

Breakdown of the mean-field approximation in a wealth distribution model

M Medo

Physics Department, University of Fribourg, Pérolles, 1700 Fribourg,
Switzerland

and

Department of Mathematics, Physics and Informatics, Mlynská dolina,
842 48 Bratislava, Slovak Republic

E-mail: matus.medo@unifr.ch

Abstract. One of the key socioeconomic phenomena to explain is the distribution of wealth. Bouchaud and Mézard (2000 *Physica A* **282** 536) have proposed an interesting model of an economy based on trade and investments of agents. In the mean-field approximation, the model produces a stationary wealth distribution with a power law tail. In this paper we examine characteristic timescales of the model and show that for any finite number of agents, the validity of the mean-field result is time-limited and the model in fact has no stationary wealth distribution. Further analysis suggests that for heterogeneous agents, the limitations are even stronger. We conclude with general implications of the results presented.

Keywords: fluctuations (theory), stochastic processes (theory), interacting agent models

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1. Introduction

Many empirical studies report broad distributions of income and wealth of individuals and these distributions are often claimed to have power law tails with exponents around 2 for most countries [1]–[5]. The first models attempting to explain the observed properties appeared over fifty years ago [6]–[8]. Much more recently, physics-motivated kinetic models based on random pairwise exchanges of wealth by agents have attracted considerable interest [9]–[13]. An alternative point of view is adopted in the wealth redistribution model (WRM) where agents continuously exchange wealth in the presence of noise [14]–[16]. There are also several specific effects which can lead to broad wealth distributions [17]–[19]. (For reviews of power laws in wealth and income distributions see [20]–[22], while for general reviews of power laws in science see [23, 24].)

In this paper we analyse the WRM with two complementary goals in mind. Firstly we investigate the simplest case when exchanges of all agents are identical, focusing on the validity of the mean-field approximation which is the standard tool for solving the model and deriving the stationary wealth distribution. In particular, we show that for any finite number of agents there is no such stationary distribution (other finite-size effects are discussed for a similar model in [18]). Secondly we investigate the model’s behaviour when the network of agent exchanges is heterogeneous. Previous attempts to investigate the influence of network topology on the model [14], [25]–[27] were all based on the mean-field approximation. We show that this is questionable because heterogeneity of the exchange network strongly limits the validity of results obtained using the mean-field approximation.

2. The model and its mean-field solution

Adopting the notation used in [14], we study a simple model of an economy which is composed of N agents with wealth v_i ($i = 1, \dots, N$). The agents are allowed to mutually exchange their wealth (representing trade) and they are also subject to multiplicative noise

(representing speculative investments). The time evolution of agents' wealth is given by the system of stochastic differential equations (SDEs)

$$dv_i(t) = \left(\sum_{j \neq i} J_{ij} v_j(t) - \sum_{j \neq i} J_{ji} v_i(t) \right) dt + \sqrt{2} \sigma v_i(t) dW_i(t), \quad (1)$$

where $\sigma \geq 0$ controls the noise strength. The coefficient J_{ij} quantifies the proportion of the current wealth $v_j(t)$ that agent j spends on the production of agent i per unit time. We assume the Itô convention for SDEs and $dW_i(t)$ is standard white noise [29, 30]. Hence, denoting averages over realizations by $\langle \cdot \rangle$, we have $\langle dW_i(t) \rangle = 0$, $\langle dW_i(t) dW_j(t) \rangle = \delta_{ij} dt$, and $\langle v_i(t) dW_i(t) \rangle = 0$. By summing $dv_i(t)$ over all agents one can see that the average wealth $v_A(t) := (1/N) \sum_{i=1}^N v_i(t)$ is not influenced by wealth exchanges and obeys the SDE $dv_A(t) = (\sqrt{2} \sigma / N) \sum_{i=1}^N v_i(t) dW_i(t)$. Therefore $\langle dv_A(t) \rangle = 0$ and $\langle v_A(t) \rangle$ is constant. For simplicity we assume $v_i(0) = 1$ ($i = 1, \dots, N$) and thus $\langle v_i(t) \rangle = 1$ and $\langle v_A(t) \rangle = 1$. (The influence of the initial conditions is discussed in section 4.1.)

The system behaviour is strongly influenced by the exchange coefficients J_{ij} . The simplest choice is $J_{ij} = J/(N-1)$ where all exchanges are equally intensive—we say that the exchange network is homogeneous. By rescaling the time we can set $J = 1$ which means that during unit time agents exchange all their wealth. Consequently, (1) simplifies to

$$dv_i(t) = (\tilde{v}_i(t) - v_i(t)) dt + \sqrt{2} \sigma v_i(t) dW_i(t), \quad (2)$$

where $\tilde{v}_i(t) := (1/(N-1)) \sum_{j \neq i} v_j$ is the average wealth of all agents but agent i . In the limit $N \rightarrow \infty$, fluctuations of $\tilde{v}_i(t)$ are negligible and one can replace $\tilde{v}_i(t) \rightarrow \langle \tilde{v}_i(t) \rangle = 1$ as in [14]. Agents then effectively interact only with the ‘mean field’ and their wealth levels are independent. Using the Fokker–Planck equation for the wealth distribution $f(v_i, t)$, the stationary solution $f(v_i)$ can be found in the form

$$f(v_i) = \frac{(\lambda - 1)^\lambda}{\Gamma(\lambda)} \exp \left[-\frac{\lambda - 1}{v_i} \right] v_i^{-1-\lambda}, \quad \lambda := 1 + 1/\sigma^2. \quad (3)$$

For $v_i \gg \lambda - 1$, $f(v_i)$ decays approximately as a power law with exponent $2 + 1/\sigma^2$, while the cumulative distribution has exponent $1 + 1/\sigma^2$. When v_i is well described by (3), we say that the system is in the *power law regime*.

The empirical studies mentioned above report power law exponents around 2, indicating that in this model, $\sigma \simeq 1$ is needed to obtain realistic power law behaviour of the wealth distribution. In our analytical calculations we assume $\sigma < 1$; strong noise ($\sigma \geq 1$) is discussed separately at the end of the following section.

3. The complete exchange network for a finite N

To examine when the power law regime is realized, we first investigate the time needed to reach the mean-field solution (3). Such relaxation times were studied very recently using kinetic models of wealth distribution [28].

Given the homogeneous initial conditions $v_i(0) = 1$ ($i = 1, \dots, N$), the exchange terms proportional to $\tilde{v}_i - v_i$ are zero at $t = 0$ and can be neglected for small times. Hence when t is small, each $v_i(t)$ evolves independently due to multiplicative noise, $v_i(t)$

is log-normally distributed, and its variance is $\text{var}[v_i](t) = \exp[2\sigma^2 t] - 1 = 2\sigma^2 t + O(t^2)$: we say that the system is in the *free regime*. From the known variance $\sigma^2/(1 - \sigma^2)$ of the mean-field solution (3), we can estimate the time of transition, t_1 , between the free regime and the power law regime as

$$t_1 = \frac{1}{2(1 - \sigma^2)}. \quad (4)$$

When $t \gg t_1$, the system has been given enough time to reach the power law regime.

We now recall the average wealth $v_A(t)$. While $\langle dv_A(t) \rangle = 0$, one can see that $\langle dv_A^2(t) \rangle$ is always positive. Hence the variance $\text{var}[v_A(t)]$ grows without limit, in contrast with the variance of (3) which is finite for $\sigma < 1$. To resolve this disagreement we have to accept that $f(v_i)$ as given by (3) is not a stationary solution. But what comes after the power law regime? Since the Fokker–Planck equation for the joint probability distribution $f(v_1, \dots, v_N)$ cannot be solved analytically, we answer this question by investigating the average quantities $\langle v_i^2(t) \rangle$ and $\langle v_i(t)v_j(t) \rangle$ ($i \neq j$); now we are considering $\sigma < 1$ and hence both are well defined. Due to the assumed homogeneous network of interactions and the chosen initial conditions, all averages $\langle v_i^2(t) \rangle$ are identical and the same holds for the cross-terms $\langle v_i(t)v_j(t) \rangle$; effectively we are left with only two variables. From the Itô lemma it follows that $d(v_i^2) = (2v_i + dv_i) dv_i$ and $d(v_i v_j) = v_i dv_j + v_j dv_i + dv_i dv_j$. After substitution of (2) and averaging over all possible realizations, we obtain the exact set of equations

$$\begin{aligned} \frac{d\langle v_i^2(t) \rangle}{dt} &= 2 \left[\langle v_i(t)v_j(t) \rangle - (1 - \sigma^2) \langle v_i^2(t) \rangle \right], \\ \frac{d\langle v_i(t)v_j(t) \rangle}{dt} &= \frac{2}{N-1} \left[\langle v_i^2(t) \rangle - \langle v_i(t)v_j(t) \rangle \right]. \end{aligned} \quad (5)$$

Since we set $v_i(0) = 1$ ($i = 1, \dots, N$), $\langle v_i(t) \rangle = 1$ and the initial conditions are $\langle v_i^2(0) \rangle = 1$ and $\langle v_i(0)v_j(0) \rangle = 1$; for the general case see section 4.1. Independently of the initial conditions, for $\sigma > 0$, (5) has only the trivial stationary solution $\langle v_i^2 \rangle = \langle v_i v_j \rangle = 0$. This confirms that for a finite N , there is no stationary distribution $f(v_i)$.

By solving (5) one obtains the variance $\text{var}[v_i](t) = \langle v_i^2(t) \rangle - \langle v_i(t) \rangle^2$ as a function of time and as a by-product also the correlation between agents i and j

$$C_{ij}(t) := \frac{\langle v_i(t)v_j(t) \rangle - \langle v_i(t) \rangle \langle v_j(t) \rangle}{\sqrt{\text{var}[v_i(t)] \text{var}[v_j(t)]}}. \quad (6)$$

Since the resulting expressions are rather complicated, here we discuss only their limiting cases. Small time expansions can be found in the form

$$\text{var}[v_i](t) = 2\sigma^2 t + O(t^2), \quad (7)$$

$$C_{ij}(t) = \frac{1}{N-1} t + O(t^2). \quad (8)$$

As can be seen, (7) agrees with our previous reasoning about the log-normal nature of $f(v_i)$ in the free regime, while (8) confirms that in the limit $N \rightarrow \infty$, wealth correlations vanish.

In the limit of large time we obtain

$$\lim_{t \rightarrow \infty} C_{ij}(t) = 1 - \sigma^2 + \frac{\sigma^2}{1 - \sigma^2} \frac{1}{N} + O(1/N^2). \quad (9)$$

Thus, as t increases, the system passes to the *synchronized regime* where the wealth of agents is strongly correlated. One can estimate the transition time by comparing the initial linear growth of C_{ij} with its stationary value, leading to

$$t_2 = (1 - \sigma^2) N + O(1). \quad (10)$$

An alternative estimate can be obtained from $\text{var}[v_i](t)$. Apart from a constant, it contains only terms proportional to $\exp[\lambda_{1,2}t]$ where

$$\lambda_{1,2} = \frac{-\sigma^2 - N(1 - \sigma^2) \pm \sqrt{N^2(1 - \sigma^2)^2 + 2N\sigma^2(3 - \sigma^2) - \sigma^2(4 - \sigma^2)}}{2(N - 1)}.$$

Since $\lambda_1 < 0$, $\lambda_2 > 0$, and for $\sigma \lesssim 1$, $|\lambda_1| \gg |\lambda_2|$, the terms proportional to $\exp[\lambda_1 t]$ cause the initial saturation of $\text{var}[v_i](t)$ but the terms proportional to $\exp[\lambda_2 t]$ eventually take over and cause the divergence of $\text{var}[v_i](t)$. The corresponding transition time can be roughly estimated by solving $\lambda_2 t = 1$, yielding

$$t_3 = \frac{1 - \sigma^2}{2\sigma^2} N + O(1). \quad (11)$$

Both t_2 and t_3 describe the transition between the power law and synchronized regimes: the former focuses on the growth of correlations, the latter on the growth of variances.

To verify the analytical results presented we investigated the model numerically. For numerical solutions of stochastic differential equations we used Milstein's method [29, 31]; random numbers were generated using the standard GSL library and the Mersenne twister generator [32], and the time increment was 10^{-4} in all simulations. In the discretization scheme used, there is a non-zero probability that the wealth $v_i(t)$ becomes negative [33]. However, thanks to the typical value of $v_i(t)$ and the small time step, in the numerical simulations presented this was not an issue. As can be seen in figure 1(a), our analytical results agree with numerical simulations of the system. Due to the small number of agents, transition times $t_{2,3}$ are small and the system goes directly from the free regime to the synchronized regime. In figure 1(b) the number of agents is large and the system behaviour is more complex. In the initial period the variance is small and correlations are negligible, while in the period $t \in [2; 300]$ the variance is almost constant and correlations are still small—the system is in the power law regime (due to large computational complexity, no numerical results are shown here). Eventually, for $t \gtrsim 10^4$, the synchronized regime is established. The transition times given by (4), (10), and (11) are shown as vertical dotted lines and agree well with the described changes of the system behaviour.

We should sound here a note of caution about the interpretation of the averages $\langle v_i^2(t) \rangle$ and $\langle v_i(t)v_j(t) \rangle$ and the wealth distribution $f(v_i, t)$. All these quantities are ensemble-based: if many copies of the system evolve independently for time t , by examining the final wealth of agent i one can estimate both the distribution $f(v_i, t)$ and the averages. By contrast, when one speaks about an empirical wealth distribution, that is based on the wealth of all agents in one realization only, it is population-based. However, when the number of realizations and the number of agents are large and the wealth correlations

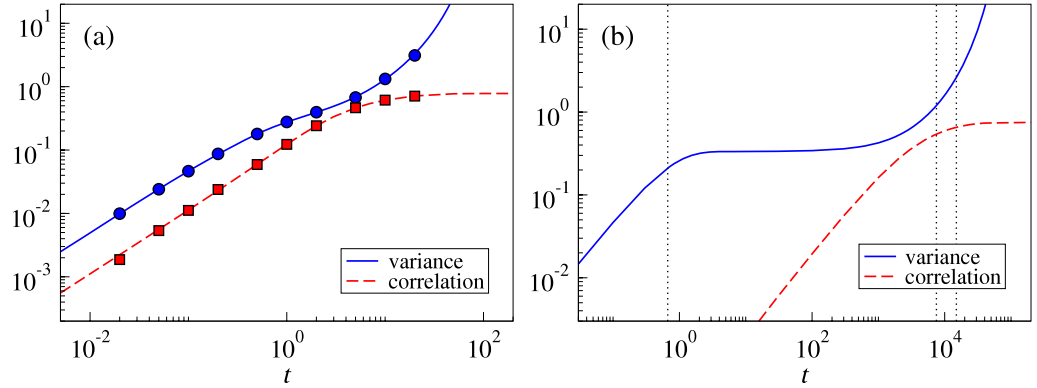


Figure 1. Time evolution of $\text{var}[v_i](t)$ and $C_{ij}(t)$ for 10 agents (a) and for 10^4 agents (b). Analytical results following from (5) are shown as lines; numerical results obtained by averaging over 10^5 realizations are shown as symbols; $\sigma^2 = 0.5$. Vertical dotted lines indicate the transition times t_1 , t_2 , and t_3 , left to right, respectively.

are small, ensemble-based and population-based quantities are alike. Such behaviour was observable also in the numerical simulations presented above. In the free and power law regimes, the variance of wealth in each realization was similar to $\text{var}[v_i(t)]$ (at various times, differences were less than 20% for $N = 10$ and less than 1% for $N = 10\,000$) and its relative fluctuations between realizations were approximately 50% for $N = 10$ and 2% for $N = 10\,000$. As time goes on, fluctuations of the population-based variance grow and so does the difference between the ensemble-based and population-based variances of wealth. In the synchronized regime, the equivalence of the two quantities breaks entirely.

The nature of the synchronized regime can be better understood by recalling the average wealth $v_A(t)$ again. As explained above, its evolution is given by a sum of multiplicative processes, $dv_A(t) = (\sqrt{2}\sigma/N) \sum_{i=1}^N v_i(t) dW_i(t)$. Despite this summation of contributions and their variable strengths ($\langle v_i^2(t) \rangle$ increases with time), according to figure 2 the distribution of v_A is approximately log-normal and, in agreement with our expectations, the variance $\text{var}[v_A]$ is increasing. In the initial regime, this increase is due to growing variances of all agents' wealth. In the power law regime, variances of wealth levels are approximately constant but their growing correlations lead to increasing $\text{var}[v_A]$. In the synchronized regime, wealth correlations are already saturated and the growth of $\text{var}[v_A]$ is caused by exponentially growing variances of wealths. Since correlations are large, ensemble-based and population-based quantities are no longer equivalent. Finally we remark that since $v_A > 0$, $\langle v_A(t) \rangle = 1$ is fixed, and $\text{var}[v_A(t)]$ grows without bound, in the course of time it is increasingly probable that $v_A(t)$ will be much smaller than its expected value $\langle v_A(t) \rangle = 1$; this can be interpreted as a high occurrence of temporal depressions of the economy.

When $\sigma \geq 1$, both $\text{var}[v_i](t)$ and $C_{ij}(t)$ diverge and must be replaced by different quantities. Instead of the variance, one can use the mean absolute deviation $\langle |v_i(t) - 1| \rangle^2$ which avoids second moments of the wealth distribution and hence can be used for any σ . Pearson's correlation coefficient can be replaced by a rank correlation coefficient (Kendall's τ or Spearman's ρ). All three proposed quantities are hard to handle in analytical

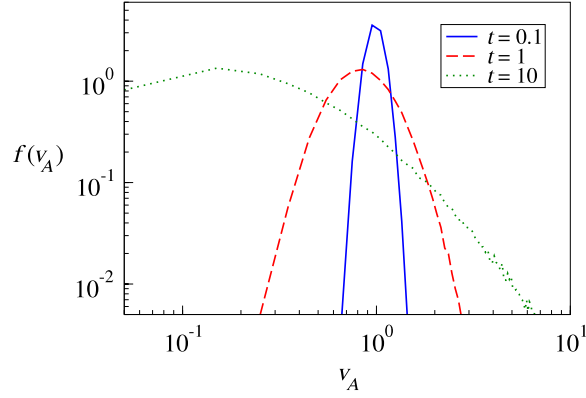


Figure 2. Probability density of the average wealth v_A at various times. Parameter values are $N = 10$ and $\sigma^2 = 0.5$; probabilities were obtained from 10^5 independent realizations of the model.

calculations, and with strong noise, numerical simulations of the system are extremely time-demanding. While we have obtained no definite results yet, preliminary outcomes suggest that in this case too the transition from the power law regime occurs at a time proportional to the number of agents N .

4. General exchange network

Now we generalize the exchange network to an arbitrary graph: denoting the set of neighbours of agent i by \mathcal{N}_i , the number of neighbours by k_i , the average number of neighbours by z . We assume that each agent interacts equally with all neighbours and per unit time exchanges the whole wealth, and hence

$$J_{ij} = 1/k_j \quad \text{for } i \in \mathcal{N}_j, \quad J_{ij} = 0 \quad \text{for } i \notin \mathcal{N}_j; \quad (12)$$

notice that the matrix of exchanges J is asymmetric. Now, (1) generalizes to

$$dv_i = (\hat{v}_i - v_i) dt + \sqrt{2}\sigma v_i dW_i, \quad (13)$$

where $\hat{v}_i := \sum_{j \in \mathcal{N}_i} v_j / k_j$. By averaging over realizations we obtain the set of equations for the stationary values of the average wealths

$$\langle v_i \rangle = \sum_{m \in \mathcal{N}_i} \frac{\langle v_m \rangle}{k_m}, \quad (14)$$

which is solved by $\langle v_i \rangle \sim k_i$. Assuming average wealth equal to 1, (14) has the unique solution $\langle v_i \rangle = k_i / z$. This means that the topology of the exchange network is crucial for the distribution of wealth among the agents. Consequently, when σ is small and hence wealth fluctuations are negligible, a power law distribution of wealth can be purely a topological effect of a scale-free degree distribution in the network of agent exchanges.

To proceed, $\langle v_i^2(t) \rangle$ and $\langle v_i(t)v_j(t) \rangle$ are again the key quantities. They fulfil the equations

$$\begin{aligned}\frac{d\langle v_i^2 \rangle}{dt} &= 2 \sum_{m \in \mathcal{N}_i} \frac{\langle v_i v_m \rangle}{k_m} - 2(1 - \sigma^2) \langle v_i^2 \rangle, \\ \frac{d\langle v_i v_j \rangle}{dt} &= \sum_{m \in \mathcal{N}_i} \frac{\langle v_j v_m \rangle}{k_m} + \sum_{n \in \mathcal{N}_j} \frac{\langle v_i v_n \rangle}{k_n} - 2\langle v_i v_j \rangle,\end{aligned}\tag{15}$$

which can be derived similarly to (5). We set the initial conditions according to the stationary wealths as $v_i(0) = k_i/z$ and thus $\langle v_i^2(0) \rangle = k_i^2/z^2$ and $\langle v_i(0)v_j(0) \rangle = k_i k_j / z^2$ (the general case is studied in section 4.1). From (15) it follows that

$$\left. \frac{d\langle v_i^2 \rangle}{dt} \right|_{t=0} > 0, \quad \left. \frac{d\langle v_i v_j \rangle}{dt} \right|_{t=0} = 0,\tag{16}$$

which means that the growth of $\text{var}[v_i](t)$ precedes the growth of $C_{ij}(t)$. This gives us a way to investigate the small time behaviour of (15): assuming $\langle v_i(t)v_j(t) \rangle$ constant, we obtain $\langle v_i^2(t) \rangle$ which in turn leads to an enhanced estimate of $\langle v_i(t)v_j(t) \rangle$. For neighbouring agents i and j , the results are

$$\text{var}[v_i(t)] = \frac{2\sigma^2 k_i^2}{z^2} t + O(t^2),\tag{17}$$

$$C_{ij}(t) = \frac{k_i + k_j}{2k_i k_j} t + O(t^2).\tag{18}$$

Moreover, it can be shown that when the shortest path between agents i and j has the length L , the leading term of $C_{ij}(t)$ is proportional to t^L . These results are confirmed by figure 3 where we investigate a system of ten agents who are placed on a ring (i.e., $k_i = 2$, $i = 1, \dots, 10$). As can be seen, numerical results agree well with $C_{ij}(t)$ proportional to t^L . The system produces a ‘cascade’ of correlations: first only neighbouring agents are considerably correlated, then also agents with the distance 2, distance 3, and so forth.

For (13), the mean-field approximation yields the stationary distribution

$$f_i(v_i) = K_i \exp[-(\lambda'_i - 1)/v_i] v_i^{-2-\lambda'_i},\tag{19}$$

where $\lambda'_i = 1 + k_i/(z\sigma^2)$ and the corresponding variance is $\text{var}[v_i] = k_i^2 \sigma^2 / (z^2(1 - \sigma^2))$. By comparing this stationary variance with (17), we obtain the time of transition from the free regime to the power law regime as

$$t'_1 = \frac{1}{2(1 - \sigma^2)},\tag{20}$$

which is identical to (4). Further, from (18) we see that the time of transition from the power law regime to the synchronized regime is proportional to $k_i k_j / (k_i + k_j)$ and thus for the whole network it can be estimated as

$$t'_2 = O(z),\tag{21}$$

which is a generalization of (10). We see that for networks with a relatively small average degree, the power law regime appears only for a limited time or not at all.

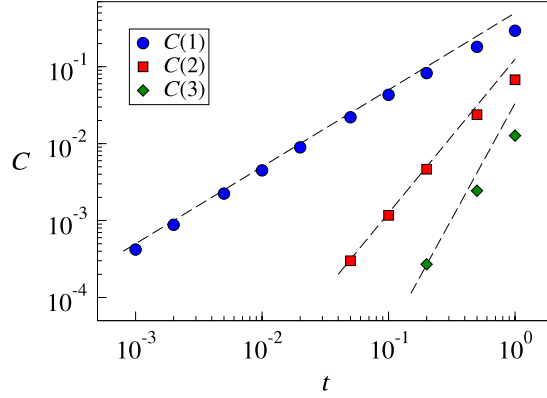


Figure 3. Time evolution of correlations for the ring network of 10 agents, $\sigma^2 = 0.25$. Symbols show numerical results for neighbouring agents (circles), agents with the distance 2 (squares), and agents with the distance 3 (diamonds), averaged over 10^7 realizations. Dashed lines have slopes 1, 2, and 3, respectively.

We were unable to obtain an equivalent of the transition time t_3 for a general network. Considering, for example, a simple star-like structure with one agent in the centre and the remaining $N - 1$ agents connected only to that agent, one can see that the transition time t'_3 is small and does not scale with N . This suggests that, like t'_2 , t'_3 , it is $O(z)$. This contradicts the findings presented in [14] (page 541) where they report stationary power law tails for $z = 4$; it is possible that their numerical results are influenced by finite-time and finite-size effects.

4.1. Influence of the initial conditions

There is still one more transition time to investigate. When the initial conditions $v_i(0)$ are not set in line with the stationary wealths given by equation (14), a certain time is needed to redistribute the excessive wealth levels over the network; we say that the system is in the *equilibration regime*. Since $\langle dW_i \rangle = 0$, noise terms do not contribute to the redistribution. Thus, (13) effectively simplifies to $dv_i = (\hat{v}_i - v_i) dt$ which leads to the exponential convergence of v_i to the stationary value k_i/z . By making the substitution $u_i := v_i - k_i/z$ we obtain

$$\dot{u}_i + u_i - \sum_{j \in \mathcal{N}_i} \frac{u_j}{k_j} = 0, \quad (22)$$

whose timescale is given by the initial terms $\dot{u}_i + u_i$ as $O(1)$. Thus, the initial wealth distribution equilibrates in time $O(1)$. Since the transition from the free regime occurs roughly at the same time, the system passes from the equilibration regime directly to the power law regime.

5. Conclusion

We have shown that for the model investigated agent wealths have no stationary distribution and the power law tailed distribution reported in previous works is only

transient. In addition, for any finite number of agents, their average wealth v_A follows a multiplicative process with a fixed expected value $\langle v_A \rangle$ and an increasing variance $\text{var}[v_A]$. Hence, as illustrated in figure 2, the probability $P(v_A < x)$ approaches 1 for any $x > 0$. We can conclude that the simple economy produced by the model is an uneasy one: the longer it evolves, the higher the probability that a given agent i has wealth much smaller than any positive fraction of the expected wealth $\langle v_i \rangle$.

There is also a more general lesson to be learnt. In essence, the mean-field approximation here anchors the agent wealths to their expected values and thus weakens the diffusive nature of the stochastic system studied. Mathematically speaking, the system behaviour depends on the order of limits $N \rightarrow \infty$ and $t \rightarrow \infty$: in the former case there is a stationary wealth distribution; in the latter case there is none. This is an undesired consequence of the mean-field approximation which, as with other stochastic models, should be used with great caution. In particular, when using it, one should check whether the nature of the system studied is not changed. To achieve this, in this paper we have used an aggregate quantity (the average wealth) and a quantity obtained using the mean-field approximation (the wealth variance).

On the other hand, in some cases an anchoring term may be appropriate. For example, a simple taxation of wealth can be achieved by introducing the term $r(1 - v_i) dt$ to (2), where $r > 0$ represents the tax rate. Then the set of equations for $\langle v_i^2(t) \rangle$ and $\langle v_i(t)v_j(t) \rangle$ has a non-trivial stationary solution for $\sigma < 1$; one can say that the proposed taxation stabilizes the system. Notably, systems of coupled stochastic equations with multiplicative noise and negative feedback are common in the study of non-equilibrium phase transitions in magnetic systems [34]. Our work shows that this negative is crucial for mean-field studies of such systems [35].

In addition to the results presented, several questions remain open. First, for large time t , the analytical form of the wealth distribution $f(v_i, t)$ is unknown. Second, for an arbitrary network of exchanges, the limiting value of the correlation $C_{ij}(t)$ and also the transition time t'_3 are of interest. Third, the strong noise case deserves more attention and perhaps an attempt for approximate analytical results. Finally, the model studied is simplistic, since it combines two ingredients of an economy—trade and speculation—in a very unrealistic way. Devising a more adequate model remains a future challenge.

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