Power Laws from Randomly-Sampled Continuous-Time Random Walks

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Abstract

It has been shown by Reed that random-sampling a Wiener process $x(t)$ at times T chosen out of an exponential distribution gives rise to power laws in the distribution $P(x(T)) \sim x(T)^{-\beta}$. We show, both theoretically and numerically, that this power-law behaviour also follows by random-sampling Lévy flights (as continuous-time random walks), having Fourier distribution $\hat{w}(k) = e^{-|k|^\alpha}$, with the exponent $\beta = \alpha$.

Key words: Continuous Time Random Walks; Population Dynamics; Power Laws.
PACS: 02.50.Cw, 87.23.2n, 89.65.Cd

1 Introduction

Many stochastic processes in various branches of the physical, economical and social sciences present empirical power-law behaviour. A much discussed mechanism in recent years for this behaviour has been that of self-organised criticality (1). Reed has argued (2), on the other hand, that for a random-walk-like stochastic process, the power-law behaviour follows most naturally when the process' measurement times T are also taken from a random distribution

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(e.g. exponential: $f_T(t) = \Lambda e^{-\Lambda t}$). This would account in a phenomenological way for many empirically-observed power laws in many stochastic phenomena.

In Economics prime examples of such laws are the distribution of incomes (Pareto's law (3)) and city sizes (Zipf's law (4)), as well as - arguably - of the standardised price returns on generic stocks or on stock indices (5; 6). In other realms of investigation, empirical size distributions for which power-law behaviour has been claimed include those of granular media's particle sizes (7); of avalanche sizes (1); of meteor impact's on the moon (8); of number of species per genus (9); of frequency of words in long sequences of text (4); of areas of burnt patches in forest fires (10); and so on. In Physics, power-laws are observed in static and dynamic correlation functions at the critical point of a second-order phase transition; in the non-equilibrium transport phenomena pertaining to the so-called 1/f-noise (11), or Barkhausen noise in the context of magnetism, (12); and so on. Within and outside the fields of application of the laws of physics the presence of power-law probability distributions $P(x) = C\,x^{-\beta}$ appears to be ubiquitous and in the overwhelming majority of cases a prediction of the value of the exponent β from the laws which determine the local behaviour of the system is lacking.

For this reason, Reed's proposal appears as an interesting explanation on a phenomenological standpoint for the ubiquity of the power-law distribution whenever a stochastic process is observed: for observations also take place at observation times which are themselves stochastically chosen.

In fact, to fix our ideas, let us consider a geometric random-walk process $x(t)$ modelled by the stochastic differential equation

$$dx = \mu x dt + \sigma x dw \tag{1}$$

where dw is the increment in time dt of a Wiener process, σ a diffusion coefficient and μ a drift bias. The distribution function of such process is lognormal, as is well known, however many stochastic processes modelled by this type of dynamics do show up power-law behaviours empirically. When Ito's calculus (13) is applied, the probability density for $y = \ln x$ is known to be, at a generic time t

$$P(y,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\{-[y - y(0) - (\mu - \sigma^2/2)t]^2/2\sigma^2 t\} \tag{2}$$

and has moment-generating function

$$M_y(s) = \langle e^{ys} \rangle_{P_y} = \exp\{y_1 s + \frac{1}{2}y_2^2 s^2\} \tag{3}$$

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where $\langle\cdot\rangle_{P_y}$ is the average with respect to the density in Eq. (2) and the first two moments are given by $y_1 = y(0) + (\mu - \sigma^2/2)t$ and $y_2 = \sigma\sqrt{t}$ as in a standard geometric random-walk process (where $y(0) = \ln x(0)$ refers to the initial conditions).

Now we assume that the observation times $t = T$ themselves are taken from a random-variable probability density $f_T(t)$, which we may assume to be of the exponential type (this being the only memoryless case):

$$f_T(t) = \Lambda e^{-\Lambda t}, \qquad \text{for} \quad \Lambda > s \tag{4}$$

where Λ is the sampling rate. The moment-generating function for the variable $\bar{y} = y(T)$ is now given by the conditional average over the observation-times distribution as well:

$$\begin{aligned} M_{\bar{y}}(s) &= \left\langle \langle e^{\bar{y}s}\rangle_{P_y} \right\rangle_{f_T} \\ &= e^{y(0)s} \frac{\Lambda}{\Lambda - (\mu - \sigma^2/2)s - \sigma^2 s^2/2} \end{aligned} \tag{5}$$

At this point, we can seek the two real poles s_1, s_2 (with $s_1 > 0$ and $s_2 < 0$) of the above expression to obtain the form

$$M_{\bar{y}}(s) = e^{y(0)s} \frac{s_1 s_2}{(s - s_1)(s - s_2)} \tag{6}$$

which, thought of as a new moment-generating function gives the density

$$P(\bar{y}) = \begin{cases} |\frac{s_1 s_2}{s_1 - s_2}| e^{-s_1(\bar{y} - y(0))} & \text{if} \quad \bar{y} > y(0) \\ |\frac{s_1 s_2}{s_1 - s_2}| e^{-s_2(\bar{y} - y(0))} & \text{if} \quad \bar{y} < y(0) \end{cases} \tag{7}$$

These exponential contributions provide, for the variable $\bar{x}$ itself, the sought power-law,

$$P(\bar{x}) = \begin{cases} |\frac{s_1 s_2}{s_1 - s_2}| \frac{1}{x(0)} \left(\frac{\bar{x}}{x(0)}\right)^{-s_1 - 1} & \text{if} \quad \bar{x} > x(0) \\ |\frac{s_1 s_2}{s_1 - s_2}| \frac{1}{x(0)} \left(\frac{\bar{x}}{x(0)}\right)^{-s_2 - 1} & \text{if} \quad 0 \le \bar{x} < x(0) \end{cases} \tag{8}$$

2 Models based on continuous-time random walks (CTRW)

Our point of view is that, for example in the case of high-frequency financial returns, assuming a geometric random walk process for the random variable

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$x(t)$ is somewhat unrealistic when looking at the available empirical data. In fact, trading in financial markets is an asynchronous process since durations between consecutive trading events are random variables (14). In such a case, $x(t)$ has the meaning of the price of an asset and $y(t)$ can be considered as the log-price or the log-return with respect to the initial price. We therefore propose an improvement of the 'microscopic' modelling by a continuous-time random-walk (CTRW), as the raw stochastic process with jumps, ξ, described by a generalised Fourier representation of the density function characterised by a form

$$\hat{w}(k) = e^{-|k|^{\alpha}} \tag{9}$$

with α a real parameter in $[0, 2]$. Eq. (9) is the characteristic function of a so-called Lévy α-stable random variable. The underlying stochastic dynamics is then characterised by randomly-spaced jumps of random amplitudes, ξ, which more faithfully mimic the actual pricing dynamics of the financial markets and other relevant situations. As another example for our model, we can consider the growth of firms and enterprises (or of atomic islands in appropriate physics contexts) where $x(t)$ is the firm's size. The CTRW jumps arising at random times are asynchronous idiosyncratic shocks, or growth rates, and our subordination to the exponential process corresponds to the fact that firms are monitored at random times. In this way, our model differs from Reed's situation in that the waiting times between two adjacent size values are also random variables (a situation closer to reality).

In this context, we recall that Gibrat has claimed the discovery (15) of a simple law that can explain the origin of inequalities within economic growth processes (16). A simple equation that takes into account the ideas of Gibrat is as follows (17):

$$x(t + 1) = \eta(t)x(t) \tag{10}$$

where $x(t)$ represents the firm's size and $\eta(t)$ is a random variable always extracted from the same probability distribution at any discrete period t. Eq. (10) is a growth model with multiplicative noise. It is the basis on which the so-called generalized Lotka-Volterra models are built (18). There are no direct or indirect interactions between the firms, and the logarithm of the size, $y(t) = \ln x(t)$, is the sum of independent and identically distributed random variables, the growth rates $\xi = \ln \eta$:

$$y(t + 1) = y(0) + \sum_{m=1}^{t} \xi(m) \tag{11}$$

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If the growth rate is independent from the size, the Central Limit Theorem and its generalisations apply in the large t limit and one gets either normal or Lévy distributed log-sizes and, therefore, log-normal or log-Lévy distributed sizes. In a situation in which the growth shocks, ξ, can arrive at random times instead of at fixed periods, Eq. (11) must be replaced by its continuous-time counterpart, given by the non-homogeneous form:

$$y(t) = y(0) + \sum_{m=1}^{n(t)} \xi(m). \tag{12}$$

where $n(t)$ is the number of shocks up to time t. These ideas have been recently applied by economists to models of firm growth (19). It is important to remark that the probability distribution for y follows either a normal or anomalous diffusive behaviour (17).

We now show that implementing the random sampling-time idea of Reed, a power-law behaviour is obtained for the distribution function of the variable $\bar{y} = y(T)$.

3 Random sampling of CTRW

3.1 Analytical approach

If we assume Poisson waiting times and Lévy distributed jumps [2], the overall displacement density is given by:

$$P(y,t) = \sum_{N=0}^{\infty} \frac{(\lambda t)^N e^{-\lambda t}}{N!} w^{*N}(y) \tag{13}$$

The Fourier Transform of (13) is

$$\hat{P}(k,t) = \sum_{N=0}^{\infty} \frac{(\lambda t)^N e^{-\lambda t}}{N!} \hat{w}^N(k) = e^{\lambda t(e^{-|k|^\alpha} - 1)} \tag{14}$$

where $\hat{w}(k) = e^{-|k|^\alpha}$ is the characteristic function of the Lévy symmetric stable distribution with exponent α and λ is the rate for the CTRW jumps. In the firm-growth interpretation, λ is the rate of arrival of the idiosyncratic shocks.

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[2] In the following we will assume that the scale factor that usually appears in the definition of the characteristic function of a Lévy distributions is simply $c = 1$

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As in Reed's case, the observation times $t = T$ are taken as a random variables distributed with rate Λ, $f_T(t) = \Lambda e^{-\Lambda t}$, and averaging over observation times Eq. (14) gives:

$$\hat{P}(k) = \int_0^\infty dt \ \Lambda e^{-\Lambda t} e^{\lambda t(\hat{w}(k)-1)} = \frac{\Lambda}{\Lambda + \lambda(1 - \hat{w}(k))} \tag{15}$$

The next step is to find the inverse Fourier Transform of (15) in order to get $P(\bar{y})$. After some algebraic manipulation, one finds:

$$\hat{P}(k) = \frac{\Lambda \lambda}{(\Lambda + \lambda)^2} \sum_{n=0}^\infty \left(\frac{\lambda}{\Lambda + \lambda}\right)^n [\hat{w}(k)]^{n+1} + \frac{\Lambda}{\Lambda + \lambda} \tag{16}$$

that implies

$$P(\bar{y}) = \frac{\Lambda \lambda}{(\Lambda + \lambda)^2} \sum_{n=0}^\infty \left(\frac{\lambda}{\Lambda + \lambda}\right)^n w^{*^{n+1}}(\bar{y}) + \frac{\Lambda}{\Lambda + \lambda}\delta(\bar{y}) \tag{17}$$

Observing that $[\hat{w}(k)]^n = e^{-n|k|^\alpha}$ it is not difficult to argue that the δ function can be formally written as $w^{*^0}(x)$. This allows us to write the expression (17) in a more compact form,

$$P(\bar{y}) = \frac{\Lambda}{\Lambda + \lambda} \sum_{n=0}^\infty \left(\frac{\lambda}{\Lambda + \lambda}\right)^n w^{*^n}(\bar{y}) \tag{18}$$

Finally, from the stable property of $\hat{w}(k)$, it is possible to express the convolution of Lévy densities in the following way:

$$w^{*^n}(\bar{y}) = n^{-\frac{1}{\alpha}} w\left(n^{-\frac{1}{\alpha}}\bar{y}\right) \tag{19}$$

3.2 Simulation vs analytical results

A closer look to eq. (15) allows us to verify the analytical results discussed in the previous section. For example, if we want the behaviour of $P(\bar{y})$ for large values of $\bar{y}$ we have to consider eq. (15) for values of k near to zero. One can check that

$$\hat{P}(k) \sim 1 - \frac{\lambda}{\Lambda}|k|^\alpha \sim e^{-\lambda|k|^\alpha/\Lambda}. \tag{20}$$

But this is exactly the Fourier Transform of a Lévy distributions with exponent α and implies that the pdf of our process has got a power-law tail with the
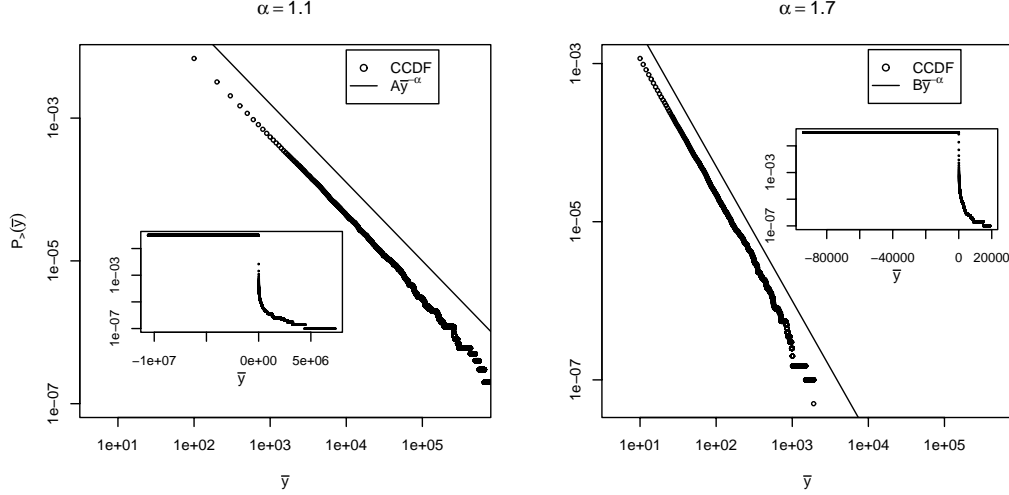
Fig. 1. Tail of the complementary cumulative distribution function (CCDF) for the variable $\bar{y}$ for $\alpha = 1.1$ and $\alpha = 1.7$. In the two plots the behaviour of $\bar{y}^{-\alpha}$ is also reported. In the two insets, the whole CCDF is plotted. The number N of points used for building the CCDF is 10^7. Values of the parameters in the simulation are: $N = 10^7$; $\Lambda = 10^{-3}$; $\lambda = 2 * 10^{-2}$ (see text for details).

same exponent. In order to check this prediction we performed extensive Monte Carlo simulation. For every simulation we sampled $N = 10^7$ points of our process. In Fig. (1) the tail of the complementary cumulative distribution function is plotted for two different values of α. In the same plot we reported also the straight lines with slope α. In the case $\alpha = 1.1$, the agreement between the analytical prediction and the Monte Carlo simulation is good. For $\alpha = 1.7$, the agreement is still satisfactory, but finite-size effects in the simulation are stronger (21).

4   Conclusions

CTRWs are suitable models of asynchronous random processes. For this reason, we computed the probability density of a CTRW with exponentially-distributed waiting times and α-Lévy-stable distributed jumps, after random sampling of the evaluation times. Our main result is given in Eq. (18). It turns out that the tail of the complementary cumulative distribution function for the variable $\bar{y} = y(T)$ follows a power-law with exponent α equal to the exponent of the jumps. When $\bar{y}$ has the meaning of log-size of, e.g., a set of firms, $\bar{y} = \ln(\bar{x})$ one should notice that the size distribution is no longer strictly power law, but a logarithmic correction is necessary:

$$P(\bar{x}) = P(\bar{y})|\frac{d\bar{y}}{d\bar{x}}| \simeq \frac{1}{|\bar{x}|[\ln(\bar{x})]^{\alpha+1}}. \tag{21}$$

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It should be noticed that our result entails for this model a super-universality for the direct stochastic variable involved, that is $\beta = 1$, when $\bar{x}$ is so large that the logarithmic correction can be neglected. This analytical result does not explain the empirical distribution of firm sizes where an exponent $\beta \simeq 2$ has been estimated (20). However, it is necessary to be careful when comparing theoretical results to estimates of power-law exponents, as the latter are delicate (21). It has been shown that a suitable CTRW model can indeed reproduce various stylised facts of firm growth distributions, but random sampling in time is not necessary (19).

A possible shortcoming of our model is implicit in the exponential distribution of the sampling times T; more realistic distributions (albeit with memory) will be taken into consideration in the future.

Acknowledgements

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