

Institut de Mathématiques
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Construction of compatible Finsler structures on locally compact length spaces

THESE

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Abstract

In 1975 Berestovskij [6] exposed how length spaces with Alexandrov curvature bounded from above and below [3] carry a $\mathcal{C}^{1,1}$ differential Riemannian structure compatible with the given length.

Later work, in particular by Berestovskij and Nicolaev [7], have shown that this Riemannian structure is at least $\mathcal{C}^{2-\epsilon}$. As there are counter-examples of length spaces that cannot carry a \mathcal{C}^2 Riemannian structure, the latter result is not likely to be improved.

Alexandrov curvature conditions are rather strong; the curvature of a normed vector space is neither bounded above nor below (provided the norm does not come from a scalar product), so that there is no hope to use them to obtain a suitable curvature condition on length spaces in order to construct a Finsler structure on them.

We found a weaker curvature condition such that length spaces verifying them carry a differential structure and a Finsler metric. In contrast to the tools used by Berestovskij (that do not apply for the latter curvature conditions), we constructed a differential structure using algebraic properties of the C^* -algebra of continuous functions on the space.

At the time, only a $\mathcal{C}^{1,1/2}$ differential structure has been constructed. It seems, in analogy to what has been achieved for the Riemannian case, that a $\mathcal{C}^{2-\epsilon}$ structure should be possible.

Résumé

En 1975, Berestovskij [6] publia un article dans lequel il construisit une structure riemannienne $\mathcal{C}^{1,1}$ sur un espace de longueur dont la courbure, au sens d'Alexandrov [3] était bornée par le haut et par le bas.

Berestovskij et Nicolaev montrèrent dans [7] que la différentiabilité de cette structure pouvait être améliorée jusqu'à $\mathcal{C}^{2-\epsilon}$. Comme il existe des contre-exemples dont l'atlas ne peut être \mathcal{C}^2 , cette dernière estimation est optimale.

Dire que la courbure selon Alexandrov est bornée est une condition fort restrictive vu que les seuls espaces vectoriels normés à les remplir sont ceux dont la norme provient d'un produit scalaire. Les autres n'ont leur courbure bornée ni par le haut, ni par le bas. Cette notion de courbure est, par conséquence, inappropriée pour construire, à partir d'espaces de longueurs à courbure bornée, une structure de Finsler compatible avec la distance.

Nous avons défini, sur les espaces de longueurs, une notion de courbure plus faible de sorte à pouvoir en faire un espace de Finsler. Les outils utilisés diffèrent de ceux de l'article de Berestovskij. Nous construisons l'atlas en passant par les propriétés algébriques de la C^* -algèbre des fonctions continues sur notre espace.

La différentiabilité de l'atlas obtenu est au moins $\mathcal{C}^{1,1/2}$. Il semble probable que, comme dans le cas riemannien, ce résultat puisse être amélioré vers $\mathcal{C}^{2-\epsilon}$.

Zusammenfassung

Im Jahre 1975 hat Berestovskij [6] bewiesen, dass Längenräume, deren Krümmung, im Sinne von Alexandrov, beschränkt ist, eine $\mathcal{C}^{1,1}$ Riemannsche Struktur besitzen, die mit der Länge kompatibel bleibt.

Später wurde, in Zusammenarbeit mit Nicolaev [7], die Differenzierbarkeit des Atlases auf $\mathcal{C}^{2-\epsilon}$ verbessert. Da es Beispiele für Längenräume mit beschränkter Krümmung gibt, die keinen \mathcal{C}^2 -Atlas haben können, ist letzteres Resultat optimal.

Alexandrovs Krümmungsbegriff ist sehr restriktiv. So ist die Krümmung normierter Vektorräume, vorausgesetzt die Norm stammt nicht von einem Skalarprodukt ab, weder nach oben noch nach unten beschränkt. Will man die Klasse der Längenräume bestimmen, die eine distanzverträgliche Finslerstruktur besitzen, ist dieser Krümmungsbegriff unbrauchbar.

In dieser Arbeit wurde eine schwächere Krümmungsbedingung definiert, so dass deren Beschränktheit auf einem Längenraum eine Finslerstruktur impliziert. Die direkte Konstruktion von Berestovskij musste algebraischen Konstrukten der C^* -Algebra der stetigen Funktionen über dem Raum weichen.

Die Differenzierbarkeit des konstruierten Atlases ist $\mathcal{C}^{1,1/2}$. Es scheint wahrscheinlich, dass auch in diesem Fall $\mathcal{C}^{2-\epsilon}$ Kartenwechsel möglich sein sollten, doch bleibt der Beweis noch aus.

Comments on the text structure

The text is divided in 3 chapters:

The first one is devoted to some general definitions and results in relation with length spaces as well as an alternative definition of space with bounded curvature. It ends with some of geometrical results related to triangles in such spaces.

Chapter 2 is devoted to the construction of analytical tools on these spaces. A tangent cone in each point as well as continuous direction fields and derivatives of scalar functions along these fields are defined.

In chapter 3 a construction of a differential structure and Finsler norm for spaces with bounded curvature is presented and the equivalence of Finsler spaces and spaces with bounded curvature is proved.

A couple of lengthy proofs have been shifted to an appendix. A proof begins with:

Proof:

and refers to the last claim, proposition, theorem or corollary. If the proof is postponed, an explicit reference is given at the beginning.

To enhance readability of long proofs, they have been divided in sub-proofs whose \triangleright *claim is given in an italic font*.

The end of such a proof is marked with a:

\triangle

References in the text are always associated with a page number in brackets were it refers to (example: theorem 3.3.3 (78)).

To help readers, several indices have been appended to the text. Beside an index for key-words, an index for notations and figures is to be found.

Moreover, an index for references is given, telling the reader on which pages a definition, proposition, remark or what ever has been referred to has been cited. This helps the reader to understand the structure behind the text-flow.

The bibliography is also found at the very end and references are given in the usual bracket notation $[n]$.

Chapter 1

Length spaces and curvature bounds

1.1 Introduction

Section 1.2 is mainly devoted to definitions of sets we will need later and to the construction of intrinsic distances and length spaces.

In section 1.3 (25) we give the definition of spaces with bounded Alexandrov curvature, as well as a new curvature definition we will need to construct a Finsler structure on length spaces.

The section ends with an example space. The proof that its curvature is, according to the new definition, bounded will be used as a model in chapter 3 for the more general Finsler space case.

The beginning of section 1.4 (37) is devoted to the definition of scale bounded spaces. These are the spaces that will turn out to be Finsler spaces.

The chapter ends with a study of the geometry of small triangles in scale bounded spaces.

1.2 Paths and lengths

1.2.1 Paths in metric spaces and parametrization

Consider a metric space (X, d) . A path in X is defined as a continuous mapping γ from an interval $[a, b]$ to X . We will often refer to $\gamma(t)$ as γ_t .

Remark 1.1 If no confusion on the used distance function is possible, \overline{pq} will denote $d(p, q)$.

Let us recall some well-known concepts and introduce some notations for path sets:

Definition 1.2.1 Let (X, d) be a metric space and let $\gamma : [a, b] \rightarrow X$ be a path.

1. The initial point of a path γ is γ_a , its final point is γ_b ,
2. it is called closed if $\gamma_a = \gamma_b$,
3. simple if $\gamma|_{[a, b]}$ is injective.

Definition 1.2.2 Let (X, d) be a metric space and let $\gamma : [a, b] \rightarrow X$ and $\gamma' : [a', b'] \rightarrow X$ be paths where $a \leq a' < b' \leq b$ and that $\gamma' = \gamma|_{[a', b']}$.

1. γ is called an extension of γ' and γ' a restriction of γ ,
2. if $a = a'$, γ is called a right extension of γ' and γ' a left restriction of γ ,
3. if $b = b'$, γ is called a left extension of γ' and γ' a right restriction of γ .

Definition 1.2.3 Let (X, d) be a metric space and $\mathcal{C}^0([a, b], X)$ ($a < b$) the set of all continuous mappings from $[a, b]$ to X . We define following path spaces:

1. $\Upsilon(X) := \bigcup_{a < b} \mathcal{C}^0([a, b], X)$.
2. $\Upsilon^p(X)$ is the set of all paths in $\Upsilon(X)$ with initial point p ,
3. $\Upsilon^{p,q}(X)$ the set of all paths in $\Upsilon(X)$ with initial point p and final point q .
4. $\Upsilon_\infty(X) := \{\gamma \in \Upsilon(X) \mid \gamma \text{ is Lipschitz continuous}\}$,
5. $\Upsilon_L(X) := \{\gamma \in \Upsilon_\infty(X) \mid \gamma \text{ with Lipschitz constant } L\}$,

$\Upsilon_L^p(X)$, $\Upsilon_L^{p,q}(X)$, $\Upsilon_\infty^p(X)$ and $\Upsilon_\infty^{p,q}(X)$ are defined by analogy.

They obviously verify the following inclusion relations:

$$\begin{array}{ccccccc}
 \Upsilon(X) & \supseteq & \Upsilon_\infty(X) & \supseteq & \Upsilon_L(X) & \supseteq & \Upsilon_{L'}(X) \\
 \cup & & \cup & & \cup & & \cup \\
 \Upsilon^p(X) & \supseteq & \Upsilon_\infty^p(X) & \supseteq & \Upsilon_L^p(X) & \supseteq & \Upsilon_{L'}^p(X) \\
 \cup & & \cup & & \cup & & \cup \\
 \Upsilon^{p,q}(X) & \supseteq & \Upsilon_\infty^{p,q}(X) & \supseteq & \Upsilon_L^{p,q}(X) & \supseteq & \Upsilon_{L'}^{p,q}(X)
 \end{array}$$

where $L \geq L' \geq 0$.

Many properties of paths are known to be independent of parametrization. This can be formally expressed by the invariance of these properties relatively to the action of the following semi-group:

Definition 1.2.4 Let (\mathcal{S}, \circ) be the semi-group of increasing continuous functions from $[0, 1]$ onto itself.

Let $\gamma \in \mathcal{C}^0([a, b], X)$ be a path. $\phi \in \mathcal{S}$ acts on $\gamma \in \Upsilon(X)$ from the right as follows:

$$\gamma \cdot \phi := \gamma \circ T_\gamma \circ \phi \circ T_\gamma^{-1}$$

where $T_\gamma : [0, 1] \rightarrow [a, b]$ is given by $t \mapsto (b - a)t + a$.

The composition of an L_1 -Lipschitz continuous function with a L_2 -Lipschitz continuous function is an $(L_1 L_2)$ -Lipschitz continuous function. This enables us to define actions on $\Upsilon_\infty(X)$ and $\Upsilon_L(X)$:

Definition 1.2.5 Let \mathcal{S}_∞ be the semi-group of all Lipschitz continuous functions in \mathcal{S} . \mathcal{S}_∞ acts on $\Upsilon_\infty(X)$.

Definition 1.2.6 Let $\gamma : [a, b] \rightarrow X$ be a path. The opposite path $-\gamma : [a, b] \rightarrow X$ of γ is the path $t \mapsto \gamma(a + b - t)$.

Remark 1.2 This operation is an involution: $-(-\gamma) = \gamma$.

1.2.2 Rectifiability and length of paths

If (X, d) is a metric space, it is possible to define the length of paths by means of the so-called "rectification" of curves (see first chapter in [12] or [16]).

The idea is roughly speaking the same as curve length definition in the Euclidean plane: one approximates the curve by a polygonal line and defines the length as the least upper bound of polygon's lengths. More formally,

Let $P([a, b])$ be the set of all finite subsets of $[a, b]$ containing at least a and b . $(P([a, b]), \subseteq)$ is a net. Let $\Delta = \{t_0 = a, t_1, \dots, t_n = b\}$ be an element in $P([a, b])$ such that $t_0 < t_2 < \dots < t_n$. We define the non-negative function

$$l(\gamma, \Delta) := \sum_{i=1}^n \overline{\gamma_{t_{i-1}} \gamma_{t_i}},$$

The triangle inequality implies $\Delta \subseteq \Delta' \implies l(\gamma, \Delta) \leq l(\gamma, \Delta')$ such that the limit of $l(\gamma, \Delta)$ over the net $(P([a, b]), \subseteq)$ is well defined.

Definition 1.2.7 $l : \Upsilon(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$(\gamma : [a, b] \rightarrow X) \mapsto l(\gamma) := \lim_{\Delta \in P([a, b])} l(\gamma, \Delta)$$

$l(\gamma)$ is called the length of γ . If this length is finite, γ is said to be rectifiable.

Proposition 1.2.8

If $\gamma : [a, b] \rightarrow X$ is in $\Upsilon_L(X)$ then $l(\gamma) \leq L(b - a)$.

Proof:

Suppose $\Delta = \{t_0, t_1, \dots, t_n\}$ with $t_0 < t_1 < \dots < t_n$. Then

$$l(\gamma, \Delta) = \sum_{i=1}^n \overline{\gamma_{t_{i-1}} \gamma_{t_i}} \leq \sum_{i=1}^n L |t_i - t_{i-1}| = L(b - a).$$

As this result is true for every Δ , it is true in the limit $l(\gamma) = \lim_{\Delta \in P([a,b])} l(\gamma, \Delta)$. \square

Corollary 1 *All paths in $\Upsilon_\infty(X)$ are rectifiable.*

Proposition 1.2.9

The length of a path is additive, i. e. if $\gamma : [a, c] \rightarrow X$ is a path and $b \in [a, c]$, then

$$l(\gamma) = l(\gamma|_{[a,b]}) + l(\gamma|_{[b,c]}).$$

Proof:

We know that if $\Delta \in P([a, c])$, $\Delta \cup \{b\} \in P([a, c])$. Moreover $l(\gamma, \Delta) \leq l(\gamma, \Delta \cup \{b\})$ so that we can restrict ourselves to elements Δ of $P([a, c])$ containing b .

From $l(\gamma, \Delta) = l(\gamma|_{[a,b]}, \Delta \cap [a, b]) + l(\gamma|_{[b,c]}, \Delta \cap [b, c])$

we conclude that $l(\gamma) \leq l(\gamma|_{[a,b]}) + l(\gamma|_{[b,c]})$.

On the other hand, if $\Delta' \in P([a, b])$ and $\Delta'' \in P([b, c])$, we have $l(\gamma, \Delta' \cup \Delta'') = l(\gamma|_{[a,b]}, \Delta') + l(\gamma|_{[b,c]}, \Delta'')$ such that $l(\gamma) \geq l(\gamma|_{[a,b]}) + l(\gamma|_{[b,c]})$.

The equality is proven. \square

Proposition 1.2.10

The length of a path is invariant under the action of \mathcal{S} .

Proof:

We suppose, without loss of generality, that γ is a path defined on $[0, 1]$.

As elements ϕ in \mathcal{S} are onto, the mapping sending a finite subset of $[0, 1]$ to its image through ϕ defines a mapping from $P([0, 1])$ onto itself such that the net $(P([0, 1]), \subseteq)$ is invariant by the action of ϕ .

We also know that ϕ is order preserving such that, if $\Delta \in P([0, 1])$ contains the points $t_0 < t_1 < \dots < t_n$, $\phi(\Delta)$ contains the points $\phi(t_0) \leq \phi(t_1) \leq \dots \leq \phi(t_n)$. On the other hand, we know, by definition, that $\gamma \cdot \phi(t) = \gamma \circ \phi(t)$ for all $t \in [0, 1]$ such that

$$l(\gamma \cdot \phi, \Delta) = \sum_{i=1}^n \overline{\gamma \circ \phi(t_{i-1}) \gamma \circ \phi(t_i)} = l(\gamma, \phi(\Delta)).$$

This implies that $l(\gamma \cdot \phi) = l(\gamma)$. \square

An other important property of rectifiable paths, which follows from proposition 1.2.9 is that they admit a length parametrization:

Definition 1.2.11 *Let $\gamma : [a, b] \rightarrow X$ be a path. γ is said to be length parameterized, if $l(\gamma|_{[t_1, t_2]}) = |t_2 - t_1|$ for every $t_1, t_2 \in [a, b]$.*

$\Upsilon_=(X)$ is the set of all length parameterized paths in X .

Corollary 1 *Let γ be a path in $\Upsilon(X)$. There exists a $\phi \in \mathcal{S}$ and a length parameterized path $\gamma' \in \Upsilon(X)$ such that $\gamma' = \gamma \cdot \phi$ if and only if γ is rectifiable.*

Proof:

Let us divide the proof in two parts.

▷ *If γ is not rectifiable, γ' is neither:* Because of proposition 1.2.10 (18), γ not rectifiable implies that γ' is not rectifiable. But a length parameterized non rectifiable path cannot have a compact domain of definition so that $\gamma' \notin \Upsilon(X)$. \triangle

We suppose, without loss of the generality, that γ is defined on $[0, 1]$.

▷ *If γ is rectifiable, the claimed ϕ exists:* Set $\phi(t) := l(\gamma|_{[0,t]})$. By proposition 1.2.9 (18), this is an increasing function. Its continuity follows from the continuity of γ itself so that $\phi \in \mathcal{S}$.

As $\gamma' = \gamma \cdot \phi = \gamma \circ \phi^{-1}$ is length parameterized by construction, ϕ does its job. \triangle

□

Every path $\gamma \in \Upsilon(X)$ from $[a, b]$ to X can be extended to an element $\hat{\gamma}$ of $\mathcal{C}^0(\mathbb{R}, X)$ with:

$$\hat{\gamma}_t := \begin{cases} \gamma_a & \text{if } t \leq a \\ \gamma_t & \text{if } t \in [a, b] \\ \gamma_b & \text{if } t \geq b \end{cases}.$$

Using this identification, we equip $\Upsilon(X)$ with the compact-open topology on $\mathcal{C}^0(\mathbb{R}, X)$ (see Chapter 7 in [20]). In this topology:

Theorem 1.2.12

$l : \Upsilon(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is lower semi-continuous.

Proof:

Let $\gamma : [a, b] \rightarrow X$ be a path in $\Upsilon(X)$ and choose ϵ such that $0 < \epsilon < l(\gamma)$.

By definition, l is lower semi-continuous, if there is a neighbourhood \mathcal{U} of γ such that for every $\gamma' \in \mathcal{U}$, $l(\gamma') > l(\gamma) - \epsilon$.

By definition of length, there is a $\Delta = \{t_0 = a, t_1, t_2, \dots, t_n = b\} \in P([a, b])$ such that $l(\gamma) - l(\gamma, \Delta) < \epsilon/2$. Without restriction of generality, we assume that $\gamma_{t_i} \neq \gamma_{t_{i+1}}$.

For every $r > 0$, $\mathcal{U}_r := \left\{ \gamma' \in \Upsilon(X) \mid \forall t \in \Delta \quad \overline{\gamma'_t \gamma_t} < r \right\}$ is an open neighbourhood of γ . Let us choose r such that $0 < r < \epsilon/(4n)$ and $0 < r < \frac{1}{2} \overline{\gamma_{t_{i-1}} \gamma_{t_i}}$ for all i . If $\gamma' \in \mathcal{U}_r$, $\overline{\gamma'_{t_i} \gamma_{t_i}} < r$ and $\overline{\gamma'_{t_{i+1}} \gamma_{t_{i+1}}} < r$, such that, by triangle inequalities, we have

$$\overline{\gamma'_{t_i} \gamma'_{t_{i+1}}} > \overline{\gamma_{t_i} \gamma_{t_{i+1}}} - 2r > \overline{\gamma_{t_i} \gamma_{t_{i+1}}} - \epsilon/(2n).$$

Adding up on both sides over i , we obtain

$$l(\gamma', \Delta) > l(\gamma, \Delta) - \epsilon/2.$$

But $l(\gamma') \geq l(\gamma', \Delta)$ and $l(\gamma, \Delta) - \epsilon/2 = l(\gamma) - (l(\gamma) - l(\gamma, \Delta)) - \epsilon/2 \geq l(\gamma) - \epsilon$ such that finally $l(\gamma') > l(\gamma) - \epsilon$. \square

1.2.3 Intrinsic metrics and length spaces

We have seen that in a metric space (X, d) , one can define the length of a path. This suggests the definition of an alternative distance on X .

Definition 1.2.13 *Let (X, d) be a metric space. We define $d_l : X \times X \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by:*

$$d_l(p_1, p_2) := \begin{cases} \inf \{l(\gamma) \mid \gamma \in \Upsilon^{p_1, p_2}(X)\} & \text{if } \Upsilon^{p_1, p_2}(X) \neq \emptyset \\ \infty & \text{if } \Upsilon^{p_1, p_2}(X) = \emptyset \end{cases}.$$

Proposition 1.2.14

For every metric space (X, d)

1. $d_l \geq d$,
2. d_l is a metric on X ,
3. $(d_l)_l \equiv d_l$.

Proof:

Let $\gamma : [a, b] \rightarrow X$ a path from p_1 to p_2 , $p_1, p_2 \in X$.

1. $d(p_1, p_2) = l(\gamma, \{a, b\})$, hence, $d(p_1, p_2) \leq l(\gamma)$. This is true for every $\gamma \in \Upsilon^{p_1, p_2}(X)$, such that $d(p_1, p_2) \leq d_l(p_1, p_2)$.
2. Symmetry and positivity of d_l are obvious. From the previous point, we know that $d_l(p_1, p_2) = 0 \implies d(p_1, p_2) = 0$. On the other hand, if $d(p_1, p_2) = 0$, id est $p_1 = p_2$, then $d_l(p_1, p_2) = 0$ as every constant path has 0 length.

Consider a path from p_1 to p_2 and another from p_2 to p_3 . Their juxtaposition defines a new path from p_1 to p_3 whose length is the sum of the length of the former. As the d_l is defined as a greatest lower bound, this property implies the triangle inequality for d_l .

3. Choose a $\Delta \in P([a, b])$ such that $l(\gamma) \leq l(\gamma, \Delta) + \epsilon$ for a given $\epsilon > 0$. Suppose $\Delta = \{t_0, \dots, t_n\}$ where $t_0 < t_1 < \dots < t_n$. We know that

$$\overline{\gamma_{t_{i-1}} \gamma_{t_i}} \leq d_l(\gamma_{t_{i-1}}, \gamma_{t_i}) \leq l(\gamma|_{[t_{i-1}, t_i]})$$

such that $l_2(\gamma) = l(\gamma)$ where l_2 is the curve length relatively to the distance d_l . Hence $(d_l)_l \leq d_l$. Using point 1, we conclude that $(d_l)_l \equiv d_l$.

□

$d_l \geq d$ implies that (X, d_l) is a topological refinement of (X, d) . (X, d_l) may be strictly finer than (X, d) :

Example 1.1 Let X be a subset of \mathbb{R}^2 which is not connected by arcs and equipped with the Euclidean distance d from \mathbb{R}^2 .

Two points in different connected components have finite distance relatively to d , but infinite distance relatively to d_l .

Example 1.2 Let

$$X = \mathbb{R}^2 \setminus \bigcup_{n \in \mathbb{Z}^*} \left(\frac{1}{n} \right) \times [-1, 1],$$

endowed with the standard distance d from \mathbb{R}^2 .

All neighbourhoods of $(0, 0)$ relatively to d have a non-empty intersection with $\mathbb{R}_+^* \times \{0\}$. But every path from $(0, 0)$ to $(x, 0)$ for $x > 0$ is longer than 2. Hence, there are neighbourhoods of $(0, 0)$ in (X, d_l) containing no points of $\mathbb{R}_+^* \times \{0\}$. The topologies are different.

Nevertheless, there is a large class of metric spaces where d and d_l are equivalent:

Definition 1.2.15 Let (X, d) be a metric space. X is said to have an *intrinsic metric* if $d_l \equiv d$.

Definition 1.2.16 Let (X, d) be a space with intrinsic metric. A subset $E \subseteq X$ is said *convex relatively to d* , if the distance in the restriction $(E, d|_{E \times E})$ is also intrinsic.

Remark 1.3 This definition of convexity has some surprising properties when thinking of convex sets in the usual sense. For example, if a set is convex, it remains convex after removing a finite subset.

The main property we want to keep with our definition is that any arc-wise connected points P and Q in a convex set can be connected by a path whose length is as close as desired to \overline{PQ} .

Definition 1.2.17 Let (X, d) be an arc-wise connected space with intrinsic metric.

If for every $p_1, p_2 \in X$ there is a path γ such that $l(\gamma) = \overline{p_1 p_2}$, X is called a *length space*.

If for every $x \in X$, there is a neighbourhood \mathcal{U} of x such that for every $p_1, p_2 \in \mathcal{U}$ there is a path γ such that $l(\gamma) = \overline{p_1 p_2}$, X is said to be *locally a length space*.

Remark 1.4 This definition of intrinsic metric is used by Berestovskij and Nikolaev (see [7]). Busemann [12] and Shiohama [24] use the equivalent notion of Menger convex space. Burago [11] defines an intrinsic metric in a more general way.

Example 1.3 \mathbb{R}^2 with its standard distance is a length space but $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not. The distance remains however an intrinsic metric.

We recall that a metric space is said to be *finitely compact*, if every closed ball of finite radius is compact in X and *locally compact*, if in every neighbourhood of a point x there is a compact neighbourhood of x .

Theorem 1.2.18

Let (X, d) be an arc-wise connected metric space.

1. If X is finitely compact, (X, d_l) is a length space,
2. If X is locally compact, (X, d_l) is locally a length space.

Example 1.4 Using the same examples as in example 1.3 (21), we can verify that \mathbb{R}^2 is finitely compact and $\mathbb{R}^2 \setminus \{(0, 0)\}$ locally compact.

We have already observed that the first is a length space. As $\mathbb{R}^2 \setminus \{(0, 0)\}$ is locally compact and following theorem 1.2.18 (21), it is locally a length space: in fact, the only couples of points that cannot be joined by a path whose length is equal to the distance between the points are those for which the origin lays on the segment joining them. Consequently, the restriction of \mathbb{R}^2 to any ball not containing $(0, 0)$ is a convex subset of $\mathbb{R}^2 \setminus \{(0, 0)\}$ and is a length space.

In order to prove theorem 1.2.18 (21), we will need the following technical lemma:

Lemma 1.2.19 *Let (X, d_l) be a space with an intrinsic metric and let $x \in X$ and $p_1, p_2 \in B(x, r)$. There is an $\epsilon > 0$ such that every path γ from p_1 to p_2 with $l(\gamma) - \overline{p_1 p_2} < \epsilon$ lies entirely in the ball $B(x, 2r)$.*

Proof:

Let $\gamma : [a, b] \rightarrow X$ be a path with $\gamma_a = p_1$, $\gamma_b = p_2$ and t be in $[a, b]$.

By triangle inequality, $\overline{x \gamma_t} \leq \overline{x p_1} + \overline{p_1 \gamma_t} \leq \overline{x p_1} + l(\gamma|_{[a, t]})$ and $\overline{x \gamma_t} \leq \overline{x p_2} + \overline{p_2 \gamma_t} \leq \overline{x p_2} + l(\gamma|_{[t, b]})$.

Adding both inequalities, we obtain:

$$\begin{aligned}
 2\overline{x \gamma_t} &\leq \overline{x p_1} + \overline{x p_2} + l(\gamma|_{[t, b]}) + l(\gamma|_{[a, t]}) \\
 &\leq \overline{x p_1} + \overline{x p_2} + l(\gamma) && \text{by proposition 1.2.9} \\
 &\leq \overline{x p_1} + \overline{x p_2} + \overline{p_1 p_2} + (l(\gamma) - \overline{p_1 p_2}) \\
 &\leq \overline{x p_1} + \overline{x p_2} + (\overline{x p_1} + \overline{x p_2}) + (l(\gamma) - \overline{p_1 p_2}) \\
 &\leq 2\overline{x p_1} + 2\overline{x p_2} + (l(\gamma) - \overline{p_1 p_2})
 \end{aligned}$$

If we choose γ such that $(l(\gamma) - \overline{p_1 p_2}) < 4r - 2\overline{x p_1} - 2\overline{x p_2} =: \epsilon > 0$ we have $\overline{x \gamma_t} \leq 2r$ for every t . \square

Proof of theorem 1.2.18:

We will prove that within a compact ball

Let x be in X and $r > 0$ be chosen in such a way that $\overline{B(x, 4r)}$ is compact. Let p_1, p_2 be points in $B(x, r)$. We will prove that there is a length parameterized path between p_1 and p_2 . This implies that both points 1 and 2 are true.

For the case we are looking at point 1 of the theorem, there is no restriction in the choice of r , such that we can put $r = 2\overline{p_1 p_2}$ and $x = p_1$.

If $\Upsilon^{p_1, p_2}(\overline{B(x, 2r)})$ is compact it follows from theorem 1.2.12 (19) that a path realizing the minimum length between p_1 and p_2 exists such that the proof is completed.

Let us prove that $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ is compact: by lemma 1.2.19, we know that there is an $\epsilon > 0$ such that all paths $\gamma \in \Upsilon_{\leq}^{p_1, p_2}(X)$ such that $l(\gamma) \leq \overline{p_1 p_2} + \epsilon$, are in $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$. Using the lower semi-continuity of l proved in theorem 1.2.12 (19), we can conclude that the greatest lower bound is realized within $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$, i. e. there is a path in $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ such that $l(\gamma) = \overline{p_1 p_2}$.

We now apply theorem 6 of chapter 7 in [20] to prove that $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ is compact: applied to our purpose, it states that our set is compact if:

1. $\overline{B(x, 2r)}$ is Hausdorff,
2. the sets $F_t := \{\gamma_t \in \overline{B(x, 2r)} \mid \gamma \in \Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})\}$ have compact closure for every $t \in \mathbb{R}$,
3. the set $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ is closed for the point-wise convergence topology on $\Upsilon_{=}(X)$,
4. let K be a compact subset of \mathbb{R} and equip $\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ with the point-wise convergence topology. The mapping defined by $(\gamma, t) \mapsto \gamma_t$ is continuous.

The first point is obviously true, $\overline{B(x, 2r)}$ being metric.

For the second point, observe that $\overline{F_t}$ is, by lemma 1.2.19 (22), a closed subset of the compact Hausdorff space $\overline{B(x, 4r)}$, so it is compact itself.

$\Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)})$ is closed in $\Upsilon_{=}(X)$ for the compact-open topology, so it must also be closed for the point-wise convergence topology, the former being coarser than the latter.

Let us show the last point: Consider a couple (γ, t) where $x := \gamma_t$ and an $\epsilon > 0$. The following set is open for the compact-open topology

$$\mathcal{U} = \left\{ \gamma' \in \Upsilon_{\leq}^{p_1, p_2}(\overline{B(x, 2r)}) \mid \gamma'(t) \subseteq B(\gamma_t, \epsilon/2) \right\},$$

such that $\mathcal{U} \times B(t, \epsilon/2)$ is a neighbourhood of (γ, t) . As all paths are length parameterized, the image of $\mathcal{U} \times B(t, \epsilon/2)$ is obviously within $B(t, \epsilon)$. \square

Example 1.5 G -spaces are length spaces (see [12] for definition).

Example 1.6 Any compact submanifold of \mathbb{R}^n with the intrinsic metric inherited from the distance in \mathbb{R}^n is a length space.

1.2.4 Geodesics in spaces with intrinsic metric

In order to do geometry on spaces with intrinsic metric, it is useful to define something that could correspond to geodesics in Finsler spaces (see section 3.3 (78)). With Busemann [12], we define these distinguished paths in the following way:

Definition 1.2.20 Let (X, d) be a space with intrinsic metric and let $\gamma : [a, b] \longrightarrow X$ be a path in $\Upsilon(X)$.

1. γ is called a segment if $l(\gamma) = \overline{\gamma_a \gamma_b}$,
2. γ is called a geodesic if, for every $t \in [a, b]$, there is a neighbourhood \mathcal{U} of t in $[a, b]$ such that $\gamma|_{\mathcal{U}}$ is a segment.

$\mathcal{G}(\mathcal{U})$ is the set of all length parameterized geodesics lying in $\mathcal{U} \subseteq X$ and $\mathcal{G}^p(\mathcal{U})$ the subset of those paths $\gamma : [a, b] \rightarrow \mathcal{U}$ of $\mathcal{G}(\mathcal{U})$ verifying $a < 0 < b$ and $\gamma_0 = p$.

Remark 1.5 By definition, any segment is a geodesic. The converse must not be true: consider S^2 with its standard metric. A closed path parameterizing a great circle is obviously a geodesic but definitely not a segment.

Remark 1.6 In finitely compact, arc-wise connected spaces with intrinsic metric, every couple of points can be joined by a segment; for locally compact spaces, this remains locally true (see theorem 1.2.18 (21)).

Remark 1.7 Every geodesic has finite length: its graph is compact, so that it can be split in a finite number of segments (see previous remark). As segments must be of finite length, geodesics also are.

Proposition 1.2.21

In spaces with intrinsic metric, restrictions of segments are segments and restrictions of geodesics are geodesics.

This result allows to consistently define:

Definition 1.2.22 Let (X, d) be a space with intrinsic metric an extended geodesic (respectively path) in X is mapping γ from an interval I of \mathbb{R} into X such that every restriction to a compact interval yields a geodesic (respectively a path).

Proof of proposition 1.2.21:

Suppose the segment given by $\gamma : [a, b] \rightarrow \mathcal{U}$ and $a \leq a' < b' \leq b$. We observe that:

$$\begin{aligned}
 l(\gamma) &= l(\gamma|_{[a, a']}) + l(\gamma|_{[a', b']}) + l(\gamma|_{[b', b]}) && \text{by proposition 1.2.9 (18)} \\
 &\quad l(\gamma|_{[a, a']}) \geq \overline{\gamma_a \gamma_{a'}} && \text{by proposition 1.2.14 (20)} \\
 &\quad l(\gamma|_{[b', b]}) \geq \overline{\gamma_{b'} \gamma_b} && \text{by proposition 1.2.14 (20)} \\
 l(\gamma) &\geq \overline{\gamma_a \gamma_{a'}} + l(\gamma|_{[a', b']}) + \overline{\gamma_{b'} \gamma_b} && \text{by the previous relations. But from} \\
 &\quad \overline{\gamma_a \gamma_{a'}} + \overline{\gamma_{a'} \gamma_{b'}} + \overline{\gamma_{b'} \gamma_b} \geq \overline{\gamma_a \gamma_b} && \text{(triangle inequality) and} \\
 &\quad l(\gamma) = \overline{\gamma_a \gamma_b} && (\gamma \text{ is a segment) we obtain} \\
 &\quad \overline{\gamma_{a'} \gamma_{b'}} \geq l(\gamma|_{[a', b']}).
 \end{aligned}$$

Recalling that $l(\gamma|_{[a', b']}) \geq \overline{\gamma_{a'} \gamma_{b'}}$ by proposition 1.2.14 (20), we conclude that $l(\gamma|_{[a', b']}) = \overline{\gamma_{a'} \gamma_{b'}}$, i. e. $\gamma|_{[a', b']}$ is a segment.

For a geodesic, we apply this argument to every segment in it. □

1.3 Spaces with bounded curvature

1.3.1 Introduction

Though certain definitions may be stated for more general spaces, we will, in this section, concentrate on local length spaces.

1.3.2 Alexandrov curvature bounds and scale curvature bounds

Alexandrov comparison criteria are defined by isometric embeddings of triangles in reference spaces of constant curvature. Let us define them as follows:

Definition 1.3.1 *Let K be an element in \mathbb{R} . We define*

$$M_K \text{ is the } \begin{cases} \text{the open half 2-sphere of radius } K^{-\frac{1}{2}} & \text{if } K > 0 \\ \text{Euclidian plane} & \text{if } K = 0 \\ \text{Lobachevskij plane of curvature } K & \text{if } K < 0 \end{cases} .$$

Definition 1.3.2 *Let (X, d) be a length space. A triangle in X is a triplet of segments a, b and c whose initial points are the end points of one of the other segments. These initial points are called vertices.*

We define the half-perimeter $\rho(\Delta)$ of a triangle Δ as $\frac{1}{2} (l(a) + l(b) + l(c))$.

Remark 1.8 In many situations the segments between the vertices of a given triangle are unique or the choice of a particular one is irrelevant for the argument. In such cases, we will often define the triangle by the triplet of its vertices (A, B, C) .

In order to simplify definition 1.3.4 (26) of curvature bound and as useful notation in proofs, we define:

Definition 1.3.3 *Consider a triangle Δ in a length space (X, d) defined by its segments a, b and c . Further we suppose that $a(0)$ is the end point of b and that A is the vertex common to b and c .*

We define $h_X(a, b, c; \kappa)$ as the height $\overline{AH_\kappa}$ where $H_\kappa = a(\kappa \cdot l(a))$ ($\kappa \in [0, 1]$).

We further define $m_X(a, b, c) := h_X(a, b, c; \frac{1}{2})$.

In case $X = M_K$, we simply write h_K and m_K rather than h_{M_K} and m_{M_K} .

Remark 1.9 As consequence of homogeneity of the reference spaces M_K , the value of h_K or m_K does only depend on the length of the triangle edges so that we will sometimes write $h_X(l(a), l(b), l(c); \kappa)$ instead of $h_X(a, b, c; \kappa)$.

Explicit algebraic relations between $l(a), l(b), l(c), \kappa$ and h_K or m_K can be found in appendix section A.2 (86).

Remark 1.10 The formulae worked out in the appendix show that $h_K(a, b, c; \kappa) \leq h_{K'}(a, b, c; \kappa)$ for all triangles (a, b, c) if and only if $K \leq K'$.

Let us now give the Alexandrov criteria for curvature bounds (see [3], [24]):

Definition 1.3.4 (Alexandrov curvature bounds) *Let (X, d) be a length space.*

Let (a, b, c) be a triangle in X and consider $(\tilde{a}, \tilde{b}, \tilde{c})$ an isometric embedding¹ of that triangle in M_K .

X is said to have its curvature bounded from below - respectively from above - by K if for any such triangle in X and any $\kappa \in [0, 1]$, $h_X(a, b, c; \kappa) \geq h_K(\tilde{a}, \tilde{b}, \tilde{c}; \kappa)$ - respectively $h_X(a, b, c; \kappa) \leq h_K(\tilde{a}, \tilde{b}, \tilde{c}; \kappa)$.

Remark 1.11 If the curvature of a space X is bounded from below (above) by K it is also bounded from below (above) by any $K' < K$ ($K' > K$) (this follows from remark 1.10).

Example 1.7 Consider a vector space V of dimension n endowed with a norm and suppose that its Alexandrov curvature is bounded.

Consider the triangle $(0, \lambda V_1, \lambda V_2)$ where $V_1, V_2 \in V$ and $\lambda > 0$. We know that

$$h_V(\|\lambda V_1\|, \|\lambda V_2\|, \|\lambda V_1 - \lambda V_2\|, \kappa) = \lambda h_V(\|V_1\|, \|V_2\|, \|V_1 - V_2\|, \kappa)$$

for any $\lambda > 0$ by the homogeneity of the norm.

On the other hand we know that, for small triangles:

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} h_K(\|\lambda V_1\|, \|\lambda V_2\|, \|\lambda V_1 - \lambda V_2\|, \kappa),$$

exists and converges to

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} h_0(\|\lambda V_1\|, \|\lambda V_2\|, \|\lambda V_1 - \lambda V_2\|, \kappa).$$

irrespective of K .

But as $\frac{1}{\lambda} h_0(\|\lambda V_1\|, \|\lambda V_2\|, \|\lambda V_1 - \lambda V_2\|, \kappa)$ does not depend on λ , all triangles in a vector space with bounded curvature must behave like Euclidean triangles. This, in turn, implies that the norm of V comes from a scalar product.

By contraposition, all other norms on V define spaces that cannot have their Alexandrov curvature bounded.

It is known that if a length space has locally Alexandrov curvature bounds, it carries a Riemannian structure ([6],[7]). In particular:

¹By isometric embedding, we mean that $(\tilde{a}, \tilde{b}, \tilde{c})$ is a triangle with $l(a) = l(\tilde{a})$, $l(b) = l(\tilde{b})$, $l(c) = l(\tilde{c})$.

Example 1.8 The Alexandrov curvature of any compact Riemannian manifold M is bounded from above and below [7]:

In fact the least upper bound respectively the greatest lower bound of the sectional curvature give the best possible values for the Alexandrov curvature bounds.

In order to obtain a similar result giving a Finsler structure, we need, as has been shown in example 1.7 (26), a weaker variant of the above curvature bounds, at least for small triangles. We will call it scale curvature bounds:

Definition 1.3.5 (Bounded scale curvature) *Let (X, d) be a length space and (a, b, c) be a triangle in X with a half perimeter $\rho \neq 0$ and consider $(\tilde{a}, \tilde{b}, \tilde{c})$ a homogeneous embedding with scaling factor ρ of that triangle in M_K (i. e. $l(\tilde{a}) = l(a)/\rho$, $l(\tilde{b}) = l(b)/\rho$ and $l(\tilde{c}) = l(c)/\rho$), see figures 1.1 and 1.2).*

The space X is said to have scale curvature bounded from below - respectively from above - by $K < \pi^2$ if for any choice of such a triangle we verify that $m_X(a, b, c) \geq \rho \cdot m_K(\tilde{a}, \tilde{b}, \tilde{c})$ - respectively $m_X(a, b, c) \leq \rho \cdot m_K(\tilde{a}, \tilde{b}, \tilde{c})$.

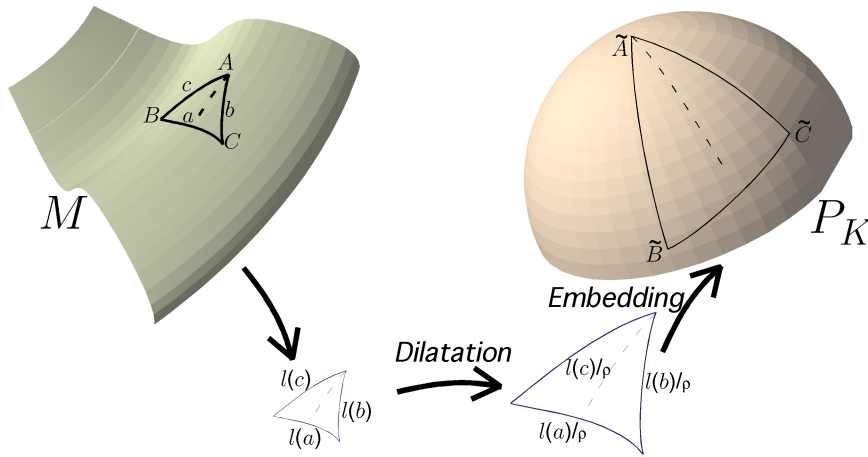


Figure 1.1: The comparison criteria for scale bounded spaces from above.

Remark 1.12 Consider M_1 , the unit 2-sphere. Obviously an embedding of a triangle with half-perimeter π into M_1 would fall on an equator. Hence, only triangles with half-perimeter $< \pi$ can possibly be embedded into M_1 . The curvature of a 2-sphere is known to be inversely proportional to the square of its diameter. Hence, if $K > 0$ triangles that are isometrically embedded into M_K must have a half perimeter smaller than π/\sqrt{K} .

For $K \leq 0$, no such limitations exist.

As, by construction of the parameterized embedding of (a, b, c) in definition 1.3.5, the half perimeter of $(\tilde{a}, \tilde{b}, \tilde{c})$ is 1, k must be chosen smaller than π^2 .

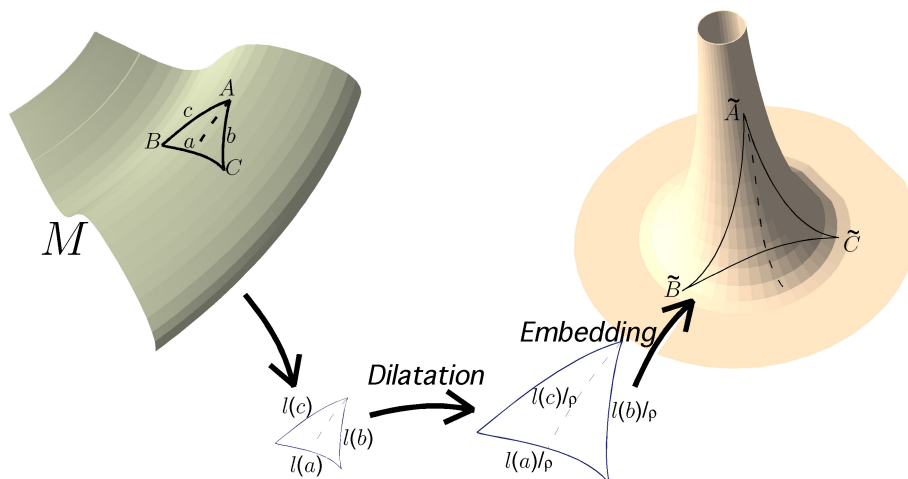


Figure 1.2: The comparison criteria for scale bounded spaces from below

Remark 1.13 An equivalent formulation is to suppose that $(\tilde{a}, \tilde{b}, \tilde{c})$ is an isometric embedding of (a, b, c) into M_K for $K = k/\rho^2$. The comparison criteria is then $m_X(a, b, c) \geq m_K(\tilde{a}, \tilde{b}, \tilde{c})$.

The proof is trivial looking at the explicit formula for m_K in section A.2 (86).

In the spirit of this other formulation, we introduce:

Definition 1.3.6 Let $k < \pi^2$ and (a, b, c) be a triangle and ρ its half perimeter. $\mathcal{M}_k(a, b, c)$ is a short form of $m_K(a, b, c)$ with $K := k/\rho^2$.

Remark 1.14 A space with bounded scale curvature from below (above) by k has also its scale curvature bounded from below (above) for any $k' < k$ ($k' > k$). (See remark 1.11 (26).)

Remark 1.15 The scale curvature of 2-sphere has no upper bound: a triangle formed by 3 points on a great circle violates the curvature condition.

However, every convex subset of M_1 , the upper half 2-sphere, admits $k = 1$ as upper and, for example $k = 0$ as lower scale curvature bound.

Remark 1.16 If a space has its Alexandrov curvature bounded from below and above by K_- and K_+ , there is, for every triangle (a, b, c) and any $\kappa \in [0, 1]$ a curvature K with $K_- \leq K \leq K_+$ such that $h_X(a, b, c; \kappa) = h_K(l(a), l(b), l(c); \kappa)$. This is also true for scale bounded curvature.

Remark 1.17 If a length space is of finite diameter D , the largest possible value for the half perimeter of a triangle is $\frac{3}{2}D$.

In the case X has its Alexandrov curvature bounded, we can assume, without loss of generality, that $K_+ > 0$ and $K_- < 0$ (see remark 1.11 (26)). The definition 1.3.5 (27) and remark 1.13 (28) tells us that this space has also the scale curvature bounded from above respectively below by any $k_+ > \frac{9}{4}D^2K_+$ and $k_- < \frac{9}{4}D^2K_-$.

In that case, the scale curvature conditions are weaker than the Alexandrov conditions.

Example 1.9 Consider the vector space V endowed with a norm and suppose that its scale curvature is bounded.

In example 1.7 (26) we have seen that, if the norm is not defined by a scalar product, the vector space cannot have bounded Alexandrov curvature. The key argument was the incompatibility of bounded curvature through norm re-scaling. This argument does not apply any more: the last curvature condition is a better candidate to construct Finsler structures on length spaces.

The scale invariance argument used in example 1.7 (26) to show that, if the norm is not defined by a scalar product, cannot have bounded Alexandrov curvature does, by the scale invariance of the curvature condition, not apply any more: the last curvature condition is a better candidate to construct Finsler structures on length spaces.

In subsection 1.3.5 (33), we will prove that for strongly convex norms (see definition 1.3.15 (33)), V really has a bounded scale curvature.

1.3.3 Behaviour of geodesics in spaces with bounded scale curvature

Spaces with Alexandrov curvature bounds have several nice properties like, for example, the absence of branch points in geodesics. A number of results extend straight ahead to spaces with bounded scale curvature:

Definition 1.3.7 Let (X, d) be a length space.

$x \in X$ is called a branch point, if there are two segments $\gamma, \gamma' : [a, b] \rightarrow X$ and a point $t_0 \in]a, b[$ so that:

1. $\gamma_{t_0} = \gamma'_{t_0} = x$,
2. $\gamma|_{[a, t_0]} \equiv \gamma'|_{[a, t_0]}$,
3. $\gamma|_{\mathcal{U}} \not\equiv \gamma'|_{\mathcal{U}}$ for every neighbourhood \mathcal{U} of t_0 .

Example 1.10 Consider a standard 2-dimensional cone whose vertex angle is larger than 2π . Its vertex is a branch point: a segment ending at the vertex can be extended to the right in infinitely many ways.

An other example is the double cone (see figure 1.3): P is a branch point. The geodesics from A to C respectively to C' have a common trace until P .

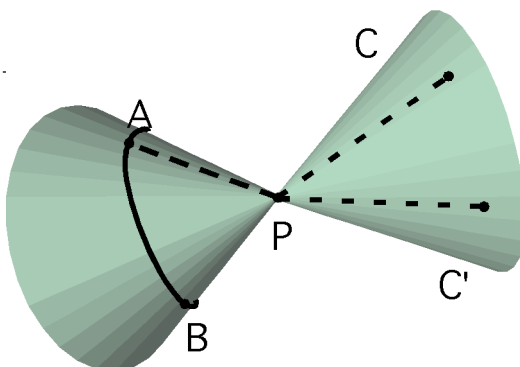


Figure 1.3: An example of lens (full curves) and of branch point (dotted curves).

Definition 1.3.8 Let (X, d) be a length space.

We say that there is a lens between p_1 and p_2 if there are two segments between p_1 and p_2 with different graphs.

Example 1.11 Consider a 2-dimensional cone whose vertex angle is smaller than 2π . Any neighbourhood of the vertex contains lenses, see figure 1.3.

Proposition 1.3.9

Every length space (X, d) whose scale curvature is bounded from below is free of branch points.

For the case of a length space with bounded Alexandrov curvature, the proof is found in [24].

Proof:

Suppose x is a branch point and γ, γ' are the segments with the properties given in definition 1.3.7 (29).

We call H the point $\gamma_{t_0} = \gamma'_{t'_0}$ and choose A on the segment γ between H and γ_b , C on γ' between H and γ'_b and B between $\gamma_a = \gamma'_a$ and H in such a way that $\epsilon := \overline{AH} = \overline{BH} = \overline{CH}$ (see figure 1.4). The third condition for branch points ensures that ϵ can be chosen such that $A \neq C$. Hence $\epsilon' := \overline{AC} \in]0, 2\epsilon]$, $\overline{AB} = \overline{BC} = 2\epsilon$ and $\overline{AH} = \epsilon$.

Let k be the lower scale curvature bound. We may assume without loss of generality (see remark 1.10 (26)) that $k < 0$. Hence the isometric embedding $(\tilde{A}, \tilde{B}, \tilde{C})$ of (A, B, C) in $M_{k/\rho^2}(A, B, C)$ exists and is a non-degenerated triangle.

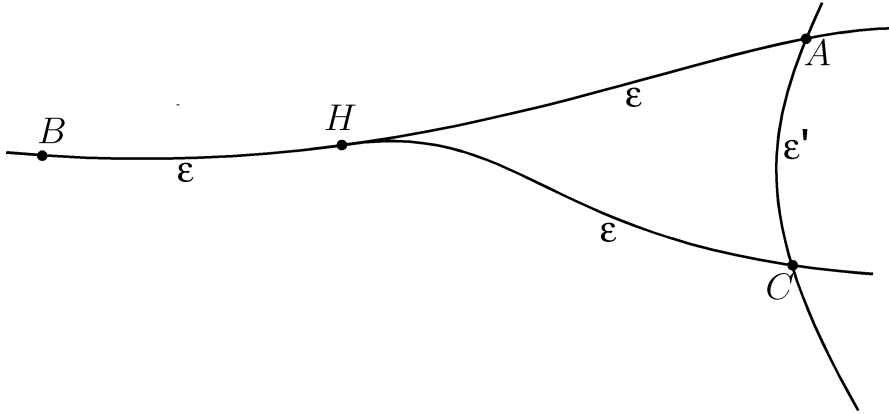


Figure 1.4: Illustration of the proof of proposition 1.3.9.

In accordance with the embedding construction, \tilde{H} lies on the segment \tilde{B} to \tilde{C} . This implies, by triangle inequality, that $\overline{\tilde{A}\tilde{H}} > \overline{\tilde{A}\tilde{C}} - \overline{\tilde{C}\tilde{H}} = \overline{AC} - \overline{CH} = \epsilon$. But $\overline{AH} = \epsilon$ by construction, giving a contradiction with the scale curvature bound from below. \square

Proposition 1.3.10

Every length space (X, d) whose scale curvature is bounded above is free of lenses.

Proof:

Let us suppose that there were a lens between p_1 and p_2 given by the segments γ and γ' . Without loss of generality, we may suppose that they are length parameterized. Let l be the distance between p_1 and p_2 . We may suppose that $\gamma(\frac{l}{2}) \neq \gamma'(\frac{l}{2})$ (if not, there is a restriction of γ and γ' forming a lens and fulfilling the condition).

Let $a = \gamma$, b be the left restriction of γ' to the length of $\frac{l}{2}$ and c the right restriction of γ' of same length. The three geodesics (a, b, c) form a triangle.

For any upper scale curvature bound k , $\mathcal{M}_k(l(a), l(b), l(c)) = 0$ because of $l(a) = l(b) + l(c)$. But, by construction, $m_X(a, b, c) = \gamma(\frac{l}{2}) \gamma'(\frac{l}{2}) > 0$.

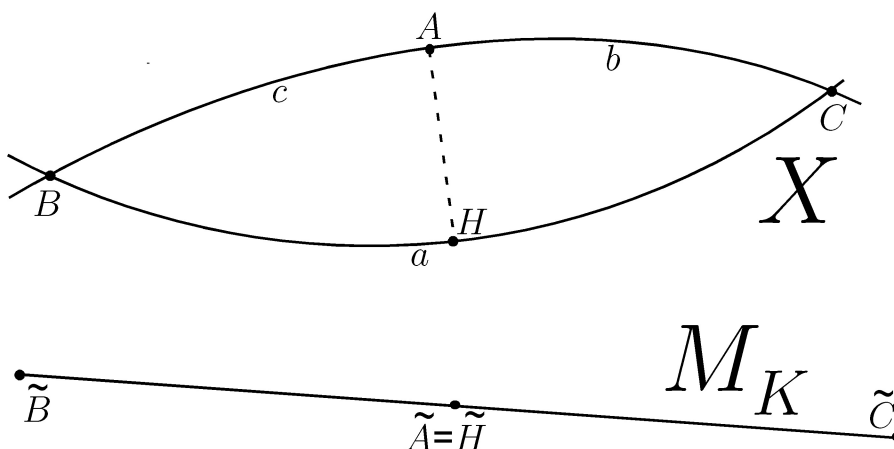
This contradicts the upper scale curvature bound condition in definition 1.3.5 (27) proving proposition 1.3.10. \square

Remark 1.18 S^n is not free of lenses. there are multiple geodesics between poles. Hence, its scale curvature is not bounded above. It is, however, bounded locally.

Proposition 1.3.11

Every closed ball in a length space (X, d) whose scale curvature is bounded from above is strictly convex.

Remark 1.19 This implies that balls in such spaces are convex.

Figure 1.5: Shape of both triangles in X and M_k .**Proof:**

It is enough to prove that, for any segment in a given ball, its midpoint lays in the same ball: Consider a ball $B(A, r)$ in X , two points $B, C \in B(A, r)$ and segments a, b, c between respectively B and C , between A, C and A, B . Because of the upper curvature condition applied on the triangle (a, b, c) , $m_X(a, b, c) \leq M_k(l(a), l(b), l(c))$.

As all balls (open or closed) in M_K are strictly convex, the above inequality implies the same on $B(A, r)$. \square

1.3.4 Quadratic convexity

The concept of quadratic convexity is key in many proofs in the upcoming chapters. This short section has been inserted here to introduce a formal and useful

Definition 1.3.12 Let (X, d) be a length space and $p \in X$. $\delta_p : B_p \rightarrow \mathbb{R}_+$ is the mapping $q \mapsto \overline{pq}$.

Definition 1.3.13 Let (X, d) be a length space.

The space is said to be quadratically convex if there is a $C > 0$ such that:

for every points $p \neq q$ and every $\gamma \in \Upsilon(B_q(\frac{1}{2}\overline{pq}))$ with $\gamma_0 = q$:

$$\delta_p(\gamma_t) = \delta_p(\gamma_0) + t \sin \phi + t^2 \cos^2 \phi \frac{R_\gamma(t)}{\delta_p(\gamma_0)}$$

for an angle $\phi \in]-\pi, \pi]$ and with the rest term $R_\gamma(t)$ confined in $[C^{-1}, C]$.

Remark 1.20 As $R_\gamma(t) < C$, $\delta_p(\gamma_t)$ is obviously $\mathcal{C}^{1,1}$ in quadratically convex spaces.

Remark 1.21 The choice of $\sin \phi$ as linear term in the development of $\delta_p(\gamma_t)$ is not a restriction in itself. Its existence is rather a consequence of remark 1.20 :

Consider the triangle $(p, \gamma_0, \gamma_\epsilon)$. Triangles inequalities imply that

$$\delta_p(\gamma_0) - |\epsilon| \leq \delta_p(\gamma_t) \leq \delta_p(\gamma_0) + |\epsilon|$$

such that

$$-1 \leq \frac{\delta_p(\gamma_t) - \delta_p(\gamma_0)}{\epsilon} \leq 1$$

for any $\epsilon \neq 0$. As $\delta_p(\gamma_t)$ is $\mathcal{C}^{1,1}$, the limit $t \rightarrow 0$ is well defined, hence

$$\left. \frac{d}{dt} \delta_p(\gamma_t) \right|_{t=0} \in [-1, 1].$$

This ensures the existence of the claimed angle.

Proposition 1.3.14

Every closed ball in a quadratically convex space (X, d) is strictly convex.

Proof:

Consider a point $p \in X$ and two points p_-, p_+ equidistant from p and connected by a segment γ . By compactness of segments, there is a point q maximizing the distance $\overline{p\gamma_t}$ on the segment. Without restriction of generality, we assume that $q = \gamma_0$.

If q were different from p_- and p_+ we would have, by symmetry, that $\frac{d}{dt} \delta_p(\gamma_t)_{t=0} = 0$. Hence,

$$\delta_p(\gamma_t) = \delta_p(\gamma_0) + t^2 \frac{R_\gamma(t)}{\delta_p(\gamma_0)}.$$

As $0 < C^{-1} < R_\gamma(t)$, $\overline{p\gamma_t}$ has a strict local minimum in $t = 0$ such that q must be either p_- or p_+ . This implies convexity of balls. \square

1.3.5 Normed vector spaces and scale curvature bounds

In this subsection, we will define vector spaces with regular convex norms and show that their scale curvature is bounded from below and above. More than just an example, it will be a model within the more general framework of Finsler spaces (see theorem 3.3.5 (82) and lemma 3.3.4 (81)).

Definition 1.3.15 *Let $(V, \|\cdot\|)$ be a finite dimensional normed vector space.*

The norm is said to be regular if:

1. *it is $\mathcal{C}^{1,1}$ on $V \setminus \{0\}$,*
2. *there is a $C > 0$ such that for every unitary v and w $f_{v,w}(\epsilon) := \frac{d}{d\epsilon} \|v + \epsilon w\|$:*

$$f_{v,w}(0) = 0 \quad \implies \quad \lim_{\epsilon \rightarrow 0} \left| \frac{f_{v,w}(\epsilon)}{\epsilon} \right| \geq C.$$

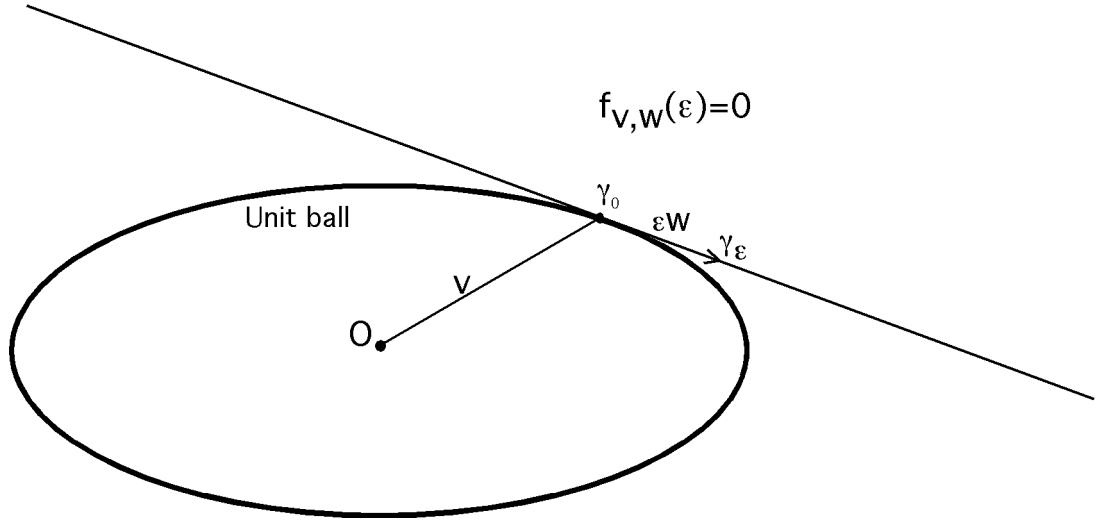


Figure 1.6: Illustration of the construction of point 2.

Example 1.12 Let $(\mathbb{R}^n, \|\cdot\|)$ be a \mathcal{C}^2 -normed space with H_v the Hessian matrix of $\|v\|^2$, a quadratic form on $\mathbb{R}^n \setminus \{0\}$. By convexity of norms, this form has to be positive defined everywhere.

Condition 2 is equivalent to require H_v to be everywhere strictly positive defined.

Counter-example 1.13 Let $(\mathbb{R}^n, \|\cdot\|_\beta)$ be a normed space with

$$\|(x_1, x_2, \dots, x_n)\| := \left(\sum_{i=1}^n |x_i|^\beta \right)^{\frac{1}{\beta}}$$

where $\beta > 2$ and let (e_1, \dots, e_n) be the canonical basis.

It is easy to see that in that case H_{e_i} is not strictly positive defined.

This implies that condition 2 in definition 1.3.15 (33) is not fulfilled for $v = e_i$.

Lemma 1.3.16 Let $(V, \|\cdot\|)$ be a finite dimensional vector space with regular norm. V is quadratically convex.

Remark 1.22 Consider two unitary vectors v and w and let γ be a segment such that $\gamma_0 = v$ and $\dot{\gamma}_0 = w$. For readability, we will write $R_{v,w}(t)$ instead of $R_\gamma(t)$.

The translation invariance of geometry in vector spaces, allows to definition 1.3.13 (32) with $p = 0$. In that case, $\delta_0(\gamma_\epsilon) = \|v + \epsilon w\|$. Moreover, norm homogeneity implies:

$$\|v + \epsilon w\| = \|v\| \left\| \frac{v}{\|v\|} + \left(\epsilon \frac{\|w\|}{\|v\|} \right) \frac{w}{\|w\|} \right\|$$

such that we can restrict ourselves to unitary v and w .

For non-unitary v and w , the condition of definition 1.3.13 (32) resumes to:

$$\|v + \epsilon w\| = \|v\| + \epsilon \|w\| \sin \phi + \epsilon^2 \frac{\|w\|^2}{\|v\|} \cos^2 \phi R_{\frac{v}{\|v\|}, \frac{w}{\|w\|}} \left(\epsilon \frac{\|w\|}{\|v\|} \right).$$

where $|\epsilon| \leq \frac{1}{2} \frac{\|v\|}{\|w\|}$ and $R_{v,w}(\epsilon) := R_{\frac{v}{\|v\|}, \frac{w}{\|w\|}} \left(\epsilon \frac{\|w\|}{\|v\|} \right)$.

Proof of lemma 1.3.16:

As the norm is $\mathcal{C}^{1,1}$ a Taylor development as claimed in definition 1.3.13 (32) exists. The non-trivial issues are the conditions on $R_{v,w}(\epsilon)$.

Referring to remark 1.22 (34) we restrict ourselves to unitary vectors v and w .

There is a decomposition $w = w_{\parallel} + w_{\perp}$ such that w_{\parallel} is collinear to v and w_{\perp} fulfills $\frac{d}{d\epsilon} \|v + \epsilon w_{\perp}\| |_{\epsilon=0} = 0$. With these notations, we now prove that:

▷ *The sign of $\sin \phi$ is positive, respectively negative, when w_{\parallel} and v are parallel, respectively anti-parallel and $\|w_{\parallel}\| = |\sin \phi|$:* Consider:

$$\frac{d}{d\epsilon} \|v + \epsilon w\|_{\epsilon=0} = \frac{\partial}{\partial \epsilon_1} \|v + \epsilon_1 w_{\parallel} + \epsilon_2 w_{\perp}\|_{\epsilon_1=\epsilon_2=0} + \frac{\partial}{\partial \epsilon_2} \|v + \epsilon_1 w_{\parallel} + \epsilon_2 w_{\perp}\|_{\epsilon_1=\epsilon_2=0}$$

By construction

$$\frac{\partial}{\partial \epsilon_2} \|v + \epsilon_1 w_{\parallel} + \epsilon_2 w_{\perp}\|_{\epsilon_1=\epsilon_2=0} = 0$$

and

$$\frac{\partial}{\partial \epsilon_1} \|v + \epsilon_1 w_{\parallel} + \epsilon_2 w_{\perp}\|_{\epsilon_1=\epsilon_2=0} = \pm \|w_{\parallel}\|$$

depending if w_{\parallel} and v are parallel or anti-parallel.

On the other hand, we know that $\frac{d}{d\epsilon} \|v + \epsilon w\|_{\epsilon=0} = \sin \phi$, hence $\|w_{\parallel}\|$ must be equal to $|\sin \phi|$. \triangle

▷ *The claim holds if $w = w_{\perp}$ and $\epsilon \in [-4, 4]$:* By construction of w_{\perp} , $\sin \phi = 0$ such that the claimed relation reduces to

$$\|v + \epsilon w_{\perp}\| = 1 + \epsilon^2 R_{v,w_{\perp}}(\epsilon)$$

or

$$R_{v,w_{\perp}}(\epsilon) = \frac{\|v + \epsilon w_{\perp}\| - 1}{\epsilon^2}$$

As the norm is $\mathcal{C}^{1,1}$, $R_{v,w_{\perp}}(\epsilon)$ is continuous in all its parameters. As v , w and ϵ run over compact sets, $R_{v,w_{\perp}}(\epsilon)$ is uniformly bounded above by the norm's Lipschitz constant. From the second condition in definition 1.3.15 (33), we conclude that $R_{v,w_{\perp}}(\epsilon)$ is also uniformly and positively bounded from below. \triangle

▷ *There is a constant $C > 0$ such that $C^{-1} \cos^2 \phi \leq \|w_{\perp}\|^2 \leq C \cos^2 \phi$:* Consider first the case where $|\sin \phi| \leq \frac{1}{2}$.

Triangle inequalities applied to $\|w_{\perp}\| = \|w - w_{\parallel}\|$ yields

$$1 - |\sin \phi| \leq \|w_{\perp}\| \leq 1 + |\sin \phi|$$

or, under our assumption

$$\frac{1}{2} \leq \|w_\perp\| \leq \frac{3}{2}.$$

At the same time, our assumption implies that $\cos^2 \phi \geq \frac{3}{4}$, hence the constant $C = 2$ will do the job.

In case $|\sin \phi| \geq \frac{1}{2}$, we consider the relation $\|w\| = \|w_\parallel + w_\perp\| = 1$ and observe that

$$\|w_\parallel + w_\perp\| = \|w_\parallel\| \cdot \left\| v + \frac{\|w_\perp\|}{\|w_\parallel\|} \cdot \frac{w_\perp}{\|w_\perp\|} \right\|.$$

As $\|w_\perp\| \leq 1 + |\sin \phi| \leq 2$, we have $\frac{\|w_\perp\|}{|\sin \phi|} \leq 4$, such that we can apply the result of last section to obtain

$$\|w\| = 1 = |\sin \phi| \left(1 + \frac{\|w_\perp\|^2}{\sin^2 \phi} R_{v, \frac{w_\perp}{\|w_\perp\|}} \left(\frac{\|w_\perp\|}{\sin \phi} \right) \right)$$

or

$$|\sin \phi| - \sin^2 \phi = \|w_\perp\|^2 R_{v, \frac{w_\perp}{\|w_\perp\|}} \left(\frac{\|w_\perp\|}{\sin \phi} \right).$$

Using the relation $\cos^2 \phi = 1 - \sin^2 \phi$, we have that

$$\frac{|\sin \phi|}{1 + |\sin \phi|} \cos^2 \phi = \|w_\perp\|^2 R_{v, \frac{w_\perp}{\|w_\perp\|}} \left(\frac{\|w_\perp\|}{\sin \phi} \right).$$

But, by our assumption on $|\sin \phi|$, $\frac{1}{3} \leq \frac{|\sin \phi|}{1 + |\sin \phi|} \leq \frac{1}{2}$. On the other hand, the rest term R is also bounded above and below by positive values, such that the claim finally holds. \triangle

Considering previous results and remark 1.22 (34), we observe that:

$$\begin{aligned} \|v + \epsilon w\| &= \|(v + \epsilon w_\parallel) + \epsilon w_\perp\| \\ &= \|v + \epsilon w_\parallel\| + \epsilon^2 \frac{\|w_\perp\|^2}{\|v + \epsilon w_\parallel\|} R_{v, \frac{w_\perp}{\|w_\perp\|}} \left(\epsilon \frac{\|w_\perp\|}{\|v + \epsilon w_\parallel\|} \right) \\ &= 1 + \epsilon \sin \phi + \epsilon^2 \cos^2 \phi \frac{\|w_\perp\|^2}{(1 + \epsilon \sin \phi) \cos^2 \phi} R_{v, \frac{w_\perp}{\|w_\perp\|}} \left(\epsilon \frac{\|w_\perp\|}{1 + \epsilon \sin \phi} \right). \end{aligned}$$

In that latter form, the lemma is obviously true in case $\sin \phi \geq 0$. If $\sin \phi < 0$, v, w_\parallel are anti-parallel. Using the fact that $\|v + \epsilon w\| = \|v - \epsilon(-w)\|$ and that v and $-w$ are now parallel, we can turn the case into one where $\sin \phi > 0$. \square

Proposition 1.3.17

Let $(V, \|\cdot\|)$ be a finite dimensional vector space with a regular norm. Then $(V, \|\cdot\|)$ is a length space with bounded scale curvature.

We need the following technical lemma, whose proof is postponed to the appendix, section B on page 89:

Lemma 1.3.18 *Let $(V, \|\cdot\|)$ be a finite dimensional vector space with a regular norm.*

Call $\Delta \subseteq V \times V \times V$ the set of all degenerated triangles in V with half-perimeter 1.

There is a neighbourhood \mathcal{U}_Δ of Δ and $K_- \leq K_+ < \pi^2$ such that K_- is a lower and K_+ an upper bound for the scale curvature of all triangles in \mathcal{U}_Δ .

Proof of proposition 1.3.17:

Let (A, B, C) be a triangle in V and H be the midpoint between B and C with non-zero half-perimeter. We are looking for constants $k_- \leq k_+ < \pi^2$ such that they are lower respectively upper scale curvature bounds for any triangle (A, B, C) as described before.

Using the fact that translations are isometries in normed spaces, we can restrict ourselves to triangles of the form $(0, B, C)$. As, by definition of scale curvature bound (see definition 1.3.5 (27)), if we the scale curvature of $(0, B, C)$ is bounded, so is $(0, \lambda B, \lambda C)$ (if $\lambda \neq 0$) such that we can further restrict ourselves to triangles with $\rho(0, B, C) = 1$. Under those restrictions, the configuration space for $(0, B, C)$ is a compact C . By lemma 1.3.18, there is an open neighbourhood \mathcal{U}_Δ containing all degenerated triangles such that all triangles in it have their scale curvature bounded.

The set $C \setminus \mathcal{U}_\Delta$ is still compact and does not contain any degenerated triangles. Point iv) of lemma A.3.2 (87) tells us that, for non degenerated triangles, the relation

$$\Psi \left(\|B - C\|, \|C\|, \|B\|, \frac{1}{2} \|B + C\| \right) (k) = 0$$

defines an implicit continuous function k as function of triangles in $C \setminus \mathcal{U}_\Delta$.

Hence, there is a minimum and a maximum for k such that scale curvature is bounded on $C \setminus \mathcal{U}_\Delta$.

Being bounded on both \mathcal{U}_Δ and $C \setminus \mathcal{U}_\Delta$, it is bounded for every triangle in V . \square

1.4 The geometry of scale bounded spaces

1.4.1 Locally bounded curvature

If we are only interested in local properties of spaces, the curvature criteria given in definition 1.3.5 (27) are far too strong, so we give the following local version:

Definition 1.4.1 (Local scale curvature bounds) *Let (X, d) be a metric space that is locally a length space.*

Let (a, b, c) be a triangle in X , ρ its half perimeter and $(\tilde{a}, \tilde{b}, \tilde{c})$ the respectively embedding with factor ρ in M_K .

X is said to have scale curvature locally bounded from below (respectively above) if for any point $x \in X$, there is a neighbourhood \mathcal{U} of x and $k < \pi^2$ such that for

any choice of triangle (a, b, c) in \mathcal{U} we verify $m_X(a, b, c) \geq m_k(\tilde{a}, \tilde{b}, \tilde{c})$ (respectively $m_X(a, b, c) \leq m_k(\tilde{a}, \tilde{b}, \tilde{c})$).

Example 1.14 From remark 1.15 (28), we immediately conclude that the scale curvature of a 2-sphere is locally bounded.

Counter-example 1.15 For a 2 dimensional cone with vertex smaller (respectively larger) than 2π the scale curvature cannot be bounded from above (respectively below).

This follows as contraposition to proposition 1.4.3 (respectively proposition 1.4.2).

Remark 1.23 If the space is a local length space with its curvature locally bounded from below (above), its scale curvature is also locally bounded from below (above). (See remark 1.17 (29))

Both propositions of the preceding paragraph can be restated in a local version:

Proposition 1.4.2

Every local length space (X, d) whose scale curvature is locally bounded from below is free of branch points.

Proposition 1.4.3

Every local length space (X, d) whose scale curvature is locally bounded from above has, in any point, a neighbourhood free of lenses.

Proof:

These result follow immediately from proposition 1.3.9 (30). and proposition 1.3.10 (31) \square

Proposition 1.4.4

For any point in a length space (X, d) whose scale curvature is locally bounded from above there is, for any point, a neighbourhood \mathcal{U} such that closed balls lying in \mathcal{U} are strictly convex.

Proof:

See proof of proposition 1.3.11 (31). \square

In [7] §12 Berestovskij and Nikolaev define their "Spaces with bounded curvature" as being spaces with curvature locally bounded from both sides, as stated in definition 1.4.1 (37) but add the local extensibility (in the sense of definition 1.2.2 (16)) of geodesics as well as local compactness.

We need similar conditions for our Finsler generalization and define:

Definition 1.4.5 Let (X, d) be a space with intrinsic metric.

X is called a scale bounded space if:

- i) X is locally compact and arc-wise connected,

- ii) X has its scale curvature locally bounded from below and above,
- iii) any geodesic in X is extensible.

Remark 1.24 As (X, d) is a space with intrinsic metric and because of point i), we know, by theorem 1.2.18 (21), that X is locally a length space such that the definition definition 1.3.5 (27) of the curvature condition mentioned in point ii) has a meaning.

Remark 1.25 A scale bounded space is a G -space [12].

1.4.2 Local geometry in scale bounded spaces

Estimations of various lengths in triangles as functions of others are key to the understanding of the geometry of scale bounded spaces. In scale bounded spaces, such an estimation is given by the scale curvature conditions: they tell us how the median length behaves as function of the edges lengths of a triangle. Some other estimations can be derived from this first example.

We first give two definitions for working with local objects:

Definition 1.4.6 Let (X, d) be a scale bounded space and $p \in X$.

With B_p we will denote an open ball centered in p such that:

- i) $\overline{B_p}$ is compact,
- ii) the scale curvature of $\overline{B_p}$ seen as restriction of X is bounded from below and above.

With B_p^* we will denote $B_p \setminus \{p\}$ and ι_p is the least upper bound for the possible radii of B_p

Remark 1.26 The existence of such balls is ensured by definition 1.4.5 (38).

Remark 1.27 There are many balls fulfilling the above conditions. B_p is to be thought as a notation. If a property is said to be true on B_p , we mean that it is true for any possible choice of B_p .

Remark 1.28 By proposition 1.4.3 (38) and proposition 1.4.4 (38) any triple of points in B_p defines a unique triangle whose edges lie entirely in B_p .

Geodesics within B_p have nice properties:

Proposition 1.4.7

Let (M, d) be a scale bounded space and $p \in M$. Any segment starting from $q \in B_p$ can be extended on the right to a segment whose length is at least $\iota_p - \overline{pq}$.

In fact, only the scale curvature bound above will be needed to prove the proposition.

Proof:

Let us choose a length parameterized segment from $q \in B_p$ to $q' \in B_p \setminus \{q\}$. As geodesics can be extended, the maximal domain of definition for an extension of this segment must be an open interval on the right², in other words the segment can be extended to a geodesic $\gamma : [0, l[\rightarrow M$ where $0 < l \leq +\infty$.

▷ Let r be the radius of B_p . Length l is larger than $r - \overline{pq}$: If l is $+\infty$ the assertion is trivial, if l is finite, the extension must have points outside B_p . If not, $\lim_{t \rightarrow l} \gamma_t$ would be defined and lie in the closure of B_p such that a further extension would be possible giving a contradiction to the maximality of the extension we constructed. \triangle

As $\gamma|_{[0, r - \overline{pq}]}$ is now well defined, the truth of the following claim ends the proof:

▷ $\gamma|_{[0, r - \overline{pq}]}$ is a segment: There is a real $0 < s < r - \overline{pq}$ and an integer $n > 0$ such that $\gamma|_{[0, s]}$ and $\gamma|_{[s(1-2^{-n}), s(1+2^{-n})]}$ are segments.

For more convenience, we call $A := \gamma_{s(1+2^{-n})}$, $B := \gamma_s$ and $C_m := \gamma_{(1-2^{-m})s}$ for $0 \leq m \leq n$. As B_p is convex (see proposition 1.3.11 (31)), (A, B, C) is a small triangle. By construction, we observe that $\overline{AB} + \overline{BC_n} = \overline{AC_n}$ such that the triangle (A, B, C_n) is degenerated.

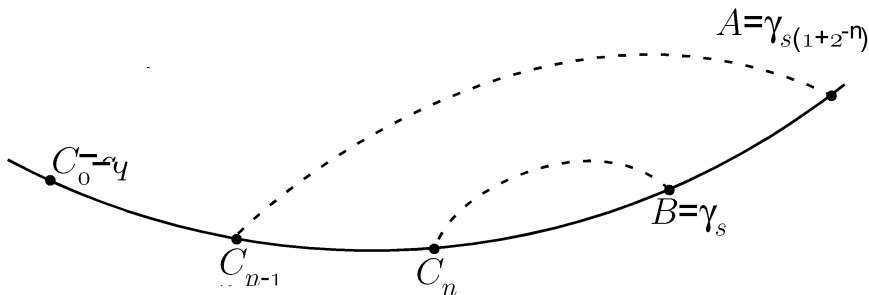


Figure 1.7: The construction of proposition 1.4.7 (39)

On the other hand, C_n is the middle point between B and C_{n-1} . Applying the scale upper bound criteria to the triangle (A, B, C_{n-1}) , the triangle (A, B, C_n) can only be degenerated if (A, B, C_{n-1}) also was. Hence $\overline{AB} + \overline{BC_{n-1}} = \overline{AC_{n-1}}$ what in turn means that $\gamma|_{[s(1-2^{-n+1}), s(1+2^{-n})]}$ is a segment.

This construction can be repeated inductively over the index n until $C_0 = q$ getting the relation $\overline{AB} + \overline{BC_0} = \overline{AC_0}$ so that $\gamma|_{[0, s(1+2^{-n})]}$ is a segment.

As this is true for every $0 < s < r - \overline{pq}$, $\gamma|_{[0, r - \overline{pq}]}$ is a segment itself. \triangle

\square

Proposition 1.4.8

Let (M, d) be a scale bounded space, $p \in M$. B_p as restriction of M is quadrat-

²We will assume here that the domain of definition of a geodesic is not a closed interval.

ically convex.

The proof of the following rather technical lemma is postponed to the appendix, section C (page 97).

Lemma 1.4.9 *Let (M, d) be a scale bounded space, $p \in M$. For every segment $\gamma \in \Upsilon_=(B_p^*)$, the application $\delta_p(\gamma_t)$ is $\mathcal{C}^{1,1}$.*

Proof of proposition 1.4.8:

In order to simplify notation, we will use $\delta(t)$ as abbreviation for $\delta_p(\gamma_t)$ and define $I_\gamma := \{t | \gamma_t \in \overline{B_p}\}$.

▷ *The angle ϕ exists.* Consider the triangles $\Delta_t := (p, \gamma_0, \gamma_t)$. Due to triangle inequalities, $-t \leq \delta(t) - \delta(0) \leq t$. It follows that

$$\frac{\delta(t) - \delta(0)}{t} \in [-1, 1]$$

As, by lemma 1.4.9, the limit $t \rightarrow 0$ exists, this limit must also lie in $[-1, 1]$. Hence, there must be a $\phi \in [-\pi/2, \pi/2]$ with the claimed property. \triangle

▷ *I_γ is a closed interval.* $\delta(t)$ is obviously continuous such that I_γ is closed as $\overline{B_p}$ is. On the other hand, by proposition 1.3.11 (31) and remark 1.19 (31), $\overline{B_p}$ is geodesically convex, hence I_γ and interval. \triangle

Let k_+ and k_- bound the scale curvature of B_p . By remark 1.14 (28), we can suppose without restriction that $k_+ > 0$ and $k_- < 0$. In order to apply the curvature condition of definition 1.3.5 (27) to the triangles Δ_t we have to embed them into M_{k_\pm/ρ_t} (see remark 1.13 (28)).

As $k_+ < \pi^2$ (see remark 1.12 (27)), there is a $k'_+ \in]k_+, \pi^2[$. Let $\eta := \frac{k'_+}{k_+} > 1$ such that $k_+/\rho_t \leq k'_+/\rho_0$ is true as long as $|t| + \delta(t) \leq \eta \delta(0)$.

A similar consideration for a lower curvature bound leads to a choice for $k'_- < k_-$. Hence, if the comparison criteria for Δ_t , with t restricted as above, is fulfilled according to definition 1.3.5 (27) for k_+ and k_- then the Alexandrov curvature criteria of definition 1.3.4 (26) is also fulfilled with $K_+ := k'_+/\rho_0$ and $K_- := k'_-/\rho_0$ as upper respectively lower curvature bound.

Consider a point \tilde{p} and a length parameterized path $\tilde{\gamma}_t$ in M_{K_+} such that $\overline{\tilde{p} \tilde{\gamma}_0} = \overline{p \gamma_0} = \delta(0)$ and that

$$\left. \frac{d}{dt} \overline{\tilde{p} \tilde{\gamma}_t} \right|_{t=0} = \sin \phi.$$

Define $\delta_+(t) := \overline{\tilde{p} \tilde{\gamma}_t}$. Similarly, we define $\delta_-(t)$ in M_{K_-} .

If $\phi \in]-\pi/2, \pi/2[$ and $\delta_+(t) < \delta_-(t)$ then proposition 1.4.8 (40) ensures that $\delta_+(t) < \delta(t) < \delta_-(t)$ is true in a neighbourhood of $t = 0$. On the other hand, if there is a $t_0 \neq 0$ such that $\delta_+(t_0) = \delta(t_0)$, $t_0 \notin I_\gamma$, otherwise, we would have a contradiction with the embedding of Δ_{t_0} into M_{K_+} .

As the same argument applies if $\delta_-(t_0) = \delta(t_0)$, the inequalities $\delta_+(t) < \delta(t) < \delta_-(t)$ are true as long as $|t| > \eta \delta(0)$.

If $\phi = \pm\pi/2$ then $\delta(t)$ is degenerated such that $\delta_+(t) = \delta(t) = \delta_-(t)$ for every t , such that, by connectedness of I_γ :

$$\delta_+(t) \leq \delta(t) \leq \delta_-(t) \quad \forall t \in I_\gamma.$$

The claim is known to be true in all spaces M_K , hence the above relation implies the claim to be true on X . \square

We close this section with a useful invariance property of the derivative of $\delta_p(\gamma_t)$:

Corollary 1 *Let (M, d) be a scale bounded space, $p \in M$ and three points $q, q', A \in B_p$ lying on a common segment. For every path $\gamma \in \Upsilon_=(B_p^*)$ with $\gamma_0 = A$:*

$$\left. \frac{d}{dt} \delta_q(\gamma_t) \right|_{t=0} = \mp \left. \frac{d}{dt} \delta_{q'}(\gamma_t) \right|_{t=0}$$

where the sign is negative if A is between q and q' and positive otherwise.

Proof:

Suppose A is between (resp. not between) q and q' . From last corollary, we know that $\delta_q(\gamma_t)$ and $\delta_{q'}(\gamma_t)$ are both at least \mathcal{C}^1 in t around 0. Hence $f(t) := \delta_q(\gamma_t) \pm \delta_{q'}(\gamma_t)$ has a defined derivative in 0.

On the other hand, as $A = \gamma_0$, q and q' are on a same segment, triangle inequalities impose $f(t)$ to have a global minimum (resp. maximum) in $t = 0$ such that $\frac{d}{dt} f(t) = 0$. \square

Chapter 2

Analysis on scale bounded spaces

2.1 Introduction

In Riemannian spaces a scalar function can be locally characterized by its value and derivative specified either by a 1-form element of the co-tangent space in that point or by its gradient, an element of the tangent space.

Though both notions are not equivalent, they are in a linear one-to-one correspondence.

In a scale bounded space, an analogous correspondence can be constructed. This is the subject of the present chapter.

To define differentiability of scalar functions, we first construct in section 2.2 (44) a bundle $C(M)$ (see definition 2.2.2 (44)) that will play a similar role as the tangent bundle in Riemannian spaces.

This allows us to define, in section 2.3 (55), what we mean by derivation along sections of $C(M)$ of scalar functions (see definition 2.3.3 (55)). The distance function δ_p turns out to be smooth (see proposition 2.3.6 (56)).

It is well known that the tangent vector bundle of a manifold M is isomorphic to the dual of $\mathfrak{m}/\mathfrak{m}^2$ where \mathfrak{m} is the maximal ideal of the locally ringed space of real-valued differentiable functions on M [21], [19].

In a similar way we construct a 1-form sheaf $V_p(M)$ in section 2.4 (58) that will be a substitute for a co-tangent bundle on our scale bounded spaces.

The section 2.5 is devoted to the construction of a gradient as an element of $C_p(M)$ (see definition 2.5.1 (63)). We will prove that we also have an one-to-one correspondence between $C_p(M)$ and $V_p(M)$ (see proposition 2.5.5 (69)).

This last section ends with definition of dimension of scale bounded spaces (see theorem 2.5.3 (64) and corollaries).

The 1-form-gradient correspondence and finite dimension of $V_p(M)$ are the key elements used in chapter 3 to construct a distance compatible Finsler structure on scale bounded spaces.

2.2 The tangent cone

2.2.1 The space of directions

The first step towards the differential structure is the definition of the space of directions as given in [2], [22] and [11].

Definition 2.2.1 *Let (M, d) be a scale bounded space and p a point in M .*

We define the directions at p as the germs of elements in $\mathcal{G}^p(\mathcal{U})$ (the set of geodesics issued from p) where \mathcal{U} is a neighbourhood of p .

More formally, as:

$$S_p(M) := \varinjlim_{\mathcal{U} \ni p} \mathcal{G}^p(\mathcal{U}).$$

$S(M)$ is the disjoint union $\bigsqcup_{p \in M} S_p(M)$ and is called the tangent direction space.

Remark 2.1 As scale bounded spaces have no branch points (see proposition 1.4.2 (38)), the extension of geodesics are unique such that if two segments define the same germ, one is a restriction of the other.

Remark 2.2 Let $v \in S_p(M)$ be a direction at p represented by γ . $-v \in S_p(M)$ is the direction represented by $-\gamma$ (see definition 1.2.6 (17)). By the local extensibility condition of scale bounded spaces, this opposite direction always exists.

Remark 2.3 There is a canonical projection from $\mathcal{G}(M)$ onto $S(M)$ sending a path γ to its germ at the beginning point. We will denote this germ by $[\gamma]$.

2.2.2 The $C(M)$ bundle over M

As constructed, for example in [11], we define the tangent bundle of M as the bundle of the cones over $S_p(M)$ constructed in all points $p \in M$. To this end, we consider \approx , a relation of equivalence defined over $S(M) \times \mathbb{R}_+$ by:

$$(s_1, t_1) \approx (s_2, t_2) \iff (s_1, t_1) = (s_2, t_2) \text{ or } (t_1 = t_2 = 0 \text{ and } \pi(s_1) = \pi(s_2)).$$

Definition 2.2.2 *Let (M, d) be a scale bounded space. Then*

$$C(M) := S(M) \times \mathbb{R}_+ / \approx$$

equipped with the projection $\bar{\pi} : C(M) \longrightarrow M$ sending (s, t) to $\pi(s)$ is a bundle over M .

The norm $\|(s, t)\|$ of $(s, t) \in C(M)$ is t . $C_p(M) := \pi^{-1}(p)$ is called the tangent cone in p .

Remark 2.4 We identify the elements of $S(M)$ with elements of $C(M)$ by the canonical embedding:

$$\begin{aligned} S(M) &\hookrightarrow C(M) \\ s &\mapsto (s, 1) \end{aligned}$$

We can define a scalar multiplication on the bundle $C(M)$:

Definition 2.2.3 Let (M, d) be a scale bounded space and $(s, t) \in C_p(M)$.

For any $\lambda \in \mathbb{R}$, we define $\lambda \cdot (s, t) \in C_p(M)$ as

$$\lambda \cdot (s, t) := \begin{cases} (s, \lambda t) & \text{if } \lambda \geq 0 \\ (-s, -\lambda t) & \text{if } \lambda \leq 0 \end{cases}.$$

Remark 2.5 Elements in $S_p(M)$ have been constructed as equivalence classes $[\gamma]$ of length parameterized geodesics. We extend this class construction as follows:

$$\lambda \cdot [\gamma] := \{\gamma' \text{ is a geodesic defined around } 0 \mid \exists \gamma'' \in [\gamma] : \gamma'_t = \gamma''_{\lambda t} \text{ where defined}\}.$$

In other words, if $t \neq 0$, the class of $(s, t) \in C_p(M)$ consists of geodesics of the germ class s whose parametrization has been changed to be proportional to the length, the proportionality factor being $\|(s, t)\| = t$. If $t = 0$ the class contains, independently of s , all constant paths $\gamma_t = p$.

2.2.3 Exponential map

Similarly to differential geometry, it is possible to define a function that has most of the properties of the exponential map.

Proposition 2.2.4 (Existence of an exponential)

Let (M, d) be a scale bounded space, $p \in M$ and $B(0, \iota_p) \subseteq C_p(M)$ be the set of all $v \in C_p(M)$ with $\|v\| < \iota_p$

The following application is well defined and one-to-one:

$$\begin{aligned} \exp_p : B(0, \iota_p) &\longrightarrow B_p \subseteq M \\ 0 &\longmapsto p \\ v &\longmapsto \gamma(\|v\|) \end{aligned}$$

where $\gamma \in v$ is a geodesic defined on $[0, \|v\|]$.

If $v \in C(M)$, the base point $\pi(v)$ is implicitly known, such that we may omit the index and write $\exp v$ instead of $\exp_{\pi(v)} v$.

Proof:

Because of proposition 1.4.7 (39) and the absence of branch points in B_p (see proposition 1.4.2 (38)), \exp_p is well defined.

We show that \exp_p is onto and injective:

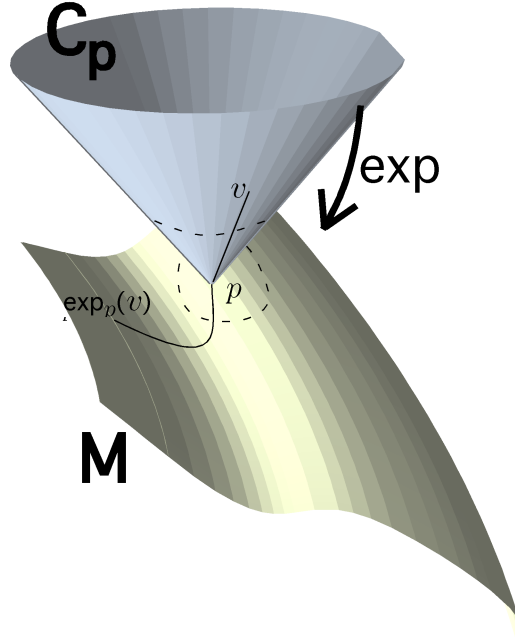


Figure 2.1: An illustration of the exponential mapping.

1. $\triangleright \exp_p$ is onto: Let q be a point in $B_p^* \subseteq M$. There is a segment from p to q (see theorem 1.2.18 (21)) such that we have a length parameterized segment γ with $\gamma_0 = p$ and $\gamma_{\overline{pq}} = q$. By construction, $\exp(\overline{pq} \cdot [\gamma]) = q$. \triangle
2. $\triangleright \exp_p$ is injective: Let (s_1, t_1) and (s_2, t_2) be points in $S_p(M) \times]0, r[$ such that $q := \exp(s_1, t_1) = \exp(s_2, t_2)$. By construction $t_1 = \overline{pq} = t_2$. If v_1 was different from v_2 , their extension to the length \overline{pq} ensured by proposition 1.4.7 (39), would form a local lens, what would contradict proposition 1.4.3 (38). \triangle

□

Lemma 2.2.5 Let (M, d) be a scale bounded space and $p, q \in M$.

ι_p is a continuous function of p such that if $\overline{pq} < \iota_p$ then $\iota_q \geq \iota_p - \overline{pq}$.

Proof:

The property that if $\overline{pq} < \iota_p$ then $\iota_q \geq \iota_p - \overline{pq}$ is a direct consequence of proposition 1.4.7 (39).

The same property implies continuity of ι_p . \square

2.2.4 Topological equivalencies over S_p

For $\epsilon > 0$ small enough, $S_p(M)$ can be equipped with a metric pulling back the metric on M through the mapping $s \mapsto \exp(\epsilon s)$. The purpose of this section is to prove their equivalence transforming the exponential map to a local homeomorphism.

Some limit constructions are also included in the following proofs; they will play a similar role as the angle defined for Alexandrov spaces in our infinitesimal constructions (see [7], [11], [24]).

Definition 2.2.6 *Let (M, d) be a scale bounded space.*

We define the following mappings from $S_p(M) \times S_p(M)$ to \mathbb{R}_+ :

1. $d_r(v_1, v_2) := \frac{1}{2r} \overline{\exp(r v_1) \exp(r v_2)}$ for $0 < r < \iota_p$,
2. $d_{r_0}^{\sup}(v_1, v_2) := \sup_{0 < r \leq r_0} d_r(v_1, v_2)$ for $0 < r_0 < \iota_p$,
3. $d^{\sup}(v_1, v_2) := \lim_{r \rightarrow 0} d_r(v_1, v_2)$,

Remark 2.6 Obviously, $d_r(v_1, v_2) \leq 1$, $d_{r_0}^{\sup}(v_1, v_2) \leq 1$ and $d^{\sup}(v_1, v_2) \leq 1$ for every couple $v_1, v_2 \in S_p(M)$.

Proposition 2.2.7

Let (M, d) be a scale bounded space, $p \in M$.

1. *For $0 < r < \iota_p$, d_r , d_r^{\sup} and d^{\sup} of definition 2.2.6 are distances, all topologically equivalent,*
2. *there is a $C > 0$ such that for every $q \in B_p$ and for every pair of distances d_1 and d_2 out of $\{d_r, d_r^{\sup} | 0 < r < |\iota_p - \iota_q|\} \cup \{d^{\sup}\}$, we have*

$$C^{-1}d_1(v, w) \leq d_2(v, w) \leq C d_1(v, w) \quad \forall v, w \in S_q(M),$$

3. *the resulting equivalent topologies are complete.*

Remark 2.7 These distances define a topology on $S_p(M)$. $C_p(M)$ inherits an own topology from the product topology of $S_p(M) \times \mathbb{R}_+$, see definition 2.2.2 (44).

Remark 2.8 In order to control curvature in G -spaces, Busemann considers pairs of geodesics issued from the same point p that we might specify with a pair $v_1, v_2 \in S_p(M)$ and compares $d_{2r}(v_1, v_2)$ and $d_r(v_1, v_2)$. Hence, the equivalence of all our distances will implies that scale bounded space have bounded curvature in the sense of Busemann [12].

Proof:

We first collect some small results. They will be summarize at the very bottom of the proof:

1. \triangleright For $0 < r < \iota_q$, d_r and d_r^{sup} are distances: Up to the factor $\frac{1}{2r}$, d_r is nothing else but the pull-back of the space distance in M through the map $v \mapsto \exp(rv)$. The supremum of every family of distances is again a distance, so are d_r^{sup} . \triangle

If $v = \pm w$ all distances are equal so that we can restrict ourself to the case $v \neq \pm w$:

Let $0 < r, r' < |\iota_p - \iota_q|$ and $v, w \in S_q(M)$, $A := \exp(rv)$, $A' := \exp(r'v)$, $B := \exp(rw)$, $B' := \exp(r'w)$ and define the length parameterized segments $\gamma_t := \exp(tw)$ and ξ going from $\xi_0 = A$ to $\xi_{t_0} = B$.

As $v \neq -w$, q is not on ξ such that the direction from q to ξ_t is well-defined. Call $\xi \in S_q(M)$ this direction.

2. \triangleright The topologies induced by d_r and $d_{r'}$ on $S_q(M)$ are equivalent: By proposition 1.4.8 (40), we can write the estimations

$$\begin{aligned} \delta_A(\gamma_t) &= r + t \sin \phi + t^2 \cos^2 \phi \frac{R_{\gamma(r)}}{r} \\ \delta_{A'}(\gamma_t) &= r' + t \sin \phi' + t^2 \cos^2 \phi' \frac{R_{\gamma(r')}}{r'} \end{aligned}$$

As q , A and A' are aligned, corollary 1 of lemma 1.4.9 (42) implies $\phi = \phi'$. Further, we know by construction that $\delta_A(\gamma_r) = 2rd_r(v, w)$ and $\delta_{A'}(\gamma_{r'}) = 2r'd_{r'}(v, w)$ such that

$$\begin{aligned} 2r d_r(v, w) &= r + r \sin \phi + r \cos^2 \phi R_{\gamma(r)} \\ 2r' d_{r'}(v, w) &= r' + r' \sin \phi + r' \cos^2 \phi R_{\gamma(r')} \end{aligned}$$

or, equivalently

$$\begin{aligned} d_r(v, w) &= \frac{1}{2} (1 + \sin \phi) (1 + (1 - \sin \phi) R_{\gamma(r)}) \\ d_{r'}(v, w) &= \frac{1}{2} (1 + \sin \phi) (1 + (1 - \sin \phi) R_{\gamma(r')}) \end{aligned}$$

As $v \neq -w$, $\sin \phi \neq -1$ (again by proposition 1.4.8 (40)), such that:

$$\frac{d_r(v, w)}{d_{r'}(v, w)} = \frac{1 + (1 - \sin \phi) R_{\gamma(r)}}{1 + (1 - \sin \phi) R_{\gamma(r')}}.$$

According to proposition 1.4.8 (40) (to avoid confusion we designate the constant C defined there with C' here), $d_r(v, w), d_{r'}(v, w) \leq \frac{1}{2}(\eta - 1)$ implies that $C'^{-1} < R_{\gamma(r)}, R_{\gamma(r')} < C'$ and as $0 \leq 1 - \sin \phi \leq 2$, we have the estimation

$$(1 + 2C')^{-1} < \frac{d_r(v, w)}{d_{r'}(v, w)} < 1 + 2C'. \quad (2.1)$$

The topological equivalence follows from the above inequalities. \triangle

3. $\triangleright \tilde{\xi}$ is continuous for topologies defined by d_r : Choose an $\epsilon > 0$ and $s \in]0, t_0[$. Let $r'' := \overline{q \xi_s}$ and $t \in]0, t_0[$ be chosen such that $|t - s| \leq \epsilon r''$ and set $F := \exp_q(r'' \tilde{\xi}_t)$.

These definitions and triangle inequality imply following relations:

$$\begin{aligned} \overline{q \xi_t} &\leq \overline{q \xi_s} + \overline{\xi_t \xi_s} = r + |t - s| \\ \overline{F \xi_t} &= |\overline{q \xi_t} - \overline{q F}| = |r + |t - s| - r| = |t - s| \\ \overline{F \xi_s} &\leq \overline{F \xi_t} + \overline{\xi_t \xi_s} = 2|t - s| \end{aligned}$$

But, by definition, $2r'' d_{r''}(\tilde{\xi}_s, \tilde{\xi}_t) = \overline{F \xi_s}$ such that

$$d_{r''}(\tilde{\xi}_s, \tilde{\xi}_t) \leq \frac{|t - s|}{r''}.$$

Hence $\tilde{\xi}$ is continuous in s with respect to the topology defined by $d_{r''}$ around $\tilde{\xi}_s$. As, by point 2, all distances d_r are equivalent, $\tilde{\xi}$ is also continuous with respect to d_r . \triangle

4. \triangleright For any $t \in [0, t_0]$ the relation $d_r(\tilde{\xi}_0, \tilde{\xi}_t) \leq 2d_r(\tilde{\xi}_0, \tilde{\xi}_{t_0})$ holds: Set $F := \exp_q(r \tilde{\xi}_t)$ and G the intersection between the segments qF and AB .

By construction and triangle inequality, the following relations hold:

$$\begin{aligned} \overline{q G} &\leq \overline{q A} + \overline{A G} \leq \overline{q A} + \overline{A B} = r + t_0 \\ \overline{F G} &= |\overline{q F} - \overline{F G}| = |r - |r + t_0|| = t_0 \\ \overline{A F} &\leq \overline{A G} + \overline{F G} \leq \overline{A B} + \overline{F G} = 2t_0 \end{aligned}$$

As $\overline{A F} = 2r d_r(\tilde{\xi}_0, \tilde{\xi}_t)$ and $t_0 = 2r d_r(\tilde{\xi}_0, \tilde{\xi}_{t_0})$, the last inequality proofs the claim. \triangle

5. \triangleright There is a $C > 0$ such that $C^{-1}d_{r'}(v, w) \leq d_r(v, w) \leq C d_{r'}(v, w)$ if $d_r(v, w) \leq \frac{1}{4} \frac{\eta-1}{1+2C'}$: Consider that v and w verify $d_r(v, w) \leq \frac{1}{4} \frac{\eta-1}{1+2C'}$ and define $f(t) := d_r(\tilde{\xi}_0, \tilde{\xi}_t)$ and $g(t) := d_{r'}(\tilde{\xi}_0, \tilde{\xi}_t)$. By point 3, $f(t)$ and $g(t)$ are both continuous for $t \in [0, t_0]$.

Suppose there is an $s \in [0, t_0]$ such that $g(s) = \frac{1}{2}(\eta - 1)$. Due to point 4 and our condition above, $f(s) < \frac{1}{2} \frac{\eta-1}{1+2C'}$. Hence the relation 2.1 applies such that $f(s) > g(s)(1+2C')^{-1} = \frac{1}{2} \frac{\eta-1}{1+2C'}$ what is a contraction. It follows that $g(t) < \frac{1}{2}(\eta - 1)$ for every $t \in [0, t_0]$.

As already seen, $f(t) < \frac{1}{2} \frac{\eta-1}{1+2C'} \leq \frac{1}{2}(\eta - 1)$ is true for $t \in [0, t_0]$ such that the condition for the relation 2.1 are always fulfilled. Hence, the claim is true for $C = 1 + 2C'$. \triangle

As the constant C in point 5 is independent of r' , the claim stays true if $d_{r'}$ is replaced by $d_{r'}^{\text{sup}}$ or d^{sup} . So we have point 2 of proposition 2.2.7 (47) as soon as the restriction $d_r(v, w) \leq \frac{1}{4} \frac{\eta-1}{1+2C'}$ is lifted.

But for $d_r(v, w) \geq \frac{1}{4} \frac{\eta-1}{1+2C'}$, $\frac{d_{r'}(v, w)}{d_r(v, w)}$ is well defined and as, by remark 2.6 (47), $d_{r'}(v, w) \leq 1$ the ratio cannot exceed $4 \frac{1+2C'}{\eta-1}$. By swapping r and r' , the above considerations show that the ratio has also a lower bound larger 0.

The first point in proposition 2.2.7 (47) followed from points 1 and 2. \square

2.2.5 Topologies on $S(M)$ and $C(M)$.

Fibers of $C(M)$ already carry a topology. With a "connection" mapping between fibers their topology can be extended to a topology on the whole bundle. The exponential map suggests a natural connection between fibers in p and q with $\exp_q^{-1} \circ \exp_p$.

However, it is easier to first define the bundle topology through a family of sections, postponing the equivalence to the above approach.

Definition 2.2.8 *Let (M, d) be a scale bounded space. $\sigma_p: B_p^* \rightarrow S(M)$ stands for the section $q \mapsto \overline{p}q^{-1} \cdot \exp_p^{-1}(q)$.*

The finest topology for $S(M)$ preserving the fiber topology and making σ_p continuous for all $p \in M$ is:

Definition 2.2.9 *Let (M, d) be a scale bounded space. The sets*

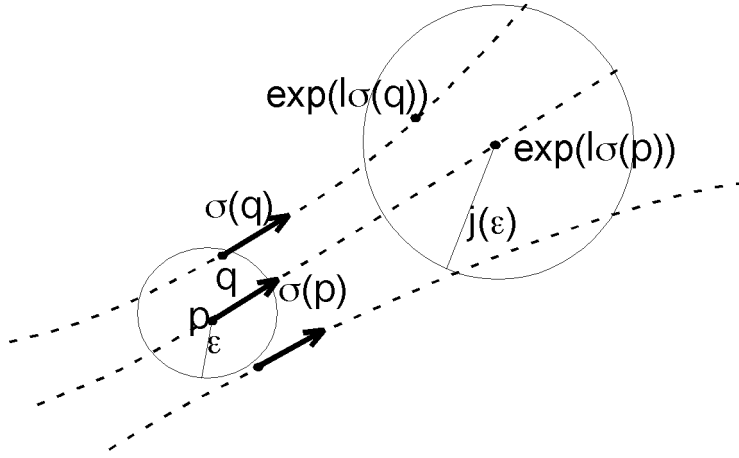
$$\left\{ s \in S(M) \mid \pi(s) \in \mathcal{U} \text{ and } d^{\text{sup}}(s, \sigma_p(\pi(s))) < \lambda \right\}$$

where p is any point in M , \mathcal{U} is an open set in B_p^ and $\lambda > 0$ form a basis for a topology of $S(M)$. $C(M)$ inherits its topology from the product topology of $S(M) \times \mathbb{R}_+$.*

$\chi(C(M))$ is the set of all continuous sections of $(C(M), \bar{\pi})$.

Remark 2.9 The canonical projection π and the embedding of remark 2.4 (45) are continuous.

Remark 2.10 By construction, $\sigma_p \in \chi(C(B_p^*))$.

Figure 2.2: Construction of $\exp(l\sigma(p))$.

Remark 2.11 Let σ be a section of $(C(M), \bar{\pi})$. Its continuity in p can be expressed as follows:

For any $0 < l < \iota_p$ and any $\epsilon > 0$ such that the ball $B_d(\exp(l\sigma(p)), \epsilon) \subseteq B_p$ there is a $j(\epsilon) > 0$ such that:

$$q \in B_d(p, j(\epsilon)) \implies \exp(l\sigma(q)) \in B_d(\exp(l\sigma(p)), \epsilon)$$

Proposition 2.2.10

Let (M, d) be a scale bounded space. $C^0(M)$, the set of continuous scalar functions on M , acts on $\chi(M)$:

If $f \in C^0(M)$ and $\sigma \in \chi(M)$, $(f \cdot \sigma)(p) := f(p) \cdot \sigma(p) \in \chi(M)$.

Proof:

The only non trivial point is to prove that if σ and f are continuous, $f(p) \cdot \sigma(p)$ is also continuous.

Suppose first that $f \in C^0(\mathcal{U})$ is positive and let $\sigma \in \chi(S(\mathcal{U}))$. The applications sending $(\sigma(p), t(p))$ to $(\sigma(p), f(p)t(p))$ respectively $(-\sigma(p), f(p)t(p))$ are obviously continuous.

In the more general case a continuous f splits M into three subsets $M_0 := \{p \in M \mid f(p) = 0\}$, $M_+ := \{p \in M \mid f(p) > 0\}$ and $M_- := \{p \in M \mid f(p) < 0\}$ where M_{\pm} are open and M_0 is closed.

As the lemma is true on the restriction to M_+ , M_- and M_0 , we have only to check continuity on their borders. As all common border points of M_+ and M_- have to be in M_0 , only continuity around points of M_0 must be proved.

Let $q \in M_0$. As f is continuous, $f(q')$ tends to 0 for q' tending to q within M_+ . From the equivalence relation \approx of definition 2.2.2 (44), we can conclude that $f(q') \cdot \sigma(q')$ must also tend to 0, ensuring continuity in q . The same argument applies to M_- . \square

Corollary 1 *The application \exp_p is a homeomorphism between $\{v \in C_p(M) \mid \|v\| < \iota_p\}$ and B_p .*

Proof:

The mapping $s \in S_p(M)$ to $\exp(\epsilon s)$ is, for $\epsilon \in]0, \iota_p[$, continuous by remark 2.7 (47) and proposition 2.2.7 (47). This mapping is also continuous in ϵ so that \exp_p is continuous.

Openness follows, by definition 2.2.9 (50), from proposition 2.2.10 (51) and definition 2.2.8 (50) as δ_p is continuous over B_p^* . \square

Remark 2.12 As scale bounded spaces are locally compact, there is an $\epsilon > 0$ such that $\overline{B_d(p, \epsilon)}$ is compact. Hence, by corollary 1 of proposition 2.2.10, $S_p(M)$ is compact and $C_p(M)$ locally compact.

For an estimation of the distance between a pair of points equidistant to a third, we can already use proposition 2.2.7 (47). In case the points are not equidistant, the following lemma will be useful:

Lemma 2.2.11 *Let (M, d) be a scale bounded space and $p \in M$.*

For every $r \in]0, \iota_p[$ there is an $L > 0$ such that for $q, q' \in B_p \setminus B_d(p, r)$ and if $q_\epsilon, q'_\epsilon \in B_p$ with $\epsilon > 0$:

$$\overline{q_\epsilon q'_\epsilon} \leq L \overline{q q'}$$

where $q_\epsilon := \exp(\epsilon \sigma_p(q))$ and $q'_\epsilon := \exp(\epsilon \sigma_p(q'))$.

Proof:

Without loss of generality, we suppose $\overline{p q} \leq \overline{p q'}$ and call q'' the point on the segment p to q' such that $\overline{p q} = \overline{p q''}$. We also set $q''_\epsilon := \exp(\epsilon \sigma_p(q''))$.

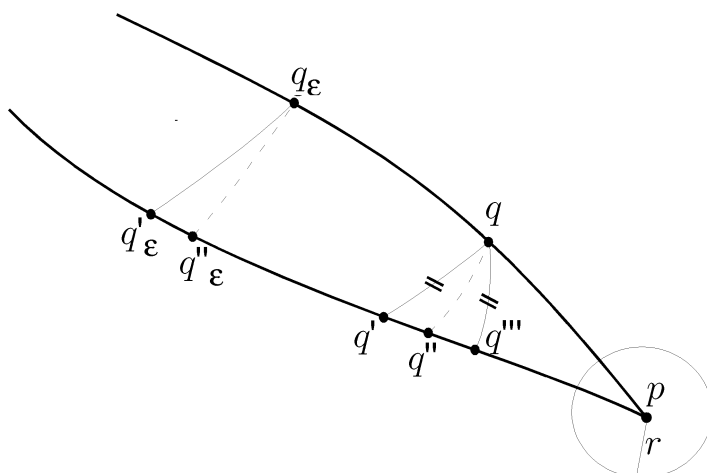


Figure 2.3: Illustration of the construction of lemma 2.2.11.

By proposition 2.2.7 (47) the distances $d_{\overline{p}\overline{q}}$ and $d_{\overline{p}\overline{q_\epsilon}}$ on S_p are equivalent. Hence, there is a constant $C > 0$ such that $(2\overline{p}\overline{q_\epsilon})^{-1} \overline{q_\epsilon q''} \leq C (2\overline{p}\overline{q})^{-1} \overline{q q''}$ or:

$$\overline{q_\epsilon q''} \leq C \frac{\overline{p}\overline{q_\epsilon}}{\overline{p}\overline{q}} \overline{q q''} \leq C \frac{\iota_p}{r} \overline{q q''}. \quad (2.2)$$

We further see, using triangle inequality, that $\overline{p q'} \leq \overline{p q} + \overline{q q'}$. As, by construction, $\overline{p q''} = \overline{p q}$, we obtain by subtraction of $\overline{p q''}$ on each side, that

$$\overline{p q'} - \overline{p q''} = \overline{q' q''} \leq \overline{q q'}. \quad (2.3)$$

This last relation implies the existence of a point q''' lying between p and q'' such that $\overline{q' q'''} = \overline{q q'}$. Considering triangle inequality in (q, q', q''') and relation 2.3, we observe that $\overline{q q'''} \leq 2\overline{q q'}$. This in turn implies, considering the triangle (q, q'', q''') and $\overline{q'' q'''} \leq \overline{q' q''}$, that

$$\overline{q q''} \leq 3\overline{q q'}. \quad (2.4)$$

We now resume this in:

$$\begin{aligned} \overline{q_\epsilon q''} &\leq \overline{q_\epsilon q''} + \overline{q'' q'} && \text{by triangle inequality} \\ &\leq \overline{q_\epsilon q''} + \overline{q'' q'} && \text{by property of exp} \\ &\leq C \frac{\iota_p}{r} \overline{q q''} + \overline{q'' q'} && \text{by inequality 2.2} \\ &\leq C \frac{\iota_p}{r} \overline{q q''} + \overline{q q'} && \text{by inequality 2.3} \\ &\leq 3C \frac{\iota_p}{r} \overline{q q'} + \overline{q q'} && \text{by inequality 2.4} \\ &\leq (1 + 3C \frac{\iota_p}{r}) \overline{q q'} \end{aligned}$$

$L := 1 + 3C \frac{\iota_p}{r}$ is the expected Lipschitz constant. □

2.2.6 Fiber homeomorphism within $S(M)$.

Proposition 2.2.12

All tangent cones of a scale bounded space are homeomorphic.

Proof:

We will prove that for every point $p \in M$, the fiber $S_p(M)$ is homeomorphic to $S_q(M)$ if $\overline{p q} \leq \iota_p/4$. This would mean that all spaces of directions in $B_d(p, \iota_p/4)$ are homeomorphic one to another. M is arc-wise connected such that, by compactness of paths, we can connect any two points in M by a finite set of balls $B_d(p_i, \iota_{p_i}/4)$ such that $B_d(p_i, \iota_{p_i}/4) \cap B_d(p_{i+1}, \iota_{p_{i+1}}/4) \neq \emptyset$. By transitivity of homeomorphism, proposition 2.2.12 would be proved.

We choose q and $r' > 0$ such that $B_d(q, r') \subseteq B_d(p, \iota_p/4)$ and define $\phi : S_p(M) \rightarrow S_q(M)$ as follows: let v be in $S_p(M)$ and be represented by a

segment γ of length $\iota_p/4$. This is possible by proposition 1.4.7 (39). $\gamma_{\iota_p/4}$ is a border point $x \in \partial B_d(p, \iota_p/4)$. Then $\phi(v)$ is the direction in $S_q(M)$ represented by the segment from q to x . As this segment is unique (see proposition 1.3.10 (31)), ϕ is well-defined.

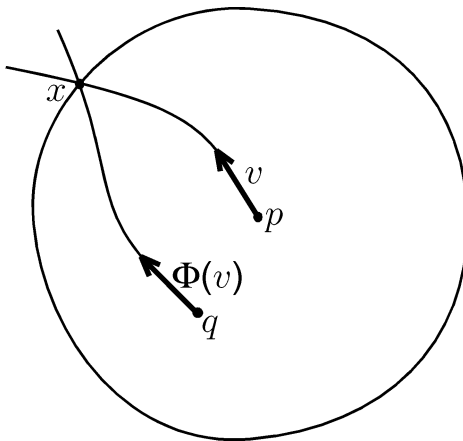


Figure 2.4: The application ϕ .

1. $\triangleright \phi$ is injective: consider $v_1, v_2 \in S_p(M)$ such that $\phi(v_1) = \phi(v_2)$. Call x_1 and x_2 the points where the representant of v_1 and v_2 respectively cut $\partial B_d(p, \iota_p/4)$ for the first time.

As $\phi(v_1) = \phi(v_2)$, the segments from q to x_1 and from q to x_2 are in the same class in $S_q(M)$, such that one is a right extension of the other. Without restriction of generality, we suppose that x_1 is between q and x_2 and call γ the segment from q to x_2 .

As $\overline{B_d(p, \iota_p/4)}$ is strictly convex (see proposition 1.4.4 (38)), the entire segment from q to x_2 lies in $\overline{B_d(p, \iota_p/4)}$ and all points between q and x_2 are in $B_d(p, \iota_p/4)$. This would contradict x_1 to be in $\partial B_d(p, \iota_p/4)$ unless $x_1 = x_2$. \triangle

2. $\triangleright \phi$ is surjective: let us choose a $w \in S_q(M)$. By proposition 1.4.7 (39) and the definition of $\iota_p/4$, there is a representing segment γ of w whose length is at least $3\iota_p/4$. But as $\overline{pq} < r_l$, by triangle inequality, the endpoint of γ is outside $B_d(p, \iota_p/4)$ such that it cuts $\partial B_d(p, \iota_p/4)$ in at least one point x . As $\phi^{-1}(w)$ contains the direction from p to x , ϕ is onto. \triangle

3. $\triangleright \phi$ is locally a homeomorphism: Let v be in $S_p(M)$ and $x := \exp(\iota_p/4 v)$. Consider the neighbourhood basis of x given by $\{B_d(x, \epsilon)\}_{0 < \epsilon < \iota_p/4 - \overline{pq}}$. Neither p nor q are elements of these neighbourhoods. By corollary 1 of proposition 2.2.10 (52) and remark 2.9 (50) the inverse images under \exp_p and \exp_q of that neighbourhood basis are a neighbourhood basis respectively for $v \in C_p(M)$ and $\phi(v) \in C_q(M)$.

As ϕ is (up to a scalar multiplication with $\iota_p/4$) the restriction to $S_p(M)$ of $\rho_p \circ \exp_q^{-1} \circ \exp_p$, ϕ is continuous.

The same argument applies to ϕ^{-1} . \triangle

Together the above points prove that ϕ is a homeomorphism. \square

2.3 The derivation

As has already been suggested in proposition 1.4.8 (40) or in proposition 1.4.8 (40), the distance function has differential properties. Having defined the bundle $C(M)$ and continuous sections on it, it is natural to define derivation along sections and select a subset of the continuous scalar functions on M which can be considered as *differentiable*.

The definition 2.3.4 (56) of that subset is directly related to the properties of the distance function given in proposition 1.4.8 (40) and those we can prove in proposition 2.3.6 (56) such that, by construction, the distance functions are $\mathcal{C}^1(\mathcal{U})$. The main result of this section is, in fact, proposition 2.3.5 (56).

2.3.1 Derivatives along sections of $C(M)$

Definition 2.3.1 Let (M, d) be a scale bounded space and \mathcal{U} an open set in M .

We denote by $\mathcal{C}^0(\mathcal{U})$ the set of all continuous functions from \mathcal{U} to \mathbb{R} , by $\mathcal{C}^{0,\alpha}(\mathcal{U})$ the subset of all local α -Hölder-continuous functions in $\mathcal{C}^0(\mathcal{U})$ ($0 < \alpha \leq 1$). They are naturally endowed with the open-compact topology.

Remark 2.13 $\mathcal{C}^{0,1}(\mathcal{U})$ is the set of local Lipschitz-continuous functions.

In order to do analysis on those functions, we will need to restrict ourselves to scalar functions that are *differentiable*, whatever this may mean. It seems natural to define it as follows:

Definition 2.3.2 Let (M, d) be a scale bounded space and \mathcal{U} an open set in M .

Let $v \in C_p(M)$ and $f \in \mathcal{C}^0(\mathcal{U})$. We set

$$\partial_v f := \left. \frac{d}{d\epsilon} f \circ \exp(\epsilon v) \right|_{\epsilon=0} \quad \text{if the expression is well-defined.}$$

With $\partial_v^+ f$ we denote the derivative on the right.

Definition 2.3.3 Let (M, d) be a scale bounded space and \mathcal{U} an open set in M , σ a section of $C(\mathcal{U})$ and $f \in \mathcal{C}^{0,1}(\mathcal{U})$.

If $\partial_{\sigma(p)} f$ is well-defined for every $p \in \mathcal{U}$, the mapping $\partial_\sigma f$ sending p to $\partial_{\sigma(p)} f$ is called the derivative of f along the section σ .

For more convenience, we will often write $\partial_\sigma f(p)$ instead of $\partial_{\sigma(p)} f$.

Remark 2.14 $\partial_\sigma f(p)$ is $\mathcal{C}^0(\mathcal{U})$ -homogeneous in σ and \mathbb{R} -homogeneous in f . As a special case, we observe that $\partial_{-v} f = -\partial_v f$ for all $v \in C_p(M)$.

We can now define derivability of a function

Definition 2.3.4 *Let (M, d) be a scale bounded space.*

We denote by $\mathcal{C}^{1+}(M)$ the set of all functions f in $\mathcal{C}^{0,1}(M)$ such that, for every point $p \in M$, the following conditions hold:

1. $\frac{d}{dt} f \circ \gamma(t)$ is Lipschitz continuous for every $\gamma \in \mathcal{G}(B_p)$,
2. for $q \in B_p^*$, $\partial_{\sigma_q} f|_{B_p^* \cap B_q^*} \in \mathcal{C}^{\frac{1}{2}}(B_p^* \cap B_q^*)$.

Proposition 2.3.5

The set $\mathcal{C}^{1+}(\mathcal{U})$ of definition 2.3.4 is a sub-algebra of $\mathcal{C}^0(\mathcal{U})$.

Let $\sigma \in \chi(\mathcal{U})$. Then ∂_σ is a derivation from the algebra $\mathcal{C}^{1+}(\mathcal{U})$ to the algebra $\mathcal{C}^0(\mathcal{U})$.

Proof:

$\mathcal{C}^0(\mathcal{U})$ is an \mathbb{R} -algebra.

Suppose $f, g \in \mathcal{C}^{1+}(\mathcal{U})$. Direct calculation from definition 2.3.2 (55) leads to the verification of the Leibniz rule $\partial_v(f \cdot g) = g \cdot \partial_v f + f \cdot \partial_v g$ such that $\partial_\sigma(f \cdot g)$ is $\mathcal{C}^{\frac{1}{2}}(B_p)$. By definition 2.3.3 (55), $f \cdot g \in \mathcal{C}^{1+}(\mathcal{U})$ such that $\mathcal{C}^{1+}(\mathcal{U})$ is also an \mathbb{R} -algebra.

Considering the definition 2.2.9 (50) for the topology of $C(M)$ and definition 2.2.8 (50), we see that the local sections $\{\sigma_p \mid p \in M\}$ define the topology, such that if a function f has a continuous derivative along every local section σ_p , it also has continuous derivative along every continuous local section. Hence, the Leibniz rule being already proved, $\partial_\sigma : \mathcal{C}^{1+}(\mathcal{U}) \longrightarrow \mathcal{C}^0(\mathcal{U})$ is a continuous derivation. \square

2.3.2 Differentiability of the distance function

Proposition 2.3.6

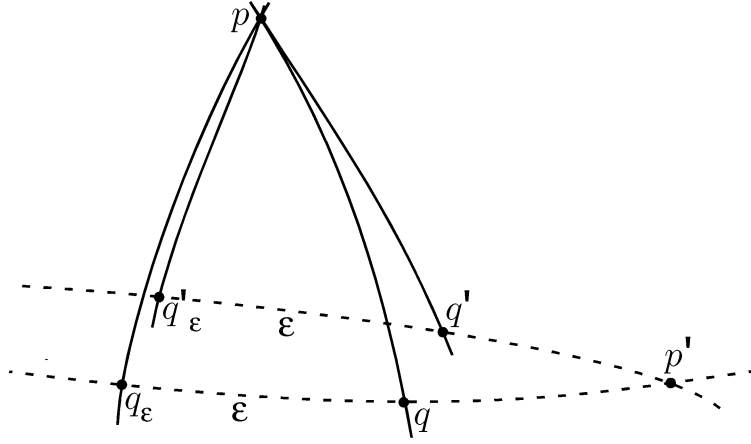
Let (M, d) be a scale bounded space and $p \in M$. The mapping sending $q \in B_p^$ to \overline{pq} is in $\mathcal{C}^{1+}(B_p^*)$.*

Proof:

We have already proved in proposition 1.4.8 (40), that the derivative of δ_p (see definition 1.3.12 (32)) is defined in any point of a neighbourhood of p outside p and that its restriction to a geodesic is $\mathcal{C}^{1,1}$. This proves that point 1) of definition 2.3.4 is fulfilled. Let us prove that point 2) is also fulfilled:

Let $p' \in B_p^*$ and $q, q' \in B_p^* \cap B_{p'}^*$. We have to prove that there is an $H > 0$ and an $\epsilon_0 > 0$ such that, if $0 < \epsilon \leq \epsilon_0$:

$$\overline{qq'} \leq \epsilon^2 \implies \left| \partial_{\sigma_{p'}} \delta_p(q') - \partial_{\sigma_{p'}} \delta_p(q) \right| \leq H \epsilon. \quad (2.5)$$



Let us consider proposition 1.4.8 (40) with the path $\gamma_t = \exp(t\sigma_{p'}(q))$. We have that

$$\delta_p(\gamma_t) - \delta_p(\gamma_0) = t \sin \phi + t^2 \frac{\cos^2 \phi}{\delta_p(\gamma_0)} R(\gamma).$$

As $\sin \phi = \partial_{\sigma_{p'}} \delta_p(q)$ and using the boundedness of $R(\gamma)$, we can find an $L > 0$ to obtain the rough estimation:

$$\left| \partial_{\sigma_{p'}} \delta_p(q) - \frac{\delta_p(q_\epsilon) - \delta_p(q)}{\epsilon} \right| \leq L \epsilon. \quad (2.6)$$

where we let $q_\epsilon := \gamma_\epsilon = \exp(\epsilon \sigma_{p'}(q))$.

Because of uniformity of the boundedness of $R(\gamma)$, we also have that:

$$\left| \partial_{\sigma_{p'}} \delta_p(q') - \frac{\delta_p(q'_\epsilon) - \delta_p(q')}{\epsilon} \right| \leq L \epsilon. \quad (2.7)$$

where $q'_\epsilon := \exp(\epsilon \sigma_{p'}(q'))$.

By lemma 2.2.11 (52), we know that there is an $L' > 0$ such that

$$\overline{q_\epsilon q'_\epsilon} \leq L' \overline{q q'}$$

and, using triangle inequality on (p, q, q') and $(p, q_\epsilon, q'_\epsilon)$, we have that

$$\begin{aligned} \left| \frac{\delta_p(q_\epsilon) - \delta_p(q)}{\epsilon} - \frac{\delta_p(q'_\epsilon) - \delta_p(q')}{\epsilon} \right| &= \\ \left| \frac{\delta_p(q_\epsilon) - \delta_p(q'_\epsilon)}{\epsilon} + \frac{\delta_p(q') - \delta_p(q)}{\epsilon} \right| &\leq \left| \frac{\overline{q_\epsilon q'_\epsilon}}{\epsilon} + \frac{\overline{q q'}}{\epsilon} \right| \\ &\leq (1 + L') \frac{\overline{q q'}}{\epsilon} \end{aligned}$$

Combining these inequalities with 2.6 and 2.7, we have the estimation:

$$\left| \partial_{\sigma_{p'}} \delta_p(q) - \partial_{\sigma_{p'}} \delta_p(q') \right| \leq 2L \epsilon + (1 + L') \frac{\overline{q q'}}{\epsilon}.$$

If $\overline{q q'} \leq \epsilon^2$,

$$\left| \partial_{\sigma_p} \delta_p(q) - \partial_{\sigma_{p'}} \delta_p(q') \right| \leq 2L\epsilon + (1 + L')\epsilon = (1 + 2L + L')\epsilon$$

such that 2.5 is true with $H := 1 + 2L + L'$. \square

Remark 2.15 If the derivative exists, it would, as consequence of triangle inequality, verify the following property:

$$-\|v\| \leq \partial_v \delta_p \leq \|v\|.$$

Remark 2.16 Let $v \in V_p(M)$. Then $\partial_v \delta_p$ is not defined if $v \neq 0$, but $\partial_v^+ \delta_p$ is. $\partial_v^+ \delta_p = \|v\|$.

Remark 2.17 If $v \in V_p(M)$, $\partial_v \delta_p^2 = 0$ such that $\delta_p^2 \in \mathcal{C}^{1+}(B_p)$.

Remark 2.18 Suppose R is a bounded function in $\mathcal{C}^{1,1}(\cdot - \epsilon_0, \epsilon_0[\setminus \{0\}])$. Then there is also a neighbourhood \mathcal{U} of p such that $\delta_p^2 \cdot R \circ \delta_p$ is in $\mathcal{C}^{1+}(\mathcal{U})$:

1. $\delta_p^2 \cdot R \circ \delta_p$ is certainly in $\mathcal{C}^{1+}(\mathcal{U} \setminus \{0\})$,
2. because of remark 2.17 and the boundedness of R , there is, for the derivative along any section σ around p , a unique extension to p ,
3. the restriction to a geodesic through p is trivially $\mathcal{C}^{1,1}$.

2.4 The 1-form sheaf

2.4.1 Locally ringed spaces

Definition 2.4.1 Let (M, d) be a scale bounded space and $p \in M$. We define the germ spaces in p of $\mathcal{C}^0(\mathcal{U})$ and $\mathcal{C}^{1+}(\mathcal{U})$ (where \mathcal{U} is a neighbourhood of p) as

$$\mathcal{A}_p^0(M) := \varinjlim_{\mathcal{U} \ni p} \mathcal{C}^0(\mathcal{U}) \quad \mathcal{A}_p^1(M) := \varinjlim_{\mathcal{U} \ni p} \mathcal{C}^{1+}(\mathcal{U})$$

and

$$\mathcal{A}^0(M) := \bigsqcup_{p \in M} \mathcal{A}_p^0(M) \quad \mathcal{A}^1(M) := \bigsqcup_{p \in M} \mathcal{A}_p^1(M)$$

The projection π_0 (respectively π_1) is sending $\mathfrak{f} \in \mathcal{A}^0$ (respectively $\mathfrak{f} \in \mathcal{A}^1$) to the unique point where all functions in the equivalence class of \mathfrak{f} are defined.

Remark 2.19 $\mathcal{A}_p^0(M)$ and $\mathcal{A}_p^1(M)$ are stalks with algebra structure of the locally ringed spaces $\mathcal{A}^0(M)$ and $\mathcal{A}^1(M)$ [19], [21].

Remark 2.20 Let \mathcal{U} be an open set of M . There is a canonical application $p_{\mathcal{U}}$ from $\mathcal{C}^s(\mathcal{U}) \times \mathcal{U}$ to $\mathcal{A}^s(M)$ sending (f, p) to the germ of f in p . We denote $[f]_p$ this germ.

Remark 2.21 Consider $\mathfrak{f} \in \mathcal{A}_p^1(\mathcal{U})$ and $v \in C_p(M)$. As the derivability is a local property, $\partial_v f = \partial_v f'$ for every couple $f, f' \in \mathfrak{f}$. This allows us to give sense to $\partial_v \mathfrak{f} := \partial_v f$ where f can be any element in \mathfrak{f} .

Lemma 2.4.2 For any $\sigma \in \chi(\mathcal{U})$, ∂_σ is a derivation operator from $\mathcal{A}^1(\mathcal{U})$ to $\mathcal{A}^0(\mathcal{U})$.

Proof:

This is a direct consequence of proposition 2.3.5 (56) and definition 2.4.1 (58). \square

Proposition 2.4.3

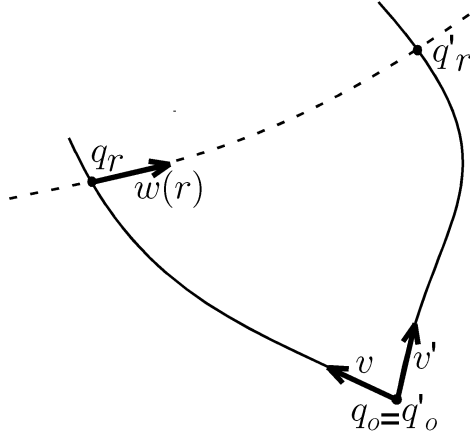
Let (M, d) be a scale bounded space $p \in M$ and $\mathfrak{f} \in \mathcal{A}_p^1(M)$.

The mapping from $(S_p(M), d^{\sup})$ to \mathbb{R} sending v to $\partial_v \mathfrak{f}$ is Lipschitz-continuous. The Lipschitz constant can be chosen in order to be locally uniform.

Remark 2.22 From proposition 2.2.7 (47) we immediately see that proposition 2.4.3 is also true if we equip $S_p(M)$ with one of the other distances defined in definition 2.2.6 (47).

Proof:

Let $f \in \mathfrak{f}$, choose $v, v' \in S_p(M)$ and call $q_r := \exp(rv)$ respectively $q'_r := \exp(rv')$ with $0 < r < \iota_p$. Then $w(r) \in C_{q_r}(M)$ is the vector defined by $\exp w(r) = q'_r$.



1. As f is locally Lipschitz continuous, we know that there is a neighbourhood \mathcal{U} of p and a constant $L > 0$ such that $f|_{\mathcal{U}}$ is L -Lipschitz continuous.
2. By definition 2.2.6 (47) and the definition of $w(r)$, we have $\lim_{r \rightarrow 0} \frac{1}{2r} \|w(r)\| \leq d^{\sup}(v, v')$.
3. By definition of $\partial_v f$ and $\partial_{v'} f$ we know that

$$\lim_{r \rightarrow 0} \frac{f(q_r) - f(q_0)}{r} = \partial_v f \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{f(q'_r) - f(q'_0)}{r} = \partial_{v'} f$$

and from $q_0 = q'_0$ follows:

$$\left| \frac{f(q'_r) - f(q'_0)}{r} - \frac{f(q_r) - f(q_0)}{r} \right| = \frac{1}{r} |f \circ \exp w(r) - f(q_r)|$$

For $r > 0$ small enough, we know by point 1 and the definition of $w(r)$ that $|f \circ \exp w(r) - f(q_r)| \leq L \exp w(r) q_r = L \|w(r)\|$ such that

$$\left| \frac{f(q'_r) - f(q'_0)}{r} - \frac{f(q_r) - f(q_0)}{r} \right| \leq \frac{L}{r} \|w(r)\|.$$

Working out the limit for $r \rightarrow 0$ on either side and considering the points 3 and 2, we obtain:

$$|\partial_{v'} f - \partial_v f| \leq 2L d^{\text{sup}}(v, v').$$

The claimed uniformity is at least verified on \mathcal{U} (see point 1). \square

Corollary 1 *For any $f \in \mathcal{A}^1(\mathcal{U})$ both $\inf_{v \in S_p(M)} \partial_v f$ and $\sup_{v \in S_p(M)} \partial_v f$ exist and are finite.*

Proof:

Let $f \in \mathfrak{f}$ and $v, v' \in S_p(M)$. We know, by remark 2.6 (47), that $d^{\text{sup}}(v, v') \leq 1$ such that $|\partial_{v'} f - \partial_v f| \leq 2L$. This implies that the range of $\partial_v f$ as function of $v \in S_p(M)$ is bounded. \square

This corollary suggests the following definition:

Definition 2.4.4 *For any $f \in \mathcal{A}_p^1(M)$, we define*

$$\|f\| := \sup_{v \in S_p(M)} \partial_v f.$$

Remark 2.23 As, by proposition 2.4.3 (59), $\|f\|$ is defined and $\partial_v f$ is continuous in $v \in S_p(M)$ and as, by remark 2.12 (52), $S_p(M)$ is compact, there are directions such that $\partial_v f = \|f\|$.

Corollary 2 *For every $p \in M$, $\|f\|$ is a semi-norm on $f \in \mathcal{A}_p^1(M)$.*

Proof:

As $\partial_{-v} f = -\partial_v f$ (see remark 2.14 (55)), $\|f\|$ is non-negative. Homogeneity follows from the homogeneity of $\partial_v f$ and triangle inequality from proposition 2.4.3 (59). \square

Corollary 3 *Let $f \in \mathcal{A}_p^1(M)$, $f \in \mathfrak{f}$ defined on a neighbourhood \mathcal{U} of p and set $L(f) = \inf \{L > 0 \mid f \text{ is } L\text{-Lipschitz continuous}\}$ relatively to d^{sup} . Then*

$$\lim_{\mathcal{U} \ni p} L(f|_{\mathcal{U}}) = 2 \|f\|.$$

Proof:

By proposition 2.4.3 (59), there are neighbourhoods \mathcal{U} of p such that $L(f|_{\mathcal{U}})$ exists and as (\mathcal{U}, \subseteq) is a net for the neighbourhoods of p , and $\mathcal{U} \subseteq \mathcal{U}'$ implies $L(f|_{\mathcal{U}}) \leq L(f|_{\mathcal{U}'})$, the limit exists and is finite.

If L is a Lipschitz constant for f as claimed by proposition 2.4.3 (59), $|\partial_{-v}f - \partial_vf| \leq L$ for every $v \in S_p(M)$. But, by definition 2.4.4 and remark 2.14 (55), $|\partial_{-v}f - \partial_vf| \leq 2\|f\|$ such that $\lim_{\mathcal{U} \ni p} L(f|_{\mathcal{U}}) \leq 2\|f\|$.

On the other hand, if $\lim_{\mathcal{U} \ni p} L(f|_{\mathcal{U}}) < 2\|f\|$, there would be a \mathcal{U} such that $L(f|_{\mathcal{U}}) < 2\|f\|$ what implies that $|f \circ \exp(\epsilon v) - f \circ \exp(-\epsilon v)| < 2\|f\|$ for every $v \in S_p(M)$ and $\epsilon > 0$ small enough. This, in turn, implies that $|\partial_vf - \partial_{-v}f| < 2\|f\|$ or $|\partial_vf| < \|f\|$.

This contradicts the definition of the norm, such that $\lim_{\mathcal{U} \ni p} L(f|_{\mathcal{U}}) \geq 2\|f\|$. \square

Corollary 4 *Let (M, d) be a scale bounded space, $p \in M$ and $f \in \mathcal{A}_p^1(M)$.*

For every couple $v, w \in S_p(M)$, $|\partial_vf - \partial_wf| \leq 2\|f\| d^{\sup}(v, w)$.

Proof:

It is a by-product of corollary 3 of proposition 2.4.3 (60). \square

2.4.2 Construction of the 1-form sheaf

The 1-form sheaf $V(M)$ will be defined as the quotient space of two ideals of $\mathcal{A}^1(M)$:

Definition 2.4.5 *Let (M, d) be a scale bounded space and $p \in M$.*

$$\ker_p := \{f \in \mathcal{A}_p^1(M) \mid \|f\| = 0\}$$

is a vectorial sub-space of $\mathcal{A}_p^1(M)$. This defines the sub-sheaves:

$$\ker = \bigsqcup_{p \in M} \ker_p$$

We define the co-tangent space on M as

$$V(M) := \mathcal{A}^1 / \ker$$

Remark 2.24 $V(M)$ is itself a locally ringed space that inherits the canonical projection π_1 from $\mathcal{A}^1(M)$. The fibers or stalks $V_p(M) := \pi_1^{-1}(p) = \mathcal{A}_p^1 / \ker_p$ are \mathbb{R} -vector spaces.

Proposition 2.4.6

Let (M, d) be a scale bounded space, $p \in M$ and $f \in \mathcal{C}^{1+}(\mathcal{U})$.

- i) The semi-norm of definition 2.4.4 (60) induces a norm on $V_p(M)$,*

- ii) the map $p \mapsto \|[f]_p\|$ is continuous,
- iii) $V(M)$ is a locally ringed space that inherits the canonical projection π_1 from $\mathcal{A}^1(M)$. The fibers or stalks $V_p(M) := \pi_1^{-1}(p) = \mathcal{A}_p^1 / \ker_p$ are Banach spaces.

Proof:

By definition \mathfrak{m}_p^0 is the vector subspace of all germs with norm 0, such that the quotient space V has exactly one germ with norm 0. The other norm properties are proved in corollary 2 of proposition 2.4.3 (60). This proves point i).

The uniformity claimed in proposition 2.4.3 (59) and corollary 3 of proposition 2.4.3 (60) prove that the semi-norm of corollary 2 of proposition 2.4.3 (60) is continuous. This implies that \mathfrak{m}_p^0 is a closed subspaces of $\mathcal{A}_p^1(M)$ [9].

It follows that our norm is also continuous, proving point ii).

On the other hand, if \mathfrak{m}_p^0 is closed, the quotient topology is complete, proving point iii). \square

Remark 2.25 $\|f\|$ is the smallest possible choice for the Lipschitz constant of proposition 2.4.3 (59) in corollary 4 of proposition 2.4.3 (61).

The function ∂ can be seen either as a coupling of $V_p(M) \times C_p(M)$ to \mathbb{R} or, equivalently, as mapping from $C_p(M)$ to $V_p^*(M)$, the Banach space dual of $V_p(M)$.

Corollary 1 *Let (M, d) be a scale bounded space and $p \in M$.*

The coupling $V_p(M) \times C_p(M)$ to \mathbb{R} defined as $([f], v) \mapsto \partial_v[f]$ is non-degenerated, i. e.

- i) *if $([f], v) = 0$ for any $v \in C_p(M)$, then $[f] = 0$ and*
- ii) *if $([f], v) = 0$ for any $[f] \in V_p(M)$, then $v = 0$.*

Remark 2.26 The above relations show that $C_p(M)$ embeds naturally into $V_p^*(M)$ and that the image is a generating system for $V_p^*(M)$.

Proof:

By definition 2.4.4 (60), $(f, v) = 0$ for any $v \in C_p(M)$ implies that $\|f\| = 0$. By proposition 2.4.6 (61), point i), this implies $f = 0$.

Suppose $v \in C_p(M)$ and $v \neq 0$ and set $A = \exp(-\epsilon v)$ with $\epsilon > 0$ such that $A \in B_p$. Then $\partial_v \delta_A = \|v\| > 0$ so that, by contraposition, $(f, v) = 0$ for any $f \in V_p(M)$ implies $v = 0$. \square

2.5 Gradient and 1-forms

2.5.1 The gradient

For any $f \in \mathcal{A}_p^1$, there are directions $v \in S_p(M)$ fulfilling $\partial_v f = \|f\|$ (see remark 2.23 (60)). If this direction was unique, it could define a gradient for f . The existence of a continuous gradient field for every function in $\mathcal{C}^{1+}(\mathcal{C})$ is the scope of that section.

Some nice results like definition of dimension for scale bounded spaces can immediately be derived.

Definition 2.5.1 Let (M, d) be a scale bounded space, $p \in M$ and $f \in \mathcal{A}_p^1$.

There are directions $v \in S_p(M)$ such that $\partial_v f = \|f\|$ (see remark 2.23 (60)).

If this direction is unique or if $\|f\| = 0$, we can define ∇f as $\|f\| \cdot v \in C_p(M)$. We call it the gradient of f .

When defined, we write $\nabla_q f$ instead of $\nabla[f]_q$ and with ∇f we mean the section $q \mapsto \nabla_q f$.

Remark 2.27 By remark 2.14 (55), $\nabla(\lambda f) = |\lambda| \nabla f$ for any $\lambda \in \mathbb{R}$.

Remark 2.28 By remark 2.14 (55), $\partial_{\nabla f} f = \|f\|^2$.

Proposition 2.5.2

Let (M, d) be a scale bounded space and $p \in M$.

Then $\nabla_q \delta_p$ exists for $q \in B_p^*$ and equals $\sigma_p(q)$.

Proof:

Suppose $v \in S_q(M)$ such that $\partial_v \delta_p(q) = 1$. Applying proposition 1.4.8 (40) to $\gamma_t := \exp(tv)$ we obtain:

$$\delta_p(\gamma_t) = \delta_p(\gamma_0) + t = \overline{pq} + t$$

where we used the fact that $\sin \phi = \partial_v \delta_p(q) = 1$ and, consequently, $\cos \phi = 0$. But this means that $\gamma_{-\overline{pq}} = p$ such that γ is a geodesic with trace in B_p through p and q . By proposition 1.4.7 (39), γ is the unique segment between p and q such that the direction v that γ defines in $t = 0$ is, following proposition 1.4.7 (39), $\sigma_p(q)$.

On the other hand we know, by remark 2.15 (58), that $\|[\delta_p]_q\| = 1$ such that finally $\nabla_q \delta_p = \sigma_p(q)$. \square

Corollary 1 For any $v \in C_p(M)$, there is an $f \in \mathcal{A}_p^1$ such that $\nabla f = v$.

More precisely, there is a point A such that $f = [\|v\| \delta_A]_p$.

Proof:

We first suppose $v \in S_p(M)$. Choose an $\epsilon \in]0, \iota_p[$ and set $A = \exp(-\epsilon v)$. By

proposition 2.3.6 (56), $[\delta_A]_p \in \mathcal{A}_p^1(M)$ and by proposition 2.5.2 its gradient is v .

The generalization to $v \in C_p(M)$ follows from remark 2.27 . \square

Theorem 2.5.3

Let (M, d) be a scale bounded space and $p \in M$.

Suppose there is a $v \in S_p(M)$ and two germs $\mathfrak{f}, \mathfrak{g} \in \mathcal{A}_p^1(M)$ such that:

$$\partial_v \mathfrak{f} = \|\mathfrak{f}\| = \|\mathfrak{g}\| = \partial_v \mathfrak{g}.$$

Then

$$\partial \mathfrak{f} = \partial \mathfrak{g}.$$

The proof is postponed to page 65.

Corollary 1 *The gradient $\nabla \mathfrak{f}$ exists.*

Proof:

By remark 2.23 (60), there are vectors v in $S_p(M)$ such that $\partial_v \mathfrak{f} = \|\mathfrak{f}\|$.

By corollary 1 of proposition 2.5.2 (63), there is an A such that $\|[\delta_A]_p\| = \partial_v \delta_A = 1$ such that, by theorem 2.5.3 , $\partial \mathfrak{f} = \|\mathfrak{f}\| \cdot \partial \delta_A$.

But, by proposition 2.5.2 (63), $\nabla_p \delta_A$ exists such that $\nabla \mathfrak{f} = \nabla_p \delta_A$ exists. \square

Corollary 2 *Let $\mathfrak{f} \in \mathcal{A}^1(M)$. $\nabla \mathfrak{f}$ defines uniquely $\partial \mathfrak{f}$.*

Proof:

This follows from corollary 1 of proposition 2.5.2 (63) and theorem 2.5.3 . \square

Corollary 3 *Let $\epsilon \in]0, \iota_p[$. Then $\partial \delta_{\exp(\epsilon v)}(p)$ does not depend on the choice of ϵ .*

This is a generalization of corollary 1 of lemma 1.4.9 (42).

Proof:

By corollary 1 of proposition 2.5.2 (63), $\nabla_p \delta_{\exp(\epsilon v)} = v$ for every $\epsilon \in]0, \iota_p[$ such that, by corollary 2 of theorem 2.5.3 , $\partial \delta_{\exp(\epsilon_1 v)} = \partial \delta_{\exp(\epsilon_2 v)}$ for every $\epsilon_1, \epsilon_2 \in]0, \iota_p[$. \square

Remark 2.29 By the above corollary, $\nabla \mathfrak{f} = \nabla \mathfrak{g}$ implies $\partial \mathfrak{f} = \partial \mathfrak{g}$ for any $\mathfrak{f}, \mathfrak{g} \in \mathcal{A}_p^1(M)$. As $\partial \mathfrak{m}^0 = \{0\}$ and consequently, $\nabla \mathfrak{m}^0 = \{0\}$, ∇ extends naturally to an application from the quotient space $V_p(M)$ to $C_p(M)$ (recall definition 2.4.5 (61)).

Corollary 4 *∇ as application from $V_p(M)$ to $C_p(M)$ is norm preserving and bijective.*

Proof:

We have to prove that:

1. $\triangleright \nabla$ is onto: This follows from corollary 1 of proposition 2.5.2 (63). \triangle
2. $\triangleright \nabla$ preserves the norm: By definition 2.5.1 (63) of ∇ and the norm definition in proposition 2.4.6 (61), $\|\nabla f\| = \|\partial f\|$ is obvious. \triangle
3. $\triangleright \nabla$ is injective: Let $v_1^*, v_2^* \in V_p(M)$ be represented by $f_1, f_2 \in \mathcal{A}_p^1(M)$ respectively. If we suppose $\nabla v_1^* = \nabla v_2^*$, we also have $\nabla f_1 = \nabla f_2$ (see remark 2.29 (64)).

By corollary 1 of theorem 2.5.3 (64) the last relation implies that $\partial f_1 = \partial f_2$ or $\partial(f_1 - f_2) = 0$ such that $f_1 - f_2 \in \mathfrak{m}_0$ and, by definition 2.4.5 (61), $v_1^* = v_2^*$: ∇ is injective. \triangle

□

Proposition 2.5.4

Let (M, d) be a scale bounded space, \mathcal{U} an open set of M .

If $f \in \mathcal{C}^{1+}(\mathcal{U})$ then ∇f is a continuous section of $C(\mathcal{U})$.

Proof:

By corollary 1 of theorem 2.5.3 (64), ∇f is defined.

1. $\triangleright \|\nabla f\|$ is continuous: By proposition 2.4.6 (61) point ii), $q \mapsto \|[f]_q\|$ is continuous. The claim follows from corollary 4 of theorem 2.5.3 (64). \triangle
2. \triangleright If $\nabla_p f = 0$, ∇f is continuous in p : As a neighbourhood basis of 0 is given by balls of the form $\{v \in C_p(M) \mid \|v\| < \epsilon\}$, continuity of $\|\nabla f\|$ implies the continuity of ∇f in p . \triangle
3. \triangleright If $\nabla_p f \neq 0$, ∇f is continuous in p : We can assume, without loss of generality, that $\|\nabla_p f\| = 1$. As $\|\nabla f\|$ is continuous, we may also assume, up to a restriction of \mathcal{U} , that $\nabla f \neq 0$ in any point of \mathcal{U} .

By corollary 1 of proposition 2.5.2 (63) and corollary 3 of theorem 2.5.3 (64), we know that if $\sigma_q(p) = \nabla_p f$ then $q = \exp(-\epsilon \nabla_p f) =: q_\epsilon$ where $\epsilon \in]0, \iota_p[$.

But, as $f \in \mathcal{C}^{1+}(\mathcal{U})$, the function $\partial_{\sigma_{q_\epsilon}} f$ is Hölder-continuous (see definition 2.3.4 (56)). This means that if q is close to p , $\partial_{\sigma_{q_\epsilon}} f(p) = 1$ is close to $\partial_{\sigma_{q_\epsilon}} f(q)$. We then also know that $\|\nabla_p f\| = 1$ is also close to $\|\nabla_q f\|$ such that, by proposition 2.4.3 (59), $\frac{1}{\|\nabla_q f\|} \nabla_q f \in S_q(M)$ is close to $\sigma_{q_\epsilon}(q)$.

Following the definition of the topology of $S(M)$ (see definition 2.2.9 (50)), this means that $\frac{1}{\|\nabla_q f\|} \nabla_q f$ as a function of q is continuous in p . By proposition 2.2.10 (51), this implies the continuity of ∇f in p . \triangle

□

Proof of theorem 2.5.3:

We will prove the theorem for the case where $\|f\| = 1$ and \mathfrak{g} is the germ of a distance function δ_A with $\nabla \mathfrak{g} = v$. By corollary 1 of proposition 2.5.2 (63), such a germ always exists.

The above restrictions do not restrict the generality of the proof: if the claim is true in the above described situation, it will be true between two germs \mathfrak{f} and \mathfrak{h} with unit norm, as $\partial\mathfrak{f}$ and $\partial\mathfrak{h}$ are both equal through a common distance function.

Once there, the result generalizes immediately to the case $\|\mathfrak{f}\| \neq 1$ giving the relation $\partial\mathfrak{f} = \|\mathfrak{f}\| \cdot \partial[\delta_A]_p$.

Let $f \in \mathfrak{f}$ be defined over \mathcal{U} . This choice is not unique; this freedom will be used as follows:

1. \triangleright Let $\gamma_t := \exp(tv)$. There is a function $f \in \mathfrak{f}$, such that $f \circ \gamma_t - f(p) = t$.

Let h be an arbitrary function in \mathfrak{f} . As $h \circ \gamma_t$ and $\overline{p\gamma_t} = \delta_p(\gamma_t)$ are $\mathcal{C}^{1,1}$ in t (see definition 2.3.4 (56) and proposition 2.3.6 (56)):

$$\overline{p\gamma_t}^2 R(\overline{p\gamma_t}) = \delta_p(\gamma_t) \partial_v h - h(\gamma_t) + h(p)$$

is also $\mathcal{C}^{1,1}$ in t .

Referring to remark 2.18 (58), we know that $\overline{p\gamma_t}^2 R(\overline{p\gamma_t})$ as a function of q is in $\mathcal{C}^{1+}(\mathcal{U})$. By remark 2.17 (58), the germ of this function in p is 0.

Consequently, $f(q) := h(q) - \overline{p\gamma_t}^2 R(\overline{p\gamma_t})$ as a function of q is also in $\mathcal{C}^{1+}(\mathcal{U})$ and f is in the same class \mathfrak{f} as h . By construction, we have $f \circ \gamma_t - f(p) = t$. \triangle

From now on, we suppose that the choice of $f \in \mathfrak{f}$ is the same as the one described in point 1).

2. \triangleright Let $w \in S_p(M)$. Then $|\partial_w \mathfrak{f}| \leq |\partial_w \delta_A|$.

In order to simplify notations, we introduce following definitions:

- i) $p_\xi := \exp(\xi v)$, such that $p_0 = p$,
 - ii) $r_0 \in]0, \iota_p[$, $B := \exp(-r_0 w)$ and $w_\xi := \sigma_B(p_\xi)$,
 - iii) $x_\xi^\epsilon := \exp(\epsilon w_\xi)$,
 - iv) $v_\xi^\epsilon \in S_p$ is the tangent vector to the geodesic from p to x_ξ^ϵ .
- We will, moreover, suppose that
- v) $\epsilon \in [-r_0, r_0]$, $\xi \in \{2^{-n}r_0 \mid n \in \mathbb{N}\} \cup \{0\}$ and $\xi \leq \epsilon$.

Following this definitions and the choice of f made in point 1, we know that:

$$f(p_\xi) - f(p) = \overline{p p_\xi} \partial_{v_\xi^0} f = \xi \quad (2.8)$$

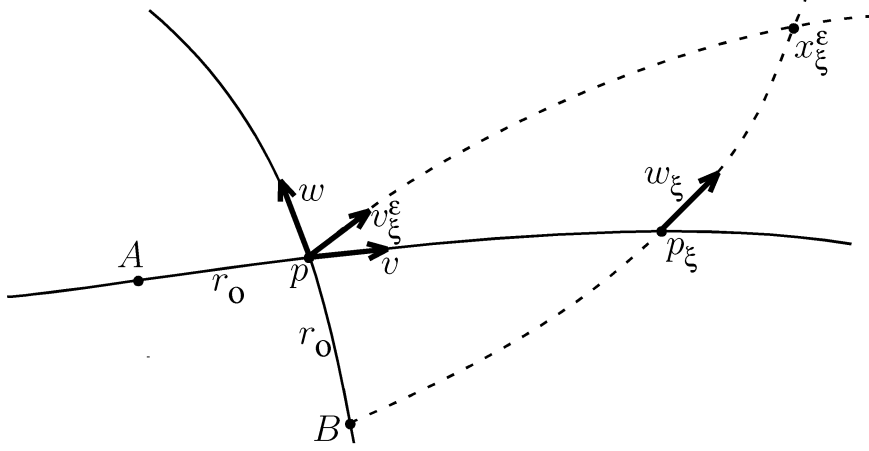
Using the estimations of proposition 1.4.8 (40), we have that:

$$f(x_\xi^\epsilon) - f(p_\xi) = \epsilon \partial_{w_\xi} f + O_\xi(\epsilon^2) \quad (2.9)$$

and

$$f(x_\xi^\epsilon) - f(p) = \overline{p x_\xi^\epsilon} \partial_{v_\xi^\epsilon} f + O\left(\overline{p x_\xi^\epsilon}^2\right).$$

As $\xi \leq \epsilon$, $\overline{p x_\xi^\epsilon}$ is of the order of ϵ so that:

Figure 2.5: Construction w_ξ , p_ξ and x_ξ^ϵ .

$$f(x_\xi^\epsilon) - f(p) = \overline{p x_\xi^\epsilon} \partial_{v_\xi^\epsilon} f + O_\xi(\epsilon^2). \quad (2.10)$$

In both equation 2.9 and 2.10 the rest terms $O_\xi(\epsilon^2)$ depend continuously on ξ .

Adding the equations 2.8 (66) and 2.9 (66) together and subtracting equation 2.10, we obtain:

$$0 = \xi + \epsilon \partial_{w_\xi} f - \overline{p x_\xi^\epsilon} \partial_{v_\xi^\epsilon} f + O_\xi(\epsilon^2) \quad (2.11)$$

But, by proposition 2.3.6 (56), we know that:

$$\begin{aligned} \overline{p x_\xi^\epsilon} &= \overline{p p_\xi} + \epsilon \partial_{w_\xi} \delta_p + O_\xi(\epsilon^2) \\ &= \xi + \epsilon \partial_{w_\xi} \delta_A + O_\xi(\epsilon^2) \end{aligned} \quad (2.12)$$

where we used corollary 1 of lemma 1.4.9 (42) which tells us that $\partial_{w_\xi} \delta_p = \partial_{w_\xi} \delta_A$.

By substitution of 2.12 in 2.11, we obtain

$$\xi + \epsilon \partial_{w_\xi} f - \left(\xi + \epsilon \partial_{w_\xi} \delta_A + O(\epsilon^2) \right) \partial_{v_\xi^\epsilon} f + O_\xi(\epsilon^2) = 0.$$

Dividing by ϵ and arranging the terms, we get:

$$\partial_{w_\xi} f - \partial_{w_\xi} \delta_A \cdot \partial_{v_\xi^\epsilon} f = \xi \frac{\partial_{v_\xi^\epsilon} f - 1}{\epsilon} + \frac{O_\xi(\epsilon^2)}{\epsilon}$$

By construction, w_ξ is continuous in ξ such that $\lim_{\xi \rightarrow 0} w_\xi = w_0 = w$. Applying the upper limit $\xi \rightarrow 0$ on both sides, we obtain:

$$\partial_w f - \partial_w \delta_A \cdot \lim_{\xi \rightarrow 0} \partial_{v_\xi^\epsilon} f = \lim_{\xi \rightarrow 0} \epsilon \frac{O_\xi(\epsilon^2)}{\epsilon^2}$$

As the rest term $\frac{O_\xi(\epsilon^2)}{\epsilon^2}$ can be homogeneous bounded relatively to ξ (this follows from the uniformity in proposition 1.4.8 (40))

$$\lim_{\epsilon \rightarrow 0} \overline{\lim}_{\xi \rightarrow 0} \epsilon \frac{O_\xi(\epsilon^2)}{\epsilon^2} = 0$$

and using the fact that $|\partial_{v_\xi} f| \leq 1$, we obtain:

$$|\partial_w f| \leq |\partial_w \delta_A|.$$

△

The last result we need to conclude is the following:

3. ▷ Let $v \in S_p(M)$, \mathcal{V} be a neighbourhood of w in $S_p(M)$ and $\mathfrak{h} \in \mathcal{A}_p^1(M)$. If $\partial_w \mathfrak{h} = 0$ and $\partial_{w'} \mathfrak{h} \leq 0 \quad \forall w' \in \mathcal{V}$ then $\|\mathfrak{h}\| = 0$.

We prove it by contraposition and suppose $\mathfrak{h} \neq 0$ such that, without restriction, we can assume that $\|\mathfrak{h}\| = 1 = \partial_v \mathfrak{h}$ where $v \in S_p(M)$. Let $h \in \mathfrak{h}$ be defined in a convex neighbourhood \mathcal{U} of p .

Let $A = \exp(-r_0 v)$ where $r_0 > 0$ such that $A \in \mathcal{U}$. By construction, in $\sigma_A(p) = v$. As $\partial_v \mathfrak{h} = 1$ and as $\partial_{\sigma_A} \mathfrak{h}(q)$ is continuous, we can suppose, without restriction of generality, that $\partial_{\sigma_A} \mathfrak{h}(q) \geq \frac{1}{2}$ if $q \in \mathcal{U}$.

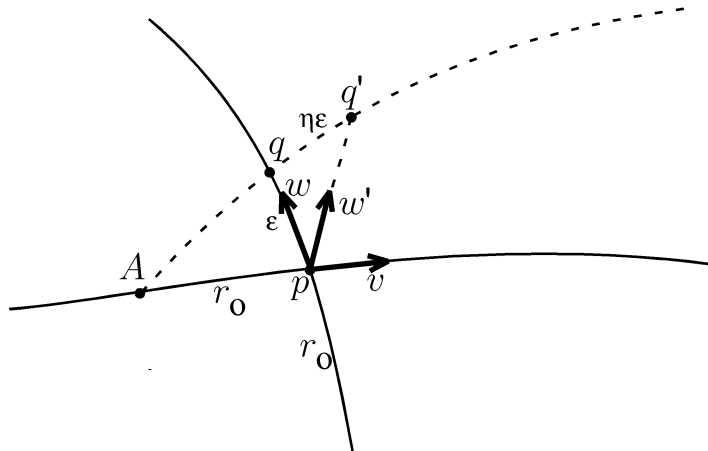


Figure 2.6: Construction A , q and q' .

Suppose $\epsilon > 0$ and $\eta \in]0, \frac{1}{2}]$ and call $q := \exp(\epsilon w)$, $q' := \exp(\eta \epsilon \sigma_A(q))$ and $w' \in V_p(M)$ such that $q' = \exp(\epsilon w')$. As $\eta \leq \frac{1}{2}$, triangle inequality implies $\|w'\| \geq \frac{1}{2}$.

We know, by proposition 1.4.8 (40), that:

$$\begin{aligned} h(q) - h(p) &= \epsilon \partial_w h + O(\epsilon^2) = O(\epsilon^2) \\ h(q') - h(q) &= \eta \epsilon \partial_{\sigma_A} h(q) + O(\epsilon^2) \\ h(q') - h(p) &= \epsilon \partial_{w'} h + O(\epsilon^2) \end{aligned} \tag{2.13}$$

where all rest terms can be uniformly bounded such that $|O(\epsilon^2)| \leq \frac{1}{3}L\epsilon^2$ for some $L > 0$.

Adding the first equation and subtraction the last, we obtain:

$$0 = \eta \epsilon \partial_{\sigma_A} h(q) - \epsilon \partial_{w'} h + O(\epsilon^2)$$

Dividing the equation by ϵ we have:

$$|\eta \partial_{\sigma_A(q)} h - \partial_{w'} h| \leq L\epsilon$$

But we know that $\partial_{\sigma_A} h(q) \geq \frac{1}{2}$, such that, for ϵ small enough and every $0 < \eta \leq \frac{1}{2}$, $\partial_{w'} h > 0$. As, for $\eta \rightarrow 0$, $w' \rightarrow w$, this construction gives, for every neighbourhood of w a w' such that $\partial_{w'} h > 0$. Hence $\|\mathfrak{h}\| = 0$. \triangle

By construction, $\partial_v \mathfrak{f} = \partial_v \delta_A$ such that $\partial_v (f - \delta_A) = 0$. From point 2 we know that $|\partial_w \mathfrak{f}| \leq |\partial_w \delta_A|$. As $\partial_v \delta_A = 1$ and by proposition 2.4.3 (59) this implies that there is a neighbourhood \mathcal{V} of v where $\partial_w (f - \delta_A) \leq 0 \quad \forall w \in \mathcal{V}$.

By point 3 and setting $\mathfrak{h} = \mathfrak{g} - \mathfrak{f}$, this implies that $\partial \mathfrak{g} = \partial \mathfrak{f}$. \square

2.5.2 Dimension of scale bounded spaces

Proposition 2.5.5

Let (M, d) be a scale bounded space and $p \in M$.

Then ∇ is a norm preserving homeomorphism¹ from $V_p(M)$ to $C_p(M)$.

The proof is postponed to page 70.

Corollary 1 Let (M, d) be a scale bounded space.

The fibers of $V(M)$ have all the same finite dimension.

Proof:

By remark 2.12 (52), $S_p(M)$ is compact. Using proposition 2.5.5, we conclude that the unit ball in $V_p(M)$ is also compact. But a normed locally compact Banach space (see proposition 2.4.6 (61)) is of finite dimension.

We further know, by proposition 2.2.12 (53), that all fibers of $C_p(M)$ are homeomorphic. So are the fibers of $V(M)$ again by proposition 2.5.5. \square

Definition 2.5.6 Let (M, d) be a scale bounded space and $p \in M$.

The dimension of M is the vectorial dimension of the fibers of $V(M)$.

Corollary 2 Let (M, d) be a scale bounded space.

The mapping ∂ from $C_p(M)$ to $V_p^*(M)$ is bijective.

¹The result is not an immediate consequence of corollary 4 of theorem 2.5.3 (64): be reminded that we have no additive structure over $C_p(M)$ yet such that the norm defines no topology.

Proof:

As seen in the proof of corollary 1 of proposition 2.5.5, $V_p(M)$ has a compact unit ball, so has its dual $V_p^*(M)$.

Hence, for every $\phi \in V_p^*(M)$, there are unitary $\mathbf{f} \in V_p(M)$ such that $\phi(\mathbf{f}) = \|\phi\|$.

▷ *This \mathbf{f} is unique:* Suppose both \mathbf{f} and \mathbf{g} are unitary and verify $\phi(\mathbf{f}) = \phi(\mathbf{g}) = \|\phi\|$ such that, by linearity, $\phi(\frac{1}{2}(\mathbf{f} + \mathbf{g})) = \|\phi\|$.

To avoid a contraction with the definition of the norm of $\phi \in V_p^*(M)$, $\|\frac{1}{2}(\mathbf{f} + \mathbf{g})\| \geq 1$. On the other hand, triangle inequalities ask for $\|\frac{1}{2}(\mathbf{f} + \mathbf{g})\| \leq 1$ such that $\|\mathbf{f} + \mathbf{g}\| = 2$.

This, in turn means that there is a $v \in S_p(M)$ such that $\partial_v(\mathbf{f} + \mathbf{g}) = 2$ (see definition 2.5.1 (63) and corollary 1 of theorem 2.5.3 (64)). But as $\partial_v \mathbf{f} \leq \|\mathbf{f}\| = 1$ and $\partial_v \mathbf{g} \leq \|\mathbf{g}\| = 1$, we must have $\partial_v \mathbf{f} = \partial_v \mathbf{g} = \|\mathbf{f}\| = \|\mathbf{g}\| = 1$ such that $\nabla \mathbf{f} = \nabla \mathbf{g}$.

As ∇ is a homeomorphism, $\mathbf{f} = \mathbf{g}$ proving the uniqueness we claimed. \triangle

This result allows us to define, similarly to what has been done in definition 2.5.1 (63), the mapping ∇^* from $V_p^*(M)$ to $V_p(M)$ defined by $\nabla^* \phi = \|\phi\| \cdot \mathbf{f}$ where ϕ and \mathbf{f} are defined as above.

Due to the uniqueness proved above ∇^* is injective and, by construction, norm preserving.

▷ *∂ as mapping from $C_p(M)$ to $V_p^*(M)$ is injective:* For each $v \in C_p(M)$, ∂_v is a linear form from $V_p(M)$ to \mathbb{R} . Continuity of the form is ensured by the definition of the norm in $V_p(M)$ given by definition 2.4.4 (60). Injectivity follows from corollary 1 of proposition 2.4.6 (62) point ii. \triangle

▷ *The composition $\partial \circ \nabla \circ \nabla^*$ is the identity mapping on $V_p^*(M)$:* Let $\phi \in V_p^*(M)$ and set $\mathbf{f} := \nabla^* \phi \in V_p(M)$, $v := \nabla \mathbf{f} \in C_p(M)$ and $\psi := \partial_v \in V_p^*(M)$.

We know that $\|\phi\| = \|\mathbf{f}\| = \|v\|$. $\psi(\mathbf{f}) = \partial_{\nabla \mathbf{f}} \mathbf{f} = \|\mathbf{f}\|^2$ but, by definition of ∇ , there no \mathbf{g} with same norm as \mathbf{f} hence $\psi(\mathbf{g}) \geq \|\mathbf{f}\|^2$ such that $\nabla^* \psi = \mathbf{f}$. As ∇^* is injective, it follows that $\psi = \phi$. \triangle

As proved previously ∂ as injective, as are ∇ and ∇^* such that $\partial \circ \nabla \circ \nabla^*$ is injective.

As, on the other hand, the above composition is the identity mapping, all 3 mappings are bijective, in particular ∂ . \square

Proof of proposition 2.5.5:

Let $W_p(M)$ be the set of unit vectors of $V_p(M)$. The space $V_p(M)$ can be seen as a real cone over that set as $C_p(M)$ is a real cone over $S_p(M)$ (see definition 2.2.2).

As, by corollary 4 of theorem 2.5.3 (64), ∇ is norm preserving and a bijective application between $V_p(M)$ and $C_p(M)$, it is also a bijective between $W_p(M)$ and $S_p(M)$.

Hence, for ∇ to be a homeomorphism it is enough to show that it is a homeomorphism on the restrictions $W_p(M)$ and $S_p(M)$.

The proof is split in two parts. The first shows that ∇ is continuous on the

defined restriction, the second that ∇ is open. In both parts we will use the following notations:

Let v_0 and v be elements in $S_p(M)$, $r \in]0, 1[$ such that $B_p(\sqrt{r}) \subseteq B_p$ is a neighbourhood as defined in proposition 1.4.8 and L the corresponding Lipschitz constant.

Let $A_0 := \exp(-r v_0)$ and $A := \exp(-r v)$. By corollary 1 of proposition 2.5.2 and corollary 3 of theorem 2.5.3, $\nabla \delta_{A_0} = v_0$ and $\nabla \delta_A = v$.

1. $\triangleright \nabla|_{W_p(M)}$ is continuous: It is enough to prove that there is a constant $C > 0$ such that

$$d^{\sup}(v_0, v) \leq \frac{1}{2}r \implies \|[\delta_{A_0}]_p - [\delta_A]_p\|^2 \leq C d^{\sup}(v_0, v).$$

Let $w \in W_p(M)$ and $\gamma_t := \exp(tw)$, define $\epsilon := \sqrt{r}$ and $r_{A_0} := \overline{A_0} \gamma_\epsilon$, $r_A := \overline{A} \gamma_\epsilon$.

Applying proposition 1.4.8 on the triangles $(p, \gamma_\epsilon, A_0)$, (p, γ_ϵ, A) and $(A_0, A, \gamma_\epsilon)$, we obtain:

$$\begin{aligned} \left| \partial_w[\delta_{A_0}] - \frac{r_{A_0} - r}{\epsilon} \right| &\leq W_p(M) \frac{L}{r - \epsilon^2} \epsilon \\ \left| \partial_w[\delta_A] - \frac{r_A - r}{\epsilon} \right| &\leq \frac{L}{r - \epsilon^2} \epsilon \\ \left| \frac{r_{A_0} - r_A}{\epsilon^2} \right| &\leq |\partial_{w'}[\delta_{p_\epsilon}]| + \frac{L}{r - \epsilon} \epsilon^2 \leq 1 + \frac{L}{r - \epsilon} \epsilon^2 \end{aligned}$$

where $w' \in C_{A_0}(M)$ is the unit vector from A_0 to A and we recall that $\epsilon^2 = \overline{A_0} \overline{A}$. The first two inequalities combine to

$$\left| (\partial_w[\delta_{A_0}] - \partial_w[\delta_A]) - \frac{r_{A_0} - r_A}{\epsilon} \right| \leq 2 \frac{L}{r - \epsilon^2} \epsilon$$

or, equivalently, to

$$|\partial_w[\delta_{A_0}] - \partial_w[\delta_A]| \leq \left(1 + \frac{L}{r - \epsilon^2} \epsilon \right) \left| \frac{r_{A_0} - r_A}{\epsilon} \right|.$$

Using the third inequality above, we end up with

$$|\partial_w[\delta_{A_0}] - \partial_w[\delta_A]| \leq \left(1 + \frac{L}{r - \epsilon^2} \epsilon \right) \left(1 + \frac{L}{r - \epsilon} \epsilon^2 \right) \epsilon.$$

By remark 2.6, there is a constant C' such that $2r\epsilon^2 = 2r\overline{A_0} \overline{A} \leq C' d^{\sup}(v_0, v)$ such that finally

$$|\partial_w[\delta_{A_0}] - \partial_w[\delta_A]| \leq C \sqrt{d^{\sup}(v_0, v)}$$

with C the non-zero infimum:

$$C := \inf_{\epsilon \in [-\sqrt{r}, \sqrt{r}]} \sqrt{\frac{C'}{2r} \left(1 + \frac{L}{r - \epsilon^2} \epsilon \right) \left(1 + \frac{L}{r - \epsilon} \epsilon^2 \right)}.$$

\triangle

2. $\triangleright \nabla|_{W_p(M)} \longrightarrow C_p(M)$ is open: We will prove that there is a constant C such that

$$\|[\delta_{A_0}]_p - [\delta_A]_p\| \leq \epsilon \implies d^{\text{sup}}(v_0, v) \leq C \epsilon.$$

Let w be the unit vector issued from p to A . As $\|[\delta_{A_0}]_p - [\delta_A]_p\| \leq \epsilon$, $|\partial_w[\delta_A]_p - \partial_w[\delta_{A_0}]_p| \leq \epsilon$ or, by construction $|-1 - \partial_w[\delta_{A_0}]_p| \leq \epsilon$.

Let $\sin(\phi) := \partial_w[\delta_{A_0}]_p$. The last inequality tells us that $\sin(\phi) \leq -(1 - \epsilon)$ and that $\cos^2(\phi) \leq 1 - (1 - \epsilon)^2 \leq 2\epsilon$.

We know by proposition 1.4.8 (40) that

$$\delta_{A_0}(A) = r + r \sin(\phi) + r^2 \cos^2(\phi) \frac{R}{r}$$

where R depends on p, A and A_0 but can be bound uniformly for any choice of them within B_p such that

$$\frac{\delta_{A_0}(A)}{r} = 1 + \sin(\phi) + \cos^2(\phi)R$$

Using the above inequalities

$$\frac{\delta_{A_0}(A)}{r} = \epsilon(1 + 2R)$$

On the other hand, we know by remark 2.6 that there is a C''

$$\frac{\delta_{A_0}(A)}{r} \geq C'' d^{\text{sup}}(v_0, v)$$

such that, finally, with $C := \frac{1+2R}{C''}$:

$$d^{\text{sup}}(v_0, v) \leq C/\epsilon.$$

□

Chapter 3

Scale bounded spaces and Finsler spaces

3.1 Introduction

In section 3.2 we will construct distance charts in a similar way as Berestovskij [6] did. As we have no vector structure on $C_p(M)$, we use $V_p(M)$ to construct them. The section ends with the proof that these charts form a $\mathcal{C}^{1, \frac{1}{2}}$ atlas on M .

Once we have defined a differential structure on scale bounded spaces, the distance function induces a norm in every tangent space such that the distance of the resulting Finsler space is compatible with that of the scale bounded space (see section 3.3 (78)).

Later in this chapter we will give a definition of Finsler spaces that suits our purpose. We will prove that scale bounded spaces are spaces of that type.

For more details on Finsler spaces, we refer to the original text [17] or the text books [23], [14], [1], [4] or [5].

3.2 The differential structure

3.2.1 Local frames

We can now construct a $\mathcal{C}^{1, \frac{1}{2}}$ atlas A for M . To that end, we first need two technical lemmas:

Remark 3.1 As shown in [25], every normed vector space of finite dimension has a basis $b = (b_1, b_2, \dots, b_n)$ such that

$$\frac{d}{d\epsilon} \|b_i + \epsilon b_j\| = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In particular, there is a basis $b = (f_1, f_2, \dots, f_n)$ of $V_p(M)$ verifying:

$$\partial_{\nabla f_i} f_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Definition 3.2.1 Let (M, d) be a scale bounded space, $p \in M$, $b := (f_1, f_2, \dots, f_n)$ a basis for $V_p(M)$ and $r \in]0, \iota_p[$.

With $\Phi_{b,r}^{(i)}$, where $1 \leq i \leq n$, we denote the application from M to \mathbb{R} defined as

$$q \mapsto d(q, \exp(r \nabla f_i)).$$

$\Phi_{b,r}$ is the application from M to \mathbb{R}^n sending q to $(\Phi_{b,r}^{(1)}, \dots, \Phi_{b,r}^{(n)})$.

The restriction of $\Phi_{b,r}$ to an open set on which it is open and injective is called a distance chart.

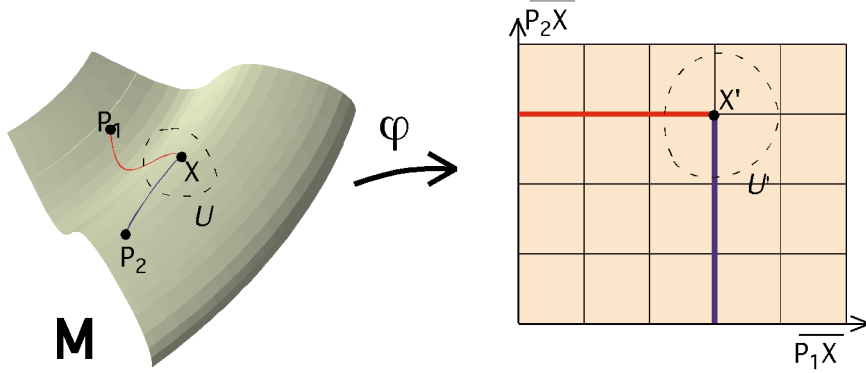


Figure 3.1: Example of distance charts. We set $P_i := \exp(r \nabla f_i)$.

$\Phi_{b,r}$ and $\Phi_{b,r}^{(i)}$ are of course continuous. This definition leads us to a technical lemma:

Lemma 3.2.2 Let (M, d) be a scale bounded space of dimension n and $p \in M$.

Suppose given an application $\Phi_{b,r}$ as defined in definition 3.2.1 such that b verifies the conditions of remark 3.1 (73). For every $\epsilon > 0$, there is a neighbourhood \mathcal{U} of p such that for $q \in \mathcal{U}$ we have:

$$\left(\left[\Phi_{b,r}^{(1)} \right]_q, \left[\Phi_{b,r}^{(2)} \right]_q, \dots, \left[\Phi_{b,r}^{(n)} \right]_q \right)$$

is a basis of $V_q(M)$ and $\left| \partial_{\nabla_q \Phi^{(i)}} \Phi^{(j)} \right| \leq \epsilon$ for $1 \leq i < j \leq n$.

Proof:

We drop, in this proof, the index b, r of $\Phi_{b,r}$. Suppose we know that for an $m < n$, there is a compact neighbourhood \mathcal{U}_m of p and an $\epsilon_m > 0$ such that:

1. for any point $q \in \mathcal{U}_m$ the vectors $\left([\Phi^{(1)}]_q, [\Phi^{(2)}]_q, \dots, [\Phi^{(m)}]_q\right)$ are linearly independent in $V_q(M)$ and
2. in any point $q \in \mathcal{U}_m$ we verify that

$$\left| \partial_{\nabla_q \Phi^{(i)}} \Phi^{(j)} \right| \leq \epsilon_m$$

for any $1 \leq i < j \leq n$.

In any point $q \in \mathcal{U}_m$ the vector list $\left([\Phi^{(1)}]_q, [\Phi^{(2)}]_q, \dots, [\Phi^{(m)}]_q\right)$ spans an m -dimensional subspace W_m of $V_q(M)$. Any vector f of this subspace is written uniquely as $f = \sum_{i=1}^m a_i [\Phi^{(i)}]_q$. We define $\|f\|_m := \sum_{i=1}^m |a_i|$. By norm equivalence on finite dimensional vector spaces, there is a $c_m > 0$ such that $\|f\|_m \leq c_m \|f\|$ for any $f \in W_m$.

As \mathcal{U}_m is compact, c_m can be chosen such that this constant is uniform over \mathcal{U}_m .

Let us suppose that in $q \in \mathcal{U}_m$, $[\Phi^{(m+1)}]_q = \sum_{i=1}^m a_i [\Phi^{(i)}]_q$ for some a_i . Using the hypothesis above, we then have:

$$\begin{aligned} \left| \partial_{\nabla_q \Phi^{(m+1)}} \Phi^{(m+1)} \right| &= \left| \sum_{i=1}^m a_i \partial_{\nabla_q \Phi^{(i)}} \Phi^{(m+1)} \right| \\ &\leq \sum_{i=1}^m |a_i| \left| \partial_{\nabla_q \Phi^{(i)}} \Phi^{(m+1)} \right| \\ &\leq \sum_{i=1}^m |a_i| \epsilon_m \\ &\leq \left\| [\Phi^{(m+1)}]_q \right\|_m \epsilon_m \leq c_m \left\| \nabla [\Phi^{(m+1)}]_q \right\| \epsilon_m \end{aligned}$$

On the other hand, we know by remark 2.28 (63) that $\partial_{\nabla_q \Phi^{(m+1)}} \Phi^{(m+1)} = 1$ and by construction $\left\| [\Phi^{(m+1)}]_q \right\| = 1$ so that we would have $1 \leq c_m \epsilon_m$.

Hence we are sure that $\left([\Phi^{(1)}]_q, [\Phi^{(2)}]_q, \dots, [\Phi^{(m+1)}]_q\right)$ is linearly independent if we choose $\epsilon_{m+1} \leq c_m^{-1}$.

Let us set $\epsilon_{m+1} = \min\{\epsilon_m, c_m^{-1}\}$. Of course $\epsilon_{m+1} > 0$.

In order to fulfill condition 2, we further set $\mathcal{U}_{m+1} \subseteq \mathcal{U}_m$ is a neighbourhood of p such that in any point $q \in \mathcal{U}_{m+1}$ and for any $1 \leq i < j \leq m+1$ we verify:

$$\left| \partial_{\nabla_q \Phi^{(i)}} \Phi^{(j)} \right| \leq \epsilon_{m+1}.$$

This neighbourhood exists as $\partial_{\nabla_q \Phi^{(i)}} \Phi^{(j)}$ depends continuously from q and as for $q = p$ we have, by hypothesis, that $\partial_{\nabla_q \Phi^{(i)}} \Phi^{(j)} = 0$.

The induction step being proved, and the case $m = 1$ being trivially true, the lemma is proven. \square

Proposition 3.2.3

Let $\Phi_{b,r}$ be defined as in lemma 3.2.2 (74).

There is a neighbourhood \mathcal{U} of p such that $\Phi_{b,r}|_{\mathcal{U}}$ is a distance chart.

Proof:

We omit the index of $\Phi_{b,r}$ and write Φ .

We choose \mathcal{U} a compact neighbourhood fulfilling the requirements of lemma 3.2.2 (74):

1. \triangleright Equip \mathbb{R}^n with a norm $\|\cdot\|$. Then

$$c := \inf_{q \in \mathcal{U} \text{ and } v \in V_q(M) \setminus \{0\}} \frac{\|\partial_v \Phi\|}{\|v\|} > 0.$$

With $\partial_v \Phi$ we mean the vector with i -th component $\partial_v \Phi^{(i)}$. Φ cannot vanish within \mathcal{U} and $\partial_v \Phi$ is homogenous in v such that

$$\inf_{v \in V_q(M) \setminus \{0\}} \frac{\|\partial_v \Phi(q)\|}{\|v\|} = \inf_{v \in S_q(M)} \|\partial_v \Phi\|.$$

hence

$$c = \inf_{v \in S(\mathcal{U})} \|\partial_v \Phi\|$$

As \mathcal{U} is compact $S(\mathcal{U})$ is compact such that there is a $v \in S(\mathcal{U})$ with $c = \|\partial_v \Phi\|$. By lemma 3.2.2 (74), $\left([\Phi^{(1)}]_q, [\Phi^{(2)}]_q, \dots, [\Phi^{(n)}]_q\right)$ is a basis of $V_q(M)$. We have seen, in the proof of remark 3.1 (73), that $\mathfrak{f} \in V_q(M) \mapsto \partial_v \mathfrak{f}$ is a form in the dual $V_q^*(M)$. As $v \neq 0$, this form is not 0 such that $\partial_v \Phi^{(i)}$ cannot vanish for every i . Hence $\partial_v \Phi \neq 0$ and $c > 0$. \triangle

2. \triangleright $\Phi_{b,r}|_{\mathcal{U}}$ is open: Let $B_d(q, \epsilon) \subseteq \mathcal{U}$ be a neighbourhood of $q \in \mathcal{U}$. We will show that $\Phi(B_d(q, \epsilon))$ is a neighbourhood of $\Phi(q)$, what would prove that Φ is open. As \exp_q is a homeomorphism around the origin (see corollary 1 of proposition 2.2.10 (52)), and as, by construction (see proposition 2.2.4 (45)) we have (for ϵ small enough) that if $\exp(v) \in B_d(q, \epsilon)$ then $\|v\| = q \exp(v)$, $\Phi(B_d(q, \epsilon))$ contains at least a ball around $\Phi(q)$ of radius r/c . \triangle
3. \triangleright There is a neighbourhood \mathcal{U}' of p such that $\Phi_{b,r}|_{\mathcal{U}'}$ is injective: Let $q, q' \in \mathcal{U}$ and $v \in V_q(M)$ such that $q' = \exp(v)$. We know that:

$$\Phi(q') - \Phi(q) = \partial_v \Phi(q) + o(\|v\|).$$

Because of compactness of \mathcal{U} and continuity of Φ and $\partial_v \Phi$ over continuous vector fields, there is for any $\epsilon > 0$, a neighbourhood $\mathcal{U}_\epsilon \subseteq \mathcal{U}$ of p such that

$$\|\Phi(q') - \Phi(q) - \partial_v \Phi(q)\| \leq \epsilon \|v\|$$

for $q, q' \in \mathcal{U}_\epsilon$.

Suppose that $\Phi(q') = \Phi(q)$ for two points in \mathcal{U}_ϵ .

We know that:

$$\|\partial_v \Phi(q)\| \leq \epsilon \|v\|$$

If v is different from 0, we know from above that

$$\frac{\|\partial_v \Phi(q)\|}{\|v\|} \geq c \quad \text{or} \quad \|\partial_v \Phi(q)\| \geq c \|v\|$$

what is a contradiction for $\epsilon < c$. In that case $\|v\| = 0$ and $q = q'$: Φ is open on $\mathcal{U}' := \mathcal{U}_\epsilon$ if $\epsilon < c$. \triangle

Continuity of Φ is obvious, as the distance functions are continuous, such that the previous points prove that $\Phi|_{\mathcal{U} \cap \mathcal{U}'}$ is a distance chart. \square

3.2.2 A $\mathcal{C}^{1, \frac{1}{2}}$ atlas for M

Theorem 3.2.4

Let (M, d) be a scale bounded space of dimension n .

Let A be the set of all distance charts. A is a $\mathcal{C}^{1, \frac{1}{2}}$ atlas over M .

Proof:

Let $\Phi = \Phi_{b,r}$ be a distance chart defined on an open neighbourhood \mathcal{U} such that lemma 3.2.2 (74) is fulfilled for an $\epsilon > 0$ we will specify later in that proof. Let $\Phi' = \Phi'_{b',r'}$ be a second charts on M also defined on \mathcal{U} . $\mathcal{V}, \mathcal{V}' \subseteq \mathbb{R}^n$ are $\Phi(\mathcal{U})$ and $\Phi'(\mathcal{U})$ respectively.

Set $p_i := \exp(r \nabla f_i)$ and $p'_i := \exp(r' \nabla f'_i)$, where f_i and f'_i are basis elements of respectively b and b' .

By proposition 2.3.6 (56) and according to point 2 of definition 2.3.4 (56), $v_i := D\Phi \circ \sigma_{p_i}$ is a $\mathcal{C}^{\frac{1}{2}}$ vector field of $T\mathcal{V}$.

▷ The $n \times n$ -matrix $M(x)$ formed by the basis $(v_1(x), \dots, v_n(x))$ expressed in the canonical basis and its inverse are $\mathcal{C}^{\frac{1}{2}}$ over $x \in \mathcal{V}$: By construction $(M(x))_{ij} = \partial_{\nabla_{\delta_{p_i}} \delta_{p_j}}(\Phi^{-1}(x))$. Hence, by proposition 2.3.6 (56), $M(x)$ is $\mathcal{C}^{\frac{1}{2}}$.

By construction, $(M(x))_{ii} = 1$ and by lemma 3.2.2 (74), $|(M(x))_{ij}| \leq \epsilon$ for $i < j$. If ϵ were 0, $\det M(x)$ would be 1 such that for ϵ chosen small enough, $\det M(x) \geq \frac{1}{2} \quad \forall x \in \mathcal{V}$.

The inverse of a matrix can be written as the inverse of its determinant times the matrix of co-factors. As $\mathcal{C}^{\frac{1}{2}}$ is an algebra, the co-factors and $\det M(x)$ are all $\mathcal{C}^{\frac{1}{2}}$, as the entries of $M(x)$.

On the other hand, if f and g are $\mathcal{C}^{\frac{1}{2}}$ and if $g \geq c > 0$, f/g is also $\mathcal{C}^{\frac{1}{2}}$. Hence, the inverse matrix is $\mathcal{C}^{\frac{1}{2}}$. \triangle

$\Phi' \circ \Phi^{-1}$ is $\mathcal{C}^{1, \frac{1}{2}}$ if $\partial_i \Phi'^{(k)} \circ \Phi^{-1}(x)$ is $\mathcal{C}^{\frac{1}{2}}$ for all $1 \leq k \leq n$ (∂_i denotes the partial derivative along the i -th canonical coordinate).

But

$$\partial_i \Phi'^{(j)} \circ \Phi^{-1}(x) = \sum_{j=1}^n (M(x)^{-1})_{ij} \partial_{v_j} \Phi'^{(j)} \circ \Phi^{-1}(x).$$

and, by definition of v_j , $\partial_{v_j} \Phi'^{(j)} \circ \Phi^{-1}(x) = \partial_{\sigma_{p_j}} \delta_{p'_k}(\Phi^{-1}(x))$ what is, by proposition 2.3.6 (56), $\mathcal{C}^{\frac{1}{2}}$. As $M(x)^{-1}$ is $\mathcal{C}^{\frac{1}{2}}$, the product above is $\mathcal{C}^{\frac{1}{2}}$. \square

Remark 3.2 The dimension of M defined in definition 2.5.6 (69) is the atlas dimension.

Remark 3.3 The bundles $C(M)$ and $V(M)$ are homeomorphic to the tangent bundles TM and T^*M respectively.

3.3 Finsler spaces

Let us define the Finsler space as follows (recall definition 1.3.15 (33) and definition 1.3.5 (27)):

Definition 3.3.1 *Let $n \geq 1$ be an integer and $\alpha \in [0; 1]$. A regular $\mathcal{C}^{n,\alpha}$ Finsler space (M, N) is a $\mathcal{C}^{n,\alpha}$ -manifold M , equipped with a norm $N : TM \longrightarrow \mathbb{R}_+$ satisfying the following properties:*

1. N is $\mathcal{C}^{n-1,\alpha}$ on TM but outside the zero section,
2. $N|_{T_p M}$ is a vector space with regular norm in every $p \in M$,
3. the constant C of point 2 in definition 1.3.15 (33) can be chosen locally uniformly.

Remark 3.4 In the book of Abate & Patrizio in [1], the given component is more restrictive. There convexity condition on the indicatrix fit the requirement given above.

This is a consequence of the example 1.12 (34).

3.3.1 Equivalences between scale bounded and regular Finsler spaces

In fact, any scale bounded space is a regular Finsler space. To a certain extend, the converse is also true. Let us precise what equivalent means:

Definition 3.3.2 *A regular Finsler space (M, N) is said to be equivalent to a length space (X, d) if M is a manifold on X and if, for every couple $x, y \in X$, \overline{xy} is the geodesical distance between x and y relatively to the Finsler metric N .*

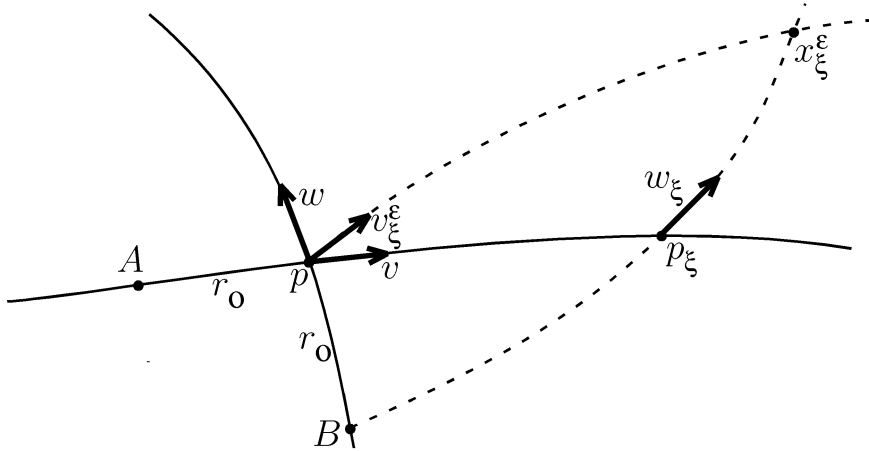
Theorem 3.3.3

A scale bounded space is equivalent to a regular $\mathcal{C}^{1,\frac{1}{2}}$ Finsler space .

Proof:

By theorem 3.2.4 (77), a scale bounded space M has a $\mathcal{C}^{1,\frac{1}{2}}$ atlas of finite dimension n . Let $N(v) := \partial_v^+ \delta_{\pi(v)}$ (see remark 2.16 (58)). N fulfills the conditions imposed by definition 3.3.1 :

- i) $p_\xi := \exp(\xi v)$, such that $p_0 = p$,
- ii) $r_0 \in]0, \iota_p[$, $B := \exp(-r_0 w)$ and $w_\xi := \sigma_B(p_\xi)$,
- iii) $x_\xi^\epsilon := \exp(\epsilon \xi w_\xi)$,
- iv) $v_\xi^\epsilon \in T_p M$ is defined by $x_\xi^\epsilon = \exp(\xi v_\xi^\epsilon)$.



As σ_A and σ_B are continuous vector fields in a neighbourhood \mathcal{U} of p , such that w_ξ and v_ξ^ϵ are continuous in $\xi \neq 0$. For w_ξ the limit w_0 obviously exists and equals w . For v_ξ^ϵ the existence of a limit vector $v_0^\epsilon = v + \epsilon w$ has to be proved.

$$(f(p_\xi) - f(p)) + (f(x_\xi^\epsilon) - f(p_\xi)) - + (f(x_\xi^\epsilon) - f(p)) \equiv 0$$
$$\partial_v f + \epsilon \partial_{w_0} f = \lim_{\xi \rightarrow 0} \partial_{v_\xi^\epsilon} f.$$

By definition of N and the previous considerations

$$N(v + \epsilon w) = \lim_{\xi \rightarrow 0} \delta_p \left(\exp(v_\xi^\epsilon) \right) = \lim_{\xi \rightarrow 0} \delta_p(x_\xi^\epsilon) / \xi.$$

For \mathcal{U} small enough (namely for $\mathcal{U} \subseteq B_p$) proposition 1.4.8 (40) tells us that

$$\delta_p(x_\xi^\epsilon) = \delta_p(p_\xi) + \epsilon \xi \sin \phi_\xi + (\epsilon \xi \cos \phi_\xi)^2 \frac{R_{\epsilon, \xi}}{\delta_p(p_\xi)}$$

or, as $\delta_p(p_\xi) = \xi$,

$$\delta_p(x_\xi^\epsilon)/\xi = 1 + \epsilon \sin \phi_\xi + (\epsilon \cos \phi_\xi)^2 R_{\epsilon, \xi}.$$

We know that there is a constant $C^{-1} \leq R_{\epsilon, \xi} \leq C$ and we know that $\lim_{\xi \rightarrow 0} \delta_p(x_\xi^\epsilon)/\xi$ exists, such that a limit angle ϕ exists and

$$N(v + \epsilon w) = 1 + \epsilon \sin \phi + (\epsilon \cos \phi)^2 R_\epsilon.$$

This implies that $N|_{T_p M}$ is $\mathcal{C}^{1,1}$ and that condition 2 (33) is fulfilled such that the norm is regular. \triangle

\triangleright *N fulfills condition 3 (78):* From proposition 1.3.17 (36) lemma 1.3.16 (34) we know that $N|_{T_p M}$ is a vector space with bounded scale curvature. From the proof of lemma 1.3.16 (34), we know that these bounds depend continuously from the Lipschitz constant and the parameter C as defined in definition 1.3.15 (33), hence, local compactness ensures the claimed uniformity. \triangle

\triangleright *N fulfills condition 1 (78):* In every distance chart a geodesic is at least a $\mathcal{C}^{1, \frac{1}{2}}$ curve such that a vector field σ_q as field of tangent vectors of geodesics is at least $\mathcal{C}^{0, \frac{1}{2}}$. However $N(\sigma_q) \equiv 1$ by construction.

This implies in return that for a canonical vector fields e_i of a distance chart $N(e_i)$ is at least $\mathcal{C}^{0, \frac{1}{2}}$. \triangle

Hence (M, N) is a regular $\mathcal{C}^{1, \frac{1}{2}}$ Finsler space. \square

3.3.2 Flag curvature and bounded curvature

If atlas and norm of a Finsler space M are smooth enough, the flag curvature F [14], [13], [15], [1] can be defined. It is a continuous field over M and plays a similar role as the sectional curvature does on Riemann manifolds. In fact, in a Riemann manifold, seen as special case of a Finsler manifold, flag curvature and sectional curvature falls together.

Flag curvature is a natural generalization of sectional curvature in relation with Jacobi fields: Let $J(t)$ be an orthogonal unitary vector field on a geodesic γ_t (with orthogonal we mean that $\partial_{J(t)} \delta_{\gamma_0} \equiv 0$). $J(t)$ is said to be a Jacobi field if the relation

$$\ddot{\lambda}(t) + F(\dot{\gamma}_t, J(t)) \lambda(t) \equiv 0$$

holds for some scalar function λ .

The Jacobi relation arises from the second variation of minimal path length problem [15]. In this context, the flag curvature can be seen as a measure of how fast close geodesics converge or diverge from on an other: $\gamma_t + \epsilon \lambda(t) J(t)$ is an ϵ -second order approximation for neighbour geodesics. From that point of view, one might expect that flag curvature and scale bounded spaces are related some how.

The special case of vector spaces with regular norm has been extensively treated in section subsection 1.3.5. The following proofs have an identical structure than their corresponding proofs in section subsection 1.3.5 except that the various estimations will depend on more parameters (in general Finsler spaces we loose homogeneity and translation invariance that made proofs easier).

Lemma 3.3.4 *If (M, N) is a $\mathcal{C}^{1,1}$ regular Finsler space with well-defined continuous flag curvature then M is locally quadratically convex.*

Proof:

The proof follows the structure of lemma 1.3.16 (34) that covers the special case where M is a vector space with regular norm.

We made extensive use of homogeneity and translation invariance of vector spaces and will have to complete our proof with arguments why it does also hold in the general case.

Let us use the same notations as in definition 1.3.13 (32): we will first restrict our proof of quadratical convexity to the case where p is given and where q 's distance to p is within $[\frac{1}{2}r, r]$. $r > 0$ will be specified later.

We will show that a constant $C > 1$ as claimed in definition 1.3.13 (32) exists and leave to the reader to check that throughout the proof, $C(p, r)$ as function of $p \in M$ and $r > 0$ is continuous and that $r_2 < r_1$ implies $C(p, r_2) \leq C(p, r_1)$. This fact and local compactness of M ensures our claim to be true.

Rather than working in a neighbourhood of q , we will work in the normed vector space $T_q M$ by mapping the neighbourhood of q to a neighbourhood of 0 through the inverse exponential map \exp_q^{-1} .

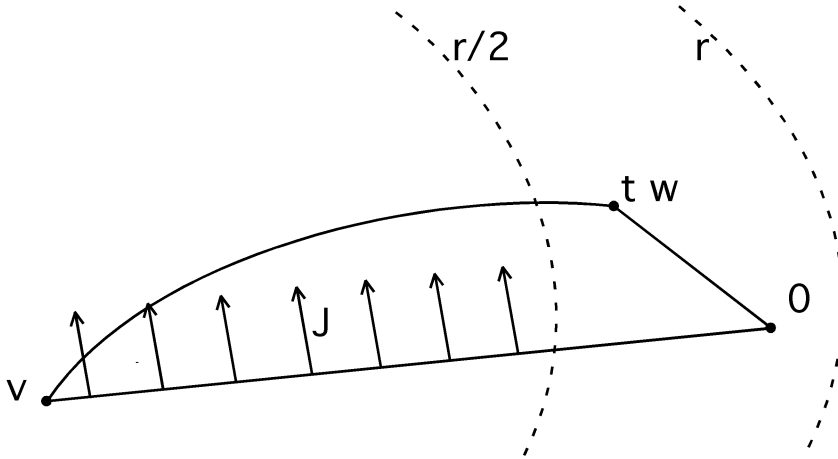


Figure 3.3: The construction in $T_q M$

In this setting all geodesics through q map to rays in $T_q M$ and all distances from q correspond to the vector norm of $T_q M$.

Let p map to v - a vector with norm between $\frac{r}{2}$ and r - and let w be an unitary vector. The vectorial segments from v to ϵw do not fall necessarily on the

geodesics between v and ϵw . The Jacobi field J on the segment 0 to v however describes, up to the second order, the behaviour of these geodesics.

As flag curvature is continuous, it can be locally bounded. Hence the Jacobi equation ensures the coordinate errors between the correct geodesic and the segment approximation are of the order of $O(t^2)$ in t and of order $O(r^2 \epsilon^2 \cos^2 \phi)$ in ϵ .

These considerations combined with the fact that norm is $\mathcal{C}^{0,1}$ as function of the base point imply that

$$\delta_v(\epsilon w) = \|v - \epsilon w\| + O(r^2 \epsilon^2 \cos^2 \phi).$$

If $\delta_v(\epsilon w) = \|v - \epsilon w\|$ space would be a vector space with regular norm and lemma 1.3.16 (34) would ensure that the constant $C > 1$ claimed in definition 1.3.13 (32) exists. Following the proof of lemma 1.3.16 (34) a reduction of C to $\frac{1+C}{2}$ would allow to absorb the error term $O(r^2 \epsilon^2 \cos^2 \phi)$ on the length of geodesics as long as $|r \epsilon \cos \phi| \leq \epsilon_0$ for $\epsilon_0 > 0$ chosen small enough.

In order to have definition 1.3.13 (32) fulfilled, the rest term $R_\gamma(t)$ must be bounded for every $\gamma \in \Upsilon = (B_q(\frac{1}{2}\bar{p}\bar{q}))$. Condition $|r \epsilon \cos \phi| \leq \epsilon_0$ does not cover all geodesics in Υ_- . However, we have already covered the cases where $\cos \phi < \epsilon_0$ or $t < \epsilon_0$. Let us consider now the cases where $|\cos \phi| \geq \epsilon_0$ and $t \geq \epsilon_0$: As

$$R_\gamma(t) = \|v\| \frac{\delta_p(\gamma_t) - \|v\| - t \sin \phi}{t^2 \cos^2 \phi}$$

is now a well-defined continuous function on a compact set of parameters, $R_\gamma(t)$ has a minimum and a maximum. If we can prove that this minimum is larger than 0 the prove is closed.

If the minimum was not larger than 0, there would be a t and a w (to define γ) such that $R_\gamma(t) = 0$. As $(0, p, t w)$ is not degenerated, $\|v - t w\|$ has to be a strictly convex function in t . A look at the relation for $\delta_p(\gamma_t)$ shows that $R_\gamma(t) = 0$ contradicts strict convexity. \square

Theorem 3.3.5

A regular $\mathcal{C}^{1,1}$ Finsler space with well-defined continuous flag curvature is equivalent to a scale bounded space.

Proof:

This proof is in the same spirit as lemma 3.3.4 (81): It is an extension of the proof of proposition 1.3.17 (36) that is based itself on lemma 1.3.18 (37).

As the same construction of tangent space through the exponential map, the same Jacobi field argument for length approximation used as in the proof of lemma 3.3.4 (81), we refer to these constructions and will limit ourselves to specific comments to extend lemma 1.3.18 (37) to our purpose.

Its proof is found in appendix B. We have to re-visit all 5 cases:

Case 1 illustrated in figure B.1 is to be considered in $T_H M$ through \exp_H^{-1} . The estimations on the rest term $R_{v,w}(\epsilon)$ have been provided by lemma 3.3.4 (81).

Case 2 illustrated in figure B.2 (91) is to be considered in $T_{C'}M$ where C' is the base point of w on segment AB .

Case 3 illustrated in figure B.3 (93) is to be considered in T_AM .

Case 4 illustrated in figure B.4 (94) is to be considered in $T_{A'}M$ where A' is the base point of w on segment BC .

Case 5 illustrated in figure B.5 (96) is to be considered in T_HM . \square

Remark 3.5 Due to the additional condition on flag curvature, theorem 3.3.5 (82) fails to be the converse of theorem 3.3.3 (78).

Flag curvature needs a high degree of differentiability to be well-defined, a degree that is obviously missing to the atlas constructed for theorem 3.3.3 (78).

In that respect, scaled bounded curvature might be seen as a generalisation for bounded flag curvature in Finsler spaces with low differentiability.

Remark 3.6 A perfect equivalence between scale bounded spaces and regular Finsler spaces could not be given such that differentiability in claims in that section has been optimised for ease of proofs; enhancements are possible.

3.3.3 Open issues

As already mentioned in the introduction, Berestovskij [6] proved that length spaces with Alexandrov curvature bounded from above and below carry a $\mathcal{C}^{1,1}$ differential Riemannian structure compatible with the given length. Later, the degree of differentiability could be improved to $\mathcal{C}^{2-\epsilon}$ [7] whereas counter-examples exist for \mathcal{C}^2 .

In the present work theorem 3.3.3 (78) provides a similar result for Finsler spaces whereas differentiability is $\mathcal{C}^{1,\frac{1}{2}}$. This some how puzzling value was carried through theorem 3.2.4 (77) by proposition 2.3.6 (56).

An improvement of differentiability in proposition 2.3.6 (56) would automatically carry forward to our main theorem 3.3.3 (78).

No known counter-examples indicate that this would not be possible, but the techniques used in the proof of proposition 2.3.6 (56) do not allow an improvement there. The issue is left open.

An other open issue of interest concerns the various notions defined and related to curvature: regular norm respectively regular Finsler space (see definition 1.3.15 (33) and definition 3.3.1 (78)) leads through lemma 1.3.16 (34) respectively lemma 3.3.4 (81) to quadratically convex spaces (see definition 1.3.13 (32)).

These, in turn, led to scale bounded spaces (see proposition 1.3.17 (36) and theorem 3.3.5 (82)).

To close the loop, scale bounded spaces are, by theorem 3.3.3 (78), regular Finsler space, spaces that might have well-defined flag curvature, if differentiability allows it.

It was not scope of that work to show equivalencies of these notions, but they seem to be at least very similar.

Moreover, definition 1.3.13 (32) of quadratically convex required inequalities to be true for every geodesical segment in a given ball. But a closer look to proofs where these features have been used or proved to be true (for example lemma 1.3.16 (34), lemma 3.3.4 (81) or lemma 1.3.18 (37)) shows that only the limit values for $t \rightarrow 0$ are crucial, the other cases been covered by a continuity argument on compact parameter sets.

This might indicate that curvature could be qualified using only conditions on the second degree developments of local distance function.

From that point of view, the choice of scale bounded curvature might not be an optimal one. However, it offered a natural transition from the Alexandrov curvature used in the original work of Berestovskij.

Appendix A

Appendix: Spheric and hyperbolic geometry

A.1 The generalization of Pythagoras theorem

In the Euclidean plane the relation:

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha) \quad (\text{A.1})$$

is known as the generalization of Pythagoras theorem.

In the case we work on S^2 equipped with a homogeneous metric of curvature $K > 0$, the relation writes:

$$\cos(a\sqrt{K}) = \cos(b\sqrt{K}) \cos(c\sqrt{K}) + \sin(b\sqrt{K}) \sin(c\sqrt{K}) \cos(\alpha) \quad (\text{A.2})$$

and in a Lobachevskij plane of curvature $K < 0$, we have:

$$\begin{aligned} \cosh(a\sqrt{-K}) &= \cosh(b\sqrt{-K}) \cosh(c\sqrt{-K}) \\ &\quad - \sinh(b\sqrt{-K}) \sinh(c\sqrt{-K}) \cos(\alpha) \end{aligned} \quad (\text{A.3})$$

Remark A.1 In fact, all relations can be deduced from A.2 ; if we put $K < 0$ in A.2 , we obtain A.3 by holomorphic extension of the trigonometric relations. A.1 is then obtained as a Taylor development of this extension in the limit $K \rightarrow 0$.

Considered as holomorphic function, we will use the form A.2 with $K \in \mathbb{R}$ to cover all cases.

A.2 Explicit relations for h_K and m_K

$h_K(a, b, c, \kappa)$ and $m_K(a, b, c, \kappa)$ can be worked out explicitly using the relations of section A.1 . We obtain:

If $K > 0$

$$\begin{aligned} \cos \left(h_K(a, b, c, \kappa) \sqrt{K} \right) &= \frac{\cos \left(b \sqrt{K} \right) \sin \left(\kappa a \sqrt{K} \right)}{\sin \left(a \sqrt{K} \right)} \\ &+ \frac{\cos \left(c \sqrt{K} \right) \sin \left((1 - \kappa) a \sqrt{K} \right)}{\sin \left(a \sqrt{K} \right)} \\ \text{and for } \kappa = \frac{1}{2} : \quad \cos \left(m_K(a, b, c) \sqrt{K} \right) &= \frac{\cos \left(b \sqrt{K} \right) + \cos \left(c \sqrt{K} \right)}{2 \cos \left(\frac{1}{2} a \sqrt{K} \right)}, \end{aligned}$$

if $K = 0$

$$h_0^2(a, b, c, \kappa) = \kappa b^2 + (1 - \kappa) c^2 - \kappa(1 - \kappa) a^2,$$

$$\text{and for } \kappa = \frac{1}{2} : \quad m_0^2(a, b, c) = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4},$$

if $K < 0$

$$\begin{aligned} \cosh \left(h_K(a, b, c, \kappa) \sqrt{-K} \right) &= \frac{\cosh \left(b \sqrt{-K} \right) \sinh \left(\kappa a \sqrt{-K} \right)}{\sinh \left(a \sqrt{-K} \right)} \\ &+ \frac{\cosh \left(c \sqrt{-K} \right) \sinh \left((1 - \kappa) a \sqrt{-K} \right)}{\sinh \left(a \sqrt{-K} \right)} \end{aligned}$$

and for

$$\kappa = \frac{1}{2} : \quad \cosh \left(m_K(a, b, c) \sqrt{-K} \right) = \frac{\cosh \left(b \sqrt{-K} \right) + \cosh \left(c \sqrt{-K} \right)}{2 \cosh \left(\frac{1}{2} a \sqrt{-K} \right)}.$$

Remark A.2 Once more, all relations can be deduced from A.2 (85) by holomorphic extension of the relations.

A.3 Implicit relation for k

It is not easy to solve algebraically the relations of section A.2 in K . However the following implicit relation will, be useful in handling with them:

Consider a triangle (A, B, C) where the point H and the lengths a, b, c, h and the half perimeter ρ are defined as we did in definition 1.3.5 (27):

Definition A.3.1 Let $(a, b, c) \neq (0, 0, 0)$ and set $\rho = \frac{1}{2}(a + b + c)$. We define the function:

$$\Psi(a, b, c, h)(k) := \frac{2 \cos\left(\frac{h}{\rho}\sqrt{k}\right) \cos\left(\frac{a}{2\rho}\sqrt{k}\right) - \cos\left(\frac{b}{\rho}\sqrt{k}\right) - \cos\left(\frac{c}{\rho}\sqrt{k}\right)}{k}.$$

This function is also well defined for $k = 0$ as holomorphic extension (see remark A.2 (86)) to expand it around $k = 0$. The limits $k \rightarrow -\infty$ and $k \rightarrow \pi^2$ also exist.

The latter relation is useful to estimate upper and lower scale curvature bounds (see definition 1.3.5 (27)). It also has a couple of nice properties:

Lemma A.3.2 Suppose $(a, b, c) \neq (0, 0, 0)$ are side lengths of a triangle, $\rho := \frac{1}{2}(a + b + c)$, and h the length of the median as considered in definition 1.3.5 (27). Ψ has the following properties:

- i) $\Psi(a, b, c, h)(k)$ is analytical in all variables.
- ii) $\Psi(\lambda a, \lambda b, \lambda c, \lambda h)(k) = \Psi(a, b, c, h)(k)$ for every $\lambda \neq 0$.
- iii) $\Psi(a, b, c, h)(k) = 0 \iff m_k(a/\rho, b/\rho, c/\rho) = h/\rho$,
- iv) if the considered triangle is not degenerated, there is exactly one value $k \in [-\infty, \pi^2]$ so that $\Psi(a, b, c, h)(k) = 0$. Within non degenerated triangles the value k depends continuously from a, b, c and h .
- v) if the considered triangle is degenerated, the value of h is given by the side lengths (a, b, c) . In that case: $\Psi(a, b, c, h)(k) = 0 \quad \forall k \in [-\infty, \pi^2]$.

Proof:

Point i) is trivial: Ψ is holomorphic in all variables for $k \neq 0$ and as it can be continuously extended to $k = 0$, it is also holomorphic in that point, what proofs analysity.

The verification of point ii is a trivial consequence of definition 1.3.2 (25) of ρ .

Point ii) is also a straight ahead consequence the explicit value of m_k worked out in section A.2 (86).

Point v) can be shown either by direct calculation or by the fact that, as for degenerated triangles, h is uniquely given and does not depend from the space in which the triangle is embedded, the relations given in section A.2 (86) must hold for our choice of a, b, c and h independently of k .

To prove point iv), we first remark that if h has the maximal possible value allowed by the definition A.3.1 of Ψ , $\Psi(a, b, c, h)(\pi^2) = 0$. If h has its smallest possible value, $\Psi(a, b, c, h)(-\infty) = 0$.

Considering remark 1.10 (26), we see that increasing k implies increasing h , if $\Psi(a, b, c, h)(k)$ has to vanish, such that there is always a k such that $\Psi(a, b, c, h)(k) = 0$.

Its uniqueness is a consequence of the fact that, if a, b, c and h are the side lengths and h the instead length of a not degenerated triangle, they cannot be isometrically embedded in two different reference spaces M_{K_1} and M_{K_2} . \square

Appendix B

Appendix: Proof of lemma 1.3.18

Lemma B.0.3 *Let $(V, \|\cdot\|)$ be a finite dimensional vector space with a regular norm.*

Call $\Delta \subseteq V \times V \times V$ the set of all degenerated triangles in V with half-perimeter 1.

There is a neighbourhood \mathcal{U}_Δ of Δ and $K_- \leq K_+ < \pi^2$ such that K_- is a lower and K_+ an upper bound for the scale curvature of all triangles in \mathcal{U}_Δ .

Proof:

To clarify the notation, we will denote the triangles we are going to consider with (A, B, C) , their side lengths by $a := \overline{BC}$, $b := \overline{AC}$ and $c := \overline{AB}$, the mid point between B and C by H and $h := \overline{AH}$ (they are similar to the notations used in component definition 1.3.5 (27)). The half-perimeter of the triangle (A, B, C) is denoted by ρ and all points and segments are supposed to be in a neighbourhood \mathcal{U} .

We will have to distinguish between 5 categories of degenerated triangles. For each case, we will show that there is a neighbourhood of those triangles in which the scaled curvatures are bounded:

1. \triangleright *Consider degenerated triangles where $B = C$ and $A \neq H$:*

Suppose A and H given and consider B as being in a ball $B_r(H)$ where $r > 0$ will be defined later. The choice of B defines implicitly the position of C , as H must be the midpoint between B and C . The family of all triangles defined as above is a neighbourhood of the degenerated triangle (A, H, H) . We have to show that, for a suitable r , this family has its scale curvature bounded from both sides:

We consider the family of triangles given by $(A, B, C) = (v, \epsilon w, -\epsilon w)$, v and w being unitary vectors. By construction, we then also have that $\overline{CH} = \epsilon h$.

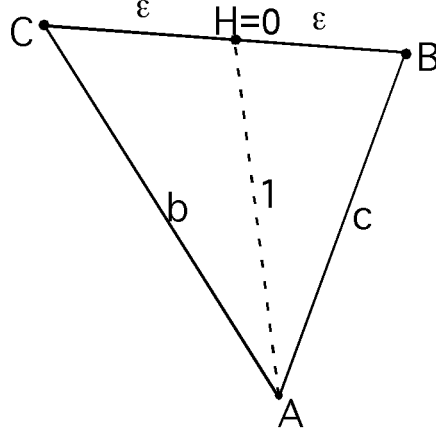


Figure B.1: The triangle family considered in 1.

Using point 1 of definition 1.3.15 (33) and lemma 1.3.16 (34), the distance from A to the length parameterized segment from B to C can be expressed as function of ϵ :

$$h_\epsilon := 1 + \epsilon \sin \phi + \epsilon^2 \cos^2 \phi R_{v,w}(\epsilon)$$

such that $\overline{AH} = h_0$, $\overline{AB} = h_\epsilon$ and $\overline{AC} = h_{-\epsilon}$. The triangle side lengths can be resumed with

$$\begin{aligned} a &= 2\epsilon \\ b &= 1 + \epsilon \sin \phi + \epsilon^2 \cos^2 \phi R_{v,w}(\epsilon) \\ c &= 1 - \epsilon \sin \phi + \epsilon^2 \cos^2 \phi R_{v,w}(-\epsilon) \\ h &= 1 \\ \rho &= 1 + \epsilon + \frac{1}{2}(R_{v,w}(\epsilon) + R_{v,w}(-\epsilon))\epsilon^2 \cos^2 \phi. \end{aligned}$$

Consider $(\tilde{A}, \tilde{B}, \tilde{C})$, a triangle in M_K with side lengths $\tilde{a} := a/\rho$, $\tilde{b} := b/\rho$ and $\tilde{c} = c/\rho$. Using the formula for m_K (see section A.2 (86)), we see that its height \tilde{h} is:

$$\tilde{h}/h = 1 + \frac{1}{2} \left(-\sqrt{K} \cot \sqrt{K} + R_{v,w}(\epsilon) + R_{v,w}(-\epsilon) \right) \epsilon^2 \cos^2 \phi + O(\epsilon^3)$$

To have the scaled curvature bounded above, we must have $\tilde{h} \geq 1$ for some K , to have it bounded below, we must ask for $\tilde{h} \leq 1$ (see definition 1.3.5 (27)).

If we remark that the application $K \mapsto \sqrt{K} \cot \sqrt{K}$ is decreasing and onto from $] -\infty, \pi^2[$ to $]0; \infty[$, the latter expansion of \tilde{h} shows obviously that these curvature conditions can be fulfilled for some K_- and K_+ , as we know (lemma 1.3.16 (34)) that there is a constant $C > 0$ such that $C^{-1} \leq R(\epsilon) \leq C$ for $|\epsilon| < \frac{1}{2}$. \triangle

2. ▷ Consider degenerated triangles where C is strictly on the segment AB :

We consider the triangle family $(A, B, C) = (0, v, (1-\lambda)v + \epsilon w)$ in V with both v and w unitary such that $\partial_w \|v\| = 0$. In order to have A strictly on the segment BC , we must ask $\lambda \in]0, 1[$.

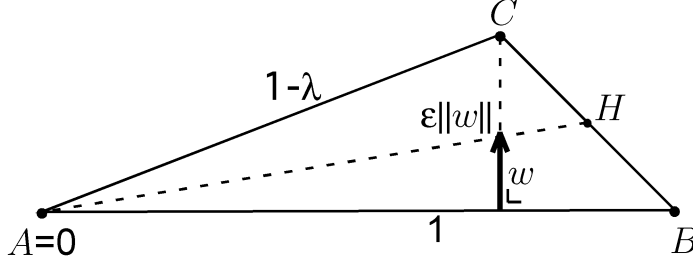


Figure B.2: The triangle family considered in 2.

We will show that for any $\lambda_0 \in]0, 1[$, there is a neighbourhood \mathcal{V} of λ_0 and an $\epsilon_0 > 0$ such that for every v , every $\lambda \in \mathcal{V}$ and every $\epsilon < \epsilon_0$ our triangle family has its scale curvature uniformly bounded above and below.

The side lengths are (using the expansion of $\|v + \epsilon w\|$ of lemma 1.3.16 (34)):

$$\begin{aligned}
 a &= \|v - (1-\lambda)(v + \epsilon w)\| \\
 &= \lambda \left\| v - \frac{1-\lambda}{\lambda} \epsilon w \right\| = \lambda \left(1 + \epsilon^2 R_{v,w} \left(-\frac{1-\lambda}{\lambda} \epsilon \right) \right) \\
 b &= \|(1-\lambda)(v + \epsilon w)\| \\
 &= (1-\lambda) \|v + \epsilon w\| = (1-\lambda) (1 + \epsilon^2 R_{v,w}(\epsilon)) \\
 c &= \|v\| = 1 \\
 h &= \left\| \left(1 - \frac{\lambda}{2} \right) v + \frac{1-\lambda}{2} \epsilon w \right\| = \left(1 - \frac{\lambda}{2} \right) \left\| v + \frac{1-\lambda}{2-\lambda} \epsilon w \right\| \\
 &= \left(1 - \frac{\lambda}{2} \right) \left(1 + \epsilon^2 R_{v,w} \left(\frac{1-\lambda}{2-\lambda} \epsilon \right) \right)
 \end{aligned}$$

For $\epsilon = 0$ the triangle (A, B, C) is degenerated, such that $a + b - c = 0$. If we expand h as function of $a + b - c$, it has the form:

$$h = 1 - \frac{\lambda}{2} - \frac{1}{2}(a + b - c)\tau_w(\lambda)$$

where $\tau_w(\lambda)$:

$$\tau_w(\lambda) = \frac{2 - \lambda - 2h}{a + b - c}$$

or, explicitly,

$$\tau_w(\lambda) = (2 - \lambda) \frac{R_{v,w} \left(\frac{1-\lambda}{2-\lambda} \epsilon \right)}{\lambda R_{v,w} \left(-\frac{1-\lambda}{\lambda} \epsilon \right) + (1-\lambda) R_{v,w}(\epsilon)}.$$

According to lemma 1.3.16 (34), this relation implies the inequalities

$$1 \frac{C^{-1}}{\lambda C + (1 - \lambda)C} \leq \tau_w(\lambda) \leq 2 \frac{C}{\lambda C^{-1} + (1 - \lambda)C^{-1}}$$

or

$$C^{-2} \leq \tau_w(\lambda) \leq 2C^2$$

for ϵ in some neighbourhood of 0. On the other hand, using the convexity of $R_{v,w}(w)$, we have that:

$$\begin{aligned} \overline{\lim}_{\epsilon \geq 0} \tau_w(\lambda) &= \overline{\lim}_{\epsilon \geq 0} \frac{2 - \lambda}{\lambda \frac{R_{v,w}(-\frac{2-\lambda}{\lambda}\epsilon w)}{R_{v,w}(\epsilon w)} + (1 - \lambda) \frac{R_{v,w}(\frac{2-\lambda}{1-\lambda}\epsilon w)}{R_{v,w}(\epsilon w)}} \\ &\leq \frac{2 - \lambda}{1 - \lambda} \overline{\lim}_{\epsilon \geq 0} \frac{R_{v,w}(\epsilon w)}{R_{v,w}(\frac{2-\lambda}{1-\lambda}\epsilon w)} \\ &\leq \frac{2 - \lambda}{1 - \lambda} < +\infty. \end{aligned}$$

These inequalities imply that there is a $C \geq 1$ and a neighbourhood of λ and of w such that $C^{-1} \leq \tau_w(\lambda) \leq C$ for λ and w in those neighbourhood.

On the other hand, considering the isometric embedding (up to a constant factor) $(\tilde{A}, \tilde{B}, \tilde{C})$ in M_K ($K < \pi^2$) of (A, B, C) , we obtain the side lengths $\tilde{a} = \lambda$, $\tilde{b} = 1 - \lambda$ and $\tilde{c} = 1 - \epsilon$ such that $\tilde{a} + \tilde{b} - \tilde{c} = \epsilon$ where $\rho(\tilde{A}, \tilde{B}, \tilde{C}) = 1 + O(\epsilon)$.

Using an expansion for \tilde{h} of the explicit formula in section A.2 (86), we obtain:

$$\begin{aligned} \tilde{h} &= \left(1 - \frac{\lambda}{2}\right) - \frac{1}{2} \tilde{\tau}_K(\lambda) \epsilon + O(\epsilon^2) \\ &= \left(1 - \frac{\lambda}{2}\right) - \frac{1}{2} \tilde{\tau}_K(\lambda) (\tilde{a} + \tilde{b} - \tilde{c}) + O((\tilde{a} + \tilde{b} - \tilde{c})^2). \end{aligned}$$

where

$$\tilde{\tau}_K(\lambda) := \frac{\sin \sqrt{K}}{4 \cos\left(\sqrt{K} \frac{\lambda}{2}\right) \sin\left(\sqrt{K} \left(1 - \frac{\lambda}{2}\right)\right)}.$$

which is a continuous function on $(K, \lambda) \in]-\infty, \pi^2[\times]0, 1[$.

We see that for every $\lambda \in]0, 1[$

$$\lim_{K \rightarrow -\infty} \tilde{\tau}_K(\lambda) = +\infty \text{ and } \lim_{K \rightarrow \pi^2} \tilde{\tau}_K(\lambda) = 0.$$

This implies that for any $C \geq 1$ and $\lambda \in]0, 1[$, there is a neighbourhood \mathcal{V} of λ and a $K_-, K_+ < \pi^2$ such that:

$$C^{-1} \leq \tilde{\tau}_{K_-}(\lambda) \leq \tilde{\tau}_{K_+}(\lambda) \leq C \quad \forall \lambda \in \mathcal{V}.$$

Combining this result with the conclusions the fact that $\tau_w(\lambda)$ is locally bounded below and above by positive constants, we conclude that any triangle of the family has a finite scale curvature.

Thereby the claim at the beginning of this section is proven. \triangle

3. \triangleright Consider the category of degenerated triangles where $A = B$ and $B \neq C$:

We look at the triangle family around the triangle $(0, 0, v)$ parametrization by $(0, \epsilon w, v)$, where v and w are unitary.

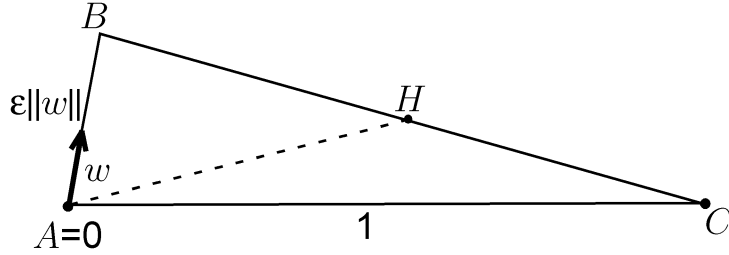


Figure B.3: The triangle family considered in 3.

Using lemma 1.3.16 (34) we can write:

$$\begin{aligned} a &= \|v - \epsilon w\| = 1 - \epsilon \sin \phi + \epsilon^2 \cos^2 \phi R_{v,w}(-\epsilon) \\ b &= \|v\| = 1 \\ c &= \|\epsilon w\| = \epsilon \\ h &= \frac{1}{2} \|v + \epsilon w\| = \frac{1}{2} + \frac{1}{2} \epsilon \sin \phi + \frac{1}{4} \epsilon^2 \cos^2 \phi R_{v,w}\left(\frac{\epsilon}{2}\right) \end{aligned}$$

If we set $\tilde{a} = a/\rho$, $\tilde{b} = b/\rho$ and $\tilde{c} = c/\rho$ and construct a triangle in M_K with side lengths $(\tilde{a}, \tilde{b}, \tilde{c})$, its half perimeter is of the form $1 + O(\epsilon)$. Using the explicit formula in section A.2 (86), \tilde{h} has the form:

$$\tilde{h} = \frac{1}{2} + \frac{1}{2} \epsilon \sin \phi + \frac{1}{2} \cos^2 \phi \left(\frac{\sqrt{K}}{\sin \sqrt{K}} - R_{v,w}(-\epsilon) \right) \epsilon^2 + O(\epsilon^3).$$

We should find a K such that $h = \tilde{h}$, in other words, such that:

$$\frac{1}{2} R_{v,w}\left(\frac{\epsilon}{2}\right) + R_{v,w}(\epsilon) = \frac{\sqrt{K}}{\sin \sqrt{K}} + O(\epsilon^3).$$

This equation has always a solution because $R_{v,w}$ is positive and

$$\lim_{K \rightarrow \pi^2} \frac{\sqrt{K}}{\sin \sqrt{K}} = +\infty, \quad \lim_{K \rightarrow -\infty} \frac{\sqrt{K}}{\sin \sqrt{K}} = 0.$$

A continuity argument allows to conclude that the scale curvature can be uniformly bounded in a neighbourhood of the triangle $(0, 0, v)$. \triangle

4. \triangleright Consider the category of degenerated triangles where A is strictly on the segment BC but $A \neq H$:

We consider the degenerated triangle $(A, B, C) = (0, \frac{1+\lambda}{2}v, -\frac{1-\lambda}{2}v)$ and the parametrization family $(\epsilon w, \frac{1+\lambda}{2}v, -\frac{1-\lambda}{2}v)$ around it, where v and w are supposed unitary and $\partial_w \|v\| = 0$.

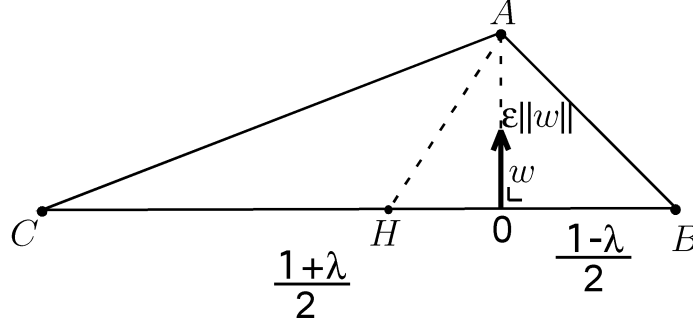


Figure B.4: The triangle family considered in 4.

In order to have A on BC for $\epsilon = 0$, $\lambda \in]-1, 1[$. $A \neq H$ implies $\lambda \neq 0$ and, considering the symmetry of exchanging B and C , we can restrict ourselves to the case $\lambda \in]0, 1[$.

The triangles side lengths, as expansion around $\epsilon = 0$ are given by (see lemma 1.3.16 (34)):

$$\begin{aligned}
 a &= \|v\| = 1 \\
 b &= \left\| \frac{1-\lambda}{2}v + \epsilon w \right\| = \frac{1-\lambda}{2} \left\| v + \frac{2\epsilon}{1-\lambda} w \right\| \\
 &= \frac{1-\lambda}{2} + \frac{2}{1-\lambda} \epsilon^2 R_{v,w} \left(\frac{2\epsilon}{1-\lambda} \right) \\
 c &= \left\| \frac{1+\lambda}{2}v - \epsilon w \right\| = \frac{1+\lambda}{2} \left\| v - \frac{2\epsilon}{1+\lambda} w \right\| \\
 &= \frac{1+\lambda}{2} + \frac{2}{1+\lambda} \epsilon^2 R_{v,w} \left(-\frac{2\epsilon}{1+\lambda} \right) \\
 h &= \left\| \frac{\lambda}{2}v - \epsilon w \right\| = \frac{\lambda}{2} \left\| v - \frac{2\epsilon}{\lambda} w \right\| \\
 &= \frac{\lambda}{2} + \frac{2}{\lambda} \epsilon^2 R_{v,w} \left(-\frac{2\epsilon}{\lambda} \right)
 \end{aligned}$$

In analogy to the proof of point 2, we observe that the triangles of the family are degenerated if and only if $b + c - a = 0$, such that h can be written as expansion in $b + c - a$:

$$h = \frac{\lambda}{2} + \frac{1}{2}(b + c - a)\tau_w(\lambda)$$

what gives, explicitly for $\tau_w(\lambda)$:

$$\tau_w(\lambda) = \frac{2h - \lambda}{b + c - a} = \frac{\frac{4}{\lambda} R_{v,w} \left(-\frac{\epsilon}{\lambda} \right)}{\frac{1}{1-\lambda} R_{v,w} \left(\frac{2\epsilon}{1-\lambda} \right) + \frac{1}{1+\lambda} R_{v,w} \left(-\frac{2\epsilon}{1+\lambda} \right)}$$

Similar to the estimations as considered in point 2, lemma 1.3.16 (34) leads to the relation

$$2 \frac{1 - \lambda^2}{\lambda} C^{-2} \leq \tau_w(\lambda) \leq 2 \frac{1 - \lambda^2}{\lambda} C.$$

that is true for every $\lambda \in]0, 1[$ with ϵ within a neighbourhood of 0. In particular, for every compact $F \subseteq]0, 1[$ there is a constant $D > 1$ such that

$$D^{-2} \leq \tau_w(\lambda) \leq D \quad \forall \lambda \in F.$$

Embedding (A, B, C) as $(\tilde{A}, \tilde{B}, \tilde{C})$ into M_K (where $K < \pi^2$), we work out \tilde{h} as function of $\tilde{b} + \tilde{c} - \tilde{a} = \epsilon$ and obtain:

$$\tilde{h} = \frac{\lambda}{2} + \frac{1}{2} \tilde{\tau}_K(\lambda) \epsilon + O(\epsilon^2) = 1 - \frac{\lambda}{2} - \frac{1}{2} \tilde{\tau}_K(\lambda) (\tilde{b} + \tilde{c} - \tilde{a}) + O((\tilde{b} + \tilde{c} - \tilde{a})^2),$$

where

$$\tilde{\tau}_K(\lambda) := \frac{\tan \left(\frac{\sqrt{K}}{2} \right)}{4 \tan \left(\frac{\sqrt{K}}{2} \lambda \right)},$$

which is a continuous function on $(K, \lambda) \in]-\infty, \pi^2[\times]0, 1[$.

We observe that $\lim_{K \rightarrow -\infty} \tilde{\tau}_K(\lambda) = 0$ and $\lim_{K \rightarrow \pi^2} \tilde{\tau}_K(\lambda) = +\infty$ for every $\lambda \in F \subseteq]0, 1[$.

Hence, there are constants $K_-, K_+ < \pi^2$ such that:

$$\tilde{\tau}_{K_+}(\lambda) \leq D \quad \forall \lambda \in F$$

and

$$\tilde{\tau}_{K_-}(\lambda) \geq D^{-1} \quad \forall \lambda \in F.$$

This implies the existence of a $K \in [K_-; K_+]$ such that $h = \tilde{h}$. As this is true for every choice of v, w and $\lambda \in F$ curvature is uniformly bound from above by K_+ and below by K_- . \triangle

5. \triangleright Consider the category of degenerated triangles where $A = H$ and $B \neq C$:

To describe triangles in the neighbourhood of $(A, B, C) = (0, \frac{1}{2}v, -\frac{1}{2}v)$, we look at the family $(0, \frac{1}{2}v + \epsilon w, -\frac{1}{2}v + \epsilon w)$ with v and w unitary.

The side-lengths are (where $\sin \phi := \partial_w \|v\|$):

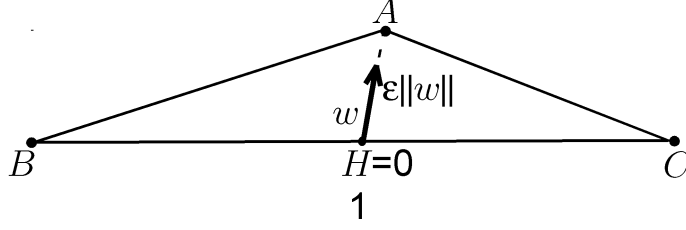


Figure B.5: The triangle family considered in 5.

$$\begin{aligned}
a &= \|v\| = 1 \\
b &= \left\| \frac{1}{2}v + \epsilon w \right\| = \frac{1}{2} \|v + 2\epsilon w\| \\
&= \frac{1}{2} + \epsilon \sin \phi + 2\epsilon^2 \cos^2 \phi R_{v,w}(2\epsilon) \\
c &= \left\| -\frac{1}{2}v + \epsilon w \right\| = \frac{1}{2} \|v - 2\epsilon w\| \\
&= \frac{1}{2} - \epsilon \sin \phi + 2\epsilon^2 \cos^2 \phi R_{v,w}(-2\epsilon) \\
h &= \|\epsilon w\| = \epsilon.
\end{aligned}$$

Consider $(\tilde{A}, \tilde{B}, \tilde{C})$, a triangle in M_K with side lengths $\tilde{a} := a/\rho$, $\tilde{b} := b/\rho$ and $\tilde{c} := c/\rho$. Using the formula for m_K (see section A.2 (86)), we see that \tilde{a} , as function of \tilde{b} , \tilde{c} and \tilde{h} is:

$$\tilde{a}/a = 1 + 4 \left(-\frac{\sqrt{K}}{2} \cot \frac{\sqrt{K}}{2} + \frac{\sqrt{1}}{2} (R_{v,w}(\epsilon) + R_{v,w}(-\epsilon)) \right) \epsilon^2 \cos^2 \phi + O(\epsilon^3)$$

To have the scaled curvature bounded above, we must have $\tilde{a} \geq a$ for some K , to have it bounded below, we must ask for $\tilde{a} \leq a$ (see definition 1.3.5 (27)).

If we remark that the application $K \mapsto \frac{\sqrt{K}}{2} \cot \frac{\sqrt{K}}{2}$ is decreasing and onto from $] -\infty, \pi^2[$ to $]0; \infty[$, the latter expansion of \tilde{a} shows obviously that these curvature conditions can be fulfilled for some K_- and K_+ , as we know (lemma 1.3.16 (34)) that there is a constant $C > 0$ such that $C^{-1} \leq R(\epsilon) \leq C$ for $|\epsilon| < \frac{1}{2}$. \triangle

The union of the neighbourhoods defined in the above points form a neighbourhood \mathcal{U}_Δ of all degenerated triangles. As, for each category, we have found uniform upper and lower bounds for the scale curvature. These maximum respectively minimum will form a suitable choices as curvature bound for \mathcal{U}_Δ . \square

Appendix C

Appendix: Proof of lemma 1.4.9

Lemma C.0.4 *Let (M, d) be a scale bounded space, $p \in M$. For every segment $\gamma \in \Upsilon_=(B_p^*)$, the application $\delta_p(\gamma_t)$ is $\mathcal{C}^{1,1}$.*

Proof of proposition 1.4.8:

Let $\gamma \in \Upsilon_=(B_p)$ defined on $] -\epsilon_0, \epsilon_0[$ and define the useful function $\eta_s(\Delta)$:

$$\eta_s(\Delta) := \frac{\overline{p\gamma_{s+\Delta}} - \overline{p\gamma_s}}{\Delta}$$

By triangle inequality, $-1 \leq \eta_s(\Delta) \leq 1$ for any $\Delta \neq 0$.

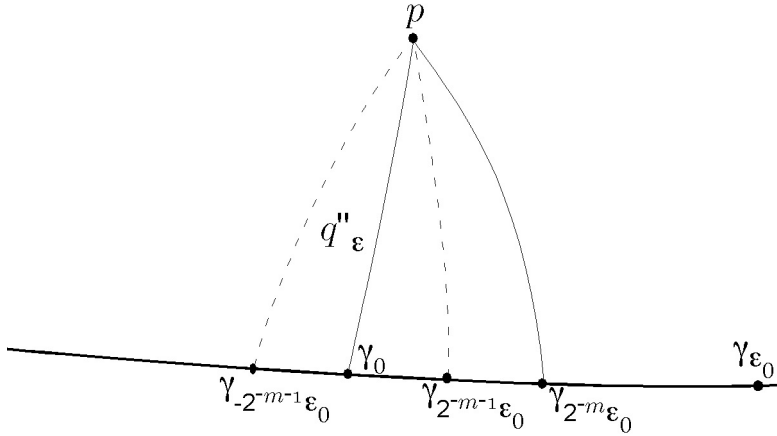


Figure C.1: Construction of $\eta_0(\Delta)$

1. \triangleright The limit $\mathcal{L}_\pm := \lim_{n \rightarrow \infty} \eta_0(\pm 2^{-n} \epsilon_0)$ exists.

Let us consider the triangle $(p, \gamma_0, \gamma_{\pm 2^{-n}\epsilon_0})$. The midpoint of the segment $\gamma_0, \gamma_{\pm 2^{-n}\epsilon_0}$ is $\gamma_{\pm 2^{-n-1}\epsilon_0}$ such that, by the scale curvature bounds k_+ and k_- , we have the estimations:

$$\begin{aligned} \mathcal{M}_{k_-} (2^{-n}\epsilon_0, \overline{p\gamma_0}, \overline{p\gamma_{\pm 2^{-n}\epsilon_0}}) \\ \leq \overline{p\gamma_{\pm 2^{-n-1}\epsilon_0}} \leq \\ \mathcal{M}_{k_+} (2^{-n}\epsilon_0, \overline{p\gamma_0}, \overline{p\gamma_{\pm 2^{-n}\epsilon_0}}) \end{aligned} \quad (\text{C.1})$$

By definition of η_0 , $\overline{p\gamma_{\pm 2^{-n}\epsilon_0}} = \overline{p\gamma_0} \pm 2^{-n}\epsilon_0\eta_0(\pm 2^{-n}\epsilon_0)$ such that:

$$\begin{aligned} \mathcal{M}_{k_-} (2^{-n}\epsilon_0, \overline{p\gamma_0}, \overline{p\gamma_0} \pm 2^{-n}\eta_0(\pm 2^{-n}\epsilon_0)\epsilon_0) \\ \leq \overline{p\gamma_{\epsilon_0 \pm 2^{-n-1}}} \leq \\ \mathcal{M}_{k_+} (2^{-n}\epsilon_0, \overline{p\gamma_0}, \overline{p\gamma_0} \pm 2^{-n}\eta_0(\pm 2^{-n}\epsilon_0)\epsilon_0) \end{aligned} \quad (\text{C.2})$$

By power expansion in ϵ (see section A.2 (86)), we observe that:

$$\mathcal{M}_k(\epsilon, 1, 1 + \zeta\epsilon) = 1 + \frac{\zeta}{2}\epsilon + O_{k,\zeta}(\epsilon^2)$$

where $O_{k,\zeta}(\epsilon^2)/\epsilon^2$ depends continuously on (k, ζ) running in $[k_-, k_+] \times [-1, 1]$. As this set is compact, there is an $L > 0$ such that, for any choice of (k, ζ) and $|\epsilon|$ small enough, we have that $|O_{k,\zeta}(\epsilon^2)| \leq \frac{1}{8}L\epsilon^2$.

From the homogeneity properties of \mathcal{M}_k (section A.2 (86)), we know that, for $x > 0$:

$$\mathcal{M}_k(x\epsilon, x, x + \zeta x\epsilon) = x \mathcal{M}_k(\epsilon, 1, 1 + \zeta\epsilon).$$

By setting $x = \overline{p\gamma_0}$, $\zeta = \eta_0(2^{-n}\epsilon_0)$ and $\epsilon = 2^{-n}\frac{\epsilon_0}{x}$ in the previous expansion, the inequalities C.2 have the form:

$$\begin{aligned} \overline{p\gamma_0} + \eta_0(2^{-n}\epsilon_0)2^{-n-1}\epsilon_0 + \overline{p\gamma_0} O_{k_-, \eta_0(2^{-n}\epsilon_0)} \left(\frac{2^{-2n}\epsilon_0^2}{\overline{p\gamma_0}^2} \right) \\ \leq \overline{p\gamma_{\epsilon_0 \pm 2^{-n-1}}} \leq \\ \overline{p\gamma_0} + \eta_0(2^{-n}\epsilon_0)2^{-n-1}\epsilon_0 + \overline{p\gamma_0} O_{k_+, \eta_0(2^{-n})} \left(\frac{2^{-2n}\epsilon_0^2}{\overline{p\gamma_0}^2} \right) \end{aligned} \quad (\text{C.3})$$

Using the approximation for $O_{k,\zeta}$, we have that:

$$|\overline{p\gamma_{\epsilon_0 \pm 2^{-n-1}}} - \overline{p\gamma_0} - \eta_0(2^{-n}\epsilon_0)2^{-n-1}\epsilon_0| \leq 2^{-2n-1} \frac{\epsilon_0^2}{\overline{p\gamma_0}} L$$

or, equivalently that:

$$|\eta_0(\pm 2^{-n-1}\epsilon_0) - \eta_0(\pm 2^{-n}\epsilon_0)| \leq 2^{-n-2} \frac{\epsilon_0^2}{\overline{p\gamma_0}} L$$

By induction over $m \in \mathbb{N}$ we obtain:

$$\begin{aligned} |\eta_0(\pm 2^{-n-m}\epsilon_0) - \eta_0(\pm 2^{-n}\epsilon_0)| &\leq 2^{-n-2}(1 + 2^{-1} + \dots + 2^{-m+1}) \frac{\epsilon_0^2}{p\gamma_0} L \\ &< 2^{-n-1} \frac{\epsilon_0^2}{p\gamma_0} L \end{aligned}$$

such that $(\eta_0(\pm 2^{-n}\epsilon_0))_{n \in \mathbb{N}}$ is a Cauchy serie proving that the limit \mathcal{L}_+ and \mathcal{L}_- exist.

△

It should be noted that L does not depend on the choice of γ .

2. ▷ \mathcal{L}_+ and \mathcal{L}_- are equal.

By construction γ_0 is the midpoint of the segment between $\gamma_{2^{-n}\epsilon_0}$ and $\gamma_{-2^{-n}\epsilon_0}$. Considering the triangle $(p, \gamma_{2^{-n}\epsilon_0}, \gamma_{-2^{-n}\epsilon_0})$, this implies:

$$\begin{aligned} \mathcal{M}_{k_-} (2^{-n+1}\epsilon_0, \overline{p\gamma_0} + 2^{-n}\eta_0(2^{-n}\epsilon_0)\epsilon_0, \overline{p\gamma_0} - 2^{-n}\eta_0(-2^{-n}\epsilon_0)\epsilon_0) \\ \leq \overline{p\gamma_0} \leq \\ \mathcal{M}_{k_+} (2^{-n+1}\epsilon_0, \overline{p\gamma_0} + 2^{-n}\eta_0(2^{-n}\epsilon_0)\epsilon_0, \overline{p\gamma_0} - 2^{-n}\eta_0(-2^{-n}\epsilon_0)\epsilon_0) \end{aligned} \quad (\text{C.4})$$

But, in the first order,

$$\mathcal{M}_k (2\epsilon, x + \zeta\epsilon, x - \zeta'\epsilon) = x + \frac{\zeta - \zeta'}{2}\epsilon + O_k(\epsilon^2)$$

such that equation C.4 can be written as

$$\begin{aligned} \overline{p\gamma_0} + \frac{\eta_0(2^{-n}\epsilon_0)\epsilon_0 - \eta_0(-2^{-n}\epsilon_0)\epsilon_0}{2} 2^{-n} + O_{k_-}(2^{-2n}) \\ \leq \overline{p\gamma_0} \leq \\ \overline{p\gamma_0} + \frac{\eta_0(2^{-n}\epsilon_0)\epsilon_0 - \eta_0(-2^{-n}\epsilon_0)\epsilon_0}{2} 2^{-n} + O_{k_+}(2^{-2n}) \end{aligned} \quad (\text{C.5})$$

These inequalities can obviously only be fulfilled, if

$$\lim_{n \rightarrow +\infty} \eta_0(2^{-n}\epsilon_0) - \eta_0(-2^{-n}\epsilon_0) = 0.$$

As, by definition of \mathcal{L}_+ and \mathcal{L}_- :

$$\lim_{n \rightarrow +\infty} \eta_0(\pm 2^{-n}\epsilon_0) = \mathcal{L}_{\pm}$$

we conclude that $\mathcal{L}_+ = \mathcal{L}_-$.

△

The value $\mathcal{L}_+ = \mathcal{L}_-$ depends on the choice of γ and on ϵ_0 . Suppose now $\gamma' \in \Upsilon_=(B_p^*)$ defined on an interval $]a, b[$. For any $t_0 \in]a, b[$ and any $\epsilon_0 > 0$ small enough, we can construct the segment $\gamma :]-\epsilon_0, \epsilon_0[\longrightarrow B_p$ defined by $\gamma_t = \gamma'_{t-t_0}$. In this way, we can work out $\mathcal{L}_+ = \mathcal{L}_-$ along every point

of a segment in B_p : the result does only depend on t_0 and ϵ_0 . We denote this value by $\mathcal{L}(t_0, \epsilon_0)$.

The restriction for ϵ_0 to be small enough is not essential to define \mathcal{L} . By the way, we notice that $\mathcal{L}(t_0, \epsilon_0) = \mathcal{L}(t_0, 2^n \epsilon_0)$ for every $n \in \mathbb{Z}$.

We suppose that the previously defined path γ has no point closer than δ to p and define \mathbb{D} as $\{m 2^{-n} \epsilon_0 \mid n \in \mathbb{N} \text{ and } m \in \mathbb{Z}\}$.

3. $\triangleright \mathcal{L}(t, \epsilon_0 - |t|)$ is (L/δ) -Lipschitz continuous on $t \in \mathbb{D} \cap] - \epsilon_0, \epsilon_0[$.

We shall prove that

$$|\mathcal{L}(t, \epsilon_0 - |t|) - \mathcal{L}(t', \epsilon_0 - |t'|)| \leq \frac{L}{\delta} |t - t'| \quad (\text{C.6})$$

for any $t, t' \in \mathbb{D} \cap] - \epsilon_0, \epsilon_0[$.

We will prove it for $t, t' \in \mathbb{D} \cap [0, \epsilon_0[$. The prove is similar for the case $t, t' \in \mathbb{D} \cap] - \epsilon_0, 0]$. If $t < 0$ and $t' > 0$, the prove follows from the above cases in the following way:

$$\begin{aligned} |\mathcal{L}(t) - \mathcal{L}(t')| &= |(\mathcal{L}(t) - \mathcal{L}(0)) - (\mathcal{L}(0) - \mathcal{L}(t'))| \\ &\leq |(\mathcal{L}(t) - \mathcal{L}(0))| + |(\mathcal{L}(0) - \mathcal{L}(t'))| \\ &\leq \frac{L}{\delta} |t - 0| + \frac{L}{\delta} |0 - t'| = \frac{L}{\delta} |t - t'|. \end{aligned}$$

We observe that any number in $\mathbb{D} \cap [0, \epsilon_0[$ can be written as $(2^{-n_1} + 2^{-n_2} + \dots + 2^{-n_l})\epsilon_0$ where $n_1, n_2, \dots, n_l \in \mathbb{N}$ and define $\mathbb{D}^{(l)}$ the set of all numbers of the form given above, $\mathbb{D}^{(0)}$ being $\{0\}$. The family $(\mathbb{D}^{(l)})_{l \in \mathbb{N}}$ is a covering of $\mathbb{D} \cap [0, \epsilon_0[$.

Suppose now that $t \in \mathbb{D}^{(0)}$ (i. e. $t = 0$) and $t' \in \mathbb{D}^{(1)}$. In part 2 we have proved that

$$|\eta_0(t') - \mathcal{L}(0, \epsilon_0)| \leq \frac{1}{2} |t' - 0| \frac{L}{\delta}$$

but if we consider the construction of $\mathcal{L}(t')$ in this same part, we obtain:

$$|\eta_0(t') - \mathcal{L}(t', \epsilon_0 - |t'|)| \leq \frac{1}{2} |t' - 0| \frac{L}{\delta}$$

such that equation 3 holds for our choice of t and t' .

This argument also applies in the case where $t \in \mathbb{D}^{(l)}$ and $t' \in \mathbb{D}^{(l+1)}$. Then, for the same reasons as above, we know that:

$$|\eta'_0(t') - \mathcal{L}(t, \epsilon_0 - |t|)| \leq \frac{1}{2} |t' - t| \frac{L}{\delta}$$

and

$$|\eta'_0(t') - \mathcal{L}(t', \epsilon_0 - |t'|)| \leq \frac{1}{2} |t' - t| \frac{L}{\delta}$$

where $\eta'_0(s)$ is defined as $\eta_0(s - t')$ such that, again, 3 holds for our choice of t and t' .

But if $t, t' \in \mathbb{D} \cap [0, \epsilon_0[$ with $t > t'$, $t - t' \in \mathbb{D} \cap [0, \epsilon_0[$ such that $t \in \mathbb{D}^{(l)}$ for some $l \in \mathbb{N}$ and $t - t' \in \mathbb{D}^{(s)}$ for some $s \in \mathbb{N}$. According to the

above notation, we set $t = (2^{-n_1} + 2^{-n_2} + \dots + 2^{-n_l})\epsilon_0$ and $t - t' = (2^{-m_1} + 2^{-m_2} + \dots + 2^{-m_s})\epsilon_0$. Call $t_r := t + (2^{-m_1} + 2^{-m_2} + \dots + 2^{-m_r})\epsilon_0$ for $r = 0, 1, \dots, s$. By construction, $t_r \in \mathbb{D}^{(l+r)}$, $t_0 = t$ and $t_s = t'$.

Meanwhile, we know that, for $r = 0, 1, \dots, s$:

$$|\mathcal{L}(t_{r+1}, \epsilon_0 - |t_{r+1}|) - \mathcal{L}(t_r, \epsilon_0 - |t_r|)| \leq |t_{r+1} - t_r| \frac{L}{\delta}$$

such that

$$\sum_{r=0}^{s-1} |\mathcal{L}(t_{r+1}, \epsilon_0 - |t_{r+1}|) - \mathcal{L}(t_r, \epsilon_0 - |t_r|)| \leq \sum_{r=0}^{s-1} |t_{r+1} - t_r| \frac{L}{\delta} = |t_s - t_0| \frac{L}{\delta}.$$

But

$$|\mathcal{L}(t_s, \epsilon_0 - |t_0|) - \mathcal{L}(t_r, \epsilon_0 - |t_r|)| \leq \sum_{r=0}^{s-1} |\mathcal{L}(t_{r+1}, \epsilon_0 - |t_{r+1}|) - \mathcal{L}(t_r, \epsilon_0 - |t_r|)|$$

such that $|\mathcal{L}(t', \epsilon_0 - |t'|) - \mathcal{L}(t, \epsilon_0 - |t|)| \leq |t' - t| \frac{L}{\delta}$ that was our claim. \triangle

4. $\triangleright \mathcal{L}(t, s)$ is continuous on $t \in]-\epsilon_0, \epsilon_0[$ and does not depend on s .

We first prove the Lipschitz continuity in 0 i.e. that

$$|\mathcal{L}(t, \epsilon_0 - |t|) - \mathcal{L}(0, \epsilon_0)| \leq |t| \frac{L}{\delta}.$$

The proof for other points and negative values of t is similar.

Let $t \in [0, \epsilon_0[$. As $\mathbb{D} \cap [0, \epsilon_0[$ is dense in $[0, \epsilon_0[$ there is a $t' \in \mathbb{D} \cap [0, \epsilon_0[$ and an $m \in \mathbb{N}$ such that $2^{-m}\epsilon_0 < t$ and $(t - 2^{-m}\epsilon_0) - (2^{-m}\epsilon_0) := \varphi \leq 2^{-3m}\epsilon_0$.

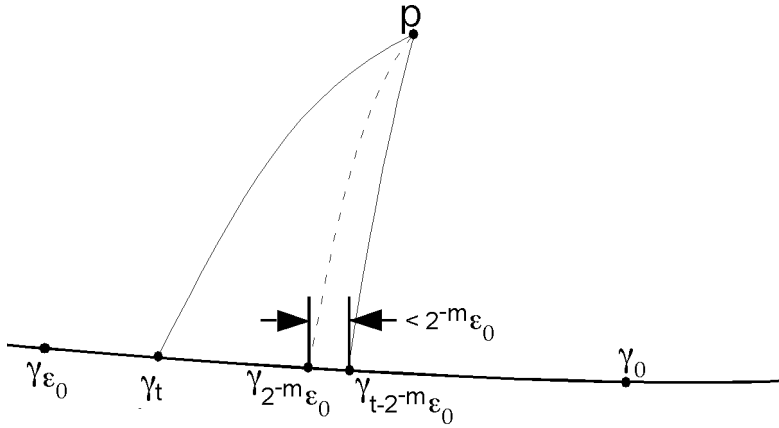


Figure C.2: Correct choice of t and m

Considering the triangle $p, \gamma_{2^{-m}\epsilon_0}, \gamma_{t-2^{-m}\epsilon_0}$, we define ζ such that

$$\overline{p \gamma_{t-2^{-m}\epsilon_0}} = \overline{p \gamma_{2^{-m}\epsilon_0}} + \varphi \zeta$$

We observe that, by triangle inequality, $|\zeta| \leq 1$ and that this definition implies the relation

$$2^{-m}\epsilon_0 \eta_0(2^{-m}\epsilon_0) + \varphi\zeta = 2^{-m}\epsilon_0 \eta'_0(-2^{-m}\epsilon_0)$$

where $\eta'_0(\Delta)$ is $(\gamma_{\Delta+t} - \gamma_\Delta) / \Delta$ such that

$$\begin{aligned} |\eta_0(2^{-m}\epsilon_0) - \eta'_0(-2^{-m}\epsilon_0)| &= \left| \frac{2^m}{\epsilon_0} \varphi\zeta \right| \\ &\leq \left| \frac{2^m}{\epsilon_0} 2^{-3m} \right| = \frac{2^{-2m}}{\epsilon_0}. \end{aligned}$$

On the other hand, we know from part 1 (97) that:

$$\begin{aligned} |\mathcal{L}(0, \epsilon_0) - \eta_0(2^{-m}\epsilon_0)| &\leq 2^{-m-1} \frac{\epsilon_0^2}{\delta} L \\ |\mathcal{L}(t, \epsilon_0 - |t|) - \eta'_0(2^{-m}\epsilon_0)| &\leq 2^{-m-1} \frac{\epsilon_0^2}{\delta} L \end{aligned}$$

Putting the last three inequalities together, we obtain that

$$|\mathcal{L}(t, \epsilon_0 - |t|) - \mathcal{L}(0, \epsilon_0)| \leq 2^{-m} \frac{\epsilon_0^2}{\delta} L + \frac{2^{-2m}}{\epsilon_0} = 2^{-m} \epsilon_0 \frac{L}{\delta} \left(1 + 2^{-m} \frac{\delta}{L\epsilon_0^2} \right)$$

As the left hand side does not depend on m and as m can be chosen as large as wanted, the inequality holds in the limit $m \rightarrow +\infty$ such that

$$|\mathcal{L}(t, \epsilon_0 - |t|) - \mathcal{L}(0, \epsilon_0)| \leq |t| \frac{L}{\delta}.$$

such that $\mathcal{L}(t, \epsilon_0 - |t|)$ is continuous. This, in turn, implies that:

$$|\mathcal{L}(0, \epsilon_0) - \eta_0(\epsilon \epsilon_0)| \leq \frac{\epsilon}{2} \frac{\epsilon_0^2}{\delta} L$$

such that

$$\lim_{\epsilon \rightarrow 0} \eta_0(\epsilon, \epsilon_0) = \mathcal{L}(0, \epsilon_0) = \lim_{\epsilon \rightarrow 0} \eta_0(\epsilon, \epsilon'_0) = \mathcal{L}(0, \epsilon'_0)$$

for any ϵ'_0 , such that $\mathcal{L}(t, s)$ does not depend on the value of $s > 0$. We denote this value by $\mathcal{L}(t)$. \triangle

\square

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