

Fourier and barycentric formulae for equidistant Hermite trigonometric interpolation [☆]

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Abstract

We consider the Hermite trigonometric interpolation problem of order 1 for equidistant nodes, i.e., the problem of finding a trigonometric polynomial t that interpolates the values of a function and of its derivative at equidistant points. We give a formula for the Fourier coefficients of t in terms of those of the two classical trigonometric polynomials interpolating the values and those of the derivative separately. This formula yields the coefficients with a single FFT. It also gives an aliasing formula for the error in the coefficients which, on its turn, yields error bounds and convergence results for differentiable as well as analytic functions. We then consider the Lagrangian formula and eliminate the unstable factor by switching to the barycentric formula. We also give simplified formulae for even and odd functions, as well as consequent formulae for Hermite interpolation between Chebyshev points.

Keywords: Hermite trigonometric interpolation; Discrete Fourier transform; Aliasing formula; Error bounds; Barycentric formula; Chebyshev points

1. Introduction

We consider the determination of the trigonometric polynomial interpolating given values at equidistant nodes and such that its derivative also takes prescribed values at those same points. More precisely, the real *Hermite trigonometric interpolation problem of order 1 for equidistant nodes* consists in finding a trigonometric polynomial of degree at most N , say

$$t(\phi) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos n\phi + b_n \sin n\phi) = \sum_{n=-N}^N d_n e^{in\phi}, \quad a_n, b_n \in \mathbb{R}, \quad d_n \in \mathbb{C}, \quad (1.1)$$

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which fulfills the $2N$ interpolation conditions

$$t(\phi_k) = f_k, \quad t'(\phi_k) = \frac{dt}{d\phi}(\phi_k) = f'_k, \quad \phi_k = k \frac{2\pi}{N}, \quad k = 0, \dots, N-1, \quad (1.2)$$

where the f_k and f'_k are given real numbers which may, but need not, be values and derivatives of a function f .

Salzer [21] has already given Lagrangian and barycentric formulae for arbitrary interpolation points and problems of arbitrarily high order (i.e., problems where higher order derivatives are also prescribed). Kress [16] has derived Lagrangian as well as remainder formulae and asymptotic convergence results for the most important case of an even number N of equidistant points. Several other authors have addressed such Hermite problems, even for arbitrary points; they were however mostly interested in existence questions [7,14], convergence results [15,20] and formulae other than Lagrange's [18].

In the present work, we give a formula for the Fourier coefficients of the Hermite trigonometric interpolating polynomial (1.1) with $a_N = 0$ in terms of those of the two polynomials interpolating the f_k and the f'_k separately. At least when one may choose N as a power of two, this formula yields the coefficients in $\mathcal{O}(N \log N)$ operations. It also gives an *aliasing formula* for the error in the coefficients. As in classical (i.e., "non-Hermite") trigonometric interpolation, we deduce from this formula error bounds and convergence results for differentiable as well as analytic f 's.

We then consider the Lagrangian formula for the interpolating polynomial. As noticed by Henrici [10], a factor in front of the formula is unstable in the vicinity of multiples of π . To eliminate this factor, we switch to the barycentric formula. We also give simplified formulae for even and odd functions, as well as consequent formulae for Hermite interpolation between Chebyshev points on $[-1, 1]$. We conclude with numerical examples.

2. Classical equidistant trigonometric interpolation

Trigonometric interpolation between equidistant nodes is a well-known efficient means of approximating a smooth real or complex periodic function f . Indeed, in contrast with piecewise interpolants, its speed of convergence with an increasing number of points is limited only by the degree of smoothness of f , not by that of the approximant (see Theorem 2.5 below). We will now recall results we shall need and extend to the Hermite case in subsequent sections. To fix ideas, let f be defined on the whole real line and L -periodic (or defined on the circle \mathbb{R}/L). For simplicity of the formulae we will take $L = 2\pi$ and the equidistant interpolation points (nodes) as $\phi_k = k \frac{2\pi}{N}$, $k = 0, 1, \dots, N-1$. We introduce the notation $M := \lfloor N/2 \rfloor$, so that $N = 2M$ or $N = 2M + 1$.

Upon choosing $b_M = 0$ for N even, there is a unique trigonometric polynomial of least degree

$$t_f(\phi) = \frac{a_0}{2} + \sum_{n=1}^M (a_n \cos n\phi + b_n \sin n\phi) \quad (2.1)$$

interpolating between the ϕ_k 's, i.e., with $t(\phi_k) = f_k := f(\phi_k)$. Its complex form is

$$t_f(\phi) = \sum_{n=-M}^M{}'' c_n e^{in\phi}, \quad (2.2)$$

where the double prime means that the terms with $|n| = M$ should be halved when N is even. The formulae for passing from the complex to the real form (2.1) are

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \quad (2.3a)$$

and

$$c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2} \quad (2.3b)$$

($b_0 = 0$; the factor $\frac{1}{2}$ in front of a_0 in (2.1) is there for the relation (2.3b) to also hold for $n = 0$). The operator which to the N -vector of the values f_k associates the N -vector of the coefficients $[c_0, c_1, \dots, c_{N-1}]^T$ is the *discrete Fourier transform* (DFT) [6]. It is given by the formula [11, p. 335]

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k w_N^{-kn}, \quad w_N := e^{\frac{2\pi i}{N}}, \quad (2.4)$$

and its inverse by $f_k = \sum_{n=0}^{N-1} c_n w_N^{kn}$. The c_n are easily seen to be N -periodic, and one usually computes them as c_0, \dots, c_{N-1} . The direct computation of a c_n requires $\mathcal{O}(N)$ flops, the DFT thus $\mathcal{O}(N^2)$ flops and so does the evaluation of t at one ϕ according to (2.1). In particular when one may choose N as a very composite number, $N = 2^\ell$, $\ell \in \mathbb{N}$, for the best, then the fast Fourier transform (FFT) computes all c_n asymptotically in $\mathcal{O}(N \log N)$ flops. However, that much overhead slows down the practical computation that the FFT becomes faster than direct evaluation only for N relatively large. For N small or when it cannot be chosen, the Lagrangian approach is often faster.

What about the approximation error $t(\phi) - f(\phi)$? One way of studying it is to first consider the error when approximating C_n , the true n th Fourier coefficient of f , with c_n , and then to use results about that error for estimating $t - f$. When $f \in L_2[0, 2\pi]$ it may be written as its Fourier series

$$f = \sum_{n=-\infty}^{\infty} C_n e^{in\phi}, \quad C_n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi. \quad (2.5)$$

If f is real, $C_{-n} = \bar{C}_n$ and $c_{-n} = \bar{c}_n$. When the Fourier series of f converges at all ϕ_k , the error in c_n is given in terms of the C_m by the remarkable *aliasing formula* ([11, p. 354], [17])

$$c_n - C_n = \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} C_{n+\ell N}, \quad (2.6)$$

which expresses that the error is the smaller the faster the decay of the C_n as $n \rightarrow \infty$. This decay is correlated with the smoothness of f , as recalled in the following two theorems. Since we work with periodic functions, we shall assume that, whenever $f^{(\ell)}$ is considered, f as well as its $\ell - 1$ first derivatives are continuous on \mathbb{R} . The most natural class of functions to consider seems to be those (with a derivative) of bounded variation [19, p. 7].

Theorem 2.1. *Let $f^{(p)}$, $p \in \mathbb{N} \cup \{0\}$, be of bounded variation on $[0, 2\pi]$ and let $V_p(a, b)$ denote the variation of $f^{(p)}$ on $[a, b]$. Then*

$$|C_n| \leq \frac{V_p(0, 2\pi)}{|n|^{p+1}}. \quad (2.7)$$

Proof. Integrating C_n from (2.5) p times by parts and using the continuity of $f^{(\ell)}$ on \mathbb{R} , $\ell = 0, \dots, p - 1$, one obtains $C_n = \frac{1}{2\pi} \frac{1}{(in)^p} \int_0^{2\pi} f^{(p)}(\phi) e^{-in\phi} d\phi$. Then consider the equidistant abscissae $\phi_k := k \frac{2\pi}{|n|}$ and the step function g defined as $g(\phi) := f^{(p)}(\phi_k)$ on $[\phi_{k-1}, \phi_k]$. One easily checks that $\int_0^{2\pi} g(\phi) e^{-in\phi} d\phi = 0$ and consequently

$$|C_n| \leq \frac{1}{2\pi} \frac{1}{|n|^p} \sum_{k=1}^n \int_{\phi_{k-1}}^{\phi_k} |f^{(p)}(\phi) - f^{(p)}(\phi_k)| d\phi.$$

But $|f^{(p)}(\phi) - f^{(p)}(\phi_k)| \leq V_p(\phi_{k-1}, \phi_k) \forall \phi \in [\phi_{k-1}, \phi_k]$ and therefore $|C_n| \leq \frac{1}{2\pi} \frac{1}{|n|^p} \sum_{k=1}^n V_p(\phi_{k-1}, \phi_k) \frac{2\pi}{|n|}$. The theorem follows from the additivity of the variation over intervals. \square

Theorem 2.2. *Let f be analytic in the strip $|\operatorname{Im} z| \leq \theta$, where $\theta > 0$. Then*

$$|C_n| \leq \mu e^{-|n|\theta}.$$

Proof. See [12, p. 20], where the period 1 adds a factor 2π to the exponent. \square

Inserting into the aliasing formula and summing over ℓ as in [11, p. 356], [6], one obtains estimates for the error committed when replacing C_n by c_n .

Theorem 2.3. *Let $f^{(p)}$, $p \in \mathbb{N}$, be of bounded variation and $N = 2M$ or $N = 2M + 1$. Then*

$$|c_n - C_n| = \mathcal{O}(M^{-(p+1)}), \quad |n| \leq M.$$

Theorem 2.4. Let f be analytic in the strip $|\operatorname{Im} z| \leq \theta$, where $\theta > 0$. Then

$$|c_n - C_n| \leq 2\mu \cosh(n\theta) \frac{e^{-N\theta}}{1 - e^{-N\theta}}, \quad |n| \leq M.$$

Also the error $t_f - f$ may be expressed by means of the aliasing formula

$$t_f(\phi) - f(\phi) = \sum_{\ell=-\infty}^{\infty} \left[(1 - e^{i\ell N\phi}) \sum_{n=-M}^M C_{n+\ell N} e^{in\tau} \right].$$

Estimating the sums yields the following results.

Theorem 2.5. If the Fourier series of f converges absolutely and if $N = 2M$ or $N = 2M + 1$ then

$$|t_f(\phi) - f(\phi)| \leq 2 \sum_{|n| \geq M} |C_n|.$$

Theorem 2.6. Let $f^{(p)}$, $p \in \mathbb{N}$, be of bounded variation $V_p := V_p(0, 2\pi)$ and let $N = 2M$ or $N = 2M + 1$. Then

$$|t_f(\phi) - f(\phi)| \leq \frac{2V_p}{M^p} \left(\frac{1}{M} + \frac{2}{p} \right) = \mathcal{O}(M^{-p}).$$

Theorem 2.7. Let f be analytic in the strip $|\operatorname{Im} z| \leq \theta$, where $\theta > 0$. Then

$$|t_f(\phi) - f(\phi)| \leq \frac{4\mu}{1 - e^{-\theta}} e^{-M\theta}.$$

The convergence is therefore algebraic—i.e., the error decreases polynomially as a function of N —for differentiable f 's and is even exponential for analytic f 's. Theorem 2.5 may be found in [12, p. 46], Theorems 2.6 and 2.7 essentially in [11, p. 365].

Besides the Fourier approach there is also the Lagrangian, “physical” space approach to the same interpolant. Indeed, t_f may be written directly in terms of the f_k as

$$t_f(\phi) = \frac{1}{N} \sin \frac{N\phi}{2} \sum_{k=0}^{N-1} (-1)^k \operatorname{cst} \frac{\phi - \phi_k}{2} f_k, \quad (2.8)$$

where we have used the notation

$$\operatorname{cst} \phi := \begin{cases} \cot \phi = [\tan \phi]^{-1}, & N \text{ even,} \\ \operatorname{cst} \phi = [\sin \phi]^{-1}, & N \text{ odd,} \end{cases}$$

introduced in [3]. Equation (2.8) is attributed to de la Vallée Poussin in [24, p. 55]; a Lagrangian proof is given in [10]. Equation (2.8) is efficient in terms of complexity, but the factor $\sin \frac{N\phi}{2}$ is ill-conditioned in the vicinity of the nodes for large N . Gautschi has recently suggested to cure the problem in the more general case of sinc-interpolation by making the factor in front of the sum depend on ϕ [9]. This dependence is often unpractical; moreover, in the trigonometric case the cst in (2.8) is the most expensive part of the computation and so the barycentric formula, which eliminates the factor without introducing a parameter depending on ϕ , seems preferable. It consists in writing (2.8) also for the function 1 and dividing the two formulae to get [10]

$$t_f(\phi) = \frac{\sum_{k=0}^{N-1} (-1)^k \operatorname{cst} \frac{\phi - \phi_k}{2} f_k}{\sum_{k=0}^{N-1} (-1)^k \operatorname{cst} \frac{\phi - \phi_k}{2}}. \quad (2.9)$$

Functions that are even with respect to the center of the interval of interpolation are important in practice. For them $c_{-n} = c_n$, $a_n = c_n$, $b_n = 0$, $C_n \in \mathbb{R}$ and

$$t_f(\phi) = \frac{a_0}{2} + \sum_{n=1}^M a_n \cos n\phi;$$

the barycentric formula reads [2]

$$t_f(\phi) = \frac{\sum_{k=0}^M \frac{(-1)^k \delta_k \eta_k}{\cos \phi - \cos \phi_k} f_k}{\sum_{k=0}^M \frac{(-1)^k \delta_k \eta_k}{\cos \phi - \cos \phi_k}}, \quad \delta_k := \begin{cases} 1/2, & \phi_k = 0 \text{ or } \phi_k = \pi, \\ 1, & \text{otherwise,} \end{cases} \quad \eta_k := \begin{cases} 1, & N \text{ even,} \\ \cos \frac{\phi_k}{2}, & N \text{ odd.} \end{cases}$$

A very important use of even trigonometric interpolation is classical polynomial interpolation between Chebyshev points. Its barycentric formulae are obtained from those just shown by replacing $\cos \phi$ by x , which transforms the $\cos n\phi$ into Chebyshev polynomials and the equidistant points ϕ_k on the circle into Chebyshev points $x_k = \cos \phi_k$. The case of the Chebyshev points of the second kind is the basic tool of the `chebfun`-system [1].

3. A formula for the coefficients of the Hermite interpolant

The conditions (1.2) form $2N$ equations, while a polynomial of degree at most N as in (1.1) contains $2N + 1$ parameters. For a reason that will become clear later on, we now write it as

$$t(\phi) = \sum_{n=-N}^N {}'' d_n e^{in\phi} = \frac{a_0}{2} + \sum_{n=1}^{N-1} (a_n \cos n\phi + b_n \sin n\phi) + \frac{1}{2} (a_N \cos N\phi + b_N \sin N\phi) \quad (3.1)$$

with the relations (2.3) between the d_k and the a_k and b_k . The double prime means that the terms with $|n| = N$ are to be halved. The conditions (1.2) read

$$\sum_{j=-N}^N {}'' d_j w_N^{jk} w_N^{jk} = f_k, \quad \sum_{j=-N}^N {}'' i n d_n w_N^{nk} = f'_k,$$

with w_N as in (2.4). Changing the summation index to j and taking linear combinations with coefficients $\frac{1}{N} w_N^{-kn}$, we obtain

$$\sum_{j=-N}^N {}'' d_j \left(\frac{1}{N} \sum_{k=0}^{N-1} w_N^{k(j-n)} \right) = c_n, \quad \sum_{j=-N}^N {}'' i j d_j \left(\frac{1}{N} \sum_{k=0}^{N-1} w_N^{k(j-n)} \right) = c'_n, \quad (3.2)$$

where we have used (2.4) to introduce the coefficients c_n , respectively c'_n , of the classical trigonometric polynomials t_f and $t_{f'}$ interpolating the f_k , respectively f'_k . Taking into account that

$$\sum_{k=0}^{N-1} w_N^{k\ell} = \begin{cases} N, & \ell \equiv 0 \pmod{N}, \\ 0, & \text{otherwise,} \end{cases}$$

one obtains from (3.2) for $n = 0$ or $n = N$

$$\frac{d_{-N}}{2} + d_0 + \frac{d_N}{2} = c_0, \quad (3.3a)$$

$$-iN \frac{d_{-N}}{2} + iN \frac{d_N}{2} = c'_0 \quad (3.3b)$$

and for $n = 1, \dots, N - 1$

$$d_{n-N} + d_n = c_n, \quad (3.3c)$$

$$i(n - N)d_{n-N} + i n d_n = c'_n. \quad (3.3d)$$

At this point one may distinguish two cases, depending on c'_0 . In the generic case $c'_0 \neq 0$, (3.3b) implies $d_{-N} \neq d_N$ and $b_N \neq 0$ by (2.3). In order to have $2N$ coefficients in (3.1), we will choose $a_N = 0$.

In the special case $c'_0 = 0$ (which applies when $t_{f'} = t'_f$), (3.3b) implies $d_{-N} = d_N$ and $b_N = 0$. Then, by (3.3a), $d_0 - c_0 = -d_N$, and it seems natural in view of the decay of the coefficients to restrict ourselves to $d_0 = c_0$, thus $d_{-N} = d_N = 0$ and $a_N = 0$ as in the generic case. So we finally consider in all cases

$$t(\phi) = \frac{a_0}{2} + \sum_{n=1}^{N-1} (a_n \cos n\phi + b_n \sin n\phi) + \frac{b_N}{2} \sin N\phi \quad (3.4)$$

with $b_N = 0$ iff $c'_0 = 0$. (This corresponds to Kress's choice in [16].) In analogy with the classical case [11], we will call (3.4) a *balanced* Hermite trigonometric polynomial of degree at most N . Solving (3.3) for the d_n , we obtain the following formulae for the coefficients of t as functions of those of t_f and $t_{f'}$:

$$\begin{aligned} d_0 &= c_0, \\ \left. \begin{aligned} d_n &= \left(1 - \frac{n}{N}\right)c_n - \frac{i}{N}c'_n \\ d_{n-N} &= \frac{n}{N}c_n + \frac{i}{N}c'_n \end{aligned} \right\} \quad n = 1, \dots, N-1, \\ d_N &= -\frac{i}{N}c'_0, \quad d_{-N} = \frac{i}{N}c'_0 (= -d_N). \end{aligned} \quad (3.5)$$

These formulae hold if (3.4) is the interpolant, which is therefore unique if it exists. And it is easily shown that (3.4) with d_n as in (3.5) satisfies the interpolation conditions (1.2).

Theorem 3.1. *Let $\{c_n\}$ and $\{c'_n\}$ be the DFT of the values $\{f_k\}_{k=0}^{N-1}$, respectively $\{f'_k\}_{k=0}^{N-1}$. Then there exists a unique trigonometric polynomial (3.4) satisfying the interpolation conditions (1.2) and its coefficients are given by (3.5). One has $b_N = 0$ iff $c'_0 = 0$.*

The operator which to $f \in C^1[0, 2\pi]$ associates its equidistant Hermite trigonometric interpolant (3.4) is therefore a projection.

Obtaining the Fourier coefficients requires two DFTs of size N to get $\{c_n\}$ and $\{c'_n\}$, plus about two flops per coefficient d_n . As the f_k and f'_k are real, one may compute both $\{c_n\}$ and $\{c'_n\}$ with a single complex FFT of size N , which is more efficient than the double call of a real DFT of length N , each calculated by means of a complex DFT of length $N/2$ and some overhead ([11, p. 337], [6, p. 129]).

Replacing $n - N$ by k in the third equation of (3.5), one obtains

$$d_k = \left(1 + \frac{k}{N}\right)c_{k+N} + \frac{i}{N}c'_{k+N}, \quad k = -N+1, \dots, -1,$$

which in view of the N -periodicity of the sequences $\{c_n\}$ and $\{c'_n\}$ reads

$$d_n = \left(1 + \frac{n}{N}\right)c_n + \frac{i}{N}c'_n.$$

For the same reason, $c'_0 = c'_N$ and (3.5) may be written for all n as

$$d_n = \left(1 - \operatorname{sign}(n)\frac{n}{N}\right)c_n - i \operatorname{sign}(n)\frac{c'_n}{N}, \quad n = -N, \dots, N, \quad (3.6)$$

with

$$\operatorname{sign}(n) = \begin{cases} 1, & n > 0, \\ 0, & n = 0, \\ -1, & n < 0. \end{cases}$$

It is to be able to write this global formula that we equipped b_N with a factor $\frac{1}{2}$ in (3.1).

One of the referees has kindly noticed that (3.6) may be considered as a replacement for the non-existing discrete Fourier transform for Hermite interpolation.

Equation (3.6) shows that $d_{-n} = \bar{d}_n$. Moreover, as $N \rightarrow \infty$, $d_n \rightarrow c_n$ and, since $c_n \rightarrow C_n$ under suitable conditions on f (e.g., f of bounded variation), the Fourier coefficients (FC) of t tend toward the true FC of f . On the other hand, the d_n are not periodic in general, so that we will not assign them a value for $|n| > N$.

The presence of $\operatorname{sign}(n)$ as a factor of the FC in (3.6) brings the *conjugation operator* or Hilbert transform (on the circle) into play. This operator maps the function $f \in L_2$ with FC $\{c_n\}$ to the function $Kf \in L_2$ with coefficients $\{-i \operatorname{sign}(n)c_n\}$ [12]. Rewriting (3.6) as

$$d_n = c_n - \frac{i \operatorname{sign}(n)}{N}[c'_n - inc_n], \quad n = -N, \dots, N, \quad (3.7)$$

and noticing that $\{inc_n\}$ are the FC of the derivative of the trigonometric polynomial with the extended coefficients $\{c_n\}_{n=-N}^N \in \mathbb{R}^{2N+1}$, we obtain the following result.

Theorem 3.2. *The balanced Hermite trigonometric polynomial t of degree at most N (3.4) satisfying the interpolation conditions (1.2) is given by*

$$t = \tilde{t}_f + \frac{1}{N} \mathbf{K}[\tilde{t}_{f'} - \tilde{t}'_f], \quad (3.8)$$

where \tilde{t}_f and $\tilde{t}_{f'}$ are the trigonometric polynomials

$$\begin{aligned} \tilde{t}_f &= \sum_{n=-N}^N c_n e^{in\tau} = t_f(\phi) + \sum_{n=0}^M \{c_n e^{i(n-N)\phi} + c_{-n} e^{-i(n-N)\phi}\}, \\ \tilde{t}_{f'} &= \sum_{n=-N}^N c'_n e^{in\tau} = t_{f'}(\phi) + \sum_{n=0}^M \{c'_n e^{i(n-N)\phi} + c'_{-n} e^{-i(n-N)\phi}\} \end{aligned}$$

and where \mathbf{K} is the conjugation operator in L_2 .

Since $\mathbf{K}g = 0$ iff g is a constant, (3.8) shows that $t = t_f$ iff $\tilde{t}_{f'} - \tilde{t}'_f = c$ for some additive constant c . This is e.g. the case when the f_k are the values of a classical balanced trigonometric polynomial of degree at most M ($c_n = c'_n = 0$ for $|n| > M$). One thus cannot improve upon t_f by computing the f'_k as the values of its derivatives.

4. The aliasing formula

We will now assume that $\{f_k\}$ and $\{f'_k\}$ are the values of a function $f \in H_p^1$, the Sobolev space of all 2π -periodic functions whose derivative also belongs to $L_2[0, 2\pi]$, so that its Fourier series (2.5) converges uniformly [22, p. 82] and that of f' is given by termwise differentiation, i.e.,

$$f' = \sum_{n=-\infty}^{\infty} in C_n e^{in\tau}.$$

If the series converges at the ϕ_k , then the aliasing formula (2.6) also holds for f'

$$c'_n = \sum_{\ell=-\infty}^{\infty} i(n + \ell N) C_{n+\ell N}.$$

Inserting with (2.6) into (3.7), we obtain

$$d_n = \sum_{\ell=-\infty}^{\infty} C_{n+\ell N} - i \frac{\text{sign}(n)}{N} \left[\sum_{\ell=-\infty}^{\infty} i(n + \ell N) C_{n+\ell N} - in \sum_{\ell=-\infty}^{\infty} C_{n+\ell N} \right]$$

which gives the following result.

Theorem 4.1. *Let $\{f_k\}_{k=0}^{N-1}$ and $\{f'_k\}_{k=0}^{N-1}$ be the values of f , respectively f' , at the equidistant nodes ϕ_k in (1.2) and let $f = \sum_{n=-\infty}^{\infty} C_n e^{in\phi} \in H_p^1$. If the Fourier series of f' converges at the ϕ_k , then the n th coefficient d_n in the equidistant Hermite trigonometric interpolating polynomial (3.4) satisfies the aliasing formula*

$$d_n = \sum_{\ell=-\infty}^{\infty} (1 + \text{sign}(n)\ell) C_{n+\ell N}, \quad n = -N, \dots, N. \quad (4.1)$$

One has in particular the simple cases

$$d_0 = \sum_{\ell=-\infty}^{\infty} C_{\ell N}, \quad d_N = \sum_{\ell=-\infty}^{\infty} (1 + \ell) C_{(1+\ell)N} = \sum_{\ell=-\infty}^{\infty} \ell C_{\ell N}, \quad d_{-N} = -d_N.$$

Using (2.3a), we also obtain aliasing formulae for the a_n and b_n . For instance, for $n \neq 0$,

$$\begin{aligned}
a_n = d_n + d_{-n} &= \sum_{\ell=-\infty}^{\infty} [(1+\ell)C_{n+\ell N} + (1-\ell)C_{-n+\ell N}] \\
&= \sum_{\ell=-\infty}^{\infty} [(1+\ell)C_{n+\ell N} + (1+\ell)C_{-n-\ell N}] = \sum_{\ell=-\infty}^{\infty} (1+\ell)A_{n+\ell N},
\end{aligned}$$

where the A_n are the (true) FC in the real form of the Fourier series of f . This yields the simple formulae

$$a_0 = \sum_{\ell=-\infty}^{\infty} A_{\ell N}, \quad a_n = \sum_{\ell=-\infty}^{\infty} (1+\ell)A_{n+\ell N}, \quad n \neq 0, \quad a_N = \sum_{\ell=-\infty}^{\infty} \ell A_{\ell N} \quad (4.2)$$

and the same for the b_n . These formulae show that the Hermite trigonometric interpolant conserves parity: as the B_n (A_n) vanish for $f \in \mathbb{R}$ even (odd), so do the b_n (a_n).

Notice that (4.2) is also the aliasing formula for the interpolant of even functions and for Hermite Chebyshev interpolation (see Section 8 below).

5. Convergence results derived from the aliasing formula

We can now insert the decay estimates of Section 2 into (4.1) to obtain upper bounds for the approximation error $t - f$.

5.1. Bounds for the coefficients

We start with a bound for the error in the Fourier coefficients d_n : (4.1) yields for every n , $|n| \leq N$,

$$d_n - C_n = \sum_{\ell \neq 0} (1 + \text{sign}(n)\ell)C_{n+\ell N} = \sum_{\ell > 0} (1 - \text{sign}(n)\ell)C_{n-\ell N} + \sum_{\ell > 0} (1 + \text{sign}(n)\ell)C_{n+\ell N}.$$

In view of $|C_{-n}| = |C_n| \forall n$, when the C_n decrease rapidly enough for the series to converge, one obtains by checking the cases $n > 0$ and $n < 0$ separately

$$|d_n - C_n| \leq \sum_{\ell=2}^{\infty} (\ell-1)|C_{\ell N-|n|}| + \sum_{\ell=1}^{\infty} (\ell+1)|C_{\ell N+|n|}|, \quad |n| \leq N, \quad (5.1)$$

a bound that does not depend on the sign of n . It therefore suffices to consider positive n . Notice that no absolute value of a C -index is less than N , and that the bound holds for all $|n| \leq N$, as compared with $|n| \leq M$ for the classical case.

5.1.1. f smooth

Let us now assume that $f^{(p)}$ is of bounded variation $V_p := V_p(0, 2\pi)$, $p \geq 2$. Inserting (2.7) into (5.1) yields

$$\begin{aligned}
|d_n - C_n| &\leq V_p \left\{ \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell N - n)^{p+1}} + \sum_{\ell=1}^{\infty} \frac{\ell+1}{(\ell N + n)^{p+1}} \right\} \\
&= \frac{V_p}{N^{p+1}} \left\{ \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell - \frac{n}{N})^{p+1}} + \sum_{\ell=1}^{\infty} \frac{\ell+1}{(\ell + \frac{n}{N})^{p+1}} \right\} \\
&\leq \frac{V_p}{N^{p+1}} \left\{ \sum_{\ell=2}^{\infty} \frac{\ell - \frac{n}{N}}{(\ell - \frac{n}{N})^{p+1}} + \sum_{\ell=1}^{\infty} \frac{\ell + \frac{n}{N} + 1 - \frac{n}{N}}{(\ell + \frac{n}{N})^{p+1}} \right\} \\
&\leq \frac{V_p}{N^{p+1}} \left\{ \sum_{\ell=2}^{\infty} \frac{1}{(\ell - \frac{n}{N})^p} + \sum_{\ell=1}^{\infty} \frac{1}{(\ell + \frac{n}{N})^p} + \sum_{\ell=1}^{\infty} \frac{1}{(\ell + \frac{n}{N})^{p+1}} \right\}.
\end{aligned}$$

As in [11], we may now bound the series by integrals which may be evaluated to yield, for $|n| \leq N$,

$$|d_n - C_n| \leq \frac{V_p}{N^{p+1}} \left\{ \frac{1}{(p-1)\left(\frac{3}{2} - \frac{n}{N}\right)^{p-1}} + \frac{1}{(p-1)\left(\frac{1}{2} + \frac{n}{N}\right)^{p-1}} + \frac{1}{p\left(\frac{1}{2} + \frac{n}{N}\right)^p} \right\}.$$

The quantity in braces is bounded by

$$\frac{1}{p-1}(2^{p-1} + 2^{p-1}) + \frac{1}{p}2^p = 2^p \left(\frac{1}{p} + \frac{1}{p-1} \right) < 2^{p+1}$$

and we have the following result.

Theorem 5.1. *Let $f^{(p)}$ be of bounded variation $V_p := V_p(0, 2\pi)$, $p \geq 2$, and let $N = 2M$ or $N = 2M + 1$. Then*

$$|d_n - C_n| \leq \frac{V_p}{M^{p+1}}, \quad |n| \leq N. \quad (5.2)$$

A comparison with Theorem 2.3 shows that the bound on the error in d_n now holds for $|n| \leq N$, not only for $|n| \leq M$ as in the classical case.

5.1.2. Functions analytic in a strip

As a second application of the aliasing formula we now assume that f is analytic (holomorphic) in the strip $|\operatorname{Im} w| \leq \theta$. Then, by Theorem 2.2, $|C_n| \leq \mu e^{-|n|\theta}$. Inserting into (5.1) yields

$$\begin{aligned} |d_n - C_n| &\leq \mu \left(\sum_{\ell=2}^{\infty} (\ell-1) e^{-(\ell N - |n|)\theta} + \sum_{\ell=1}^{\infty} (\ell+1) e^{-(\ell N + |n|)\theta} \right) \\ &= \mu \left(e^{|n|\theta} \sum_{\ell=1}^{\infty} \ell e^{-(\ell+1)N\theta} + e^{-|n|\theta} \sum_{\ell=0}^{\infty} (\ell+2) e^{-(\ell+1)N\theta} \right) \\ &= \mu e^{-N\theta} \left[e^{|n|\theta} \sum_{\ell=0}^{\infty} \ell e^{-\ell N\theta} + e^{-|n|\theta} \left(\sum_{\ell=0}^{\infty} \ell e^{-\ell N\theta} + 2 \sum_{\ell=0}^{\infty} e^{-\ell N\theta} \right) \right] \end{aligned}$$

and, using $\sum_{\ell=0}^{\infty} e^{-\ell N\theta} = \frac{1}{1-e^{-N\theta}}$ and $\sum_{\ell=0}^{\infty} \ell e^{-\ell N\theta} = \left(\sum_{\ell=0}^{\infty} e^{-\ell N\theta} \right)^2 = \frac{1}{(1-e^{-N\theta})^2}$,

$$|d_n - C_n| \leq 2\mu e^{-N\theta} \left[\frac{\cosh n\theta}{(1-e^{-N\theta})^2} + \frac{e^{-|n|\theta}}{1-e^{-N\theta}} \right], \quad |n| \leq N. \quad (5.3)$$

Theorem 5.2. *If $f(w)$ is analytic in the strip $|\operatorname{Im} w| \leq \theta$, then the coefficients d_n of its balanced Hermite trigonometric interpolating polynomial tend to those of f as given in (5.3); the convergence is $\mathcal{O}(e^{-N\theta})$.*

Again, the advantage with respect to classical trigonometric interpolation is that the bound holds for all $|n| \leq N$, not only for $|n| \leq M$.

5.2. Bounds for the approximation error

The aliasing formula (4.1) also yields bounds for $t - f$ as a function of growing N . Equation (3.1) becomes with (4.1)

$$t(\phi) = \sum_{n=-N}^N {}'' d_n e^{in\phi} = \sum_{n=-N}^N {}'' e^{in\phi} \left(\sum_{\ell=-\infty}^{\infty} (1 + \operatorname{sign}(n)\ell) C_{n+\ell N} \right). \quad (5.4)$$

Now assume that $f \in H_p^1$, so that its Fourier series (2.5) converges absolutely, and reorder the terms of the latter as in (5.4)

$$f(\phi) = \sum_{n=-N}^N {}'' \sum_{\ell=-\infty}^{\infty} C_{n+2\ell N} e^{i(n+2\ell N)\phi} = \sum_{n=-N}^N {}'' e^{in\phi} \sum_{\ell=-\infty}^{\infty} C_{n+2\ell N} e^{i2\ell N\phi}. \quad (5.5)$$

(Notice that the terms $C_{n+2\ell N}$ are the same for $n = -N$ and $n = N$, so that they both arise twice and “compensate” the primes in the sum for N even.) Subtracting (5.4) yields

$$f(\phi) - t(\phi) = \sum_{n=-N}^N \overset{''}{e^{in\phi}} \sum_{\ell=-\infty}^{\infty} [C_{n+2\ell N} e^{i2\ell N\phi} - (1 + \text{sign}(n)\ell)C_{n+\ell N}].$$

The expression in brackets vanishes for $\ell = 0$, and this yields convergence. Taking absolute values, one gets as for (5.1)

$$|t(\phi) - f(\phi)| \leq \sum_{n=-N}^N \overset{''}{\sum_{\ell \neq 0}} |C_{n+\ell N}| + \sum_{n=-N}^N \overset{''}{\left(\sum_{\ell=2}^{\infty} |C_{\ell N - |n|}| + \sum_{\ell=1}^{\infty} (\ell + 1) |C_{\ell N + |n|}| \right)}.$$

As in the classical case, the first term may be written as a single sum and in the second the terms with positive and negative n are the same. Putting the indices $\ell N \pm |n|$ on an axis, one sees that

$$|t(\phi) - f(\phi)| \leq 2 \sum_{|n| \geq N} \overset{''}{|C_n|} + 2 \sum_{|n| \geq N} \overset{''}{\left(\frac{n}{N} + 1 \right)} |C_n| \leq 6 \sum_{|n| \geq N} \overset{''}{|C_n|}.$$

Theorem 5.3. *Let $f \in H_p^1$ and let t be its balanced equidistant Hermite trigonometric interpolating polynomial. Then*

$$|t(\phi) - f(\phi)| \leq 6 \sum_{|n| \geq N} \overset{''}{|C_n|}. \quad (5.6)$$

Taking the derivatives into account thus moves the lower limit of the sum from M to N , with a moderate increase of the constant from 2 to 6.

This result will now be used in the same cases as above to estimate the speed of convergence.

5.2.1. f smooth

The same proof as that of Theorem 2.6 yields the following bound.

Theorem 5.4. *Let $f^{(p)}$ be of bounded variation $V_p := V_p(0, 2\pi)$, $p \geq 2$. Then*

$$|t(\phi) - f(\phi)| \leq \frac{6V_p}{N^p} \left(\frac{1}{N} + \frac{2}{p} \right) = \mathcal{O}(N^{-p}).$$

The bound therefore converges $\frac{2p}{3}$ times faster to zero than that of classical rational interpolation.

5.2.2. f analytic in a strip

As in the classical case, (5.6) guarantees the exponential convergence of t toward f .

Theorem 5.5. *Let f be analytic in the strip $|\text{Im } w| \leq \theta$. Then its balanced equidistant Hermite trigonometric interpolating polynomial tends to f as*

$$|t(\phi) - f(\phi)| \leq \frac{12\mu}{1 - e^{-\theta}} e^{-N\theta}.$$

Since $0 < e^{-\theta} < 1$, one has geometric convergence with respect to N . With the sequence $N = 2^\ell$, $\ell = 1, 2, \dots$, the convergence is quadratic with respect to ℓ . Compared with the classical case, the bound has N instead of M in the exponential, thus requires about half as many points for the same error, up to a factor of 3.

6. The barycentric formula for t

The Jackson polynomial and the trigonometric polynomial from (6-9) in [24, p. 23] may be obtained by the formulae (3.3). By combining them, we obtain the Lagrangian formula for the balanced Hermite trigonometric polynomial of degree at most N interpolating between equidistant points

$$t(\phi) = \frac{2}{N^2} \sin^2 \frac{N\phi}{2} \sum_{k=0}^{N-1} \left(f'_k \cot \frac{\phi - \phi_k}{2} - f_k \frac{d}{d\phi} \cot \frac{\phi - \phi_k}{2} \right). \quad (6.1)$$

Kress [16] has also given this formula (and corresponding ones for higher-order derivatives) for N even.

In contrast with the polynomial case [13], the factor $\sin \frac{N}{2}\phi$ is ill-conditioned in the vicinity of the $\phi_k \neq 0$ for large N [10]. As mentioned already, one way of coping with this difficulty is to rewrite (6.1) in dependence on the evaluation point ϕ by changing in the factor the variable $\phi = \phi - \phi_0$ to $\phi - \phi_\ell$, where ϕ_ℓ is the node closest to ϕ [9]. To avoid determining this ϕ_ℓ for every ϕ , one can go over to the barycentric formula [10]. Dividing (6.1) by the corresponding formula

$$1 = -\frac{2}{N^2} \sin^2 \frac{N\phi}{2} \sum_{k=0}^{N-1} \frac{d}{d\phi} \cot \frac{\phi - \phi_k}{2}$$

for the function $f \equiv 1$, we obtain

$$t(\phi) = \frac{\sum_{k=0}^{N-1} (f_k \frac{d}{d\phi} \cot \frac{\phi - \phi_k}{2} - f'_k \cot \frac{\phi - \phi_k}{2})}{\sum_{k=0}^{N-1} \frac{d}{d\phi} \cot \frac{\phi - \phi_k}{2}}. \quad (6.2)$$

This is the barycentric formula for equidistant Hermite trigonometric interpolation when 0 is one of the nodes.

7. Interpolating even and odd functions

In practice the functions to be interpolated are often even or odd, as in the case of Chebyshev interpolation on the interval (see below); it is then preferable to limit oneself to the interval $[0, \pi]$ and to interpolate with a cosine or a sine polynomial. There are four ways of distributing N equispaced nodes on the circle symmetrically with respect to 0 (and π) [2]. Considering only the points in $[0, \pi]$, these are the cases when

- (1) none of 0 and π is a node;
- (2) 0 is a node, π is not;
- (3) π is a node, 0 is not;
- (4) both 0 and π are nodes.

The cases (2) and (4) are covered by (6.2). The others can be handled by a simple change of variable in the same formula.

7.1. Interpolation of an even function with a cosine polynomial

Let $f \in H_p^1$ with f' defined at the nodes be even. We wish to find the balanced Hermite trigonometric polynomial (6.1) with all $b_n = 0$ that interpolates equidistant data obtained from even functions. Let first $N = 2M$. Then $f_{N-k} = f_k$ (in particular $f_N = f_0$), f' is odd, $f'_{N-k} = -f'_k$, $k = 0(1)M$, which implies $f'_0 = 0$ and $f'_M = 0$. Inserting into (6.1), summing separately over $k = 0, k = 1, \dots, M-1, k = M$ and $k = M+1, \dots, N-1$ and using that $\phi_{N-k} = 2\pi - \phi_k$, $\cot \frac{\phi - \phi_k}{2} \pm \cot \frac{\phi + \phi_k}{2} = \frac{-2 \sin \phi_k}{\cos \phi - \cos \phi_k}$, $(2 \sin^2 \frac{\phi}{2})^{-1} = \frac{1 - \cos \phi \cos \phi_0}{(\cos \phi - \cos \phi_0)^2}$ and $(2 \cos^2 \frac{\phi}{2})^{-1} = \frac{1 - \cos \phi \cos \phi_{N/2}}{(\cos \phi - \cos \phi_{N/2})^2}$, one obtains

$$t(\phi) = \frac{4}{N^2} \sin^2 \frac{N\phi}{2} \sum_{k=0}^M \left(\frac{\delta_k (1 - \cos \phi \cos \phi_k)}{(\cos \phi - \cos \phi_k)^2} f_k - \frac{\sin \phi_k}{\cos \phi - \cos \phi_k} f'_k \right) \quad (7.1)$$

with the δ_k as defined at the end of Section 2. When interpolating $f \equiv 1$ the second term disappears and we obtain the barycentric formula for N even as

$$t(\phi) = \frac{\sum_{k=0}^M \left(\frac{\delta_k(1-\cos\phi\cos\phi_k)}{(\cos\phi-\cos\phi_k)^2} f_k - \frac{\sin\phi_k}{\cos\phi-\cos\phi_k} f'_k \right)}{\sum_{k=0}^M \frac{\delta_k(1-\cos\phi\cos\phi_k)}{(\cos\phi-\cos\phi_k)^2}}. \quad (7.2)$$

Equation (7.2) halves the number of operations as compared with (6.2).

Similar calculations show that formula (7.1) and thus formula (7.2) also hold when N is odd, $N = 2M + 1$. We have thus treated the cases (2) and (4). Notice that, since the information always consists of an even number of values, the formulae are simpler than those of the classical case given in [2].

Case (3) may be addressed from case (2) by means of the change of variable $\tilde{\phi} := \pi - \phi$, $\tilde{\phi}_k := \pi - \phi_k$, $\tilde{f}(\tilde{\phi}) := f(\phi)$, where the ϕ_k 's are the nodes of case (2). Short calculations with trigonometric identities show that (7.1) and (7.2) hold in this case too (with δ_k moving the factor 1/2 from 0 to π).

The $2M$ points on the full circle in case (1) are just points of the general case (1.2) rotated (in the positive sense) by the angle $\phi_0 = \frac{\pi}{N}$. Then formula (6.1) is still valid with $\sin^2 \frac{N}{2}(\phi - \phi_0)$ instead of $\sin^2 \frac{N}{2}\phi$ (since $\sin^2 \frac{N}{2}(\phi - \phi_0) = \sin^2 \frac{N}{2}(\phi - \pi) = \cos^2 \frac{N}{2}\phi$, the parity is maintained and t still has $a_N = 0$ —see the proof of Lemma 1.1 in [16]). Going barycentric shows that (6.2) holds in this case too and specializing to even functions as above yields (7.2) again (with all $\delta_k = 1$).

7.2. Hermite polynomial interpolation between Chebyshev points

Interpolation between Chebyshev points $x_k = \cos\phi_k$ on the interval $[-1, 1]$ is but a special case of even trigonometric interpolation at the equidistant nodes ϕ_k : roughly speaking, the change of variable $x = \cos\phi$ transplants any function $F(x)$ defined on $[-1, 1]$ into an even 2π -periodic function $f(\phi) := F(\cos\phi)$ with the same differentiability properties as f (they are conserved at 0 and π because of the vanishing inner derivatives in the chain rule). Hence the barycentric formula for Chebyshev points is given by (7.2).

Most important in practical applications is case (4), i.e., the *Chebyshev points of the second kind* $x_k = \cos k \frac{\pi}{M}$, $k = 0, \dots, M$, which encompass the extremities -1 and 1 [8,23]. Specializing (7.2) for these points and replacing $\cos\phi$ and $\cos\phi_k$ with x , respectively x_k yields

$$P(x) = \frac{\sum_{k=0}^M \left(\frac{1-xx_k}{(x-x_k)^2} F(x_k) - \frac{\sin\phi_k}{x-x_k} F'(x_k) \right)}{\sum_{k=0}^M \frac{1-xx_k}{(x-x_k)^2}}$$

since $\sin\phi_k = 0$ for $k = 0$ and $k = M$. This formula does not seem to have been written down before. The corresponding formula for the Chebyshev points of the first kind is given in [11, p. 257].

7.3. Interpolation of an odd function with a sine polynomial

Let again $f \in H_p^1$ with f' defined at the nodes, but now let it be odd and let $N = 2M$. We now aim at finding barycentric formulae for the balanced Hermite trigonometric polynomial (6.1) with $a_n = 0$, all n , that interpolates odd data. Here f' is even, so that $f_{N-k} = -f_k$, $f'_{N-k} = f'_k$ and in particular $f_0 = 0$, $f'_{\frac{N}{2}} = 0$. Using the same relations for sines and cosines as in Section 7.1, as well as $\cot \frac{\phi}{2} = -\frac{\sin\phi}{\cos\phi-\cos\phi_0}$ and $\tan \frac{\phi}{2} = \frac{\sin\phi}{\cos\phi-\cos\phi_{N/2}}$, one obtains

$$t(\phi) = \frac{4}{N^2} \sin^2 \frac{N}{2} \phi \sin\phi \sum_{k=0}^M \left(\frac{\sin\phi_k}{(\cos\phi - \cos\phi_k)^2} f_k - \frac{\delta_k}{\cos\phi - \cos\phi_k} f'_k \right). \quad (7.3)$$

To go barycentric, we interpolate the odd function $\sin\phi$ [2], use the fact that $\sin^2\phi_k = 0$ and $\cos^2\phi_k = 1$ whenever $\phi_k = 0$ or π and divide (7.3) by the result to get

$$t(\phi) = \sin\phi \cdot \frac{\sum_{k=0}^M \left(\frac{\sin\phi_k}{(\cos\phi - \cos\phi_k)^2} f_k - \frac{\delta_k}{\cos\phi - \cos\phi_k} f'_k \right)}{\sum_{k=0}^M \frac{\delta_k(1-\cos\phi\cos\phi_k)}{(\cos\phi - \cos\phi_k)^2}} \quad (7.4)$$

Table 1
Convergence of c_2 and d_2 toward C_2 in Example 1

N	$c_2 - C_2$	Order est.	$d_2 - C_2$	Order est.
4	-0.24767604552648		0.00232395447352	
8	0.00232395447352	6.736	0.00042832410540	2.440
16	0.00007614729831	4.932	0.00004007323086	3.418
32	0.00000412894457	4.205	0.00000311075045	3.687
64	0.00000024926361	4.050	0.00000021821258	3.833
128	0.00000001544524	4.012	0.00000001448074	3.914
256	0.00000000096325	4.003	0.00000000093316	3.956
512	0.00000000006017	4.001	0.00000000005923	3.978
1024	0.00000000000376	4.000	0.00000000000373	3.989
2048	0.00000000000024	3.991	0.00000000000024	3.985

Table 2
Convergence of c_{M-2} and d_{M-2} toward C_{M-2} in Example 2

N	C_{M-2}	$c_{M-2} - C_{M-2}$	$d_{M-2} - C_{M-2}$
8	-0.13180542594924	0.10857918943538	-0.05705642379224
16	-0.05953511746105	-0.03064518542934	-0.00406522809637
32	-0.01214655347676	-0.00551707528260	-0.00003266258596
64	-0.00050560664143	-0.00022837937825	-0.00000000235023
128	-0.0000087605662	-0.00000039570552	-0.00000000000000
256	-0.0000000000263	-0.0000000000119	-0.00000000000000

with, again, the δ_k as defined at the end of Section 2. Similar considerations as in the former paragraph show the validity of this formula in all four cases (with the δ_k putting the factors 1/2 at the right places).

8. Numerical illustration

The convergence results of Section 5 describe how completing the data at equidistant nodes with the values of the derivative improves the approximation. We shall now demonstrate this in two practical cases.

Example 1. Let

$$f(\phi) := \begin{cases} \sin^p \phi, & 0 \leq \phi < \pi, \\ \sin^{p+1} \phi, & \pi \leq \phi < 2\pi, \end{cases} \quad p \in \mathbb{N};$$

$f \in C^{p-1}$, with $f^{(p)}$ of bounded variation. f does not display obvious symmetries. To test Theorem 5.1, we have computed the errors for $n = 2$ (fixed) and increasing N . Table 1 displays the results for $p = 3$. The first column gives N , the second and fourth $c_n - C_n$ and $d_n - C_n$, respectively, and the third and fifth the experimental orders of convergence of the two approximations, as described in [4,5]. Here and in Table 2 the exact coefficients C_n , which are real, were computed with Maple; this yields $C_2 = 0.25232395447352$. The results show that d_n is significantly better than c_n merely for relatively small N and that the advantage diminishes with growing N . Both orders of convergence are $p + 1$, as anticipated in Theorems 2.3 and 5.1.

Example 2. Here we consider $f(\phi) = \frac{1}{1+a^2 \cos^2 \phi}$, which is analytic in the strip of halfwidth $\theta = \frac{1}{a}$ about \mathbb{R} . First we have approximated C_{M-2} (computed with the trapezoidal rule and a number of nodes large enough for machine precision) for $a = 5$, i.e., the periodized Runge function, and increasing N . In view of the aliasing formula and the rapid decay of the c_n , c_{M-2} is of the order of the discretization error in the DFT. It thus has an error close to 100% and is useless as an approximation to C_{M-2} , as documented in the third column of Table 2 which displays $c_{M-2} - C_{M-2}$. The fourth column shows how taking the information about the derivatives into account massively improves the coefficients: $d_{M-2} - C_{M-2}$ is much smaller.

Table 3
Convergence of the classical and Hermite interpolants in Example 2

n	$\ t_f - f\ _\infty$	$\ t - f\ _\infty$
4	0.81061320792792	0.81061320792792
8	0.69162214692591	0.61506609005530
16	0.47854965956575	0.34035709769107
32	0.20898140000595	0.09808600222645
64	0.03681191244580	0.00535748377324
128	0.00141519367568	0.00001064663187
256	0.00000270457899	0.00000000002652
512	0.00000000000786	0.00000000000000

Finally we have compared the errors $\|t_f - f\|_\infty$ and $\|t - f\|_\infty$, with a larger a ($a = 10$) to slow down convergence. t_f and t have been evaluated by means of the barycentric formulae (2.9) and (6.2), respectively. Table 3 shows how doubling N squares the error and that the Hermite interpolant t requires about half as many points to yield the same accuracy as t_f , up to a factor of 3, as explained by Theorems 5.5 and 2.6.

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