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VALUATIONS WITH CROFTON FORMULA AND FINSLER GEOMETRY (REVISED VERSION)

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Abstract. Valuations admitting a smooth Crofton formula are studied using Geometric Measure Theory and Rumin's cohomology of contact manifolds. The main technical result is a current representation of a valuation with a smooth Crofton formula. A geometric interpretation of Alesker's product is given for such valuations. As a first application in Finsler geometry, a short proof of the theorem of Gelfand-Smirnov that Crofton densities are projective is derived. The Holmes-Thompson volumes in a projective Finsler space are studied. It is shown that they induce in a natural way valuations and that the Alesker product of the k-dimensional and the l-dimensional Holmes-Thompson valuation is the $k+l$-dimensional Holmes-Thompson valuation.

Introduction

The classical Crofton formula computes the length of a curve in the plane by averaging the number of intersection points of the curve and a straight line. Higher dimensional generalizations, where straight lines are replaced by affine planes of a fixed dimension, are known under the name *Linear Kinematic Formulas*. These formulas were proved by Blaschke and his school. They can be used to compute the so-called intrinsic volumes of subsets of Euclidean space. Quite recently, it was shown that similar formulas also hold in a Finsler setting. A Finsler metric on a manifold is, roughly speaking, the assignment of a norm in each tangent space. There are various definitions of volume for a Finsler manifold; the two best-known examples are the Busemann volume (which is the Hausdorff measure of the underlying metric space) and the Holmes-Thompson volume (which comes from symplectic geometry). The reader is referred to [11] for more information on volumes on Finsler spaces.

A Finsler metric on a finite-dimensional vector space is called *projective* if its geodesics are straight lines. As Álvarez Paiva and Fernandes showed, the Holmes-Thompson volume of a compact submanifold of a projective Finsler space can be computed by a Crofton formula [9].

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The aim of this paper is to put these formulas into the more general context of valuations. Recall that a (convex) valuation on an n-dimensional oriented vector space V is a real-valued map Ψ on the space $\mathcal{K}(V)$ of compact convex subsets such that the following Euler additivity holds true for all $K, L \in \mathcal{K}(V)$ with $K \cup L \in \mathcal{K}(V)$:

$$\Psi(K \cup L) + \Psi(K \cap L) = \Psi(K) + \Psi(L).$$

Subanalytic valuations are defined similarly, replacing the word "convex" by "subanalytic".

Valuations on general manifolds were defined by Alesker in a series of papers [2]-[6]. Compact convex sets are replaced by a convenient system of subsets, like the space of compact submanifolds with corners or differentiable polyhedra. It turns out that, under an additional and important smoothness condition which is given below, an Alesker valuation on a finite-dimensional vector space restricts to a convex valuation and to a subanalytic valuation. Moreover, both restriction maps are isomorphisms. Keeping this in mind, we will be a bit sloppy and switch between the convex, the subanalytic and the manifolds with corners setting. In the first sections of this paper, we will state the results in terms of subanalytic valuations. In later sections, it will be more convenient to work with convex valuations.

Let us say that a subanalytic valuation admits a smooth Crofton formula of degree k if there exists a smooth (signed) measure μ on the manifold $\mathrm{AGr}_{n-k}^{+}(V)$ of oriented affine $n-k$-planes in V such that

$$\Psi(X) = \int_{\mathrm{AGr}_{n-k}^{+}(V)} \chi(X \cap H) d\mu(H), \quad X \text{ compact, subanalytic.} \tag{1}$$

It is easy to see that the right hand side defines a valuation for every μ, but not every valuation is of this type.

Since the measure μ is smooth, it is not surprising that the valuation Ψ has some smoothness properties.

In order to define smoothness of a valuation, we need the notion of the conormal cycle of a compact subanalytic set. Its definition is recalled in Section 1, for the moment it is enough to know that there is a canonical way to associate to each compact subanalytic set a Legendrian cycle $\mathrm{cnc}(X)$ in the cosphere bundle S^*V in such a way that

$$\mathrm{cnc}(X \cup Y) + \mathrm{cnc}(X \cap Y) = \mathrm{cnc}(X) + \mathrm{cnc}(Y) \quad X, Y \text{compact, subanalytic.}$$

It follows that each smooth $n-1$-form ω on S^*V induces a valuation Ψ_ω by setting

$$\Psi_\omega(X) := \mathrm{cnc}(X)(\omega).$$

Valuations which can be represented in this way are called *smooth*, compare [3]. The space of smooth valuations on V is denoted by $\mathcal{V}^\infty(V)$.

A given smooth valuation Ψ may be represented by different forms ω. But there exists a second order differential operator D, called the *Rumin* *operator*, such that $D\omega$ is uniquely associated to Ψ [16]. This operator is related to the cohomology of contact manifolds. Its construction will be sketched in Section 2.

We say that Ψ is of *pure degree* k if $k = 0$ and Ψ is a real multiple of the Euler characteristic or if $k > 0$, $D\omega$ is of bidegree $(k, n-k)$ (w.r.t. the product $S^*V = V \times S^*(V)$ and Ψ vanishes on points.

Before stating our main results, we have to recall that there is an involution on the space of smooth valuations, called the *Euler-Verdier involution*. It was introduced by Alesker [3] and is induced (up to some factor) by the natural involution of S^*V.

The heart of the paper is the proof of the following theorem.

Theorem 1. *Let μ be a smooth (signed) measure on $\mathrm{AGr}_{n-k}^+(V)$. Then the valuation Ψ defined by* (1) *is smooth, of pure degree k and belongs to the $(-1)^k$ eigenspace of the Euler-Verdier involution.*

In fact, we will show a bit more. Namely, we will see that $\Psi = \Psi_\omega$ for a form ω with the property that $D\omega$ is the Gelfand transform of μ for some double fibration.

This theorem will be used together with the following uniqueness result.

Theorem 2. *Let Ψ be a smooth valuation of pure degree k which belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution. If $\Psi(M) = 0$ for all k-dimensional submanifolds with boundary, then $\Psi = 0$.*

Let us illustrate these results in the translation invariant case.

Recall that a valuation Ψ on V is called translation invariant if $\Psi(x+X) = \Psi(X)$ for all $x \in V$ and all X. If $\Psi = \Psi_\omega$ is smooth, then Ψ is translation invariant if and only if $D\omega$ is translation invariant [16].

A translation invariant valuation Ψ is called of degree k if $\Psi(tX) = t^d\Psi(X)$ for all $t > 0$. By a result of McMullen [24], a non-zero valuation can be uniquely written as a sum of homogeneous components of degrees $0, 1, \ldots, n$. In the smooth, translation invariant case Ψ is of degree k if and only if Ψ is of pure degree k.

A translation invariant valuation Ψ is called even if $\Psi(-X) = \Psi(X)$ for all X. A smooth translation invariant valuation of degree k is even if and only if it belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution (compare Theorem 3.3.2 in [3]).

On a Euclidean vector space V, translation invariant, even valuations of degree k can be described by their Klain functions. Given such a valuation Ψ, its Klain function is the function on the Grassmannian $\mathrm{Gr}_k(V)$ which associates to $L \in \mathrm{Gr}_k(V)$ the real number $\Psi(D_L)$, where D_L is the unit ball in L. We thus get a map (called the Klain embedding) from the space of smooth, translation invariant, even valuations of degree k to $C^\infty(\mathrm{Gr}_k(V))$. By a theorem of Klain [23], this map is injective. In fact, Theorem 2 is the generalization of Klain's injectivity result to the non translation invariant situation.

The description of the image of the Klain imbedding was provided by Alesker-Bernstein [7] in terms of the cosine transform. It implies a partial converse to Theorem 1: A smooth, even, translation invariant valuation of degree k admits a smooth translation invariant Crofton measure.

Let us return to the general situation.

Given smooth (signed) measures μ_1 on $\mathrm{AGr}^+_{n-k_1}(V)$ and μ_2 on $\mathrm{AGr}^+_{n-k_2}(V)$, let μ be the push-forward under the natural intersection map

$$\mathrm{AGr}^+_{n-k_1}(V) \times \mathrm{AGr}^+_{n-k_2}(V) \setminus \Delta_{k_1,k_2} \to \mathrm{AGr}^+_{n-k_1-k_2}(V).$$

Here Δ_{k_1,k_2} is the null set of pairs of affine planes of dimensions $n-k_1$ and $n-k_2$ such that their intersection is not of dimension $n-k_1-k_2$. This construction appears in [9] and we call μ the *Álvarez-Fernandes product* of μ_1 and μ_2. If Ψ_1, Ψ_2 and Ψ are the valuations with Crofton measures μ_1, μ_2 and μ, then we also say that Ψ is the Álvarez-Fernandes product of Ψ_1 and Ψ_2. However, at this point it is not clear that this product is well-defined, since a smooth valuation can have different Crofton formulas.

On the other hand, there is another product, called the *Alesker product* on the space of smooth valuations [2].

Theorem 3. *Suppose that $\Psi_1, \Psi_2 \in \mathcal{V}^\infty(V)$ admit smooth Crofton measures. Then the Álvarez-Fernandes product of Ψ_1 and Ψ_2 equals the Alesker product of Ψ_1 and Ψ_2. In particular, the Álvarez-Fernandes product is well-defined.*

Using Theorems 1, 2 and 3 we will derive the following application in Finsler geometry.

Theorem 4. *Let V be an n-dimensional vector space with a projective Finsler metric. Then the Holmes-Thompson volume of k-dimensional submanifolds (with or without boundary) extends to a unique smooth valuation $\Psi_k^{HT} \in \mathcal{V}^\infty(V)$ of pure degree k and belonging to the $(-1)^k$-eigenspace of the Euler-Verdier involution. Moreover, for $k+l \leq n$ the Alesker product of Ψ_k^{HT} and Ψ_l^{HT} is Ψ_{k+l}^{HT}.*

Our second application in Finsler geometry concerns projective densities. The definition of a smooth k-density on a manifold M will be recalled in Section 6. A smooth k-density ϕ can be integrated over a (not necessarily oriented) k-dimensional submanifold $N \subset M$.

A smooth density ϕ on $\mathbb{RP}^n$ is called *projective* if k-dimensional projective subspaces are extremal for the variational problem $N \mapsto \int_N \phi$. ϕ is called *Crofton density* if there exists a smooth (signed) measure μ on the space of $n-k$-dimensional projective subspaces such that

$$\int_N \phi = \int \#\{N \cap L\} d\mu(L)$$

for all submanifolds N of dimension k.

Theorem 5. *([22], [10])*
Let ϕ be a Crofton k-density on $\mathbb{RP}^n$. Then ϕ is projective.

This theorem was stated by Gelfand-Smirnov [22]. A short proof, using a PDE characterization of projective densities, was recently given by Álvarez Paiva-Fernandes [10]. We will give a short, geometric proof of the same result.

Plan of the paper. The paper is organized as follows. In Section 1 we introduce the necessary notation from Geometric Measure Theory. Then we present a short introduction to the theory of support functions and conormal cycles. The construction of the Rumin-de Rham complex of a contact manifold, in particular Rumin's operator D, is recalled in Section 2. The definition of a valuation of pure degree k is introduced and Alesker's definition of the Euler-Verdier involution is recalled. Then Klain's injectivity result is used to prove Theorem 2. Section 3 is about Gelfand transforms of forms and currents and contains some technical lemmas. The heart of the paper is Section 4, where Theorem 1 is proved using the Gelfand transform of the conormal cycle under a particular double fibration. The product structure on valuations is studied in Section 5. In particular, Theorem 3 is proved. The proof of Theorem 5 is contained in Section 6. In Section 7 we recall the definition of the Holmes-Thompson volume of a Finsler manifold and prove Theorem 4.

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1. The conormal cycle

We give here a short introduction to the theory of conormal cycles. We will define conormal cycles of compact convex and of compact subanalytic sets without fixing an Euclidean metric on V. We do not give proofs and refer to [21], [13], [14], [15] for details.

1.1. **Notation from Geometric Measure Theory.** We adopt the following convention: a projection map $\pi : A \to B$ between manifolds will just be denoted by π_B. There will be no risk of confusion, since all projections will be natural ones.

We follow [19] for the notation on Geometric Measure Theory.

The *boundary* ∂T of a (Federer-Fleming-) current T is defined by $\partial T(\omega) = T(d\omega)$. A current T with ∂T is called a *cycle*.

A rectifiable current T such that ∂T is also rectifiable is called an *integral current* and the space of k-dimensional integral currents is denoted by $\mathcal{I}_k(M)$. In particular, $\mathcal{I}_0(M)$ consists of finite linear combinations of Dirac measures with integer coefficients.

Let $f : M \to M'$ be a Lipschitz map between (Riemannian) manifolds. Given a current T such that f is proper on $\operatorname{spt} T$, the *push-forward* of T under f is denoted by $f_* T$.

Let $T \in \mathcal{I}_k(M)$ and $f : M \to M'$, where M' is an oriented n'-dimensional manifold. Then the *slice* $\langle T, f, y\rangle$ is defined and belongs to $\mathcal{I}_{k-n'}(M)$ for almost all $y \in M'$. Given a smooth map $g : N \to M$ and a current $T \in \mathcal{I}_k(N)$, the following equation holds for almost all $y \in M'$:

$$\langle g_* T, f, y\rangle = g_* \langle T, f \circ g, y\rangle. \tag{2}$$

1.2. **Cosphere-bundle.** Let V be an oriented n-dimensional vector space and $S^*(V) := (V^* \setminus \{0\})/\mathbb{R}_+$ its cosphere. An element $[\xi] \in S^*(V)$ can be identified with the oriented hyperplane $\ker \xi$ in V. We set $S^*V := V \times S^*(V)$, the *cosphere bundle* over V. With $\pi_V : S^*V \to V$ and $\pi_{S^*(V)} : S^*V \to S^*(V)$ denoting the natural projections, the contact structure of S^*V is defined by $Q_{(x,[\xi])} := d(\pi_V)^{-1}_{(x,[\xi])}(\ker \xi)$. A form whose restriction to Q vanishes is called *vertical*.

1.3. **Support functions.** Let $\mathcal{B}$ be the oriented line bundle over $S^*(V)$ such that the fiber over a point $[\xi] \in S^*(V)$ is given by the line $\mathcal{B}_{[\xi]} := V/\ker \xi$, with induced orientation. There is a natural map $u : S^*V \to \mathcal{B}, (x,[\xi]) \mapsto x/\ker \xi \in \mathcal{B}_{[\xi]}$.

Let T be an integral, Legendrian cycle on S^*V, i.e. an integral cycle of dimension $n-1$ which vanishes on vertical forms. The map

$$\begin{aligned} h_T : S^*(V) &\to \mathcal{I}_0(\mathcal{B}) \\ [\xi] &\mapsto u_* \langle T, \pi_2, [\xi]\rangle \end{aligned}$$

is called the support function of T. Note that, since we take slices, h_T is only defined for almost every $[\xi] \in S^*(V)$. The value $h_T([\xi])$ is a 0-dimensional integral current in $\mathcal{B}$.

One version of Fu's uniqueness theorem [21] states that T is uniquely determined by h_T.

Next, we describe the support function of a compact convex set $K \subset V$ and of a compact subanalytic set $X \subset V$. Denote the canonical projection by $\pi_{[\xi]} : V \to \mathcal{B}_{[\xi]}$.

We set

$$h_K([\xi]) := \sum_{s \in \mathcal{B}_{[\xi]}} \left(\lim_{\epsilon \to 0} \chi(\pi_{[\xi]}^{-1}(s) \cap K) - \chi(\pi_{[\xi]}^{-1}(s+\epsilon) \cap K) \right) \delta_{([\xi],s)} \in \mathcal{I}_0(\mathcal{B})$$

and similarly

$$h_X([\xi]) := \sum_{s \in \mathcal{B}_{[\xi]}} \left(\lim_{\epsilon \to 0} \chi(\pi_{[\xi]}^{-1}(s) \cap X) - \chi(\pi_{[\xi]}^{-1}(s+\epsilon) \cap X) \right) \delta_{([\xi],s)} \in \mathcal{I}_0(\mathcal{B}).$$

Note that in the convex case, there is exactly one value of s where the corresponding coefficient is non-zero. In the subanalytic set, there are finitely many such values of s. In both cases, $h([\xi]) \in \mathcal{I}_0(\mathcal{B})$.

It is well-known that h_K uniquely determines K. By a result of Bröcker [17], h_X uniquely determines X.

1.4. **Conormal cycles.** An integral Legendrian cycle T is called the conormal cycle of K (resp. X) if $h_T([\xi]) = h_K([\xi])$ (resp. $h_T([\xi]) = h_X([\xi])$) for almost all $[\xi] \in S^*(V)$. The uniqueness of the conormal cycle follows from the above mentioned theorem by Fu. The existence is easy to prove in the convex case. In the subanalytic case, the conormal cycle was constructed by Fu [21]. A more elementary approach is contained in [15].

It is easily checked that for compact convex sets K_1, K_2 such that $K_1 \cup K_2$ is also convex, we have

$$\operatorname{cnc}(K_1 \cup K_2) + \operatorname{cnc}(K_1 \cap K_2) = \operatorname{cnc}(K_1) + \operatorname{cnc}(K_2). \tag{3}$$

In the same way, for compact subanalytic sets X, Y we have

$$\operatorname{cnc}(X \cup Y) + \operatorname{cnc}(X \cap Y) = \operatorname{cnc}(X) + \operatorname{cnc}(Y). \tag{4}$$

1.5. **Projections.** Let $L_0 \subset V$ be an $n-k$-dimensional oriented subspace. Let $V_0 := V/L_0$ with the induced orientation. Let $\mathcal{B}_0$ be the oriented line bundle over $S^*(V_0)$ such that the fiber over a point $\bar{E} \in S^*(V_0)$ is given by the oriented line $V_0/\bar{E}$. We set $u_0 : S^*V_0 \to \mathcal{B}_0, (\bar{x}, \bar{E}) \mapsto \bar{x}/\bar{E} \in V_0/\bar{E}$.

The projection $\pi_{V_0} : V \to V_0$ induces a natural inclusion $\tau : S^*(V_0) \to S^*(V)$ and a map $\tau_B : \mathcal{B}_0 \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{B}_0 & \xrightarrow{\tau_B} & \mathcal{B} \\ \downarrow & & \downarrow \\ S^*(V_0) & \xrightarrow{\tau} & S^*(V) \end{array}$$

commutes.

If X is a compact subanalytic set, then the projection of X to V_0 is again compact subanalytic. However, it is better to work with the push-forward $\pi_{V_0}(X)$. This is no longer a set, but a constructible function on V. At a point $\bar{x} \in V_0$, its value is by definition the Euler characteristic of the fiber $\pi_{V_0}^{-1}(\bar{x}) \cap X$. The theory of support functions and conormal cycles can be extended to compactly supported constructible functions [15]. In particular, the following equation holds (and can be used as an ad hoc definition of $h_{\pi_{V_0}(X)}$):

$$(\tau_B)_* h_{\pi_{V_0}(X)} = h_X \circ \tau. \tag{5}$$

2. Smooth valuations and the Rumin-de Rham complex

Let ω be a smooth differential form of degree $n-1$ on S^*V. By (4), the map $X \mapsto \operatorname{cnc}(X)(\omega)$ defines a subanalytic valuation Ψ_ω. Such valuations are called *smooth*.

The kernel of the map $\omega \mapsto \Psi_\omega$ is nontrivial and can be described in terms of the Rumin operator D. Let us first recall the Rumin-de Rham complex [25].

Let (N,Q) be a contact manifold of dimension $2n-1$. For simplicity, we suppose that there exists a global contact form α, i.e. $Q=\ker\alpha$. This global contact form is not unique, since multiplication by any non-vanishing smooth function on N yields again a contact form. However, the following spaces only depend on (N,Q) and not on the particular choice of α.

$$\mathcal{V}^k(N) = \{\omega \in \Omega^k(N) : \omega = \alpha \wedge \xi, \xi \in \Omega^{k-1}(N)\};$$
$$\mathcal{I}^k(N) = \{\omega \in \Omega^k(N) : \omega = \alpha \wedge \xi + d\alpha \wedge \psi, \xi \in \Omega^{k-1}(N), \psi \in \Omega^{k-2}(N)\};$$
$$\mathcal{J}^k(N) = \{\omega \in \Omega^k(N) : \alpha \wedge \omega = d\alpha \wedge \omega = 0\}.$$

Forms in $\mathcal{V}^k(N)$ are called *vertical* and characterized by the fact that they vanish on the contact distribution.

Since $d\mathcal{I}^k \subset \mathcal{I}^{k+1}$, there exists an induced operator $d_Q : \Omega^k/\mathcal{I}^k \to \Omega^{k+1}/\mathcal{I}^{k+1}$.

Similarly, $d\mathcal{J}^k \subset \mathcal{J}^{k+1}$ and the restriction of d to $\mathcal{J}^k$ yields an operator $d_Q : \mathcal{J}^k \to \mathcal{J}^{k+1}$.

In the middle dimension, there is a further operator, which we call *Rumin operator*, defined as follows. Let $\omega \in \Omega^{n-1}(N)$. Then $\mathcal{J}^n$ contains a unique element of the form $d(\omega + \alpha \wedge \nu), \nu \in \Omega^{n-2}(N)$ and $D\omega$ is defined to be this element. The operator D is a second order differential operator. It can be checked that $D|_{\mathcal{I}^{n-1}} = 0$, hence there is an induced operator $D : \Omega^{n-1}/\mathcal{I}^{n-1} \to \mathcal{J}_n$.

The Rumin-de Rham complex of the contact manifold (N,Q) is given by

$$0 \to C^\infty(N) \xrightarrow{d_Q} \Omega^1/\mathcal{I}^1 \xrightarrow{d_Q} \dots \xrightarrow{d_Q} \Omega^{n-2}/\mathcal{I}^{n-2} \xrightarrow{d_Q} \Omega^{n-1}/\mathcal{I}^{n-1} \xrightarrow{D} \mathcal{J}_n \xrightarrow{d_Q}$$
$$\xrightarrow{d_Q} \mathcal{J}_{n+1} \xrightarrow{d_Q} \dots \xrightarrow{d_Q} \mathcal{J}_{2n-1} \to 0.$$

The cohomology of this complex is called *Rumin cohomology* and is denoted by $H^*_Q(N,\mathbb{R})$. By [25], there exists a natural isomorphism between the Rumin cohomology and the de Rham cohomology:

$$H^*_Q(N,\mathbb{R}) \xrightarrow{\cong} H^*_{dR}(N,\mathbb{R}). \tag{6}$$

The next theorem (which is a weak version of Theorem 1 in [16]) provides a link between the Rumin cohomology of the contact manifold S^*V and smooth valuations.

Theorem 2.1. *([16])*
Let ω be a smooth $n-1$-form on S^*V. Then $\Psi_\omega = 0$ if and only if

(1) *$D\omega = 0$ and*
(2) *$\int_{S^*_x V} \omega = 0$ for all $x \in V$.*

If $D\omega = 0$, then $r := \int_{S^*_x M} \omega \in \mathbb{R}$ is independent of $x \in V$ and

$$\Psi_\omega = r\chi$$

where χ denotes the Euler characteristic.

Note that the condition $D\omega = 0$ means that, up to a vertical form, ω is closed.

Definition 2.2. *A smooth valuation $\Psi = \Psi_\omega$ is said to have pure degree $k \geq 1$ if Ψ vanishes on points and if the bidegree of $D\omega$ (w.r.t. to the product structure $S^*V = V \times S^*(V)$) is $(k, n-k)$. Ψ has pure degree 0 if it is a multiple of the Euler characteristic (i.e. $D\omega = 0$).*

Alesker introduced an involution on the space of smooth valuations, called the Euler-Verdier involution. Let $s : S^*V \to S^*V$ be the natural involution, i.e. the map that sends $(x, E) \in S^*V$ to $(x, \bar{E}) \in S^*V$, where the bar means change of orientation.

Definition 2.3. *Let $\Psi = \Psi_\omega$ be a smooth valuation. Then the Euler-Verdier involution is defined as $(-1)^n \Psi_{s^*\omega}$.*

Of course, one has to check that this operation is well-defined, i.e. independent of the choice of ω. This is easily done using Theorem 2.1.

Proof of Theorem 2. The statement is trivial if $k = 0$. Suppose that Ψ is a smooth valuation of pure degree $k > 0$ which belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution and which vanishes on k-dimensional submanifolds with boundary.

We can write $\Psi = \Psi_\omega$ with $D\omega$ of bidegree $(k, n-k)$. By Theorem 2.1 it is enough to show that $D\omega = 0$. The argument we give here follows the proof of the more general statement Prop. 3.1.5 in [2].

Since Ψ is smooth, we can define for each $x_0 \in V$ the valuation Ψ_{x_0} by

$$\Psi_{x_0}(X) := \frac{1}{k!} \left. \frac{d^k}{dt^k} \right|_{t=0} \Psi(tX + x_0).$$

With $\phi_{x_0,t} : S^*V \to S^*V, (x, E) \mapsto (tx + x_0, E)$, Ψ_{x_0} is represented as $\Psi_{x_0} = \Psi_{\omega_{x_0}}$ with

$$\omega_{x_0} := \frac{1}{k!} \left. \frac{d^k}{dt^k} \right|_{t=0} \phi^*_{x_0,t}\omega.$$

The exterior derivative d commutes with $\frac{d}{dt}$ and $\phi^*_{x_0,t}$. Hence we get

$$d\omega_{x_0} = \frac{1}{k!} \left. \frac{d^k}{dt^k} \right|_{t=0} \phi^*_{x_0,t} D\omega.$$

Let us use linear coordinates $(x_1,\ldots,x_n)$ on V and (local) coordinates $(y_1,\ldots,y_{n-1})$ on $S^*(V)$. Since Ψ is of pure degree k, we can locally write $D\omega = \sum_{I,J;\#I=k,\#J=n-k} a_{IJ} dx_I \wedge dy_J$ with smooth functions a_{IJ} on S^*V (here I and J range over multi-indices with $\#I=k$ and $\#J=n-k$). It follows that $d\omega_{x_0} = \sum_{I,J} a_{IJ}(x_0,\cdot) dx_I \wedge dy_J$. In particular, $d\omega_{x_0}$ is translation invariant, of bidegree $(k,n-k)$ and

$$d\omega_{x_0}(v_1,\ldots,v_n) = D\omega(v_1,\ldots,v_n), \quad \forall E \in S^*(V), v_1,\ldots,v_n \in T_{(x_0,E)}S^*V. \tag{7}$$

Since $D\omega$ is vertical, the translation invariance of $d\omega_{x_0}$ and (7) imply that $d\omega_{x_0}$ is vertical, i.e. $d\omega_{x_0} = D\omega_{x_0}$.

We conclude that Ψ_{x_0} is translation invariant, smooth, of degree k and even. By the assumption on Ψ, the Klain function of Ψ_{x_0} vanishes. The injectivity of the Klain imbedding implies that $\Psi_{x_0}=0$, i.e. $D\omega_{x_0}=0$. Since this holds true for all $x_0 \in V$, (7) implies that $D\omega = 0$. By assumption, $\Psi = \Psi_\omega$ vanishes on points, hence $\int_{S_x^*V} \omega = 0$ for all $x \in V$. Theorem 2.1 gives $\Psi = 0$. $\square$

3. Double fibrations and Gelfand transform

3.1. **Double fibrations.**

Definition 3.1. *A double fibration is a diagram of manifolds*

$$A \xleftarrow{\pi_A} M \xrightarrow{\pi_B} B$$

where

(1) *$\pi_A : M \to A$ and $\pi_B : M \to B$ are smooth fiber bundles;*
(2) *$\pi_A \times \pi_B : M \to A \times B$ is a smooth embedding and*
(3) *the sets $A_b := \pi_A(\pi_B^{-1}(b)), b \in B$ and $B_a := \pi_B(\pi_A^{-1}(a)), a \in A$ are smooth submanifolds.*

A morphism between double fibrations is a commutative diagram of fibrations

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & M & \xrightarrow{\pi_B} & B \\ \rho_A \downarrow & & \rho_M \downarrow & & \rho_B \downarrow \\ A' & \xleftarrow{\pi_{A'}} & M' & \xrightarrow{\pi_{B'}} & B' \end{array}$$

3.2. **Gelfand transform of a differential form.**

Now suppose that A and M are oriented and that the fiber of π_A is compact. The fiber integration $(\pi_A)_* : \Omega^*(M) \to \Omega^*(A)$ decreases the degree of a form μ by the dimension of the fiber, i.e. by $l := \dim M - \dim A$. It is defined by

$$\int_A \alpha \wedge (\pi_A)_*\mu = \int_M \pi_A^*\alpha \wedge \mu$$

for all compactly supported differential forms α on A. Other sign conventions can be found in the literature (e.g. [12]); the above one corresponds to the one in [10].

It is easily checked that

$$d(\pi_A)_*\mu = (\pi_A)_* d\mu \tag{8}$$

and that for a form α on A the following projection formula holds:

$$(\pi_A)_*(\mu \wedge \pi_A^*\alpha) = (-1)^{l \deg \alpha}(\pi_A)_*\mu \wedge \alpha. \tag{9}$$

The *Gelfand transform* of a differential form β on B is the form $\mathrm{GT}(\beta) := (\pi_A)_*\pi_B^*\beta$.

We will need the following functorial property of the Gelfand transform ([9], Thm. 2.2). Let

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & M & \xrightarrow{\pi_B} & B \\ \rho_A \downarrow & & \rho_M \downarrow & & \rho_B \downarrow \\ A' & \xleftarrow{\pi_{A'}} & M' & \xrightarrow{\pi_{B'}} & B' \end{array}$$

be a morphism of double fibrations such that

$$\xi = \pi_B|_{\rho_M^{-1}(m')} : \rho_M^{-1}(m') \to \rho_B^{-1}(\pi_{B'}(m'))$$

is a diffeomorphism for all $m' \in M'$. Then

$$(\rho_A)_* \mathrm{GT}(\beta) = \deg(\xi)\, \mathrm{GT}((\rho_B)_*\beta) \tag{10}$$

where $\deg(\xi)$ equals $+1$ if ξ is orientation preserving and -1 else.

3.3. **Gelfand transform of a current.** Given a current T in A, the current π_A^*T on M defined by

$$\pi_A^*T(\omega) := T((\pi_A)_*\omega)$$

is called the *lift* of T and was studied by Brothers [18] and Fu [20]. In the case of a product bundle $M = A \times F$, the lift of T is simply $T \times [[F]]$. Moreover, lifting currents is natural with respect to bundle operations, increases the dimension by the dimension of the fiber and commutes with the boundary operator ∂.

Definition 3.2. *The Gelfand transform of T is the current* $\mathrm{GT}(T) := (\pi_B)_*\pi_A^*T$ *in B.*

We will need the following two easy lemmas.

Lemma 3.3. *Let $\pi_A : M \to A$ and $\pi_B : N \to B$ be oriented fiber bundles with diffeomorphic compact fibers F. Let $\tilde{f} : M \to N$ and $f : A \to B$ be smooth maps such that the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & N \\ \downarrow \pi_A & & \downarrow \pi_B \\ A & \xrightarrow{f} & B \end{array}$$

Suppose that the induced map $\tilde{f}_a : \pi_A^{-1}(a) \to \pi_B^{-1}(f(a))$ is an orientation preserving diffeomorphism for each $a \in A$.

Then for an integral k-current T on A and almost all $y \in N$ we have

$$(\pi_A)_* \left\langle \pi_A^* T, \tilde{f}, y \right\rangle = (-1)^{l(k-\dim B)} \langle T, f, \pi_B(y) \rangle.$$

Proof. Let ν be a smooth non-vanishing form of top-degree on N which induces the orientation of N. From the assumption it follows easily that $f^*(\pi_B)_* \nu = (\pi_A)_* \tilde{f}^* \nu$.

Let α be a compactly supported smooth form on A of degree $k - \dim B$. Using the projection formula (9) and the Slicing Theorem ([19], Thm. 4.3.2. (1)), we get

$$\begin{aligned}
\int_N \left(\left\langle \pi_A^* T, \tilde{f}, \cdot \right\rangle (\pi_A^* \alpha) \right) \wedge \nu &= \pi_A^* T(\tilde{f}^* \nu \wedge \pi_A^* \alpha) \\
&= T((\pi_A)_* (\tilde{f}^* \nu \wedge \pi_A^* \alpha)) \\
&= (-1)^{l \deg \alpha} T((\pi_A)_* \tilde{f}^* \nu \wedge \alpha) \\
&= (-1)^{l \deg \alpha} T(f^* (\pi_B)_* \nu \wedge \alpha) \\
&= (-1)^{l \deg \alpha} \int_B (\langle T, f, \cdot \rangle \alpha) \wedge (\pi_B)_* \nu \\
&= (-1)^{l \deg \alpha} \int_N (\langle T, f, \pi_B(\cdot) \rangle \, \alpha) \wedge \nu,
\end{aligned}$$

and the equation follows. □

Lemma 3.4. *Let $\pi_B : N \to B$ be an oriented fiber bundle with fiber F and let $\tilde{f} : M \to N$ be a smooth map. Let $\psi : N \to B \times F$ be an orientation preserving trivialization of N and $\pi_1 : B \times F \to B$, $\pi_2 : B \times F \to F$ the canonical projection maps. Let T be an integral current on M. Then for almost all $y \in N$ we have*

$$\left\langle \langle T, \pi_B \circ \tilde{f}, \pi_B(y) \rangle, \pi_2 \circ \psi \circ \tilde{f}, \pi_2 \circ \psi(y) \right\rangle = \langle T, \tilde{f}, y \rangle.$$

Proof. By ([19], Thm. 4.3.2 (6)), $\langle T, \tilde{f}, y \rangle = \langle T, \psi \circ \tilde{f}, \psi(y) \rangle$ for almost all $y \in N$. The assertion of the lemma follows from $\pi_B = \pi_1 \circ \psi$ and ([19], Thm. 4.3.5). □

4. Current representation of a valuation with Crofton formula

Let V be an oriented, n-dimensional vector space. The Grassmannian $\mathrm{Gr}_{n-k}^+(V)$ of oriented $n-k$-planes in V has an induced orientation which can be described as follows.

The tangent space $T_L \mathrm{Gr}_{n-k}^+(V)$ can be identified with the space of linear maps from L to some complementary subspace L'. We orient L' in the natural way. Let $e_1, \ldots, e_{n-k}$ be a positive base of L and let $e_{n-k+1}, \ldots, e_n$ be a positive base of L'. Let $A_{i,j}, i = 1, \ldots, n-k; j = n-k+1, \ldots, n$ be the linear map which sends e_i to e_j. Then we define

$$A_{1,n-k+1}, A_{1,n-k+2}, \ldots, A_{1,n}, \ldots, A_{n-k,n}$$

to be a positive base of $T_L \operatorname{Gr}_{n-k}^+(V)$.

The oriented affine Grassmannian $\operatorname{AGr}_{n-k}^+(V)$ is a fiber bundle over $\operatorname{Gr}_{n-k}^+(V)$ with canonically oriented fibers V/L and has therefore an induced orientation.

Let $1 \leq k \leq n$ and set

$$M := \left((x,E,L) \in V \times S^*(V) \times \operatorname{Gr}_{n-k}^+(V) : L \subset E\right).$$

Then the natural projection $M \to S^*V$ is a fiber bundle whose fiber above (x,E) is the oriented Grassmannian $\operatorname{Gr}_{n-k}^+(E)$. The natural orientations of S^*V and $\operatorname{Gr}_{n-k}^+(E)$ induce an orientation on M. There is also a natural projection map from M to $\operatorname{AGr}_{n-k}^+(V)$ defined by $(x,E,L) \mapsto x+L$.

Proposition 4.1. *Let $X \subset V$ be compact and subanalytic and $1 \leq k \leq n$. Define an integral current $H_{n-k}(X)$ on $\operatorname{AGr}_{n-k}^+(V)$ by integration over $\operatorname{AGr}_{n-k}^+(V)$ with multiplicity function $H \mapsto \chi(H \cap X)$. Then the Gelfand transform of* $\operatorname{cnc}(X)$ *for the double fibration*

$$S^*V \longleftarrow M \longrightarrow \operatorname{AGr}_{n-k}^+(V)$$

is $(-1)^{k(n-k)}\partial H_{n-k}(X)$.

Proof. Let $T = \operatorname{GT}(\operatorname{cnc}(X))$ be the Gelfand transform of $\operatorname{cnc}(X)$ for the above double fibration. Let $p : \operatorname{AGr}_{n-k}^+(V) \to \operatorname{Gr}_{n-k}^+(V)$ be the natural projection map.

It suffices to show that

$$\langle T,p,L\rangle = (-1)^{k(n-k)}\langle \partial H_{n-k}(X),p,L\rangle \tag{11}$$

for almost all $L \in \operatorname{Gr}_{n-k}^+(V)$ and that $\partial H_{n-k}(X)$ and T have no vertical components with respect to p.

Claim 1: The restriction of dp to an approximate tangent plane of T is surjective.

Let $\tilde{T} := \pi_{S^*V}^* \operatorname{cnc}(X)$. It is an integral cycle of dimension $(n-1)+(n-k)(k-1) = \dim \operatorname{AGr}_{n-k}^+(V) - 1$.

Let $w \in T_{(x,E,L)}M$ be such that $d\pi_{S^*V}(w)$ is horizontal and such that $dp \circ d\pi_{\operatorname{AGr}_{n-k}^+(V)}(w) = 0$. Then $d\pi_{\operatorname{AGr}_{n-k}^+(V)}(w)$ is tangential to the $k-1$-dimensional manifold of affine $n-k$-planes in E parallel to L.

Let $W \subset T_{(x,E,L)}M$ be an approximate tangent plane of $\tilde{T}$ such that $d\pi_{\operatorname{AGr}_{n-k}^+(V)}(W)$ is not degenerated. Since the kernel of $dp|_{x+L}$ has dimension k, it follows that $d\pi_{\operatorname{AGr}_{n-k}^+(V)}(W)$ and $\ker dp|_{x+L}$ intersect transversally, which implies that $dp\left(d\pi_{\operatorname{AGr}_{n-k}^+(V)}(W)\right) = T_L \operatorname{Gr}_{n-k}^+(V)$.

Claim 2: The restriction of dp to an approximate tangent plane of $\partial H_{n-k}(X)$ is surjectiv.

By definition of $H_{n-k}^+(V)$, an $n-k$-plane $H \in \operatorname{AGr}_{n-k}^+(V)$ can be in the support of $\partial H_{n-k}^+(V)$ only if the Euler characteristic $\chi(H' \cap X)$ is not constant for H' near H.

For $L \in \mathrm{Gr}_{n-k}^+(V)$, a generic H in $p^{-1}(L)$ intersects X transversally. By Thom's isotopy lemma, the Euler characteristic $\chi(H' \cap X)$ is constant for all H' near H in $\mathrm{AGr}_{n-k}^+(V)$. Hence $H \notin \operatorname{spt} \partial H_{n-k}^+(V)$. It follows that the codimension of $p^{-1}(L) \cap \operatorname{spt} \partial H_{n-k}^+(V) \subset p^{-1}(L)$ is positive. Since ∂H_{n-k} is a current of codimension 1, the claim follows easily (for instance using a subanalytic stratification of p compatible with $\operatorname{spt} H_{n-k}$).

Let N be the flag manifold of all pairs (E, L) where $E \subset V$ is an oriented hyperplane and $L \subset E$ is an oriented $n-k$-plane. Then N is a fiber bundle over $S^*(V)$ with fiber $\mathrm{Gr}_{n-k}^+(\mathbb{R}^{n-1})$ and the induced orientation. We can also consider N as a fiber bundle over $\mathrm{Gr}_{n-k}^+(V)$ with fiber $\mathrm{Gr}_{k-1}^+(\mathbb{R}^k)$. The induced orientation of N is the same as the one introduced above, because $\dim \mathrm{Gr}_{n-k}^+(\mathbb{R}^n) \dim \mathrm{Gr}_{k-1}^+(\mathbb{R}^k) = k(n-k)(k-1)$ is even.

The projection $\pi_{S^*(V)} : S^*V \to S^*(V)$ lifts to a projection $\pi_N : M \to N, (x, E, L) \mapsto (E, L)$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi_N} & N \\ \downarrow \pi_{S^*V} & & \downarrow \pi_{S^*(V)} \\ S^*V & \xrightarrow{\pi_{S^*(V)}} & S^*(V) \end{array}$$

commutes.

For fixed L_0, we let $V_0 := V/L_0$ and $M_0 := \{(x, E, L) \in M : L = L_0\}$. There is a map $\pi_{S^*V_0} : M_0 \to S^*V_0, (x, E, L_0) \to (x/L_0, E/L_0)$.

Claim 3: For almost all $L_0 \in \mathrm{Gr}_{n-k}^+(V)$ we have

$$(\pi_{S^*V_0})_* \langle \tilde{T}, \pi_{\mathrm{Gr}_{n-k}^+(V)}, L_0 \rangle = \operatorname{cnc}(\pi_{V_0}(X)). \tag{12}$$

Let T_0 be the current on the left hand side. Since $\langle \tilde{T}, \pi_{\mathrm{Gr}_{n-k}^+(V)}, L_0 \rangle$ is an integral current with support in M_0, T_0 is a well-defined integral cycle thanks to Federer's flatness theorem.

Let $W \subset T_{(x,E,L_0)} \mathrm{Gr}_{n-k}^+(V)$ be a linear subspace such that $d\pi_V(W) \subset E$. Then

$$d\pi_{V_0} \circ d\pi_{S^*V_0}(W) = d\pi_{V_0} \circ d\pi_V(W) \subset d\pi_{V_0}(E) = E/L_0,$$

which means that $d\pi_{S^*V_0}(W)$ is a horizontal plane in $T_{(x/L_0, E/L_0)} S^*V_0$. Since $\operatorname{cnc}(X)$ is Legendrian, we can apply this to the generalized tangent planes of the support of $\langle \tilde{T}, \pi_{\mathrm{Gr}_{n-k}^+(V)}, L_0 \rangle$ and obtain that T_0 is a Legendrian cycle.

Let us compute the support function of T_0. Let $\pi_{S^*(V_0)} : S^*V_0 \to S^*(V_0)$ be the projection on the second factor. For almost all $\bar{E}_0 = E_0/L_0 \in S^*(V_0)$ we have

$$\begin{aligned} \langle T_0, \pi_{S^*(V_0)}, \bar{E}_0 \rangle &= (\pi_{S^*V_0})_* \left\langle \langle \tilde{T}, \pi_{\mathrm{Gr}_{n-k}^+(V)}, L_0 \rangle, \pi_{S^*(V_0)}, \bar{E}_0 \right\rangle \text{ by (2)} \\ &= (\pi_{S^*V_0})_* \langle \tilde{T}, \pi_N, (E_0, L_0) \rangle \quad \text{by Lemma 3.4.} \end{aligned}$$

Let $\mathcal{B}_0$, $u_0 : S^*V_0 \to \mathcal{B}_0$ and $\tau_B : \mathcal{B}_0 \to \mathcal{B}$ be as in Section 1. We then have (on M_0)

$$\tau_B \circ u_0 \circ \pi_{S^*V_0} = u \circ \pi_{S^*V}.$$

It follows that

$$\begin{aligned}(\tau_B)_* h_{T_0}(\bar{E}_0) &= (\tau_B \circ u_0 \circ \pi_{S^*V_0})_* \left\langle \tilde{T}, \pi_N, (E_0, L_0) \right\rangle \\ &= (u \circ \pi_{S^*V})_* \left\langle (\pi_{S^*V})^* \operatorname{cnc}(X), \pi_N, (E_0, L_0) \right\rangle \\ &= u_* \langle \operatorname{cnc}(X), \pi_{S^*(V)}, E_0 \rangle \quad \text{by Lemma 3.3} \\ &= h_X(E_0),\end{aligned}$$

which implies, by Equation (5), that T_0 is the conormal cycle of $\pi_{L_0}(X)$.

Claim 4: The slices of T and $(-1)^{k(n-k)} \partial H_{n-k}(X)$ agree for almost all $L_0 \in \operatorname{Gr}^+_{n-k}(V)$.

Let $\tau : V_0 \to \operatorname{AGr}^+_{n-k}(V), x/L_0 \mapsto x + L_0$. Then $\tau \circ \pi_{V_0} \circ \pi_{S^*V_0} = \pi_{\operatorname{AGr}^+_{n-k}(V)}$ on M_0.

We now compute that

$$\begin{aligned}\langle T, p, L_0 \rangle &= \left\langle (\pi_{\operatorname{AGr}^+_{n-k}(V)})_* \tilde{T}, p, L_0 \right\rangle \\ &= (\pi_{\operatorname{AGr}^+_{n-k}(V)})_* \langle \tilde{T}, \pi_{\operatorname{Gr}^+_{n-k}(V)}, L_0 \rangle \\ &= \tau_* \circ (\pi_{V_0})_* \circ (\pi_{S^*V_0})_* \langle \tilde{T}, \pi_{\operatorname{Gr}^+_{n-k}(V)}, L_0 \rangle \\ &= \tau_* \circ (\pi_{V_0})_* \operatorname{cnc}(\pi_{V_0}(X)) \\ &= \tau_* \partial [[\pi_{V_0}(X)]] \\ &= \partial \tau_* [[\pi_{V_0}(X)]].\end{aligned}$$

By definition, $[[\pi_{V_0}(X)]]$ is the integral current which is given by integration over V_0 with multiplicity function $x/L_0 \mapsto \chi((x + L_0) \cap X)$. Hence $\tau_*[[\pi_{V_0}(X)]]$ is given by integration over $p^{-1}(L)$ with the multiplicity function $x + L_0 \mapsto \chi((x + L_0) \cap X)$. It follows that

$$\tau_*[[\pi_{V_0}(X)]] = \langle H_{n-k}(X), p, L \rangle.$$

Since $\partial \langle H_{n-k}(X), p, L \rangle = (-1)^{k(n-k)} \langle \partial H_{n-k}(X), p, L \rangle$, the claim follows. $\square$

The next theorem clearly implies Theorem 1.

Theorem 4.2. *Let ϕ be a smooth form of top degree on $\operatorname{AGr}^+_{n-k}(V), 1 \leq k \leq n$. Define a valuation Ψ on V by*

$$\Psi(X) := \int_{H \in \operatorname{AGr}^+_{n-k}(V)} \chi(X \cap H) \phi(H).$$

Then Ψ is represented by an $n-1$-form ω on S^*V with $D\omega = (-1)^{k(n-k)} \operatorname{GT}(\phi)$ and $\int_{S^*_xV} \omega = 0$ for all $x \in V$. In particular, Ψ is smooth. Moreover, Ψ is of pure degree k and belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution.

Proof. Since $\mathrm{AGr}_{n-k}^+(V)$ is a non-compact manifold of dimension $k(n-k+1)$, $H_{dR}^{k(n-k+1)}(\mathrm{AGr}_{n-k}^+(V)) = 0$. Let τ be a $k(n-k+1)-1$-form with $d\tau = \phi$. We claim that $\Psi = \Psi_\omega$ with $\omega := (-1)^{k(n-k)} \mathrm{GT}(\tau)$.

Let us first check that $D\omega = d\omega$. For $(x,E,L) \in M$, let $W \subset T_{(x,E,L)}M$ be a linear subspace such that $d\pi_{S^*V}(W)$ is horizontal. Let v be a vector not contained in E. Then the derivative at 0 of the smooth curve $t \mapsto x + tv + L \in \mathrm{AGr}_{n-k}^+(V)$ is not contained in $d\pi_{\mathrm{AGr}_{n-k}^+(V)}(W)$. Therefore $d\pi_{\mathrm{AGr}_{n-k}^+(V)}(W)$ is a proper subspace of $T_{x+L}\,\mathrm{AGr}_{n-k}^+(V)$. It follows that $d\omega = (-1)^{k(n-k)} \mathrm{GT}(\phi)$ vanishes on horizontal n-planes. From the definition of D it follows that $D\omega = d\omega$.

Now we compute

$$\begin{aligned}
\Psi(X) &= \int_{H \in \mathrm{AGr}_{n-k}^+(V)} \chi(X \cap H)\phi \\
&= H_{n-k}(X)(\phi) \\
&= \partial H_{n-k}(\tau) \\
&= (-1)^{k(n-k)} \mathrm{GT}(\mathrm{cnc}(X))(\tau) \quad \text{by Proposition 4.1} \\
&= \mathrm{cnc}(X)(\omega) \\
&= \Psi_\omega(X).
\end{aligned}$$

Since the space of affine $n-k$-planes through $x \in V$ has measure zero, we get

$$\int_{S_x^*V} \omega = \Psi_\omega(\{x\}) = \Psi(\{x\}) = 0.$$

Let us check that $D\omega$ is of bidegree $(k, n-k)$ with respect to the product decomposition $S^*V = V \times S^*(V)$.

Given $v \in T_xV \times \{0\} \subset T_{(x,E)}S^*V$ and $(x,E,L) \in M$, there exists a unique lift $\tilde{v} \in T_{(x,E,L)}M$ with $d\pi_{\mathrm{Gr}_{n-k}^+(V)}(\tilde{v}) = 0$. Then $d\pi_{\mathrm{AGr}_{n-k}^+(V)}(\tilde{v})$ is tangential to the k-dimensional submanifold in $\mathrm{AGr}_{n-k}^+(V)$ consisting of affine $n-k$-planes parallel to L. Therefore, replacing more than k such vectors into $D\omega = \mathrm{GT}(\phi)$ gives 0.

For each vector $w \in T_{(x,E,L)}M$ with $d\pi_{S^*V}(w) \in \{0\} \times T_ES^*(V)$ we have that $d\pi_{\mathrm{AGr}_{n-k}^+(V)}(w)$ is tangential to the $k(n-k)$-dimensional manifold of affine planes containing x. Replacing more that $k(n-k)$ such vectors into $\pi^*_{\mathrm{AGr}_{n-k}^+(V)}\phi$ thus yields 0. Since the dimension of the fibers of π_{S^*V} is $(k-1)(n-k)$, this means that replacing more than $n-k$ vectors of $\{0\} \times T_ES^*(V)$ into $D\omega$ yields 0. This shows that $D\omega$ is of bidegree $(k, n-k)$.

Let us now show that Ψ belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution. Let $s : S^*V \to S^*V$, $\tilde{s} : M \to M, (x,E,L) \mapsto (x, \bar{E}, \bar{L})$ and $s' : \mathrm{AGr}_{n-k}^+(V) \to \mathrm{AGr}_{n-k}^+(V), x + L \mapsto x + \bar{L}$ denote the canonical involutions.

These maps define a morphism of double fibrations, i.e. a commutative diagram of fiber bundles

$$\begin{array}{ccccc} S^*V & \stackrel{\pi_{S^*V}}{\longleftarrow} & M & \stackrel{\pi_{\mathrm{AGr}^+_{n-k}(V)}}{\longrightarrow} & \mathrm{AGr}^+_{n-k}(V) \\ s\downarrow & & \tilde{s}\downarrow & & s'\downarrow \\ S^*V & \stackrel{\pi_{S^*V}}{\longleftarrow} & M & \stackrel{\pi_{\mathrm{AGr}^+_{n-k}(V)}}{\longrightarrow} & \mathrm{AGr}^+_{n-k}(V) \end{array}$$

The natural involution on the Grassmannian $\mathrm{Gr}^+_{n-k}(V)$ has degree $(-1)^{k+n-1}$. It follows that the degree of s' is $(-1)^{k+n-1+k} = (-1)^{n-1}$. The degree of $\tilde{s}$ is $(-1)^{n+n-k-1} = (-1)^{k-1}$. The restriction of $\pi_{\mathrm{AGr}^+_{n-k}(V)}$ to the fibers of $\tilde{s}$ has thus degree $(-1)^{n+k}$.

Using the functorial properties of the Gelfand transform (Equation (10)) and the fact that s is an involution of degree $(-1)^n$, we get

$$s^* \,\mathrm{GT}(\phi) = (-1)^n s_* \,\mathrm{GT}(\phi) = (-1)^k \,\mathrm{GT}(\phi),$$

which implies that Ψ belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution. $\square$

5. Products of smooth valuations

5.1. **Álvarez-Fernandes product.** Let $\Psi_1 \in \mathcal{V}^\infty(V)$ be represented by a smooth Crofton measure μ_1 on $\mathrm{AGr}^+_{n-k_1}(V)$ and let $\Psi_2 \in \mathcal{V}^\infty(V)$ be represented by a smooth Crofton measure μ_2 on $\mathrm{AGr}^+_{n-k_2}(V)$, where $k_1 + k_2 \leq n$.

Let Δ_{k_1,k_2} be the set of pairs $(E_1, E_2) \in \mathrm{AGr}^+_{n-k_1}(V) \times \mathrm{AGr}^+_{n-k_2}(V)$ such that $\dim E_1 \cap E_2 > n - k_1 - k_2$. Then Δ_{k_1,k_2} is a null set.

Given $(E_1, E_2) \in \mathrm{AGr}^+_{n-k_1}(V) \times \mathrm{AGr}^+_{n-k_2}(V) \setminus \Delta_{k_1,k_2}$, the intersection $E_1 \cap E_2$ has a canonical orientation and belongs to $\mathrm{AGr}^+_{n-k_1-k_2}$. We thus get a map

$$\mathrm{AGr}^+_{n-k_1}(V) \times \mathrm{AGr}^+_{n-k_2}(V) \setminus \Delta_{k_1,k_2} \to \mathrm{AGr}^+_{n-k_1-k_2}\,. \tag{13}$$

Let μ be the push-forward of $\mu_1 \times \mu_2$ under this map. Then μ defines a valuation, which we call the *Álvarez-Fernandes product* of Ψ_1 and Ψ_2 (compare [9]).

At this point it is not clear that this product is well-defined, i.e. does not depend on the choices of μ_1 and μ_2.

5.2. **Alesker product.** In this section, we consider for simplicity only convex valuations, although everything also works in the subanalytic case with Minkowski addition replaced by the convolution product of constructible functions [17], [13], [15].

Alesker introduced a product of smooth valuations on a finite-dimensional vector space V ([1], [2]). Let Ψ_1 be the smooth convex valuation defined by

$$\Psi_1(K) := \nu_1(K + A_1),$$

where ν_1 is a smooth measure on V and A_1 a convex body with strictly convex and smooth boundary. Similarly, let $\Psi_2(K) = \nu_2(K + A_2)$. Let $\nu_1 \times \nu_2$ be the product measure on $V \times V$ and $\Delta : V \to V \times V, x \mapsto (x,x)$ the diagonal imbedding. Then the Alesker product $\Psi_1 \cdot \Psi_2$ is defined to be the valuation

$$\begin{aligned}\Psi_1 \cdot \Psi_2(K) &= (\nu_1 \times \nu_2)(\Delta(K) + A_1 \times A_2)\\ &= \int_V \Psi_1(K \cap (x - A_2)) d\nu_2(x).\end{aligned}$$

The valuation extends by distributivity to linear combinations of valuations of the above form and then by continuity to all smooth valuation.

Lemma 5.1. *Let Ψ_1 be represented by the smooth Crofton measure μ_1 on $\mathrm{AGr}^+_{n-k_1}(V)$ and let Ψ_2 be any smooth valuation on V. Then for all $K \in \mathcal{K}(V)$*

$$\Psi_1 \cdot \Psi_2(K) = \int_{\mathrm{AGr}^+_{n-k_1}(V)} \Psi_2(K \cap E) d\mu_1(E).$$

Proof. Both sides of the equation are additive and continuous in Ψ_2. Therefore it suffices to show the equation in the case where $\Psi_2(K) = \nu(K + A)$, with ν a smooth measure on V and A a strictly convex body with smooth boundary.

We then get, by definition of the product

$$\begin{aligned}\Psi_1 \cdot \Psi_2(K) &= \int_V \Psi_1(K \cap (x - A)) d\nu(x)\\ &= \int_V \int_{\mathrm{AGr}^+_{n-k_1}(V)} \chi(K \cap (x - A) \cap E) d\mu_1(E) d\nu(x)\\ &= \int_{\mathrm{AGr}^+_{n-k_1}(V)} \int_V \chi(K \cap E \cap (x - A)) d\nu(x) d\mu_1(E)\\ &= \int_{\mathrm{AGr}^+_{n-k_1}(V)} \Psi_2(K \cap E) d\mu_1(E).\end{aligned}$$

□

Proof of Theorem 3. By Lemma 5.1 we get

$$\begin{aligned}\Psi_1 \cdot \Psi_2(K) &= \int_{\mathrm{AGr}^+_{n-k_1}(V)} \Psi_2(K \cap E_1) d\mu_1(E_1)\\ &= \int_{\mathrm{AGr}^+_{n-k_1}(V)} \int_{\mathrm{AGr}^+_{n-k_2}(V)} \chi(K \cap E_1 \cap E_2) d\mu_1(E_1) d\mu_2(E_2)\\ &= \int_{\mathrm{AGr}^+_{n-k_1-k_2}(V)} \chi(K \cap E) d\mu(E).\end{aligned}$$

□

6. Crofton densities and projective densities

Recall that a k-density on a vector space V is a smooth homogeneous real-valued function ϕ on the cone $\Lambda_s^k(V)$ of simple k-vectors in V. Here the word *homogeneous* means that $\phi(\lambda w) = |\lambda|\phi(w)$ for all $\lambda \neq 0$ and all $w \in \Lambda_s^k(V)$. A smooth k-density on a manifold M is a smooth map ϕ that assigns to $x \in M$ a k-density ϕ_x on T_xM.

Given a smooth k-density ϕ on a manifold M and a (not necessarily oriented) k-dimensional submanifold $N \subset M$, the integral $\int_N \phi$ is defined in the usual way (take charts and use homogeneity to prove that the integral is independent of the choice of the charts).

A k-density ϕ on $\mathbb{RP}^n$ is called *projective* if k-dimensional projective subspaces are extremal for the variational problem $N \mapsto \int_N \phi$. ϕ is called a *Crofton density* if there exists a smooth (signed) measure μ on the space of $n-k$-dimensional projective subspaces such that

$$\int_N \phi = \int \#\{N \cap L\} d\mu(L)$$

for all submanifolds N of dimension k.

These notions appear in [22], where it is stated that a k-density is a Crofton density if and only if it is projective. However, it turns out that the proof of "projective implies Crofton" is incomplete (cf. [10]). The implication that a Crofton density is projective is true; a short proof is presented in [10]. The inverse implication holds true for $k = 1$ and $k = n-1$.

The current representation of Section 4 can be used for a short proof that Crofton densities are projective.

Theorem 6.1. *(*[22]*,* [10]*)*
Let ϕ be a Crofton k-density on $\mathbb{RP}^n$. Then ϕ is projective.

This theorem follows from the definition of Crofton densities, Theorem 1 and the next theorem.

Theorem 6.2. *Let Ψ be a valuation of pure degree k on V. Then affine k-dimensional subspaces are locally extremal in the following sense. Let X be a bounded, open and subanalytic subset of a k-dimensional affine subspace in V. Then X is extremal for Ψ under variations fixing a neighborhood of ∂X.*

Proof. The theorem is trivial if $k = 0$, so let us suppose $k > 0$. Let $\Psi = \Psi_\omega$ with $D\omega$ of bidegree $(k, n-k)$.

Let W be a smooth vector field on V which is zero on a neighborhood of ∂X. The flow Φ on V generated by W lifts in a canonical way to a Legendrian flow on S^*V, which is generated by the *complete lift* W^c of W.

The following equality (which in fact holds for every bounded subanalytic set X) was established in the proof of Theorem 1 in [16]:

$$\left.\frac{d}{dt}\right|_{t=0} \Psi(\Phi_t X) = \Psi_{i_{W^c} D\omega}(X). \tag{14}$$

Since $D\omega$ is a vertical form of bidegree $(k, n-k)$, $i_{W^c} D\omega$ is the sum of a form of bidegree $(k-1, n-k)$ and a vertical form. Each tangent vector of $\operatorname{cnc}(X)$ lying above an inner point of X is of bidegree $(k, n-k-1)$. It follows that the right hand side of (14) vanishes, showing that X is extremal. $\square$

7. Holmes-Thompson volumes of projective Finsler metrics

Recall that a norm on a finite-dimensional vector space is called a *Minkowski norm* if the unit sphere and its dual are smooth. A *Finsler metric* on a manifold is a function $F : TM \to \mathbb{R}$ which is smooth outside the zero-section such that its restriction to each tangent space $T_pM, p \in M$ is a Minkowski norm. A Finsler metric on a vector space V is called *projective* if straight lines are geodesics.

Using the Finsler metric on a Finsler manifold, we can measure the length of a curve in the usual way. However, it is less clear how to measure the volume of higher-dimensional manifolds, including M itself. One possibility is to use the Hausdorff measure as definition of volume. This volume, called the *Busemann volume* lacks several good properties (compare [11], [8]). In some contexts, it is better to work with another volume, called the *Holmes-Thompson volume*, whose definition we would like to recall.

Note first that T^*M is a symplectic manifold. The Holmes-Thompson volume of M is defined as the symplectic volume of the unit codisc bundle $D^*M \subset T^*M$. Since submanifolds carry induced Finsler metrics, we can thus measure their Holmes-Thompson volumes. It turns out that the Holmes-Thompson volume has in many (although not all) respects better properties than the Busemann volume.

Theorem 4 is another hint that the Holmes-Thompson volume is more natural than the Busemann volume. Not only do these volumes extend to smooth valuations, they also behave naturally with respect to Alesker's product.

Proof of Theorem 4. The fact that the k-dimensional Holmes-Thompson volume admits a smooth Crofton formula of degree k was proved in [9]. From Theorem 1 it follows that it extends to a valuation Ψ_k^{HT} of pure degree k which belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution.

Uniqueness of the extension follows from Theorem 2. Finally, the equation

$$\Psi_k^{HT} \cdot \Psi_l^{HT} = \Psi_{k+l}^{HT}$$

follows from Theorem 3, since the Crofton measure of the k-dimensional Holmes-Thompson volume is the k-th power (with respect to the Álvarez-Fernandes product) of the Crofton measure of the Finsler metric [9]. $\square$

We remark that, in general, the Crofton measure of the k-dimensional Holmes-Thompson volume in a projective Finsler space need not be positive. The question under which conditions it is positive seems to be a difficult one. If the metric is invariant under translations (i.e. a norm on V), then positivity of these measures is equivalent to the metric being a hypermetric (cf. [27]).

In [26], the Holmes-Thompson valuations are introduced and studied in the translation invariant case. Their behavior is similar to that of *intrinsic volumes* on Euclidean vector spaces.

References

[1] Alesker, S.: The multiplicative structure on polynomial valuations. *Geom. Funct. Anal.* **14** (2004), 1–26.

[2] Alesker, S.: Theory of valuations on manifolds I. Linear spaces. To appear in *Israel J. Math.*

[3] Alesker, S.: Theory of valuations on manifolds II. To appear in *Adv. Math.*

[4] Alesker, S.: Theory of valuations on manifolds IV. New properties of the multiplicative structure. To appear in *GAFA seminar notes*

[5] Alesker, S.: Valuations on manifolds: a survey. Preprint 2006.

[6] Alesker, S., Fu, J. H. G.: Theory of valuations on manifolds III. Multiplicative structure in the general case. To appear in *Trans. Amer. Math. Soc.*

[7] Alesker, S., Bernstein, J.: Range characterization of the cosine transform on higher Grassmannians. *Adv. Math.* **184** (2004), 367–379.

[8] Álvarez Paiva, J. C., Berck, G.: What is wrong with the Hausdorff measure in Finsler spaces? Preprint CRM 599, 2004.

[9] Álvarez Paiva, J. C., Fernandes, E.: Crofton formulas in projective Finsler spaces. *Electron. Res. Announc. Am. Math. Soc.* **4** (1998), 91–100.

[10] Álvarez Paiva, J. C., Fernandes, E.: Crofton formulas and Gelfand transforms. To appear in *Selecta Math.*

[11] Álvarez Paiva, J. C., Thompson, A. C.: Volumes on Normed and Finsler Spaces. In: *A sampler of Riemann Finsler geometry* (D. Bao, R. Bryant, S. S. Chern, Z. Shen, eds.) Cambridge University Press. MSRI publications **50** (2004), 1–48.

[12] Berline, N., Getzler, E., Vergne, M.: *Heat kernels and Dirac operators.* Springer-Verlag Berlin, 1992.

[13] Bernig, A.: Support functions, projections and Minkowski addition of Legendrian cycles. *Indiana Univ. Math. J.* 55 (2006), 443–464.

[14] Bernig, A.: Legendrian currents, support functions and tensor-valued measures on singular spaces. Habilitationsschrift, Université de Fribourg, 2005.

[15] Bernig, A.: The normal cycle of a compact definable set. To appear in *Israel J. Math.*

[16] Bernig, A., Bröcker, L.: Valuations on manifolds and Rumin cohomology. Preprint 2005.

[17] Bröcker, L.: Euler integration and Euler multiplication. *Adv. Geom.* **5** (2005), 145–169.

[18] Brothers, J. E.: Integral geometry in homogeneous spaces. *Trans. Amer. Math. Soc.* **124** (1966), 480–517.

[19] Federer, H.: *Geometric Measure Theory.* Springer-Verlag New York 1969.

[20] Fu, J. H. G.: Kinematic formulas in integral geometry. *Indiana Univ. Math. J.* **39** (1990), 1115–1154.

[21] Fu, J. H. G.: Curvature measures of subanalytic sets. *Amer. J. Math.* **116** (1994), 819–880.

[22] Gelfand, I. M., Smirnov, M. M.: Lagrangians satisfying Crofton formulas, Radon transforms and nonlocal differentials. *Adv. Math.* **109** (1994), 188–227.

[23] Klain, D.: Even valuations on convex bodies. *Trans. Amer. Math. Soc.* **352** (2000), 71–93.

[24] McMullen, P.: Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **35** (1978), 113–135.
[25] Rumin, M.: Formes différentielles sur les variétés de contact. *J. Differential Geom.* **39** (1994), 281–330.
[26] Schneider, R.: Intrinsic volumes in Minkowski spaces. *Rendiconti del circolo matematico di Palermo* Suppl. **50** (1997), 355–373.
[27] Schneider, R., Wieacker, J. A.: Integral geometry in Minkowski spaces. *Adv. Math.* **129** (1997), 222–260.

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