. Substituting this approximation into Eq. (10), we find

$$\tau = \frac{-w}{2\chi\varphi}. \tag{13}$$

As a second step, we substitute Eq. (13) into Eq. (11) and arrive at the first order partial differential equation

$$\partial_w \chi + \frac{\alpha w}{2\chi^2 \varphi} \partial_I \chi + \frac{\alpha w}{2\chi\varphi^2} \partial_I \varphi = 0, \tag{14}$$

the integration of which gives two integration invariants

$$\chi\varphi = \Psi_1 \tag{15}$$

$$\frac{\alpha w^2}{4} - \chi^2 \varphi^2 \int \frac{dI}{\varphi} = \Phi_2, \tag{16}$$

Accounting for Eq. (13), we rewrite this result as

$$\chi\varphi = \Psi_1, \qquad \int \frac{dI}{\varphi} - \alpha\tau^2 = \Psi_2. \tag{17}$$

Now the desired solution for the boundary value problem Eqs. (7-8) (in case when $\varphi(I) \neq I$) can be found in a standard way: i) we express I and χ as functions of integration invariants $I = I(\Psi_1, \Psi_2)$ and $\chi = \chi(\Psi_1, \Psi_2)$ at the boundary $\tau = 0$, and construct the following relation between the integration invariants $I(\Psi_1, \Psi_2) = \exp(-\chi(\Psi_1, \Psi_2)^2)$; ii) substituting expressions (17) into this relation, we arrive at the desired solution to the boundary value problem.

Let us now consider several models of nonlinearity which are of physical interest [12].

3. Nonlinear Models

3.1. Kerr nonlinearity

For the case of cubic Kerr nonlinearity, $\varphi = 1$, (see Eq. (4)) we get

$$\tau = \frac{-w}{2\chi}. \tag{18}$$

Substituting this result into Eq. (11), we obtain a first order partial differential equation $\partial_w \chi + (\alpha w/2\chi^2)\partial_I \chi = 0$. Its' solution reads $(\alpha w^2/4\chi^2) - I = F(\chi)$, where $F(\chi)$ is an arbitrary function which should be constructed in a way to fulfill the boundary condition (12). We have

$$\ln(I - \alpha\tau^2) = -\chi^2 \tag{19}$$

Returning to the original variables in Eq. (18) and Eq. (19), we obtain

$$\begin{aligned} \frac{-x^2}{(1-2\alpha z^2 I)^2} &= \ln\left(I[1-\alpha z^2 I]\right), \\ v &= -\frac{2\alpha z x I}{1-2\alpha z^2 I}. \end{aligned} \tag{20}$$

This result was previously obtained by Kovalev on the basis of the Lie and renormalization group symmetries analysis [33, 37]. Differentiating these expressions by x and z, resolving the obtained system of four algebraic equations with respect to $\partial_x I$, $\partial_z v$ etc. and substituting the resulting expressions into (7-8), one can verify that the solutions (20) are exact on the beam axis ($x = 0$, $v|_{x=0} = 0$). These calculations can be easily done by an analytical package, for instance by Maple. All the solutions studied further have been probed in a similar way as well.

One can see that the solution (20) becomes singular at the point $z_{\rm sf} = 1/(2\sqrt{\alpha})$ that provides the self-focusing distance of a Gaussian beam. At this point, a derivative $\partial_z I|_{z_{\rm sf}}$ goes to infinity, however, the magnitude of the on-axial intensity reminds finite and equal to $I_{\rm ext} = 2$. Such a singular behavior of a solution was first observed in Ref. [1] for hyperbolic secant beam profile and later studied for different boundary conditions in Refs. [33]. We notice that all of them scale as $1/\sqrt{\alpha}$, $(\alpha = I_0 n_2)$ as functions of the initial beam intensity. In such a way, a particular form of the initial intensity distribution changes only prefactor for $1/\sqrt{I_0}$. The functional dependence on the I_0 reminds.

Let us now start an investigation of the higher order nonlinearities.

3.2. Power nonlinearity.

We consider the refractive index which in the original (dimensional) form reads $n(I) = n_{2k} I^k$, k being arbitrary integer positive number ($k \neq 2$). Then $\varphi = kI^{k-1}$, and $\alpha = n_{2k} I_0^k$. On the basis of Eq. (17) we obtain

$$\Psi_1 = kI^{k-1}\chi, \quad \Psi_2 = \frac{I^{2-k}}{k(2-k)} - \alpha\tau^2. \tag{21}$$

At the entry plane of nonlinear medium $\tau = 0$ we express n and χ as functions of these integration invariants:

$$I = \Psi_2 (2k - k^2)^{1/(2-k)}, \ \ \chi = \frac{\Psi_1}{(\Psi_2(2k-k^2))^{\frac{k-1}{2-k}} k} \tag{22}$$

Taking the boundary conditions into account, one gets

$$\Psi_2 (2k - k^2)^{1/(2-k)} = \exp\left(\frac{-\Psi_1^2}{(\Psi_2(2k-k^2))^{\frac{k-1}{2-k}} k^2} \right),$$

that together with (21), gives

$$\begin{aligned} &[I^{2-k} + (k^2 - 2k)\alpha I^2 z^2]^{\frac{1}{2-k}} = \\ &\exp\left(\frac{-X^2 I^{2k-2}}{[I^{2-k} + (k^2-2k)\alpha I^2 z^2]^{\frac{2k-2}{2-k}}} \right), \\ &v = -2\alpha z X k I^k, \end{aligned} \tag{23}$$

where $X = x(1 - 2\alpha z^2 k I^k)^{-1}$. Similarly to the previous case, it is possible to verify that this solution is exact at the beam axis.

The on-axial intensity distribution is given by the expressions:

$$I^2[I^{-k} + k(k-2)\alpha z^2] = 1, \tag{24}$$

and presented in Fig. 1 for $\alpha = 1$ and various values of k.

For a negative α, Eq. (24) describes nonlinear defocusing with monotonic decrease of the on-axial intensity. For a positive α the expression (24) exhibits a singularity at a certain point z_{sf}. This point corresponds to the beam collapse or, from the physical point of view, compression of the beam into a single filament [24]. We find z_{sf} in the following way: first, express $z(I)$ explicitly

$$z = \pm \sqrt{\frac{I^{-2} - I^{-k}}{\alpha k (k-2)}}, \tag{25}$$

where $k > 2$. Differentiate Eq. (25) with respect to I and using condition $\partial z(I)|_{I_{\text{ext}}} = 0$, we determine the "turned point" of the curve $z(I)$ $I_{\text{ext}} =$

$(2/k)^{1/(2-k)}$. Substituting this value into Eq. (25), we obtain the self-focusing position for the given form of the refractive index:

$$z_{\rm sf} = \left[\frac{(2/k)^{\frac{2}{k-2}} - (2/k)^{\frac{k}{k-2}}}{\alpha k(k-2)}\right]^{1/2}. \tag{26}$$

An increase of the power k leads to a decrease of the self-focusing distance. As expected, an increase of the incident intensity of the beam ($\alpha = n_{2k} I_0^k$) leads to a decrease of the self-focusing distance.

In several publications (see e.g. [18, 33]) it has been demonstrated both numerically and analytically that for the Kerr nonlinearity the self-focusing position scales as $\sim 1/\sqrt{I_0}$. From the presented result one can see that this scaling law is valid only for this particular nonlinearity. For an arbitrary k, $z_{\rm sf}$, however scales as $z_{\rm sf} \sim I_0^{-k+1/2}$ for power nonlinearities. We demonstrate in the following, that for other forms of nonlinearity the scaling law becomes even more complicated.

3.3. Saturating nonlinearities.

From the physical point of view, the latter form of refractive index does not have many applications. It is mostly considered in papers devoted to the general problem of the wave collapse [36, 30]. A more physical form of the nonlinearity has to manifest a saturating behavior: for example, refraction indices of the form $n = n_2 I/(1+n_s I)$ or $n = n_2/(1+n_s I)$ appear in problems of the laser beam propagation in vapors of metals and dielectrics [12, 6]

Let us consider the first case. Then $\varphi = 1/(1+\beta I)^2$, $\beta = n_s I_0$, $\alpha = n_2 I_0$ and from Eqs. (15,16) we have

$$\Psi_2 = (1+\beta I)^3/3\beta - \alpha\tau^2, \;\; \Psi_1 = \frac{\chi}{(1+\beta I)^2}.$$

The boundary conditions provide us with the following relation:

$$(3\beta\Psi_2)^{1/3} - 1 = \beta\exp(-\Psi_1^2(3\beta\Psi_2)^{4/3}),$$

the solution being

$$\begin{aligned} &[(1+\beta I)^3 - \alpha 3\beta I^2 z^2]^{1/3} - 1 = \\ &\beta\exp\left(\frac{-x^2((1+\beta I)^3 - 3\alpha\beta z^2 I^2)^{4/3}}{[(1+\beta I)^2 - 2\alpha I z^2]^2}\right), \\ &v = \frac{-2\alpha I z x}{(1+\beta I)^2 - 2\alpha I z^2}. \end{aligned} \tag{27}$$

As it was expected, these expressions turn into Eq. (20) under the limit $\beta \to 0$.

At the beam axis we have an expression for the intensity distribution

$$[(1+\beta I)^3 - \alpha 3\beta I^2 z^2]^{1/3} - 1 = \beta. \tag{28}$$

The expression (28) can be analyzed similarly to the case considered above. Resolving z as a function of I we obtain

$$z = \pm\sqrt{\frac{(1+\beta I)^3 - (1+\beta)^3}{\alpha\beta 3 I^2}}. \tag{29}$$

In contrast to the case considered above, the equation $\partial_I z(I) = 0$ gives us now three complex values of I_{ext}, (except for the case $\beta = -1$ when only one complex root exists). Obtained values of I_{ext} do not depend on α. Substitution of each value I_{ext} into (29) yields, generally speaking, a complex quantity. Obviously, if the obtained value of z_{sf} is not purely real, no beam collapse takes place for the chosen set of parameters. However, if the value of z_{sf} is real, a singularity at this point will be observed. In case of two positive real roots z_1 and z_2, the intensity monotonically increases from $z = 0$ to the point z_2 and at $z_2 = z_{\text{sf}}$ the self-focusing will be observed. The intensity distribution in between the points z_1 to z_2 is not unique. This uncertainty corresponds to the developing beam instabilities before the self-focusing at the point z_2. Due to this uncertainty, under the same experimental conditions, one can either observe appearance of several light channels before z_{sf} with their subsequent merging into a single filament, or only beam self-focusing at the point z_2 [18].

In Fig. 2 we present all these cases. The curve c corresponds to the absence of the self-focusing: there are no points where $\partial_I z(I)$ vanishes. The curve b is constructed for a single positive value $z_{\text{sf}} \sim 1.6$. Curve a represents a case of two critical values $z_1 \sim 0.3$, and $z_2 \sim 1$. The segment $[z_1, z_2]$ corresponds to the region of the beam instability before the self-focusing point $z_2 \sim 1$.

The fact that the on-axial intensity goes to infinity and, consequently, the diameter of the beam approaches zero under $z \to \infty$ originates from the semiclassical approximation. In reality, infinite beam compression will be arrested by the diffraction.

Position where intensity increase is arrested could be estimated from the assumption that this happens when the beam radius to be comparable to the wave length λ. The rough estimate give us that for laser beam with waste $w_0 = 1$ mm and $\lambda = 800$ nm the arrest occurs at the point where intensity increases about 100 times with respect to its initial value.

3.4. Two-terms nonlinearity.

Now let us apply obtained results to the case of refractive index consisting of two terms; usually the first one is related to the Kerr nonlinearity and reads as $n_2 I$ whilst the second one is a higher order power function of intensity. Physically this term can be attributed to the fifth order nonlinear susceptibility $n_4 I^2$ or the material ionization $\sigma_K I^K$, where K is the number of photons required for the simultaneous absorption.

Consider the case $n(I) = n_2 I - n_4 I^2$ first. Then dimensionless $\varphi = 1-\beta I$, where $\beta \equiv 2n_4 I_0/n_2$. Substituting φ into (17) one gets

$$\Psi_2 = -\frac{1}{\beta}\ln(1-\beta I) - \alpha\tau^2, \quad \Psi_1 = \chi(1-\beta I). \tag{30}$$

From the boundary conditions it follows

$$1 - \exp(-\beta\Psi_2) = \beta\exp\left(-\Psi_1^2 \mathrm{e}^{2\beta\Psi_2}\right) \tag{31}$$

$$\begin{aligned} &1 - (1-\beta I)\mathrm{e}^{\alpha I^2 z^2 \beta} = \beta\exp\left(-X^2\mathrm{e}^{-2\alpha\beta I^2 z^2}\right), \\ &v = -2\alpha I z X(1-\beta I), \end{aligned}$$

where $X = x(1 - 2\alpha I z^2(1-\beta I))^{-1}$.

On-axial intensity distribution is given by the implicit expression:

$$\alpha I^2 z^2 = \frac{1}{\beta}\ln\left[\frac{1-\beta}{1-\beta I}\right], \tag{32}$$

and is presented in Fig. 3 for a positive α and several positive values of β.

Let α be positive, then for the self-focusing we obtain

$$z_{\mathrm{sf}} = \sqrt{\frac{\beta(-4L(h)-2)}{\alpha}\frac{L(h)}{2L(h)+1}} \tag{33}$$

where $h \equiv \mathrm{e}^{-1/2}/(2\beta-2)$, and $L(h)$ denotes the Lambert function. The latest is defined as a solution to equation $L(h)\exp(L(h)) = h$.

Looking on Eq. (33) and keeping in mind that the dimensionless coefficients $\alpha = n_2 I_0$ and $\beta = 2n_4 I_0/n_2$ are function of the initial beam intensity, we see that functional dependence of the self-focusing position on the beam intensity is again rather complicated and differs from the scaling law $1/\sqrt{I_0}$.

Similar to the case of saturating nonlinearity, the following three subcases can be distinguished depending on the magnitude of β: i) there is no beam collapse: intensity monotonically achieves a saturated value (curve a at the Fig. 3); ii) one critical point (curve b at the Fig. 3); iii) solution is not unique at a certain interval. This means that the self-focusing is prevented by a distance where the beam is unstable and can decay into several filaments (curve c at the Fig. 3). Let us study the cases above in detail.

The function $z(I)$ in Eq. (32) has only one critical point $\partial_I z(I) = 0$ if $\beta = \beta_c \sim 0.175$. For $\beta < 0.175$, two critical points z_1 and z_2 exist and a region of beam instability before the self-focusing appears. For $\beta > 0.175$, the Eq. (32) has no special points, the on-axial intensity monotonically increases approaching a saturation value $I_{\rm sat}$. The self-focusing position $z_{\rm sf}$ as a function of β for various α is presented in Fig. 4. We see that for the value of β bigger then β_c the is no self-focusing collapse: derivatives $\partial_z I$ does not turn into infinity.

By studying the asymptotic behavior of $I = I(z)$ one finds that $I_{\rm sat} = 1/\beta$. Note that the saturated value fulfills the condition $1-\beta I_{\rm sat} = \varphi(I_{\rm sat}) = 0$.

Another case for which an explicit expression for $I = I(\Psi_2)$ can be found at the boundary corresponds to the third power dependence of the refractive index on intensity $n(I) = n_2 I - n_6 I^3$. We have $\varphi = 1 - \beta I^2$, $\beta = 3n_6 I_0^2/n_2$ and

$$\begin{aligned} \Psi_2 &= \frac{\operatorname{arctanh}(\sqrt{\beta} I)}{\sqrt{\beta}} - \alpha\tau^2, \\ \Psi_1 &= \chi(1-\beta I^2) \end{aligned} \tag{34}$$

whose solution reads

$$\begin{aligned} &\tanh\left(\operatorname{arctanh}(\sqrt{\beta}I) - \alpha I^2 z^2 \sqrt{\beta}\right) = \\ &\sqrt{\beta}\exp\left(\frac{-\chi^2(1-\beta I^2)^2}{(1-\tanh[\operatorname{arctanh}(\sqrt{\beta}I) - \alpha I^2 z^2\sqrt{\beta}])^2}\right), \\ &v = -2aIzX(1-\beta I^2), \end{aligned}$$

where $X = x(1 - 2aIz^2(1-\beta I^2)^{-1}$, with the on-axial intensity

$$2 \text{ arctanh}(\sqrt{\beta}I) - \alpha I^2 z^2 \sqrt{\beta} = 2 \text{ arctanh}(\sqrt{\beta}). \tag{35}$$

The features of these solutions are similar to the cases considered before. The value of the saturating intensity is equal to $I_{\text{sat}} = 1/\sqrt{\beta}$.

Notice that the values of I_{sat} for $K = 2$ and 3 obtained here are different from previous theoretical estimates [18, 11, 39], which were obtained assuming that the intensity in the filament saturates when the nonlinear terms in $n(I)$ compensate each other [18]. From the present results we see, however, that this is not the case. Upon propagation the beam tends to reach the on-axial value of the intensity which *maximizes* the index of refraction at the beam axis. In other words, not the nonlinear refractive index itself but its variation should be zero:

$$\partial_I n(I)|_{I_{\text{sat}}} = 0. \tag{36}$$

This condition on the saturated filament intensity is general, and independent of the nonlinear medium. It can serve as a basis for future calculations. Note that such a general mathematical condition cannot be obtained from the numerical simulations.

4. Beam with arbitrary initial intensity distribution

Previously we considered in detail influence of the media nonlinearity on the features of the light propagation. Let us now assume that the medium exhibits only Kerr nonlinear response but initial beam intensity distribution is an arbitrary convex and positive function of the form $I(0, \chi) = \exp(F(\chi))$.

As previously, we start from equations (10,11), which now should be supplemented with the boundary conditions of a general form

$$\tau(0, I) = 0, \qquad \chi(0, I) = H(I). \tag{37}$$

Evidently, $H(I)$ is the inverse function to F. Again, due to smallness of α, we construct an approximate solution in two steps.

Differentiate the boundary condition $I = \exp(F(\chi))$ with respect to I, we get $\chi_I = 1/(F_\chi I)$, where F_χ means $\partial_\chi F(\chi)$. Substituting χ_I into Eq. (10), we get

$$\tau = wF_\chi^{-1}. \tag{38}$$

Substituting this function into Eq. (11) after integration and taking boundary conditions into account, we get

$$\frac{\alpha\tau^2}{2}F_{\chi\chi} + I = \exp(F).$$

In such a way, returning to the original variables, we get the solution:

$$\begin{aligned}\frac{\alpha I^2 z^2}{2}F_{\chi\chi} + I &= \exp(F),\\ v &= \alpha I z F_{\chi}.\end{aligned} \tag{39}$$

We notice that, if $F(\chi)$ is a polynomial function, and for physical reasons is a convex and positive, the highest power term should be negative.

Let us assume that at the boundary we have two-peak profile $I = \exp(-x^4 + bx^2)$. Then from Eq. (39), we get

$$\begin{aligned}I + aI^2z^2(b - 6(x - vz)^2) &= \exp(-x^4 + bx^2),\\ v &= 2aIz(b(x - vz) - 2(x - vz)^3).\end{aligned} \tag{40}$$

Function $I(x)$ for each z has three extremum under $x = 0$, and $x = \pm\sqrt{b/2}$. For $x = 0$ the intensity distribution is given by formula $I(0, z) = (-1 + \sqrt{1 + 4\alpha bz^2})/2\alpha bz^2$, which demonstrates that the on-axial intensity monotonically decreases.

For $x = \pm\sqrt{b/2}$, the intensity distribution along the propagation is governed by the expression

$$I(\pm\sqrt{b/2}, z) = \frac{1 - \sqrt{1 - 8\alpha bz^2 \mathrm{e}^{b^2/4}}}{4\alpha bz^2}, \tag{41}$$

which provides the nonlinear collapse at the point $z_{\mathrm{sf}} = (8\alpha b\mathrm{e}^{b^2/4})^{-1}$, where, as previously, $\alpha = n_2 I_0$.

In such a way, we can conclude that, if at the boundary intensity distribution has several maxima, each of these peaks collapses independently. In Ref. [22] this effect was studied numerically and experimentally and assumed to be one of the possible mechanisms for multiply filamentation. From our analytical solutions we see, that for the Kerr nonlinearity, considered in this section, dependence of the self-focusing position on the initial beam intensity scales as $z_{\mathrm{sf}} \sim 1/\sqrt{I_0}$ for each peaks.

5. Solutions in (1+2) dimensions

Previously constructed solutions in (1+1) dimensions are actual for such physical problems as pulse propagation in fibers or beam propagation in a planar geometry. However, a description of laser beam propagation in three dimensions in cylindrical geometry naturally arises from many experiments. It is therefore very desirable to construct analytical solutions in (1+2) dimensions as well. Let us now consider this case. Again we denote z a propagation direction and x a radial variable. Instead of Eqs. (7-8), we then arrive at following equations

$$\partial_z v + v\partial_x v - \alpha\varphi\partial_x I = 0, \tag{42}$$

$$\partial_z I + v\partial_x I + I\partial_x v + \frac{vI}{x} = 0, \tag{43}$$

$$v(0,x) = 0, \qquad I(0,x) = I_0(x),$$

Because of the last term in Eq. (43), it is no longer convenient to use the hodograph transformation, which was turned out to be very fruitful in the case of (1+1) dimensions. However, as we can see from Appendix Eq. (62), the approximation of Eq. (13) does not depend on the dimensionality of the problem. Based on this approximation we can express v as a function of other variables:

$$v = \frac{-2\alpha\varphi I z x}{1 - 2\alpha\varphi I z^2}. \tag{44}$$

Now we differentiate Eq. (44) with respect to x, and substitute the result together with Eq. (44) into Eq. (43). Keeping only main terms of the power expansion of α, one obtains a first order partial differential equation:

$$(1 - 2\alpha\varphi I z^2)\partial_z I = 2\alpha I x z(2\varphi + \varphi_I I)\partial_x I + 4\alpha\varphi I^2 z. \tag{45}$$

Eq. (45) can be integrated in a standard way for each particular form of the nonlinear refractive index. Solution to (45) together with Eq. (44) provides us with desired approximate analytical solution to Eqs. (42-42). We have to notice here that due to approximations made so far, we do not any longer expect our solutions to be exact at the beam axis. Nevertheless, as we shall later demonstrate, constructed approximate solutions still provide a good accuracy.

5.1. Kerr nonlinearity

First of all, we are going to check the accuracy of the solutions that were constructed based on the suggested approach in (1+2) dimensions. To do this, let us consider a case of Kerr nonlinear refractive index, $\varphi = 1$. Substituting (44) into (45) and keeping only main power terms of α we obtain the following characteristic equation

$$\frac{dz}{1-2\alpha\varphi I z^2} = \frac{-dx}{4\alpha I z x} = \frac{dI}{4\alpha I^2 z}. \tag{46}$$

Equation (46) yields two integration invariants:

$$\Psi_1 = 2I\alpha z^2 - \ln I, \qquad \Psi_2 = Ix$$

Then, taking the initial conditions into account, one obtains the solution

$$v = \frac{-2\alpha I z x}{1-2\alpha I z^2}, \qquad I\mathrm{e}^{-2I\alpha z^2} = \exp\left(\frac{-x^2}{\mathrm{e}^{-4I\alpha z^2}}\right). \tag{47}$$

Both of these expressions become singular at a point

$$z_{\mathrm{sf}} = 1/\sqrt{2\alpha \mathrm{e}}, \tag{48}$$

corresponding to the self-focusing position of the beam. Let us now compare this result with the Marburger formula Ref. [35] traditionally used for estimates of the self-focusing position for Kerr nonlinearity (see e.g. [11, 18]). Obtained as an interpolation of results of numerical simulations, the Marburger formula defines the self-focusing position as

$$z_{\mathrm{M}} = 0.367 z_0 \left((\sqrt{P_{\mathrm{in}}/P_{\mathrm{cr}}} - 0.852)^2 - 0.0219 \right)^{-1/2}, \tag{49}$$

where $P_{\mathrm{cr}} \simeq \frac{3.72\lambda^2}{8\pi n_2}$, $P_{in} = w_0^2 \pi I_0/2$ and $z_0 = \pi w_0^2/\lambda$. Rewriting our result, Eq. (48), in dimensional variables we get

$$z_{\mathrm{sf}} = \frac{0.316 w_0}{\sqrt{n_2 I_0}}. \tag{50}$$

It should be stressed that this expression was obtained under the geometrical optics approximation. Therefore the comparison between two predictions should be performed in the limit $P_{in}/P_{cr} \gg 1$ for Marburger formula. That condition is indeed actual for many experimental conditions. Under this limit from Eq. (49) one obtains

$$z_M = \frac{0.356 w_0}{\sqrt{n_2 I_0}}. \tag{51}$$

One can see that difference between predictions of the Marburger formula Eq. (51) and Eq. (50) is about 10%. Thus, to the best of our knowledge, the accuracy of Eq. (48) is significantly better than predictions of other (semi-)analytical methods (see e.g. Ref. [8]). Moreover, in contrast to the Marburger result, our approach allows not only to find the self-focusing position for the Kerr nonlinearity, but to get analytical expressions for the intensity and phase distribution upon propagation for various forms of the refractive index as well. Now we shall turn to a case of more complicated then Kerr-type media response.

5.2. Two-term nonlinearity

Let us consider the case of cubic-quintic nonlinearity, $\varphi = 1 - \beta I$, $\varphi_I = -\beta$. Substituting these expressions into Eq. (45) after integration one obtains

$$\Psi_1 = -\ln\left(\frac{I}{\beta I - 1}\right), \qquad \Psi_2 = x^2 I^2 (\beta I - 1).$$

The solutions read:

$$v = \frac{-2\alpha(1-\beta I) I z x}{1 - 2\alpha(1-\beta I) I z^2},$$

$$\frac{\mathrm{e}^{-2\alpha z^2 I}}{\beta I \mathrm{e}^{-2\alpha z^2 I} + 1 - \beta I} = \tag{52}$$

$$\exp\left(-x^2 (\beta I \mathrm{e}^{-\alpha z^2 I} + 1 - \beta I)^3 \mathrm{e}^{4\alpha z^2 I}\right). \tag{53}$$

The on-axial intensity distribution is given by the approximate formula:

$$2\alpha z^2 I = \ln\left(\frac{I - \beta I}{1 - \beta I}\right). \tag{54}$$

One can see that the Eq. (54) provides the same value of the saturated intensity $I_{\text{sat}} = 1/\beta$ as Eq. (32) as in (1+1) dimensions. We therefore conclude that the general condition for the saturated intensity given by Eq. (36) does not depend on the dimension of the problem. Let us now compare obtained solutions with previous detailed numerical simulations. For this goal the Ref. [9] is used, where a numerical simulation for the refractive index in the form $n = n_2 I - n_4 I^4$ is presented. In our notations the model parameters from Ref. [9] read $\alpha = 12.73$, $\beta = 0.05$, $w_{in} = 1/\sqrt{2}$. Using these values we get: $z_{\rm sf} = 0.07$, $\sqrt{I_{sat}} = 16$ while the results of numerical simulations are $z_{\rm sf} = 0.05$, and $\sqrt{I_{sat}}$ varies form 15 to 22. (All variables are in units chosen in Ref. [9].) One can see that our analytical prediction turns out to be in good agrement with results of numerical simulations. Moreover, in Ref. [9] the formation of a ring pattern around the beam axis was predicted. Solutions (52-53) reflect this feature as well. From Eq. (53) it is easily seen that x is not necessarily an uniquely defined function of the intensity. Solving the equations $\partial_I x(I) = 0$ and $\partial_{II} x(I) = 0$ numerically one can find special points corresponding to the positions of the ring patterns. A similar investigation already has been performed in detail in Ref. [23].

6. Summary

In the present paper, we succeeded in constructing analytical solutions of the eikonal equations in (1+1) and (1+2) dimensions for several forms of refractive index. For each particular form, we were able to definitely predict global features of the light intensity distribution in the media: if it would be a self-focusing, saturation or a beam instability at a certain distance. For any form of the refractive index it was possible to predict the intensity magnitude in the light channel. For several forms of $n(I)$, we also predicted the self-focusing positions. A comparison with results of numerical simulation of Ref. [35, 9] is presented in Sec. 5. We have to stress that this result is completely new, any previous semi-analytical methods were incapable of solving this problem with such an accuracy for higher order nonlinearities. We demonstrated that the functional dependence of the self-focusing position on the initial light intensity is strongly dependent on the form of the nonlinear refractive index and is not sensible to the beam profile at the boundary.

Evidently, accurate theoretical predictions of both the self-focusing point and the filament intensity is actual for many applications. Apart from these, our result can assist in solving the inverse problem: calculation of the media response on the very short laser pulse irradiation which is a complicated theoretical problem. However, the magnitude of the higher order response terms in $n(I)$ can be found from experiments on the basis of Eq. (36) and dynamics of the self-focusing position for the varied beam power.

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8. Appendix

Let us present here derivation of Eqs. (10,11) on the basis of so called hodigraph transformation. The key point of the hodograph transformation is exchange of dependent and independent variables. In order to do this, we write down the total derivatives:

$$dv = v_x dx + v_z dz, \qquad dI = I_x dx + I_z dz. \tag{55}$$

dividing both of Eqs. (55) into dv and dI, we get system of four equations

$$1 = v_x \frac{\partial x}{\partial v} + v_z \frac{\partial z}{\partial v}, \qquad 1 = I_x \frac{\partial x}{\partial I} + I_z \frac{\partial z}{\partial I}, \tag{56}$$

$$0 = v_x \frac{\partial x}{\partial I} + v_z \frac{\partial z}{\partial I}, \qquad 0 = I_x \frac{\partial x}{\partial v} + I_z \frac{\partial z}{\partial v}. \tag{57}$$

Resolving system of algebraic Eqs. (56-57) with respect to v_x, v_z, I_x and I_z, we get

$$v_x = \frac{1}{J}\frac{\partial z}{\partial I} \qquad v_z = -\frac{1}{J}\frac{\partial x}{\partial I}, \tag{58}$$

$$I_x = -\frac{1}{J}\frac{\partial z}{\partial v} \qquad I_z = \frac{1}{J}\frac{\partial x}{\partial v}, \tag{59}$$

where J is the Jacobian of the transition

$$J = \frac{\partial x}{\partial v}\frac{\partial z}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial z}{\partial v}$$

Let us now introduce new variables $\tau \equiv Iz$, $\chi \equiv x - vz$, $w \equiv v/\alpha$. Then, we get

$$J = \frac{\partial_w \chi \partial_I \tau}{\alpha I} - \frac{\partial_w \tau \partial_I \chi}{\alpha I} - \frac{\tau \partial_w \chi}{\alpha I^2} + \frac{\tau \partial_I \tau}{I^2} - \frac{\tau^3}{I^3}.$$

$$\frac{\partial z}{\partial I} = \frac{\partial_I \tau}{I} - \frac{\tau}{I^2}, \qquad \frac{\partial x}{\partial v} = \frac{1}{\alpha}\partial_w \chi + \frac{\tau}{I} + \frac{w}{I}\partial_w \tau, \tag{60}$$

$$\frac{\partial x}{\partial I} = \partial_I \chi + \frac{w}{\alpha}\frac{\partial_I \tau}{I} - \frac{w}{\alpha}\frac{\tau}{I^2} \qquad \frac{\partial z}{\partial v} = \frac{\partial_v \tau}{I}. \tag{61}$$

Substituting Eqs. (58,59,60,61) into Eqs. (7,8), as a result, we get in (1+1) dimensions system of equations (10,11), and in (1+2) dimensions:

$$\partial_w \tau - \frac{I}{\varphi(I)}\partial_I \chi = 0, \tag{62}$$

$$\partial_w \chi + \alpha \partial_I \tau + \frac{\alpha w}{\chi I + \alpha \tau w}\left[\partial_w \chi \partial_I \tau - \partial_w \tau \partial_I \chi - \frac{\tau \partial_w \chi}{I} + \frac{\alpha \tau \partial_I \tau}{I} - \frac{\alpha \tau^3}{I^2}\right] = 0, \tag{63}$$

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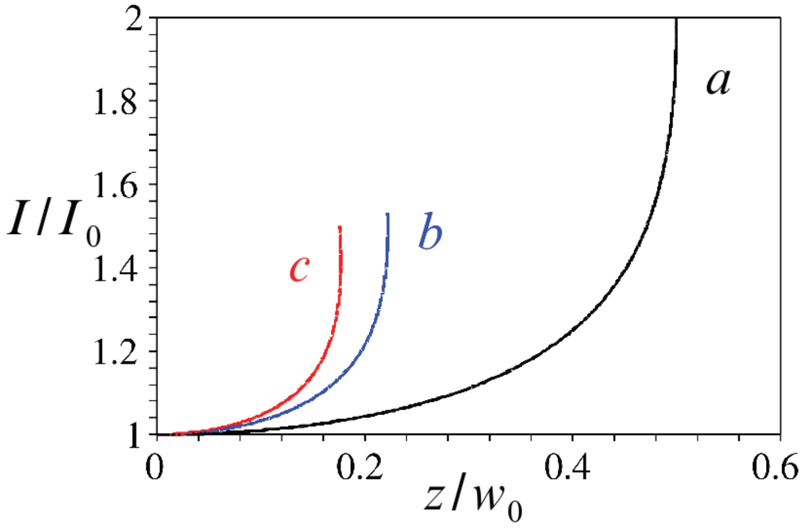

Figure 1: (Color online.) On-axial intensity distribution I/I_0 as a function of the propagation distance z/w_0 for the refractive index of the form $n(I) = n_{2k}I^k$ and various k. $\alpha = n_{2k}I^k = 1$: black curve (a) $k = 1$; blue curve (b) $k = 3$; red curve (c) $k = 4$. For each k, the beam propagates only up to certain point, both its coordinate $z_{\rm sf}/w_0$ and the value of intensity $I_{\rm ext}/I_0$ at this point are discussed in the text.

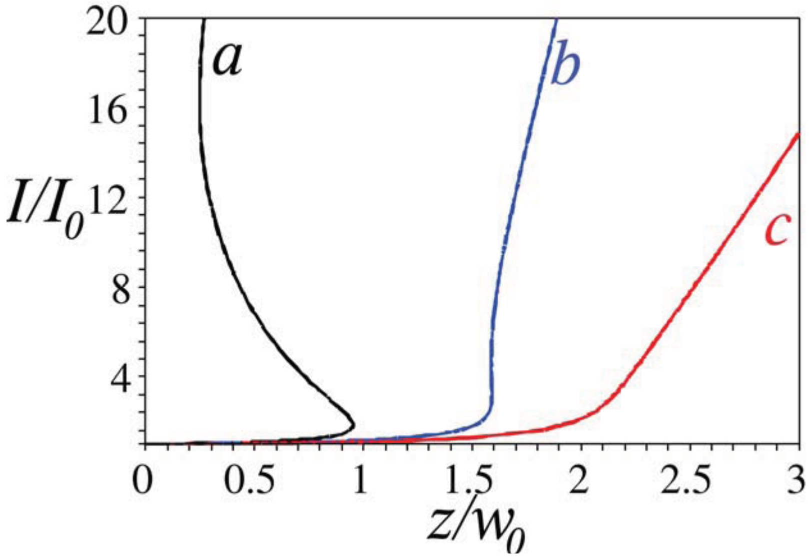

Figure 2: (Color online.) On-axial intensity distribution versus the propagation distance for nonlinear refractive index of the form $n = n_2 I/(1 + n_s I)$, $\alpha = n_2 I_0 = 0.2$, $\beta = n_s I_0$: black curve (a) $\beta = -0.1$; blue curve (b) $\beta = 0.25$; red curve (c) $\beta = 0.5$. The behavior of the intensity under the limit $z \to \infty$ is discussed in the text.

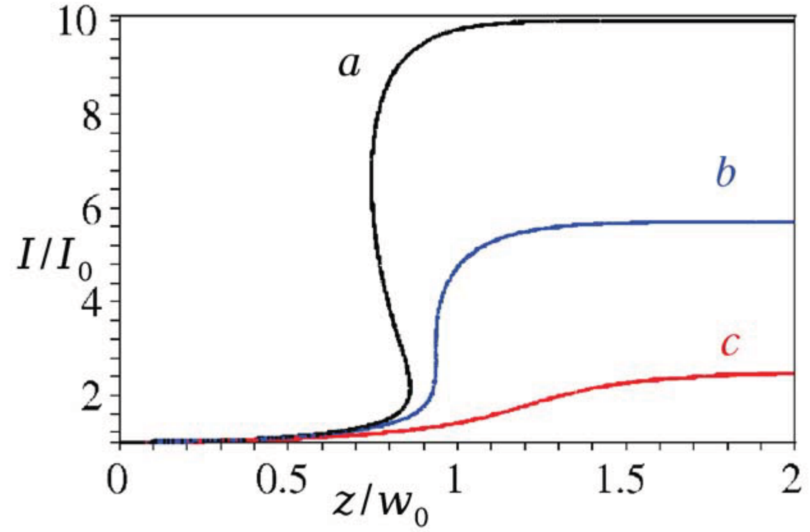

Figure 3: On-axial intensity distribution versus the propagation distance obtained for the refractive index $n = n_2 I - n_4 I^2$, $\beta = 2n_4 I_0/n_2$ $\alpha = n_2 I_0 = 0.4$: black curve (a) $\beta = 0.1$; blue curve (b) $\beta = 0.125$; red curve (c) $\beta = 0.4$.

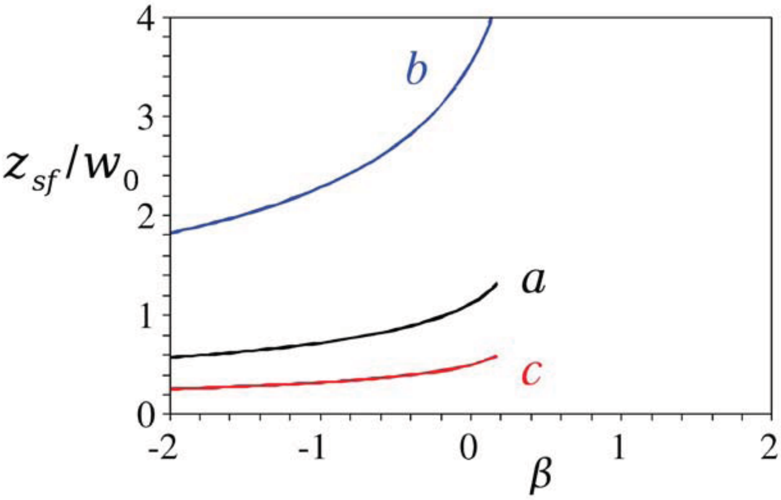

Figure 4: The self-focusing position as a function of $\beta = 2n_4 I_0/n_2$ for various $\alpha = n_2 I_0$: black curve (a) $\alpha = 0.2$; blue curve (b) $\alpha = 0.02$; red curve (c) $\alpha = 1$. All curves end at the point $\beta_c = 0.175$, for the value of β bigger then β_c the is no self-focusing collapse: derivatives $\partial_z I$ does not turn into infinity.

RESEARCH HIGHLIGHTS

Analytical solutions for the eikonal equations with high order refractive index are constructed.

Expressions for the beam collapse position are obtained and analyzed.

It is demonstrated that the functional dependence of the self-focusing position on the initial light intensity is strongly dependent on the form of the nonlinear refractive index.