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# Feedback and efficiency in limit order markets

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**Abstract**

A consistency criterion for price impact functions in limit order markets is proposed that prohibits chain arbitrage exploitation. Both the bid-ask spread and the feedback of sequential market orders of the same kind onto both sides of the order book are essential to ensure consistency at the smallest time scale. All the stocks investigated in Paris Stock Exchange have consistent price impact functions.

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## 1. Introduction

Mainstream finance and mathematical finance suppose that the price dynamics follows a random walk [1,5]. Extreme assumptions are indeed most useful in a theoretical framework. This is why the opposite is worth considering Ref. [4]: suppose that trader 0 is active at time $t$; he buys/sells a given amount of shares $n_0$, leading to (log-)price change $r(t) = r_0$, where $t$ is in transaction time, $t$ being the $t$-th transaction. Assume that trader 1 has perfect information about $t$ and $r_0$ and exploits it accordingly.

A related situation is found in Ref. [6] whose main result is that a arbitrage opportunity, when exploited, does not disappear but is spread around $t$. It is a counter-intuitive outcome, that raises two questions: how to accommodate the never-disappearing arbitrage, and how microscopic arbitrage removal is possible at all at this time scale. This proceeding, a short version of Ref. [4], suggests that real markets remove arbitrage on a single transaction basis by a double feedback of the last transactions on the order book.

The price impact function $I(n)$ is by definition the relative price change caused by a transaction of $n$ (integer) shares ($n > 0$ for buying, $n < 0$ for selling); mathematically,

$$p(t+1) = p(t) + I(n), \tag{1}$$

where $p(t)$ is the *log*-price and $t$ is in transaction time.

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The generic impact function that will be studied is $I(|x|) = \gamma \log |x|$. The above notation misleadingly suggests that $I$ does not depend on time. In reality, $I$ is not only subject to random fluctuations (which will be neglected here), but also, for instance, to feedback from the type of market orders which has a long memory (see e.g. Refs. [3,8,2,7] for discussions about the dynamic nature of market impact). Neglecting the dynamics of $I$ requires us to consider specific shapes for $I$ that enforce some properties of price impact for each transaction, whereas in reality they only hold on average. For example, we should restrict ourselves to the class of functions that makes it impossible to obtain round-trip positive gains [6]. But the inappropriateness of constant price impact functions is all the more obvious as soon as we consider how price predictability is removed by speculation, which is intertemporal by nature.

The most intuitive (but wrong) view of market inefficiency is to regard price predictability as a scalar deviation from the unpredictable case: if there were a relative price deviation $r_0$ caused by a transaction of $n_0$ shares at some time $t$, according to this view, we should exchange $n_1$ shares so as to cancel perfectly this anomaly, where $n_1$ is such that $I(n_1) = -r_0$. This view amounts to regarding predictability as something that can be remedied with a single trade. However, the people that would try to cancel $r_0$ would not gain anything by doing it unless they are market makers who try to stabilize the price.

It is most instructive to understand how constant price impact functions are paradoxical by considering a simple example. Trader 1, a perfectly (and possibly illegally) informed speculator, will take advantage of his/her knowledge by opening a position at time $t-1$ and closing it at time $t+1$. It is important to be aware that if one places an order at time $t$, the transaction takes place at price $p(t+1)$. Provided that trader 0 buys/sells $n_0$ shares irrespective of the price that he obtains, the round trip of trader 1 yields a monetary gain of

$$g_1 = n_1[\mathrm{e}^{p(t+2)} - \mathrm{e}^{p(t)}] = n_1 \mathrm{e}^{p_0}[\mathrm{e}^{I(n_0)} - \mathrm{e}^{I(n_1)}]$$

where $p_0$ is the log-price before any trader considered here makes a transaction. Since $I(n)$ generally increases with $n$, there is an optimal $n_1^*$ number of shares that maximizes $g_1$. The discussion so far is a simplification, in real-money instead of log-money space, of the one found in Ref. [6]. One should note that far from diminishing price predictability, the intervention of trader 1 increases the fluctuations. Therefore, in the framework of constant price impact functions, an isolated arbitrage opportunity never vanishes but becomes less and less exploitable because of the fluctuations, thus the reduction, of signal-to-noise ratio caused by the speculators.

It seems that trader 1 cannot achieve a better gain than by holding $n_1^*$ shares at time $t$. Since the actions of trader 1 do not modify in any way the arbitrage opportunity between $t-2$ and $t+2$, he can inform a fully trusted friend, trader 2, of the gain opportunity on the condition that the latter opens his position before $t-1$ and closes it after $t+1$ so as to avoid modifying the relative gain of trader 1.[1] For instance, trader 2 informs trader 1 when he/she has opened his position and trader 1 tells trader 2 when he has closed his/her position. From the point of view of trader 2, this is very reasonable because the resulting action of trader 1 is to leave the arbitrage opportunity unchanged to $r_0$ since $p(t+1) - p(t-1) = r_0$. Trader 2 will consequently buy $n_2^* = n_1^*$ shares at time $t-2$ and sell them at time $t+2$, earning the same return as trader 1. This can go on until trader $i$ has no fully trusted friend. Note that the advantage of trader 1 is that he/she holds a position over a smaller time interval, thereby increasing his return rate; in addition, since trader 2 increases the opening price of trader 1, the absolute monetary gain of trader 1 actually *increases* provided that he/she has enough capital to invest. Before explaining why this situation is paradoxical, it makes sense to emphasize that the gains of traders $i > 0$ are, of course, obtained at the expense of trader 0, and that the result of the particular order of the traders' actions is to create a bubble which peaks at time $t+1$.

The paradox is the following: if trader 1 is alone, the best return that can be extracted from his/her perfect knowledge is $\hat{g}_1(n_1^*)$ according to the above reasoning. When there are $N$ traders in the ring of trust, the total return extracted is $N$ times the optimal gain of a single trader. Now, assume that trader 1 has two brokering accounts; he/she can use each account, respecting the order in which to open and close his/her positions, effectively earning the optimal return on each account. The paradox is that his/her actions would be completely equivalent to investing $n_1^*$ and then $n_1^*$ from the same account. In particular, in the case of $I(n) = n$, this seems *a priori* exactly similar to grouping the two transactions into $2n_1^*$, but this results of course in a return smaller than the optimal return for a doubled investment. Hence, in this framework, trader 1 can earn as much as he/she pleases provided that he/she splits the investment into subparts of $n_1^*$ shares whatever $I$ is, as long as it is constant.

[1] If trader 2 were not a good friend, trader 1 could in principle ask trader 2 to open his/her position after him/her and to close it afterwards, thus earning more. But relationships with real friends are supposed to be egalitarian in this paper.

This paradox seems too good to be present in real markets. As a consequence, we should rather consider its impossibility as an *a contrario* consistency criterion for price impact functions. Let us introduce the two relevant mechanisms that are at work in real markets. Half of the solution lies in the dynamics of the order book, particularly the reaction of the order book to a sequence of market orders of the same kind. Generically, the impact of a second market order of the same kind and size is smaller by a factor $\kappa_1$ than that of the first one, and similarly by a factor $\kappa_2$ for a third one, etc. [9,7]. To this contraction of market impact on one side also corresponds an increase of market impact on the other side for the next market order of opposite type [9]; therefore, we shall assume that the impact function on the other side is divided by $\theta_1$ after the first market order, by $\theta_2\theta_1$ after the second, etc. As shown in the next section, $\kappa_1 \simeq \kappa_2$ is a very good approximation when $\kappa_1$ and $\kappa_2$ are averaged over all the stocks, hence we shall only use $\kappa$; for the same reason, we assume that $\theta_1 = \theta_2 = \theta$.

## 2. Feedback

In order to investigate whether the feedback restricted on the side on which the first sequential market orders are placed is enough to make price impact consistent, one sets $\theta = 1$. In the case of log price impact functions, the optimal number of shares and gain of trader 1 are

$$n_1^* = \frac{n_0^\kappa}{(\gamma+1)^{1/\gamma}} \tag{2}$$

and

$$g_1^* = \mathrm{e}^{p_0} n_0^{\kappa(\gamma+1)} \frac{\gamma}{(\gamma+1)^{1+1/\gamma}}. \tag{3}$$

These two equations already show that the reaction of the limit order book reduces the gain opportunity of player 1. Adding trader 2 will reduce further the impact of trader 0, hence the gain of trader 1, and, as before, trader 2 should pay for it. In this case, the reduction of gain of trader 1 is

$$\frac{\Delta g_1}{\mathrm{e}^{p_0}} = [g_1^* - g_1(n_1^*, n_2)]\mathrm{e}^{-p_0} = n_0^{\kappa(\gamma+1)} \frac{\gamma}{(\gamma+1)^{1+1/\gamma}} - n_2^\gamma \frac{n_0^{\gamma(1+\kappa\gamma)}}{(\gamma+1)^{1/\gamma+\kappa}} [n_0^{-\gamma\kappa(1-\kappa)}(\gamma+1) - 1], \tag{4}$$

while the gain that trader 2 optimizes is

$$\frac{G_2}{\mathrm{e}^{p_0}} = n_2^{\gamma+1} \left[ \frac{1}{n_2^{\kappa\gamma}} \frac{n_0^{\kappa\gamma(2\kappa-1)}}{(1+\gamma)^{\kappa-1}} - 1 \right] - \frac{\Delta g_1}{\mathrm{e}^{p_0}}. \tag{5}$$

Trader 1's impact functions are $\kappa I$ when he opens his/her position and $I$ when he/she closes it: this is an additional cause of loss for trader 1 which must be also compensated for by trader 2. Fortunately for the latter, impact functions are $I$ when opening and $\kappa I$ when closing his/her position. Therefore, provided that $\kappa$ is large enough so as not to make $\kappa^2 I(n_0)$ too small, trader 2 can earn more than trader 1 in some circumstances. Impact functions are inconsistent when $G_2^* > 0$, $n_1^* > 1$ and $n_2^* > 1$ for log impact functions. It turns out that the regions in which $G_2^* > 0$ while $n_1^* > 1$ are disjoint if $\kappa < \kappa_c \simeq 0.5$ for log impact functions.

As previously mentioned, the feedback acts on both order book sides. Assuming that $\theta_1 = \theta_2 = \kappa$ in order to be able to use available market measurements, one finds a critical value of $\kappa$ of about 0.83 for log price impact functions. For this value, only a small area of inconsistent impact functions, corresponding to $n_0 \simeq 1.5$, still exists in the $(n_0, \gamma)$ plane, but cannot be reached since both $n_0 = 1$ and 2 are outside of the inconsistent region. Therefore, even double feedback does not guarantee consistency.

## 3. Empirical data

The values of $\kappa_1$ and $\kappa_2$ can be measured in real markets. The response function $R(\delta t, V) = \langle (p(t+\delta t) - p(t))\epsilon(t)\rangle|_{V(t)=V}$ is the average price change after $\delta t$ trades, conditional on the sign of the trade $\epsilon(t)$ and on volume $V$; similarly, one defines the response function conditional on two trades of the same sign $R^+(\delta t, V) = \langle (p(t+\delta t) - p(t))\epsilon(t)\rangle|_{\epsilon(t)=\epsilon(t-1), V(t)=V}$, and $R^{++}(\delta t) = \langle (p(t+\delta t) - p(t))\epsilon(t)\rangle|_{\epsilon(t)=\epsilon(t-1)=\epsilon(t-2)}, V(t) = V$.

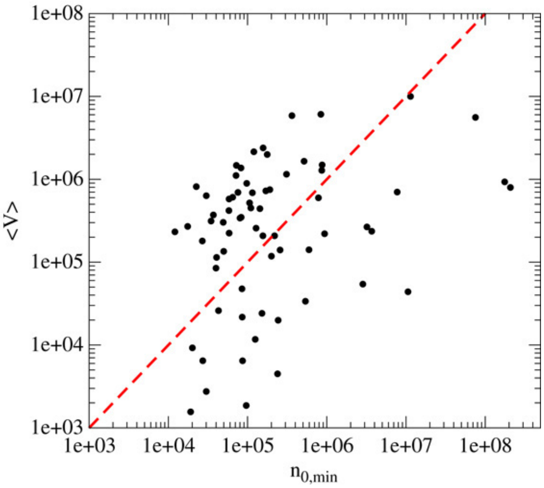


Fig. 1. Average daily volume versus $n_{0,\min}$ (Paris Stock Exchange, 2006).

A key finding of Refs. [3,2] is that $R$ factorizes into $R(\delta t)F(V)$. Thus we will be interested in $R(\delta t)$, $R^+(\delta t)$ and $R^{++}(\delta t)$.

Using measures kindly provided by J.-Ph. Bouchaud and J. Kockelkoren, we find that the estimate of this ratio $\hat{\kappa} = \langle R^+(1)\rangle/\langle R(1)\rangle \in [0.86, 1.02]$, while the averages over all the stocks of the Paris Stock Exchange in 2006 was $\overline{\hat{\kappa}_1} = 0.97 \pm 0.04$ and $\hat{\kappa}_2 = \langle R^{++}(1)\rangle/\langle R^+(1)\rangle \in [0.87, 1.01]$, while $\overline{\hat{\kappa}_2} = 0.97 \pm 0.03$. ; for a given stock, there is some correlation between $\kappa_1$ and $\kappa_2$; the data presented here do not contain error bars for the measures of $R$, $R^+$ and $R^{++}$. The approximation $\kappa_1 \simeq \kappa_2$ is reasonable, and we shall from now on call $\kappa = (\kappa_1 + \kappa_2)/2$ and replace $\kappa_1$ and $\kappa_2$ by $\kappa$ everywhere. In other words, there is some variation between the stocks, some of them being less sensitive to successive market orders of the same kind. The values of estimated $\kappa$ start at 0.86. Therefore, even feedback on both book sides does not yield consistent log impact functions. We conclude that the feedback of the order book considered here is not enough to make price impact functions consistent.

## 4. Spread

The above discussion neglects the bid-ask spread $s$. It is of great importance in practice, as the impact of one trade is on average of the same order of magnitude as the spread [9]. This means that $n_0$ must be large enough in order to make the knowledge of trader 1 valuable. It is easy to convince oneself that it is enough to replace $n_0$ by $n_0' = I^{-1}[I(n_0) + \langle s\rangle]$ in the relevant equations, and multiply all the gains by $\mathrm{e}^{\langle s\rangle/2}$. For example, the optimal number of shares that trader 1 invests if trader 0 has infinite capital is

$$n_{1,s}^* = \mathrm{e}^{-\langle s\rangle/\gamma} \frac{n_0}{(1+\gamma)^{1/\gamma}}. \tag{6}$$

From this equation we see that $n_0' = n_0 \exp(-\langle s\rangle/\gamma)$. Since $\langle s\rangle/\gamma \sim 10$ in practice, the minimal amount of shares needed to create an arbitrage, denoted by $n_{0,\min}$ is increased about 20,000 folds by the spread. The respective values of $\gamma$ and $\langle s\rangle$ are not independent, and can be measured in real markets for a given stock. In the language of Ref. [3], $\gamma = \langle \log(n)\rangle/R(1)$ where $\langle \log(n)\rangle$ is the average of the logarithm of transaction size and $R(1)$ is the response function after one time step. Using $\gamma$ and $\langle s\rangle$ measured in Paris Stock Exchange one finds that $n_{0,\min} \in [1.210^4, 2.010^8]$, with median of $1.210^5$ (Fig. 1), which is not unrealistic for very liquid stocks. Indeed, the fraction of $n_{0,\min}$ with respect to the average daily volume of each stock ranges from $\simeq$2% to more than 100%, with a median of about 42%. Therefore, for most of the stocks, trader 0 needs to trade less than two fifths of the daily average volume in one transaction in order to leave an exploitable arbitrage; for 12 stocks (18%), trading less than 10% of the average daily volume suffices. It is unlikely that a single trade is larger than the average daily volume, hence, 29 stocks (43%) do not allow *on average* a single large trade to be exploitable by a simple round trip. Interestingly, stocks with average daily volumes smaller than about $10^5$ are all consistent from that point of view (Fig. 1). In addition, the stocks for which $n_{0,\min} < \langle V\rangle$ all have a $\kappa > 0.945$.

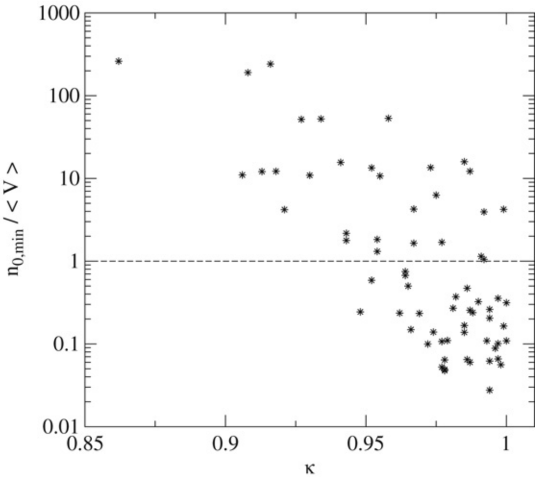


Fig. 2. Fraction of daily volume needed to inject an exploitable arbitrage versus $\kappa$ (same data set).

Therefore, the role of the spread is to increase considerably the minimum size of the trade, which in some cases remain within reasonable bounds. The mathematical discussions of the previous sections on log price impact functions are therefore still valid, provided that we replace $n_0$ with $n_0' = I^{-1}[I(n_0) + s]$, which is equivalent to rescaling $n_0$ by $\exp(\langle s\rangle/\gamma)$. Therefore, the spread must be taken into account, but does not yield systematically consistent impact functions for some stocks with a high enough daily volume.

## 5. Spread and feedback

The question is whether the feedback and the spread combined make real market impact functions systematically consistent. The stocks that are the most likely to become consistent are those whose $n_{0,\min}/\langle V\rangle < 1$ is large while having a strong feedback. According to Fig. 2, these properties are compatible. Using for each stock $\langle s\rangle$, $\kappa$, and $\gamma$ from the data, we find that three additional stocks are made consistent by feedback on trader 0's market order side alone: the feedback limited to one side of the order book, even when the spread is taken into account, is insufficient. However, adding finally the feedback on both book sides makes consistent *all* the stocks, even in the case of infinite capital. Therefore, both the spread and the feedback are crucial ingredients of consistency at the smallest time scale.

## 6. Conclusion

The paradox proposed in this paper provides an *a contrario* simple and necessary condition of consistency for price impact functions. Indeed, financial markets ensure consistent market price impact functions at the most microscopic dynamic level by two essential ingredients: the spread and the dynamics of the order book.


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