

Schläfli numbers and reduction formula

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Abstract

We define so-called poset-polynomials of a graded poset and use it to give an explicit and general description of the combinatorial numbers in Schläfli's (combinatorial) reduction formula. For simplicial and simple polytopes these combinatorial numbers can be written as functions of the numbers of faces of the polytope and the tangent numbers. We use the constructed formulas to determine the volume of 4-dimensional Coxeter polytopes.

1. Introduction

The fundamental qualitative difference between topological spaces of even and odd dimensions is naturally reflected in many results of geometry. Important examples are the *Descartes–Euler–Schläfli formula* (the generalization of the well-known Euler formula for polyhedrons) which shows, that the Euler characteristic of a polytope (the higher-dimensional analog of a polyhedron) only depends on the parity of the dimension of the polytope; the so-called *hairy ball theorem*, which proves the non-existence of continuously differentiable fields of unit tangent vectors on even-dimensional spheres and the *Gauss–Bonnet theorem*, which presents the Euler characteristic of a compact even-dimensional Riemannian manifold as the integral of a certain function (derived from the curvature of the manifold).

The solution of the volume problem for polytopes in spaces \mathbb{X}^n of constant curvature $\kappa \neq 0$ requires a strategy adapted to the parity of the dimension. An elegant starting-point to illuminate the difference of parity in this connection is *Schläfli's differential formula*, which links the volume differential of an n -dimensional polytope P to the volumes of the $(n - 2)$ -dimensional faces of P . This formula can be used to deduce inductively volume functions for a fixed parity

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of the dimension. Starting with the assumption $\mathbf{vol}(P^0) = 1$, we obtain for a $2m$ -dimensional polytope P in $\mathbb{X}^{2m} = \mathbb{S}^{2m}$ with constant curvature $\kappa = 1$, \mathbb{E}^{2m} with $\kappa = 0$ or \mathbb{H}^{2m} with $\kappa = -1$ the so-called *Schläfli (combinatorial) reduction formula*, which can be written as

$$2\kappa^m c_{2m}^{-1} \mathbf{vol}(P) = \sum_{\substack{P^{2d} \in \Omega^{2d}(P) \\ d=0, \dots, m}} \sigma^{2d}(P^{2d}) \alpha(P^{2d}|P).$$

Here c_{2m} denotes the volume of the $2m$ -dimensional unit sphere, $\Omega^{2d}(P)$ is the set of all $2d$ -dimensional ordinary faces of P and $\alpha(P^{2d}|P)$ is the $(2m - 2d - 1)$ -dimensional normalized angular measure at the apex P^{2d} . Furthermore, it can be viewed as a higher-dimensional analog of the well-known fact that the sum of the (non-normalized) angular measures in a Euclidean polygon is equal to $\pi(f^0(P) - 2)$ and that the area of a spherical and hyperbolic polygon is equal to the absolute value of its angular excess. This formula reduces the volume problem in spherical and hyperbolic space of even dimension to the determination of odd-dimensional spherical volumes. The coefficients $\sigma^{2d}(P^{2d})$ are rational numbers which only depend on the combinatorics of the face P^{2d} . In honour of Schläfli, these numbers are called *Schläfli numbers* and Schläfli identified these constants for simplices to be the tangent numbers. Furthermore, he noticed that the Schläfli numbers can also be determined by recursion formulas.

In the case where P is a fundamental polytope for a discrete subgroup Γ of isometries of the underlying space \mathbb{X}^{2m} of even dimension (for instance, if P is a Coxeter polytope), the number $2\kappa^m c_{2m}^{-1} \mathbf{vol}(P)$ is nothing else than the Euler characteristic $\chi(\Gamma)$ of the group Γ (or of the geometric orbifold \mathbb{X}^n/Γ). An insight into the algebraic aspects of the Euler characteristic of discrete groups gives [11]. For an overview of geometric orbifolds (and their Euler characteristics) we recommend [12] (chapter 13).

The main problem is the explicit determination of the rational numbers $\sigma^{2d}(P^{2d})$ which reflect in some mysterious way the combinatorics of the polytope P . The aim of this paper is to give an explicit description of the numbers $\sigma^{2j}(P^{2j})$.

We remark that Schläfli (see [10], page 255) also proved another type of reduction formula for so-called spherical orthoschemes (of degree 0). This (analytic) reduction formula was generalized by Kellerhals (see [5]) to hyperbolic orthoschemes of all degrees. For a general overview of the volume problem in hyperbolic space we recommend the general survey article [6] to the interested reader.

In Section 2 we deal with partially ordered sets or posets. In particular, we define the so-called poset polynomial.

Each coefficient of this polynomial is the number of chains of a fixed length. The evaluation of this polynomial at $1/2$, called the Schläfli number of the poset, can be expressed in terms of poset-polynomials of intervals.

In Section 3 we explain the basics about polytopes and polytopal complexes in spaces of constant curvature, angles and decompositions. We give a short summary of the (linear) relations between the angular measures of polytopes in spaces of constant curvature. Especially, we explain two different constructions of Schläfli's reduction formula.

In Section 4 we prove the main theorem which gives an explicit description of the Schläfli numbers for even-dimensional polytopes P as the poset-polynomial of the even face lattice of P evaluated at the point $1/2$.

For simplicial polytopes there is another way to describe the Schläfli numbers. These polytopes can be decomposed into simplices in a canonical way (cone-decomposition via an inner point). We use Schläfli's reduction formula for simplices and add the formulas of the

decomposition blocks. So we get a reduction formula for simplicial polytopes and can read off the Schläfli numbers which are functions of the numbers of faces and the so-called tangent numbers.

In Section 5, we consider specific formulas for simple and simplicial polytopes.

Finally, the [Appendix](#) contains a complete list of the 4-dimensional hyperbolic Coxeter polytopes, that are pyramids over a 3-dimensional prism, with their volumes.

2. The Schläfli number of a graded poset

Let $\mathcal{P} = (\mathcal{P}, \geq)$ be a poset. The *length* of \mathcal{P} is defined by $\ell(\mathcal{P}) = \max\{\ell(\mathcal{C}) : \mathcal{C} \text{ is a chain of } \mathcal{P}\}$ and the length of an interval $[x, y] := \{z \in \mathcal{P} : x \leq z \leq y\}$ is denoted by $\ell(x, y)$. We denote by $\hat{\mathcal{P}}$ the poset obtained from \mathcal{P} by adjoining a minimal and a maximal element $\hat{0}$ and $\hat{1}$. If \mathcal{P} is a poset with minimal and maximal element, we denote by \mathcal{P}_r the poset obtained from \mathcal{P} by removing $\hat{0}$ and $\hat{1}$.

Furthermore, we denote by $c^i(\mathcal{P})$ the number of chains in \mathcal{P} of length i , for $i = 0, \dots, \ell(\mathcal{P})$; by convention $c^{-1}(\mathcal{P}) := 1$ and $c^i(\mathcal{P}) := 0$ for all $i \leq -2$.

Now we define the so-called *poset-polynomial* of a poset \mathcal{P} as

$$\mathbf{f}(\mathcal{P}, \lambda) := \sum_{i \geq -1} (-1)^{i+1} c^i(\mathcal{P}) \lambda^{i+1}.$$

The number $\mathbf{f}(\mathcal{P}, \frac{1}{2})$ is called the *Schläfli number* of the poset \mathcal{P} .

In the following let \mathcal{P} always be a *graded poset of rank n* with rank function $\rho : \mathcal{P} \rightarrow \{0, 1, 2, \dots, n\}$ and with minimal and maximal element $\hat{0}$ and $\hat{1}$. For all $0 \leq i \leq n$ let $\mathcal{P}^i := \{x \in \mathcal{P} : \rho(x) = i\}$ be the set of all elements of rank i . Let $\mu \in \mathbb{N}$ with $0 \leq \mu \leq n-2$ and $(l_0, \dots, l_{\mu-1}, l_\mu)$ be a $(\mu+1)$ -tuple of natural numbers with $0 < l_0 < \dots < l_{\mu-1} < l_\mu < n$. Then a $(l_0, \dots, l_{\mu-1}, l_\mu)$ -chain in \mathcal{P}_r is a chain $x_0 < \dots < x_{\mu-1} < x_\mu$ in \mathcal{P} such that $\rho(x_i) = l_i$ for all $0 \leq i \leq \mu$.

Of course, the length of a $(l_0, \dots, l_{\mu-1}, l_\mu)$ -chain is equal to μ . Furthermore, let $A(l_0, \dots, l_{\mu-1}, l_\mu)(\mathcal{P}_r)$ be the number of all $(l_0, \dots, l_{\mu-1}, l_\mu)$ -chains in \mathcal{P}_r . For all $(\mu+1)$ -tuples which do not satisfy $0 < l_0 < \dots < l_{\mu-1} < l_\mu < n$ we set $A(l_0, \dots, l_{\mu-1}, l_\mu)(\mathcal{P}_r) := 0$. These numbers are related to the numbers of chains of \mathcal{P}_r by the following equations

$$c^\mu(\mathcal{P}_r) = \sum_{0 < l_0 < \dots < l_{\mu-1} < l_\mu < n} A(l_0, \dots, l_{\mu-1}, l_\mu)(\mathcal{P}_r) \quad (1)$$

for all $0 \leq \mu \leq n-2$.

By a simple calculation we get

$$\begin{aligned} A(l_0)(\mathcal{P}_r) &= \sum_{x \in \mathcal{P}^{l_0}} 1 \quad \text{and} \\ A(l_0, \dots, l_{\mu-1}, l_\mu)(\mathcal{P}_r) &= \sum_{x \in \mathcal{P}^{l_\mu}} A(l_0, \dots, l_{\mu-1})([\hat{0}, x]_r) \end{aligned} \quad (2)$$

for all $0 < l_0 < \dots < l_{\mu-1} < l_\mu < n$ and $0 \leq \mu \leq n-2$.

Lemma 1. *Let \mathcal{P} be a graded poset of rank $n = \ell(\mathcal{P})$ with minimal and maximal element $\hat{0}$ and $\hat{1}$. Then*

$$c^\mu(\mathcal{P}_r) = \sum_{j=1}^{n-1} \sum_{x \in \mathcal{P}^j} c^{\mu-1}([\hat{0}, x]_r) \quad (3)$$

$$\mathbf{f}\left(\mathcal{P}_r, \frac{1}{2}\right) = 1 - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{x \in \mathcal{P}^j} \mathbf{f}\left([\hat{0}, x]_r, \frac{1}{2}\right) \quad (4)$$

for all $0 \leq \mu \leq n-2$.

Proof. For $n = 0$ and 1 there is nothing to show. Now let $n \geq 2$. We have by (1) and (2)

$$\begin{aligned} c^\mu(\mathcal{P}_r) &= \sum_{j=1}^{n-1} \sum_{0 < l_0 < \dots < l_{\mu-1} < j} A(l_0, \dots, l_{\mu-1}, j)(\mathcal{P}_r) \\ &= \sum_{j=1}^{n-1} \sum_{x \in \mathcal{P}^j} \sum_{0 < l_0 < \dots < l_{\mu-1} < j} A(l_0, \dots, l_{\mu-1})([\hat{0}, x]_r) \\ &= \sum_{j=1}^{n-1} \sum_{x \in \mathcal{P}^j} c^{\mu-1}([\hat{0}, x]_r), \end{aligned}$$

where we have used the formulas (1) to describe the numbers of chains in the poset $[\hat{0}, x]_r$. This proves the first part of the lemma from which we further get by a straightforward calculation

$$\begin{aligned} \mathbf{f}\left(\mathcal{P}_r, \frac{1}{2}\right) &= \sum_{i \geq -1} (-1)^{i+1} c^i(\mathcal{P}) \left(\frac{1}{2}\right)^{i+1} \\ &= 1 - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{x \in \mathcal{P}^j} \mathbf{f}\left([\hat{0}, x]_r, \frac{1}{2}\right). \quad \square \end{aligned}$$

3. Polytopal complexes

Throughout this chapter let \mathbb{X}^n be \mathbb{S}^n , \mathbb{E}^n or \mathbb{H}^n , if not specified otherwise.

3.1. Definitions

An n -dimensional (convex and generalized) polytope P in \mathbb{X}^n is for $\mathbb{X}^n = \mathbb{S}^n$ the \mathbb{S}^n -convex hull of finitely many points which are contained in an open hemisphere; the \mathbb{E}^n -convex hull of finitely many points for $\mathbb{X}^n = \mathbb{E}^n$ and for $\mathbb{X}^n = \mathbb{H}^n$ the \mathbb{H}^n -convex hull of finitely many ordinary points and points at infinity, which contains an open set of \mathbb{X}^n . Then P may not be compact but it is always of finite volume. It turns out that P is the intersection of finitely many closed half-spaces \overline{H}_i (see [16], Theorem 1.1, page 29): $P = \bigcap_{i \in I} \overline{H}_i$. Furthermore, we define

$$P_J := P \cap \left(\bigcap_{j \in J} H_j \right)$$

for all subsets $J \subset I$; H_j denotes the hyperplane corresponding to the closed half-space \overline{H}_j .

A polytopal complex Π in \mathbb{X}^n is a finite set of n -dimensional polytopes in \mathbb{X}^n such that if a polytope belongs to Π then so do all its faces and the intersection of two polytopes in Π is a face of both polytopes. For all d with $1 \leq d \leq n$ let $f^d(\Pi)$ be the number of d -dimensional faces, $f_{\text{ord}}^0(\Pi)$ the number of ordinary 0-dimensional faces and $f_{\text{inf}}^0(\Pi)$ the number of points at infinity which are contained in Π . Clearly, for $\Pi \subseteq \mathbb{S}^n$ we have $f_{\text{inf}}^0(\Pi) = 0$.

Furthermore let $f^0(\Pi) := f_{\text{ord}}^0(\Pi) + f_{\text{inf}}^0(\Pi)$ and $f^{-1}(\Pi) := 1$. For all d with $0 \leq d \leq n$ let $\Omega^d(\Pi) := \{P_1^d, P_2^d, \dots, P_{f^d(\Pi)}^d\}$ be the set of d -dimensional faces of Π . For $d = 0$ this means that $\Omega^0(\Pi)$ is the set of ordinary vertices and vertices at infinity in Π .

Let Π be a polytopal complex in \mathbb{X}^n . A *polytopal decomposition* $\mathcal{D} = \mathcal{D}(\Pi)$ is a polytopal complex, such that $|\mathcal{D}| = \Pi$ and each element of \mathcal{D} is contained in an element of Π .

3.2. Combinatoric of polytopal complexes

We denote by \mathbf{P}^n the set of all n -dimensional polytopes in the spaces \mathbb{S}^n , \mathbb{E}^n and \mathbb{H}^n . Let P be an element of \mathbf{P}^n . The *face poset* $\mathcal{F}(P)$ of P is the set of all faces of P , partially ordered by inclusion. Of course, $\mathcal{F}(P)$ is graded and we have $\rho(F) = \dim(F) - 1$ for all $F \in \mathcal{F}(P)$. Let P_1 and P_2 be elements in \mathbf{P}^n . Then P_1 and P_2 are called *combinatorially isomorphic*, denoted by $P_1 \sim P_2$, if the two face posets $\mathcal{F}(P_1)$ and $\mathcal{F}(P_2)$ are isomorphic (as partially ordered sets). This is an equivalence relation on the set \mathbf{P}^n and we denote by \mathbf{P}^n_{\sim} the set of all equivalence classes.

Furthermore, we write $\mathcal{E}(P)$ for the *even face poset* of P ; this is the subposet of $\mathcal{F}(P)$ consisting of all proper even-dimensional faces of P , with minimal and maximal element added. Clearly, $\mathcal{E}(P)$ is graded. Let $P \in \mathbf{P}^{2m}$. Then we have a canonical map

$$\begin{aligned} h : \Omega^{2d}(P) &\longrightarrow \mathcal{E}(P) \\ P^{2d} &\longmapsto h(P^{2d}) \end{aligned}$$

which maps a $2d$ -dimensional face of P on the corresponding element $h(P^{2d})$ in $\mathcal{E}(P)$ for all $d = 0, \dots, m$. This is a bijective map between $\Omega^{2d}(P)$ and the set of elements in $\mathcal{E}(P)$ of rank $d + 1$. Furthermore, the even face poset $\mathcal{E}(P^{2d})$ is isomorphic to the interval $[\hat{0}, h(P^{2d})]$ in $\mathcal{E}(P)$.

3.3. Angles

Let P be an n -dimensional polytope in \mathbb{X}^n . In the following we will define the notion of an $(n - k - 1)$ -dimensional angle of P at a face P^k . So let P^k be an element in $\Omega^k(P)$, for $0 \leq k \leq n - 1$, that is not a vertex at infinity of P . Further let x be an interior point of P^k and let N be the $(n - k)$ -dimensional plane, passing through x and orthogonal to the k -dimensional plane determined by P^k . The $((n - k)$ -dimensional) *angle* of P at a face P^k , denoted by $P^k|P$, is defined as the cone $C_x(P) \subset T_x N$, formed by the tangent rays to (geodesic) segments xy for all $y \in N \cap P$. It is a convex polytopal cone in the space $T_x N$, which does not depend on the choice of the point x . A 1-dimensional angle of P is also called a *dihedral angle*. The face P^k of P is called the *apex* of the angle $P^k|P$.

The *angular measure* $\mu(P^k|P)$ of the angle $P^k|P$ is the volume of its intersection with the unit sphere $S^{n-k-1}(x, 1) \subset T_x N$. This is nothing else than the (spherical) volume of the spherical polytope, defined by the cone $P^k|P$ and we denote this polytope by $r(P^k|P)$.

The computation of this measure can be viewed as a problem of integration.

The $(n - k - 1)$ -dimensional *normalized angular measure* of P at a face P^k is defined as

$$\alpha(P^k|P) := \frac{\mu(P^k|P)}{c_{n-k-1}}$$

for $0 \leq k \leq n - 1$ and $\alpha(P|P) := 1$. Furthermore, if $k = 0$ and P^k is a vertex at infinity we define $\alpha(P^k|P) := 0$. The constant c_m denotes the volume of the m -dimensional unit sphere.

The notion of an angle and the normalized angular measure of P at a face P^k can be generalized to polytopal complexes Π of dimension n in \mathbb{X}^n as follows. Let P^k be an element in $\Omega^k(\Pi)$ for $0 \leq k \leq n$. Then P^k is included in a finite number of elements P_1^n, \dots, P_h^n in $\Omega^n(\Pi)$ of maximal dimension such that $P^k = P_1^n \cap \dots \cap P_h^n$. Furthermore, the number h is maximal, which means that there are no other elements in $\Omega^n(\Pi)$ containing P^k . The $(n-k-1)$ -dimensional normalized angular measure of Π at a face P^k is defined as

$$\alpha(P^k|\Pi) := \sum_{i=1}^h \alpha(P^k|P_i^n) = \sum_{P^n \in \Omega^n(\Pi)} \alpha(P^k|P^n)$$

for $0 \leq k \leq n$, where we have defined $\alpha(P^k|P^n) = 0$ if P^k is not a face of the polytope P^n .

3.4. Schläfli's differential formula

Let P be an n -dimensional polytope in the space $\mathbb{X}^n = \mathbb{S}^n$ or \mathbb{H}^n . If it is deformed differentially in such a way, that the combinatorial structure does not change, then the volume of P changes differentially and we have

$$\kappa \mathbf{dvol}(P) = \frac{1}{n-1} \sum_{P^{n-2} \in \Omega^{n-2}(P)} \mathbf{vol}(P^{n-2}) d\alpha(P^{n-2}|P). \quad (5)$$

This formula, which links the volume differential of a polytope to the volume of the faces of codimension 2, was established by Schläfli for spherical simplices (see [10], page 235). Kneser (see [4]) gave a second proof for both spherical and hyperbolic simplices. Furthermore, these results can easily be generalized to polytopes by simplicial decomposition.

3.5. Poincaré's formula

Let P be an n -dimensional polytope in \mathbb{X}^n . Then

$$\sum_{\substack{P^d \in \Omega^d(P) \\ d=0, \dots, n}} (-1)^d \alpha(P^d|P) = \begin{cases} 2\kappa^m c_{2m}^{-1} \mathbf{vol}(P), & n = 2m \text{ even} \\ 0, & n = 2m + 1 \text{ odd.} \end{cases}$$

Poincaré (see [9]) proved this formula for spherical simplices and Hopf (see [3]) generalized this result to simplices in all spaces of constant curvature. A generalization to arbitrary polytopes is easily done by decomposition into simplices.

If we restrict to the case $\mathbb{X}^n = \mathbb{E}^n$, which means $\kappa = 0$, we get for all dimensions

$$\sum_{\substack{P^d \in \Omega^d(P) \\ d=0, \dots, n}} (-1)^d \alpha(P^d|P) = 0 \quad \text{or} \quad \sum_{\substack{P^d \in \Omega^d(P) \\ d=0, \dots, n-1}} (-1)^d \alpha(P^d|P) = (-1)^{n+1}.$$

This is the Gram–Sommerville formula for Euclidean polytopes (see [2], Section 14.1).

3.6. Schläfli's reduction formula

Let P be a $2m$ -dimensional polytope in \mathbb{X}^{2m} . Then there exist rational coefficients $\sigma^{2d}(P^{2d})$ for each $P^{2d} \in \Omega^{2d}(P)$ ($d = 0, \dots, m$), depending only on the combinatorics of the face P^{2d} ,

such that

$$2\kappa^m c_{2m}^{-1} \text{vol}(P) = \sum_{\substack{P^{2d} \in \Omega^{2d}(P) \\ d=0, \dots, m}} \sigma^{2d}(P^{2d}) \alpha(P^{2d}|P). \quad (6)$$

The rational number $\sigma^{2d}(P^{2d})$ is called *Schläfli number* of the polytope P^{2d} .

This formula was proved by Schläfli (see [10], page 280) for spherical polytopes by using his differential formula and it can be generalized by results of Hopf (see [3]) to all spaces of constant curvature. We remark that this proof does not give a method to determine the Schläfli numbers explicitly.

However, Schläfli identified the Schläfli numbers if P^{2d} is a simplex $\sigma^{2d}(P^{2d}) = 2(-1)^d T_{2d+1}$, where T_{2d+1} ($d = 0, 1, \dots$) are so-called tangent numbers and he described shortly an algorithm to determine the Schläfli numbers for arbitrary polytopes (see [10], page 281). See Section 4 for details.

Another, more combinatorial way to prove Schläfli's reduction formula and to determine the Schläfli numbers of simplices is due to Peschl (see [8]).

We start with Poincaré's formula for the volume of P . Each term $\alpha(P^d|P)$ corresponding to a face P^d of odd codimension (which means, that $2m - d - 1$ is even) is the (normalized) volume of the even-dimensional polytope $r(P^d|P)$. Hence, we can express $\alpha(P^d|P)$, by using a (lower-dimensional) Poincaré formula, in terms of lower dimensional angles of the polytope $r(P^d|P)$. Finally, all angles of $r(P^d|P)$ are angles of P too and we can eliminate step by step all terms $\alpha(P^d|P)$ in ascending order $d = 1, 3, \dots, 2m - 1$ in Poincaré's formula for the volume of P (for details see [15]).

This construction shows that the Schläfli numbers $\sigma^{2d}(P^{2d})$ depend only on the combinatorics of the face P^{2d} and are independent of the underlying geometry. Furthermore, this method can be used to determine the Schläfli number for a fixed combinatorial type.

3.7. The cone decomposition of a polytope

Let P be an element in \mathbf{P}^n . Then we denote by $\mathcal{K} = \mathcal{K}(P)$ the polytopal complex we get from P by cone decomposition. This means that all k -dimensional elements in \mathcal{K} are of the form $\text{conv}(b, P^k)$ where $b = b(P)$ is the barycenter of P and $P^k \in \Omega^k(P)$ for each $k = 0, \dots, n$. Furthermore, the set $\Omega^k(\mathcal{K})$ splits into two disjoint subsets

$$\begin{aligned} \Omega_{\text{bnd}}^k(\mathcal{K}) &= \{K \in \Omega^k(\mathcal{K}) : |K| \subset |\partial P|\} \\ \Omega_{\text{int}}^k(\mathcal{K}) &= \{K \in \Omega^k(\mathcal{K}) : |K| \not\subset |\partial P|\}, \end{aligned}$$

where $|K|$ and $|\partial P|$ denote the geometrical realization of the face K and of the boundary ∂P of P , respectively. Of course, it is easy to see that $\Omega_{\text{bnd}}^k(\mathcal{K}) = \Omega^k(P)$ and so $f_{\text{bnd}}^k(\mathcal{K}) := \sharp \Omega_{\text{bnd}}^k(\mathcal{K}) = f^k(P)$ for $k = 0, \dots, n - 1$. Furthermore, each k -dimensional decomposition element in the interior of P is a cone with basis in a $(k - 1)$ -dimensional face of P . We see that $f_{\text{int}}^k(\mathcal{K}) := \sharp \Omega_{\text{int}}^k(\mathcal{K}) = f^{k-1}(P)$ for $k = 1, \dots, n$ and $f_{\text{int}}^0(\mathcal{K}) = 1 = f^{-1}(P)$. For the normalized angular measures we get

$$\alpha(K|\mathcal{K}) = \begin{cases} 1, & K \in \Omega_{\text{int}}^k(\mathcal{K}) \\ \alpha(P^k|P), & K \in \Omega_{\text{bnd}}^k(\mathcal{K}) \end{cases} \quad \text{with } K = P^k.$$

3.8. \mathbb{T} -hemispheres in \mathbb{S}^n

A *tessellation* of the space \mathbb{X}^n is a collection \mathbb{T} of polytopes in \mathbb{X}^n such that

1. the interiors of the elements in \mathbb{T} are mutually disjoint;
2. the union of the elements in \mathbb{T} is \mathbb{X}^n ; and
3. the collection \mathbb{T} is locally finite.

In the following we will define a new type of subsets of \mathbb{S}^n , which have the same combinatorial properties as polytopes. A \mathbb{T} -hemisphere $P_{\mathbb{T}}$ in \mathbb{S}^n is a closed hemisphere \overline{H}^- in \mathbb{S}^n with a tessellation \mathbb{T} of the boundary H . The hyperplane H is isometric to \mathbb{S}^{n-1} . All notations (combinatorics, faces and angles) can be translated directly from polytopes.

Of course, all \mathbb{T} -hemispheres in \mathbb{S}^n have the volume $c_n/2$ and all of its normalized angular measures are $1/2$. Furthermore, Poincaré's formula (and Schläfli's reduction formula) remains true for all \mathbb{T} -hemispheres $P_{\mathbb{T}}$ in \mathbb{S}^n . We have

$$\begin{aligned} \sum_{\substack{P^d \in \Omega^d(P_{\mathbb{T}}) \\ d=0, \dots, n}} (-1)^d \alpha(P^d | P_{\mathbb{T}}) &= \frac{1}{2} \sum_{\substack{P^d \in \Omega^d(P_{\mathbb{T}}) \\ d=0, \dots, n}} (-1)^d \\ &= \frac{1}{2} (f_0(P_{\mathbb{T}}) - f_1(P_{\mathbb{T}}) + \dots + (-1)^n f_n(P_{\mathbb{T}})) \\ &= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases} \end{aligned}$$

4. Schläfli numbers

In this section we give an explicit description of the Schläfli numbers for an arbitrary even-dimensional polytope P . Following an idea of Schläfli (see [10], page 281), we develop a recursion formula for the Schläfli numbers of an even-dimensional spherical polytope.

Lemma 2. *Let P be an n -dimensional polytope in \mathbb{S}^n . Then there exists a \mathbb{T} -hemisphere $P_{\mathbb{T}}$ in \mathbb{S}^n which is combinatorially equivalent to P .*

Proof. Let P be an n -dimensional polytope in \mathbb{S}^n , contained in an open hemisphere \mathcal{H}^- with boundary \mathcal{H} . Now we construct a \mathbb{T} -hemisphere $P_{\mathbb{T}}$ in \mathbb{S}^n which has the same combinatorial structure as P .

Let b be an inner point of P (for instance the barycentre of P) and P^{n-1} a facet of P . The point b is not contained in the hyperplane $\langle P^{n-1} \rangle$ determined by P^{n-1} . Each facet F_i , $i = 1, \dots, f^{n-2}(P^{n-1})$, of P^{n-1} is an $(n-2)$ -dimensional face of P too and contains at least $n-1$ vertices in general position. The vertices of F_i in connection with the point b determine a unique hyperplane H_i in \mathbb{S}^n and let \overline{H}_i^- be the corresponding closed hemisphere with $P^{n-1} \subset \overline{H}_i^-$ for $i = 1, \dots, f^{n-2}(P^{n-1})$. Let $C(P^{n-1})$ be the cone in \mathbb{S}^n , determined by b and P^{n-1} :

$$C(P^{n-1}) := \bigcap_{i=1}^{f^{n-2}(P^{n-1})} \overline{H}_i^-.$$

We have $C(P^{n-1}) \cap \langle P^{n-1} \rangle = P^{n-1}$ by construction and let $Q(P^{n-1}) := C(P^{n-1}) \cap \mathcal{H}$. This set is an $(n-1)$ -dimensional polytope in \mathcal{H} combinatorially equivalent to P^{n-1} . The set

$\mathbb{T} := \{Q(P^{n-1}) : P^{n-1} \in \Omega^{n-1}(P)\}$ is a tessellation of \mathcal{H} and the closed hemisphere $\overline{\mathcal{H}}$ with this tessellation is a \mathbb{T} -hemisphere in \mathbb{S}^n which has the same combinatorial structure as P . \square

Lemma 3. *Let P be a $2m$ -dimensional spherical polytope. Then the Schläfli numbers satisfy the recursion $\sigma^0(P) := 1$ and*

$$\sigma^{2m}(P) = 1 - \frac{1}{2} \sum_{d=0}^{m-1} \sum_{P^{2d} \in \Omega^{2d}(P)} \sigma^{2d}(P^{2d}) \quad (7)$$

for all $m \geq 1$.

Proof. Let P be a $2m$ -dimensional polytope in \mathbb{S}^{2m} and $P_{\mathbb{T}}$ a \mathbb{T} -hemisphere which is combinatorially equivalent to P . We have

$$2\kappa^m c_{2m}^{-1} \text{vol}(P_{\mathbb{T}}) = 1 = \frac{1}{2} \sum_{d=0}^{m-1} \sum_{P^{2d} \in \Omega^{2d}(P_{\mathbb{T}})} \sigma^{2d}(P^{2d}) + \sigma^{2m}(P_{\mathbb{T}}),$$

where we have used that $\alpha(P^{2d}|P_{\mathbb{T}}) = 1/2$ for $d = 0, 1, \dots, m-1$ and $\alpha(P_{\mathbb{T}}|P_{\mathbb{T}}) = 1$. Now P has the same combinatorial structure as $P_{\mathbb{T}}$. So we get the recursion

$$\sigma^{2m}(P) = 1 - \frac{1}{2} \sum_{d=0}^{m-1} \sum_{P^{2d} \in \Omega^{2d}(P)} \sigma^{2d}(P^{2d})$$

for all $m \geq 1$ and $\sigma^0(P) := 1$. \square

Theorem 1 (Schläfli numbers). *For all $2m$ -dimensional polytopes P in a space \mathbb{X}^{2m} of constant curvature κ we have*

$$\sigma^{2m}(P) = \mathbf{f}\left(\mathcal{E}(P)_r, \frac{1}{2}\right). \quad (8)$$

Proof. The (combinatorial) numbers $\sigma^{2d}(P^{2d})$ are independent of the underlying geometry of the space \mathbb{X}^{2m} and therefore it suffices to prove the theorem for the case $\mathbb{X}^{2m} = \mathbb{S}^{2m}$ only.

For $m = 0$ and $P \in \mathbf{P}^0$ we have $\mathbf{f}(\mathcal{E}(P)_r, \frac{1}{2}) = c^{-1}(\emptyset) = 1 = \sigma^0(P)$.

For $m = 1$ and $P \in \mathbf{P}^2$ we have $\mathbf{f}(\mathcal{E}(P)_r, \frac{1}{2}) = 1 - \frac{1}{2}f^0(P) = \sigma^2(P)$ and this value is nothing else than the constant in the spherical excess formula for polygons in the sphere.

Let $m \geq 2$, $P \in \mathbf{P}^{2m}$ and $\mathcal{E} := \mathcal{E}(P)$ be the even face poset of P . Firstly we observe that there exists the canonical bijection $\Omega^{2d}(P) \ni P^{2d} \longleftrightarrow x = h(P^{2d}) \in \mathcal{E}(P)^{d+1}$ for $d = 0, 1, \dots, m-1$ and the even face poset $\mathcal{E}(P^{2d})$ is equal to the interval $[\hat{0}, x]$ in \mathcal{E} . Furthermore, we see that $\ell(\mathcal{E}) = m+1$. By using the induction hypothesis, [Lemmas 3](#) and [1](#) we get

$$\sigma^{2m}(P) = 1 - \frac{1}{2} \sum_{d=1}^{\ell(\mathcal{E})-1} \sum_{x \in \mathcal{E}^d} \mathbf{f}\left([\hat{0}, x]_r, \frac{1}{2}\right) = \mathbf{f}\left(\mathcal{E}_r, \frac{1}{2}\right)$$

and the theorem is proved. \square

For a $2m$ -dimensional polytope $P \in \mathbf{P}^{2m}$ and the values $m = 0, 1, 2$ we can write the Schläfli numbers as a function of the face numbers of P and the face numbers of faces of P in the

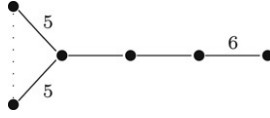


Fig. 1. The scheme $\Sigma(P)$.

following way (see [10], page 276).

$$\sigma^0(P = P^0) = 1$$

$$\sigma^2(P = P^2) = 1 - \frac{1}{2}f^0(P)$$

$$\sigma^4(P = P^4) = 1 - \frac{1}{2}(f^0(P) + f^2(P)) + \frac{1}{4} \sum_{P^2 \in \Omega^2(P)} f^0(P^2).$$

Furthermore, we define the $2d$ -dimensional weighted normalized angular measure sum $\omega^{2d}(P)$ of P as $\omega^{2d}(P) := \sum \sigma^{2d}(P^{2d})\alpha(P^{2d}|P)$, for all $0 \leq d \leq m$, where the sum is taken over all $2d$ -dimensional faces of P . Then formula (8) can be written as

$$\kappa^m \mathbf{vol}(P) = \frac{c_{2m}}{2} \sum_{d=0}^m \omega^{2d}(P).$$

5. Application to Coxeter polytopes

Of special interest are polytopes which have angles of simple measure. A *Coxeter polytope* in \mathbb{X}^n is a polytope whose dihedral angles have a normalized angular measure of the form $1/2q$ with $q \in \mathbb{N}$ and $q \geq 2$. Let H_i^- ($i \in I$) be a family of closed half-spaces in \mathbb{X}^n such that $P = \cap_{i \in I} H_i^-$ is a Coxeter n -polytope. Then P can be described by a Coxeter scheme $\Sigma(P)$ which is a labelled undirected graph with vertex set I . We connect and label two elements $i, j \in I$ in the following way:

- If $P_{\{i,j\}} = \emptyset$, then we connect i and j by a dotted line.
- If $P_{\{i,j\}}$ is a $(n-2)$ -dimensional face of P , then we distinguish cases as follows. In the case where $\alpha(P_{\{i,j\}}|P) = 1/4$ we do not connect i and j . In the other cases where $\alpha(P_{\{i,j\}}|P) = 1/2q$ with $q > 2$ we connect i and j and if $q > 3$ we label this edge q .

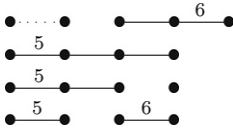
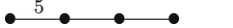


The subscheme of $\Sigma(P)$ associated to a subset $J \subset I$ is denoted by $\Sigma(P_J|P)$. If P_J is a face of P then $\Sigma(P_J|P)$ is the scheme of the corresponding face figure.







We want to compute the (exact) volumes of the 4-dimensional hyperbolic Coxeter polytopes, classified by Tumarkin [13]. From the combinatorial point of view each of these polytopes P is a pyramid over a 3-dimensional simplicial prism. We find: $f^0(P) = 7$, $f^1(P) = 15$, $f^2(P) = 14$ and $f^3(P) = 6$. The set of 2-dimensional faces consists of 11 triangles and 3 rectangles; so we get $\sigma^4(P) = \omega^4 = 7/4$. If we decode the connection between normalized angular measures and combinatorics we get the results stated in the [Appendix](#).

Example. We will compute the volume of the Coxeter 4-polytope $P = \cap_{i=1}^6 H_i^-$ with Coxeter scheme $\Sigma(P)$ as in [Fig. 1](#).

We number the vertices of the scheme so that every vertex corresponds to a hyperplane $H_i = v_i^\perp$ for all $i = 1, \dots, 6$. Now we can construct the face lattice $\mathcal{F}(P)$ of P by studying all

sub-schemes of $\Sigma(P)$ (actually all parabolic and elliptic sub-schemes). Hence we can determine the whole combinatorics of the polytope P . In the following two tables we give the sets of parabolic and elliptic sub-schemes $\Sigma(P_J|P)$ of $\Sigma(P)$ of orders 5, 4 and 2, its corresponding face P_J , the combinatorial type and the related normalized angular measure $\alpha(P_J|P)$.

$\Sigma(P_J P)$	Multiplicity	$\alpha(P_J P)$
	1	0
	2	1/14 400
	2	1/240
	2	1/120
$\omega^0(P) = 181/7200$		

$\Sigma(P_J P)$	Multiplicity	$\alpha(P_J P)$	Type of P_J
	1	1/12	3-gon
	2	1/10	3-gon
	1	1/6	3-gon
	1	1/6	4-gon
	7	1/4	3-gon
	2	1/4	4-gon
$\omega^2(P) = 53/30$			

So we see

$$\mathbf{vol}(P) = \frac{4}{3}\pi^2(\omega^0(P) - \omega^2(P) + \omega^4(P)) = \frac{61}{5400}\pi^2.$$

6. The case of simple and simplicial polytopes

First we will record some well-known facts about tangent and Euler numbers. For an overview you may see [1]. Then we give a new description of the Schläfli numbers for simplicial (and simple) polytopes.

The (modified) tangent numbers $T_{2m+1}(m \geq 0)$ are defined by the recursion

$$T_1 = \frac{1}{2} \quad \text{and} \quad (-1)^m T_{2m+1} = -\frac{1}{2m} \sum_{d=0}^{m-1} (-1)^d \binom{2m+1}{2d} T_{2d+1}. \quad (9)$$

Indeed these numbers are the coefficients of the Taylor series of $\tan(z/2)$.

$$\tan\left(\frac{z}{2}\right) = \sum_{m=0}^{\infty} T_{2m+1} \frac{z^{2m+1}}{(2m+1)!}.$$

Furthermore, the Euler numbers $E_{2m}(m \geq 0)$ are defined as the coefficients of the Taylor series of $1/\cos(z)$

$$\frac{1}{\cos(z)} = \sum_{m=0}^{\infty} E_{2m} \frac{z^{2m}}{(2m)!}.$$

We have the following connection between the tangent and the Euler numbers (see [7], page 196)

$$(-1)^m E_{2m} = \sum_{d=0}^m (-1)^d \binom{2m}{2d} 2^{2d+1} T_{2d+1}. \quad (10)$$

Let P be a $2m$ -dimensional polytope in \mathbb{X}^n . Then the $2m$ -dimensional *simplicial Schläfli number* $\diamond^{2m}(P)$, resp. *simple Schläfli number* $\square^{2m}(P)$, is defined by

$$\begin{aligned} \diamond^{2m}(P) &:= 2 \sum_{k=0}^m (-1)^k T_{2k+1} f^{2k-1}(P) \\ \square^{2m}(P) &:= 2 \sum_{k=0}^m (-1)^k T_{2k+1} f^{2m-2k}(P). \end{aligned}$$

If P^* is the dual polytope of P it is easy to see that $\diamond^{2m}(P) = \square^{2m}(P^*)$ and $\diamond^{2m}(P^*) = \square^{2m}(P)$. If $P \in \mathbf{P}^0$, then $\diamond^0(P) = \square^0(P) = 1$. If $P \in \mathbf{P}^2$, then $\diamond^2(P) = \square^2(P) = 1 - \frac{1}{2}f^0(P)$. Furthermore, if $S \in \mathbf{P}^{2m}$ is a simplex, which means that S is simplicial and simple (see [16], page 67), we see that $f^{2k-1}(S) = \binom{2m+1}{2k} = \binom{2m+1}{2m-2k+1} = f^{2m-2k}(S)$ for all $k = 0, \dots, m$. Hence by using (9) we get $\diamond^{2m}(S) = \square^{2m}(S) = 2(-1)^m T_{2m+1}$.

Now we can give an explicit description of the Schläfli numbers for simplicial polytopes by using the cone decomposition.

Theorem 2 (*Schläfli numbers of simplicial polytopes*). *Let P be a $2m$ -dimensional simplicial polytope in \mathbb{X}^{2m} . Then $\sigma^{2m}(P) = \diamond^{2m}(P)$. In particular, if P is a simplex, then $\sigma^{2m}(P) = 2(-1)^m T_{2m+1}$.*

Proof. Let P be a simplicial polytope in \mathbf{P}^{2m} and $\mathcal{K} = \mathcal{K}(P)$ the cone decomposition of P . Of course, \mathcal{K} is a simplicial complex because each face of P is a simplex. Firstly we know that formula (6) is valid. Secondly, we see that the volume of the polytope P is equal to the sum of all decomposition simplices in \mathcal{K} and formula (6) holds for each decomposition simplex too. Hence we get

$$\begin{aligned} 2\kappa^m c_{2m}^{-1} \text{vol}(P) &= \sum_{S \in \Omega^{2m}(\mathcal{K})} 2\kappa^m c_{2m}^{-1} \text{vol}(S) \\ &= \sum_{S \in \Omega^{2m}(\mathcal{K})} \left(\sum_{\substack{S^{2k} \in \Omega^{2k}(S) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \alpha(S^{2k}|S) \right) \\ &= \sum_{\substack{S^{2k} \in \Omega^{2k}(\mathcal{K}) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \sum_{S \in \Omega^{2m}(\mathcal{K})} \alpha(S^{2k}|S) \\ &= \sum_{\substack{S^{2k} \in \Omega^{2k}(\mathcal{K}) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \alpha(S^{2k}|\mathcal{K}) \\ &= \sum_{\substack{S^{2k} \in \Omega_{\text{bnd}}^{2k}(\mathcal{K}) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \alpha(S^{2k}|\mathcal{K}) + \sum_{\substack{S^{2k} \in \Omega_{\text{int}}^{2k}(\mathcal{K}) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \alpha(S^{2k}|\mathcal{K}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{P^{2k} \in \Omega^{2k}(P) \\ k=0, \dots, m-1}} \sigma^{2k}(P^{2k}) \alpha(P^{2k}|P) + \sum_{\substack{S^{2k} \in \Omega_{\text{int}}^{2k}(\mathcal{K}) \\ k=0, \dots, m}} \sigma^{2k}(S^{2k}) \\
&= \sum_{\substack{P^{2k} \in \Omega^{2k}(P) \\ k=0, \dots, m-1}} \sigma^{2k}(P^{2k}) \alpha(P^{2k}|P) + \sum_{k=0}^m f^{2k-1}(P) \sigma^{2k}(S^{2k})
\end{aligned}$$

where we have used the definition of the normalized angular measure, that each S^{2k} is a $2k$ -dimensional simplex, that $\Omega^{2k}(\mathcal{K})$ is a disjoint union of $\Omega_{\text{bnd}}^{2k}(\mathcal{K})$ and $\Omega_{\text{int}}^{2k}(\mathcal{K})$, that $\Omega_{\text{bnd}}^{2k}(\mathcal{K}) = \Omega^{2k}(P)$ for $k = 0, \dots, m-1$ ($\Omega_{\text{bnd}}^{2m}(\mathcal{K}) = \emptyset$) and that $\# \Omega_{\text{int}}^{2k}(\mathcal{K}) = f^{2k-1}(P)$ for all $k = 0, \dots, m$.

We get by formula (6)

$$\begin{aligned}
2\kappa^m c_{2m}^{-1} \text{vol}(P) &= \sum_{\substack{P^{2k} \in \Omega^{2k}(P) \\ k=0, \dots, m-1}} \sigma^{2k}(P^{2k}) \alpha(P^{2k}|P) + \sigma^{2m}(P^{2m}) \\
&= \sum_{\substack{P^{2k} \in \Omega^{2k}(P) \\ k=0, \dots, m-1}} \sigma^{2k}(P^{2k}) \alpha(P^{2k}|P) + \sum_{k=0}^m f^{2k-1}(P) \sigma^{2k}(S^{2k}).
\end{aligned}$$

We deduce a formula which expresses the number $\sigma^{2m}(P^{2m})$ in terms the numbers $\sigma^{2k}(S^{2k})$ for $k = 0, \dots, m$.

$$\sigma^{2m}(P^{2m}) = \sigma^{2m}(P) = \sum_{k=0}^m f^{2k-1}(P) \sigma^{2k}(S^{2k}). \quad (11)$$

Now we have to determine the numbers $\sigma^{2k}(S^{2k})$ for $2k$ -dimensional simplices. Of course, we have $\sigma^0(S^0) = 1$ because $\text{vol}_{\mathbb{X}^0}(S^0) = 1$. Let $P = S^{2m}$ be a $2m$ -dimensional simplex. We use formula (11) and get the relation

$$\sigma^{2m}(S^{2m}) = \sum_{k=0}^{m-1} \binom{2m+1}{2k} \sigma^{2k}(S^{2k}) + (2m+1) \sigma^{2m}(S^{2m}).$$

This is equivalent to

$$\sigma^{2m}(S^{2m}) = -\frac{1}{2m} \sum_{k=0}^{m-1} \binom{2m+1}{2k} \sigma^{2k}(S^{2k}).$$

Comparing this with the equalities (9) we see that the Schläfli numbers of an even-dimensional simplex can be written as $\sigma^0(S^0) = 2T_1$ and $\sigma^{2m}(S^{2m}) = 2(-1)^m T_{2m+1}$ for all $m \geq 1$. We return to the arbitrary simplicial polytope P and we immediately see that

$$\sigma^{2m}(P) = 2 \sum_{k=0}^m (-1)^k f^{2k-1}(P) T_{2k+1} = \diamond^{2m}(P)$$

and the theorem is proved. \square

In connection with results of Vinberg (see [14], page 122) we get an explicit description of the Schläfli numbers for simplicial and simple polytopes

$$\sigma^{2m}(P) = \begin{cases} \diamond^{2m}(P), & P \in \mathbf{P}^{2m} \text{ simplicial} \\ \square^{2m}(P), & P \in \mathbf{P}^{2m} \text{ simple.} \end{cases}$$

The cross polytope and the cube are well-known simplicial and simple polytopes. Let P be a cross polytope. Then the dual polytope P^* is a cube and we have by [Theorem 2](#)

$$\begin{aligned}\sigma^{2m}(P) &= \diamond^{2m}(P) = \square^{2m}(P^*) \\ &= 2 \sum_{k=0}^m (-1)^k 2^{2k} \binom{2m}{2k} T_{2k+1} \\ &= (-1)^m E_{2m},\end{aligned}$$

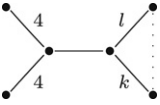
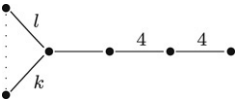
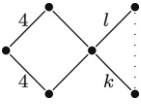
where we have used that $f^{2k-1}(P) = 2^{2k} \binom{2m}{2k}$ for all $0 \leq k \leq m$ and identity (10). So we get

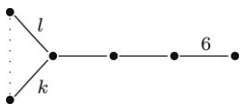
Corollary 6.1. *Let P be a $2m$ -dimensional cross polytope or cube. Then $\sigma^{2m}(P) = (-1)^m E_{2m}$.*


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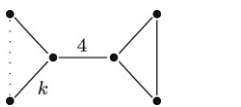
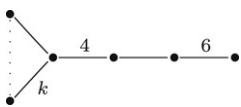
Appendix

Scheme	k	l	Volume	Value
	2	3	$\frac{1}{144}\pi^2$	0.0685389195
	2	4	$\frac{1}{72}\pi^2$	0.1370778390
	3	3	$\frac{1}{72}\pi^2$	0.1370778390
	3	4	$\frac{1}{48}\pi^2$	0.2056167584
	4	4	$\frac{1}{36}\pi^2$	0.2741556779
	2	3	$\frac{1}{288}\pi^2$	0.0342694597
	2	4	$\frac{1}{144}\pi^2$	0.0685389195
	3	3	$\frac{1}{144}\pi^2$	0.0685389195
	3	4	$\frac{1}{96}\pi^2$	0.1028083792
	4	4	$\frac{1}{72}\pi^2$	0.1370778390
	2	3	$\frac{1}{72}\pi^2$	0.1370778390
	2	4	$\frac{1}{36}\pi^2$	0.2741556779

Scheme	k	l	Volume	Value
	3	3	$\frac{1}{36}\pi^2$	0.2741556779
	3	4	$\frac{1}{24}\pi^2$	0.4112335169
	4	4	$\frac{1}{18}\pi^2$	0.5483113558

	2	3	$\frac{1}{540}\pi^2$	0.0182770452
	2	4	$\frac{1}{288}\pi^2$	0.0342694598
	2	5	$\frac{61}{10800}\pi^2$	0.0557449879
	3	3	$\frac{1}{270}\pi^2$	0.0365540904
	3	4	$\frac{23}{4320}\pi^2$	0.0525465050
	3	5	$\frac{3}{400}\pi^2$	0.0740220330
	4	4	$\frac{1}{144}\pi^2$	0.0685389195
	4	5	$\frac{197}{21600}\pi^2$	0.0900144476
	5	5	$\frac{61}{5400}\pi^2$	0.1114899757

	2	3	$\frac{1}{270}\pi^2$	0.0365540904
	2	4	$\frac{1}{144}\pi^2$	0.0685389195
	2	5	$\frac{61}{5400}\pi^2$	0.1114899757
	3	3	$\frac{1}{135}\pi^2$	0.0731081808
	3	4	$\frac{23}{2160}\pi^2$	0.1050930099
	3	5	$\frac{3}{200}\pi^2$	0.1480440661
	4	4	$\frac{1}{72}\pi^2$	0.1370778390
	4	5	$\frac{197}{10800}\pi^2$	0.1800288951
	5	5	$\frac{61}{2700}\pi^2$	0.2229799513

Scheme	k	Volume	Value
	2	$\frac{5}{432}\pi^2$	0.1142315324
	3	$\frac{5}{216}\pi^2$	0.2284630649
	2	$\frac{5}{864}\pi^2$	0.0571157662
	3	$\frac{5}{432}\pi^2$	0.1142315324

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