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Heterogeneous network with distance dependent connectivity

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Abstract. We investigate a network model based on an infinite regular square lattice embedded in the Euclidean plane where the node connection probability is given by the geometrical distance of nodes. We show that the degree distribution in the basic model is sharply peaked around its mean value. Since the model was originally developed to mimic the social network of acquaintances, to broaden the degree distribution we propose its generalization. We show that when heterogeneity is introduced to the model, it is possible to obtain fat tails of the degree distribution. Meanwhile, the small-world phenomenon present in the basic model is not affected. To support our claims, both analytical and numerical results are obtained.

PACS. 64.60.aq Networks – 89.75.Hc Networks and genealogical trees – 01.75.+m Science and society

1 Introduction

Networks are powerful tools for representation of many diverse systems arising in physics, biology, and sociology. Progress in this field is rapid; good reviews of our current knowledge are presented in [1–3]. In this work we investigate a network which is embedded in an Euclidean space where the probability that two nodes are connected by a link depends on their mutual distance. A similar model was first proposed by Kleinberg in [4]. Later, a model based on a regular underlying lattice was proposed and some numerical results were obtained [5]; very recently, this work has been generalized by introducing hidden variables [6]. In [7,8], similar models with wiring costs depending on distances are studied; in [9–13], the interplay between geographical distance and node degree is investigated.

Models mentioned above have one feature in common: resulting networks consist of many short links and long distance connections are less numerous. Notice that this corresponds to the picture widely accepted by sociologists investigating networks of acquaintances [14,15]. Their key phrase "Strength of weak ties" has a straightforward interpretation here: the probability of connecting two nodes must decrease with the distance slow enough to enable multiple long links. Then the resulting network resembles the structure observed in the human society. Our present understanding of this phenomenon agrees with early mathematical insights presented in [16] where importance of multiple edge length scales was discussed. Notice that also the classical model of Watts and Strogatz with two types of links [17] follows a similar pattern.

In this paper we deal with the network model based on the distance dependent connectivity which was investigated in [18] and is similar to [5]. This model was developed to mimic the acquaintance network in a human society. It allows us to estimate the typical degree of separation between distant vertices in the network – the results show that with a proper choice of the dependence between the linking probability and the nodes distance, the network exhibits the small-world phenomenon.

However, in the original work the degree distribution $P(k)$ was not a matter of interest. In this paper we show that it can be approximated by a Gaussian distribution. This result is not surprising because the model relies on the edges whose presence is mutually independent in the same way as it is in the classical random graph of Erdös and Rényi. Moreover, we show that the distribution of $P(k)$ is rather narrow. By contrast, when we investigate the number of persons's acquaintances in a real society, the distribution decays slowly. This observation and the lack of diversity in the original model were our main motivations for the presented work. The basic "homogeneous" model is generalized by introducing hidden variables which is a common approach in various network models [19,20], a similar attempt was recently presented in [6]. We investigate the tail behavior of the degree distribution and show that the resulting network exhibits the small-world phenomenon.

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2 The basic model

We assume that nodes of the graph form an infinite square lattice in the Euclidean plane with the side length of the elementary square equal to 1. When modeling a society, each node represents one person and thus in this way we assume a homogeneous distribution of population. The probability that two vertices with the distance d are connected by an edge we label as $Q(d)$. Notice that this is the point where we introduce homogeneity to the network: the probability $Q(d)$ is the same for every pair of nodes separated by the distance d. The degree k of a node is defined as the number of edges connected to this node. Consequently, the average node degree $z := \langle k\rangle$ is given by the linking probability $Q(d_i)$ summing over all nodes i. When $Q(d)$ changes slowly on the scale of 1, the summation can be replaced by an integration and thus

$$z = \sum_i Q(d_i) \approx \int Q(r)\,\mathrm{d}\boldsymbol{r} = \int_0^\infty Q(r) 2\pi r\,\mathrm{d}r. \tag{1}$$

Notice that with z given, equation (1) represents a normalization condition for $Q(d)$.

For a node with the degree k, the clustering coefficient C is defined as the ratio $C := n/\binom{k}{2}$ where n is the number of edges between the neighbours of the given node. Notice that $0 \leq C \leq 1$. For a particular node X and a given function $Q(d)$, the average value of n is $\langle n\rangle = \frac{1}{2}\sum_{i\neq j} Q(d_{Xi})Q(d_{ij})Q(d_{jX})$. Here the factor $\frac{1}{2}$ corresponds to the fact that by a plain summation over all $i \neq j$ a doublecounting occurs ($i \leftrightarrow j$). Consequently, the average clustering coefficient of the network can be approximated as

$$\langle C\rangle \approx \frac{1}{z(z-1)} \iint_{i,j} Q(d_{Xi})Q(d_{ij})Q(d_{jX})\,\mathrm{d}\boldsymbol{r}_i \mathrm{d}\boldsymbol{r}_j. \tag{2}$$

Here we again assumed that $Q(d)$ changes slowly on the scale of 1.

In [18], $Q(d)$ was assumed to have the form

$$Q(d) = \frac{1}{1 + bd^\alpha} \tag{3}$$

with $\alpha > 2$ to allow a proper normalization according to equation (1). This choice was motivated by the following observations of human society:

1. when two persons live close to each other, they probably know each other. Thus we require $Q(0) = 1$;
2. the greater is the distance between two persons, the smaller is the probability that they know each other. Thus $Q(d)$ should be a decreasing function of d;
3. we define the average number of distant people that every person knows as $N_d \equiv \int_R^\infty Q(r)2\pi r\,\mathrm{d}r$ where R is large and fixed. We demand N_d sufficiently high to reflect the observation that many people have distant friends (e.g. living on the opposite Earth hemisphere).

For example, $Q(d) = \exp[-bd]$ satisfies (i) and (ii) but if we choose R that covers half of human population ($\pi R^2 \approx 3\times 10^9$) and $z \approx 200$ (which is a reasonable value to model real acquaintances), we obtain $N_d \approx 10^{-11}$ which is effectively zero. Consequently, equation (3) represents a simple choice of $Q(d)$ which for α in the range 2.5–3.5 complies with the requirements written above. Yet, we do not claim that these three observations allow us to guess the precise form of $Q(d)$. We merely suppose that our choice is able to capture basic features of the human acquaintances network. More detailed discussion on the nature of $Q(d)$ can be found in [18].

In addition to the clustering coefficient defined above, another important characteristics of random networks is the degree of separation (or equally the shortest path length). It is defined as the minimal number of vertices along the shortest path between two given nodes. Denoting the geometrical distance of these two nodes as l, in the original paper it was shown that for $Q(d)$ given by equation (3), the typical degree of separation of distant nodes is

$$\tilde{D}(l) \approx -\frac{\ln Q(l)}{\ln z}. \tag{4}$$

Since in two dimensions, the typical distance l scales with the network size S as $\sqrt{S}$. Consequently, for the distance dependence given by equation (3), the typical topological distance of two nodes in the network scales as $\tilde{D} \sim \ln S$. For the human acquaintances network is $S = 6\times 10^9$ and hence $l \approx 80\,000$; when α lies in the range 2.5–3.5, and z in the range 50–500, we obtain $\tilde{D}$ in the range 3–10. In addition, by numerical integration of equation (2), for the described parameters the mean clustering coefficient lies in the range 0.05–0.30. We can conclude that the given network exhibits the small-world phenomenon.

3 The degree distribution

Let's choose one node of the network, we label it as X. The plane can be divided into thin concentric rings centered at X. If the ring radius is r and its width is w, it covers approximately $N = 2\pi rw$ vertices. Meanwhile, all vertices in one ring have approximately the same distance from X. Therefore they also have approximately the same probability $Q(r) := p$ to be connected with X. Since links are drawn independently, the number of neighbours of X in the ring with radius r, $n(r)$, is a random quantity with the binomial distribution whose mean is Np and the variance is $Np(1-p) = 2\pi rwQ(r)[1-Q(r)]$.

The degree k of node X is obtained by summing $n(r)$ over all rings. The central limit theorem applies here and thus k is normally distributed and its variance σ_k^2 is the sum of variances of all contributions $n(r)$. Replacing the summation over all rings by the integration we obtain

$$\sigma_k^2 \approx \int_0^\infty 2\pi r Q(r)\big[1 - Q(r)\big]\,\mathrm{d}r = 2z/n. \tag{5}$$

To confirm this result numerically, in Table 1 the quantity $n\sigma_k^2/z$ is shown for various values of z and n. As can be seen, the numerical results are well approximated by

Table 1. Numerical estimates of $n\sigma_k^2/z$ for various values of α and z on the square lattice with the dimensions 1000×1000 (4000×4000 for $\alpha = 2.5$), the variance σ_k^2 is obtained from 10 000 realisations of the model.

	$z = 50$	$z = 150$	$z = 500$
$\alpha = 2.5$	1.96	1.94	1.88
$\alpha = 3.0$	1.97	2.03	1.93
$\alpha = 3.5$	2.00	2.05	2.00

the analytical prediction $n\sigma_k^2/z = 2$ for a wide range of parameters.

We can conclude that the node degree k has approximately the Gaussian distribution with the mean z and the variance $2z/n$. For values of z resembling a real society (z of the order of hundreds) it follows that $\sigma_k \ll z$ and thus the degree distribution is sharply peaked around its mean value (narrowness of the degree distribution is clearly visible in Fig. 1). This is in a clear contradiction with the empirical studies [21–24] which suggest power-law behavior. The resulting social network is strongly homogeneous – it lacks nodes exceeding others in degree by orders of magnitude. In the following section we investigate how this basic model can be modified to produce a heterogeneous network and exhibit a broad degree distribution.

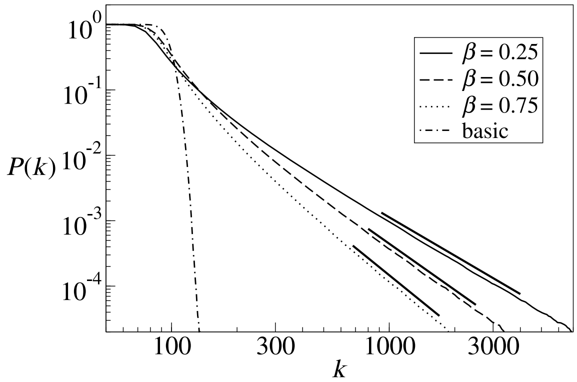

Fig. 1. The cumulative degree distribution $P(k)$ plotted in a logarithmic scale. The thick lines have slopes 1.95, 2.35, and 2.70 respectively (values predicted by equation (9) are 1.88, 2.25, and 2.63 respectively). The probability distributions were obtained by 20 realisations of the model on the lattice with dimensions 600×600 for $\alpha = 3$ and $z = 100$. For a comparison, the degree distribution of the basic homogeneous model is also shown.

4 Heterogeneous network model

The probability distribution $Q(d)$ given by equation (3), fundamental for this model, has two natural parameters: b and α. Heterogeneity can be introduced to the network by assigning random values of these parameters to each node (with the constraints $b > 0$, $\alpha > 2$). To keep the acquaintance relation symmetric we symmetrize the probability $Q_{ij}(d)$ that persons i and j with the distance d know each other by the relation

$$Q_{ij}(d; b_i, \alpha_i, b_j, \alpha_j) := \frac{Q(d; b_i, \alpha_i) + Q(d; b_j, \alpha_j)}{2}. \quad (6)$$

To simplify our calculations we assume that α is fixed in the network and only b is a random quantity drawn from the distribution $\varrho(b)$. The parameter b we call the node solitariness (as b grows, the number of acquaintances is decreasing and their average distance is getting smaller). The average degree of a vertex with the solitariness b is now

$$\begin{aligned} z(b) &= \int_0^\infty \frac{Q(r; b) + Q(r; b')}{2}\, 2\pi r \varrho(b')\, \mathrm{d}r\, \mathrm{d}b' \\ &= \int_0^\infty Q(r; b) \pi r\, \mathrm{d}r + \frac{\langle z(b)\rangle}{2} = \frac{\pi^2 b^{-2/\alpha}}{\alpha \sin(2\pi/\alpha)} + \frac{z}{2}. \end{aligned} \quad (7)$$

Here we again replaced the summation by an integration; $\langle z(b)\rangle$ is the average connectivity in the network which we already labeled as z. As the solitariness b of a vertex goes to zero, $z(b)$ goes to infinity. By contrast, as b increases to infinity, $z(b)$ has a lower bound which is equal to $z/2$.

According to our previous discussions, we would like to generalize the model to exhibit a wide connectivity distribution. To achieve this, high-degree nodes with small values of b must be present. However, the value $b = 0$ is pathological for it makes the probability distribution $Q(d; b)$ flat and creates a node with an infinite degree (if the network itself is infinite). Thus for the distribution $\varrho(b)$ we require $\varrho(0) = 0$. The simplest possible choice is $\varrho(b) = K b^\beta$ for $b \in (0; B]$, $\beta > 0$. Values of K and B are fixed by the condition $\langle z(b)\rangle = z$ and by the normalization of $\varrho(b)$, leading to

$$\begin{aligned} B &= \left[\frac{(\alpha + \alpha\beta - 2) z \sin(2\pi/\alpha)}{(1+\beta) 2\pi^2}\right]^{-\alpha/2}, \\ K &= (1+\beta)\, B^{-(1+\beta)}. \end{aligned} \quad (8)$$

Since high degrees are due to small values of b, the results derived below hold for all $\varrho(b)$ which can be approximated by $\varrho(b) \sim b^\gamma$ for b small. Thanks to the constraint $\varrho(0) = 0$ and the Taylor expansion, this is already a quite general class of functions. However, in this paper we focus on the power-law $\varrho(b)$ which allows us to investigate the model analytically.

First we show that the chosen form of $\varrho(b)$ leads to the desired fat distribution of connectivities. For the distribution of $z(b) := \overline{k}$ we have

$$g(\overline{k}) = \varrho(b) \Big/ \left|\frac{\mathrm{d}\overline{k}}{\mathrm{d}b}\right| = \frac{2K}{\alpha}\, b^{1+\beta+2/\alpha}.$$

Consequently, using equation (7) and assuming $\overline{k} \gg z$ we obtain

$$g(\overline{k}) \sim \overline{k}^{\,-(1+\alpha/2+\alpha\beta/2)}.$$

We already know that when b is given, the probability distribution of the vertex degree is sharply peaked. Therefore we can approximate the distribution $f(k)$ which we are searching for by the distribution $g(\overline{k})$ of the mean degree. Then we obtain

$$f(k) \sim k^{-(\alpha\beta+\alpha+2)/2}, \quad P(k) \sim k^{-(\alpha\beta+\alpha)/2}. \tag{9}$$

Here $P(k)$ is the cumulative probability distribution of the vertex degree. We see, that for the chosen $\varrho(b)$, the degree distribution has a power-law tail. In Figure 1, this analytical result is compared with a numerical simulation of the model for $z = 100$ and $\alpha = 3$. The power-law character of $P(k)$ is clearly visible for $k \gtrsim 300$ and the approximate values of the power-law exponents confirm equation (9).

Now we show that the modified network model still exhibits the small world phenomenon. The probability that two vertices with a fixed distance l have the degree of separation D we label as $P(D)$. We can examine this quantity by techniques similar to those presented in [18]. There it was shown that in the resulting homogeneous network, the first approximation of $P(D)$ has the form

$$P(D)_{\mathrm{HO}} \approx (D+1)z^D Q(l). \tag{10}$$

The derivation of a similar result $P(D)_{\mathrm{HE}}$ for the heterogeneous network model proposed here can be found in Appendix A; it is well defined only when $\alpha\beta > 2$. In Figure 2, the resulting ratio $\xi(D) := P(D)_{\mathrm{HE}}/P(D)_{\mathrm{HO}}$ is shown as a function of β. Notice that in the limit $\beta \to \infty$ all ratios $\xi(D)$ approach to 1. This is because as β increases, a higher weight is given to values of b close to the upper bound B. In particular, in the limit $\beta \to \infty$ all nodes share the same value of solitariness, B. Thus we can say that the proposed generalization is in the limit $\beta \to \infty$ equivalent to the original model. One can notice that $\xi(D) > 1$ for all D. This means that in the proposed heterogeneous network the probability to find a path of a certain length is higher than in the homogeneous network. In other words, hubs (nodes with a high degree) present in the heterogeneous network facilitate formation of short paths. Therefore, for the typical degrees of separation the inequality $\tilde{D}_{\mathrm{HE}} < \tilde{D}_{\mathrm{HO}}$ holds. On the other hand, since the ratios $P(D+1)/P(D)$ (which are of order of z) are much larger then the ratios $\xi(D)$ shown in Figure 2, we can also say that the introduction of heterogeneity to the network does not change the typical degree of separation substantially and $\tilde{D}_{\mathrm{HE}} \approx \tilde{D}_{\mathrm{HO}}$.

The average clustering coefficient $\langle C \rangle$ cannot be treated analytically and therefore in Figure 2 we present only numerical results. They confirm the expected fact that $\langle C \rangle$ is little sensitive to changes of the model parameters and thus it is almost as high as in the original model. One can also notice that with increasing β, $\langle C \rangle$ approaches to the value 0.161 valid for the original model (this value is taken from [18], in Figure 2 it is shown as a dashed line). This limit behavior is similar to the limit behavior of $\xi(D)$. Since we observe both a small typical degree of separation and a high average clustering coefficient, the heterogeneous network exhibits the small world phenomenon.

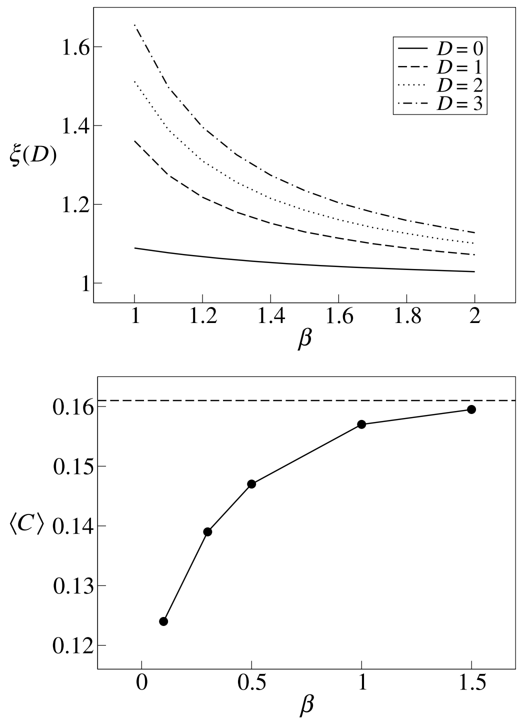

Fig. 2. Changes of the the main network properties with β for $z = 300$ and $\alpha = 3$. In the upper figure, the ratio $\xi(D) := P(D)_{\mathrm{HE}}/P(D)_{\mathrm{HO}}$ is drawn according to equations (10) and (11). The course of $\langle C \rangle$ in the lower figure has been obtained by a numerical simulation of the model with averaging over 1000 realizations; the dashed line presents the limiting value of the clustering coefficient in the limit $\beta \to \infty$.

5 Conclusion

In this paper we investigated a network model where links are drawn according to nodes distances. Building on the basic model [18], we proposed a generalization aiming to introduce heterogeneity to the network and also fat tails to the degree distribution. First, a hidden random parameter b is assigned randomly to each vertex of the network. Then between a pair of nodes, a link is drawn with the probability depending on the hidden parameter values of these two nodes. As a result we obtain highly heterogeneous network which exhibits a power-law distribution $P(k)$ over a large range of connectivities. With respect to the social interpretation of the model, one can say that it produces a social network where highly sociable party goers are present along with loners. The proportion of highly connected nodes can be adjusted by the distribution, from which the hidden parameter b is drawn – in this work we focused on the simple distribution $\varrho(b) = Kb^{\beta}$. We also showed that for the resulting network, the typical degree of separation is small and the average clustering coefficient is high; both are approximately equal to the corresponding values for the homogeneous network with same z and α. Thus we conclude the small world phenomenon is present in the networks produced by the proposed model.

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Appendix A: Calculation of P(D) in the heterogeneous network

To obtain an approximate expression for $P(D)$, the treatment is similar to the treatment of the homogeneous network model in [18]. As we will see, differences and complications arise from the additional averaging over possible values of the solitariness b with $\varrho(b)$.

We pick two nodes with a large geometrical distance l, let's label them X and Y. As an illustrative example we examine the probability $P(2)$. That is, we examine paths between X and Y that have two intermediate vertices (Fig. A.1, left). Notice that $D = 2$ requires that links $X2$, $1Y$, and XY are not present. Since probabilities of these links are small, to obtain a first approximation of $P(D)$ we neglect that such shortcuts may occur. Consequently, the diagram for $P(2)$ is simplified (Fig. A.1, right) to the existence of edges $X1$, 12, and $2Y$.

Another simplification comes from the form of $Q(d)$ given by equation (3). It is easy to check that when l is large, for $d \ll l$ holds $Q(l-d)Q(d) \gg Q(l/2)Q(l/2)$. This means that among all paths $X12Y$, the fundamental contribution comes from those which contain only one long link. Moreover, for $d \ll l$ we have also $Q(l-d) \approx Q(l)$ and therefore the probability of the long link can be approximated by $Q(l)$. As a result we can further simplify the right diagram in Figures A.1 to A.2 where the only three diagrams substantially contributing to $P(2)$ are shown (three different possibilities appear because there are three ways to choose the long link in the path $X12Y$; since probability of the long link is always approximately equal to $Q(l)$, the link is drawn between X and Y).

In the basic network model, the contribution of the left most diagram in Figure A.2 to $P(2)$ is

$$P_1 = \iint_{1,2} Q(d_{X1})Q(d_{12})Q(l)\,\mathrm{d}\boldsymbol{r}_{X1}\mathrm{d}\boldsymbol{r}_{12} = Q(l)z^2.$$

Since the remaining two diagrams give the same result, together we have $P(2) \approx 3z^2Q(l)$ which agrees with equation (10). In the modified model of a heterogeneous network, $Q(d)$ is generalized to $Q(d;b)$ and the connection probability is symmetrized by $[Q(d;b_1)+Q(d;b_2)]/2$. Then for the left most diagram shown in Figure A.2 we encounter the complex expression $[Q(d_{X1};b_X) + Q(d_{X1};b_1)]\times[Q(d_{12};b_1)+Q(d_{12};b_2)]\times[Q(l;b_2)+Q(l;b_Y)]$. Moreover, in addition to the integration over $\boldsymbol{r}_{X1},\boldsymbol{r}_{12}$, we also have to integrate over b_X, b_Y, b_1, b_2. Then we encounter the following integrals

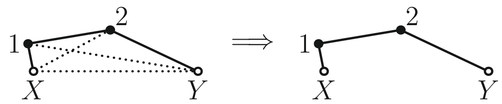

Fig. A.1. The exact diagram for $P(2)$ (left) and the approximate one utilizing the properties of $Q(d)$ (right).

Fig. A.2. Three diagrams contributing substantially to $P(2)$.

$$\int_0^B \overline{k}(b)\varrho(b)\,\mathrm{d}b :=z,$$

$$\int_0^B Q(l;b)\varrho(b)\,\mathrm{d}b \approx \frac{\beta+1}{\beta}\,Q(l;B) := K_1Q(l;B)$$

$$\int_0^B\int_1 Q(r_1;b)Q(l;b)\varrho(b)\,\mathrm{d}b\,\mathrm{d}\boldsymbol{r}_1 \approx \frac{\alpha+\alpha\beta-2}{\alpha\beta-2}\,zQ(l;B) := K_2zQ(l;B),$$

$$\int_0^B\iint_{1,2} Q(r_1;b)Q(r_2;b)\varrho(b)\,\mathrm{d}b\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \approx \frac{(\alpha+\alpha\beta-2)^2}{\alpha(1+\beta)(\alpha+\alpha\beta-4)}\,z^2 := K_3z^2.$$

The third integral converges when $\alpha\beta > 2$, the fourth when $\alpha+\alpha\beta > 4$ (since $\alpha > 2$, this is a weaker restriction).

Using the steps and notation introduced above we finally obtain the approximate result

$$P(D)_{\mathrm{HE}} \approx \frac{L(D)}{2^D}\,Q(l;B)z^D \qquad (11)$$

where

$$L(0) = K_1,$$
$$L(1) = 3K_1 + K_2,$$
$$L(2) = 6K_1 + 4K_2 + 2K_1K_3,$$
$$L(3) = 10K_1 + 10K_2 + 10K_1K_3 + 2K_2K_3,$$
$$L(4) = 15K_1 + 20K_2 + 30K_1K_3 + 12K_2K_3 + 3K_1K_3^2, \ldots$$

For $D > 4$ we obtain even more complicated expressions. Nevertheless, since the typical degree of separation is small in the discussed model, this is not a crucial complication and the solution is tractable. We can notice that in the limit $\beta \to \infty$ we have $K_1, K_2, K_3 \to 1$ and therefore $P(D)_{\mathrm{HE}} = P(D)_{\mathrm{HO}}$ as expected.

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