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Almost Extrinsically Homogeneous Submanifolds of Euclidean Space

Dedicated to Ernst A. Ruh on the occasion of his 70th birthday

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Abstract. Consider a closed manifold M immersed in $\mathbb{R}^m$. Suppose that the trivial bundle $M \times \mathbb{R}^m = TM \otimes \nu M$ is equipped with an almost metric connection $\tilde{\nabla}$ which almost preserves the decomposition of $M \times \mathbb{R}^m$ into the tangent and the normal bundle. Assume moreover that the difference $\Gamma = \partial - \tilde{\nabla}$ with the usual derivative ∂ in $\mathbb{R}^m$ is almost $\tilde{\nabla}$-parallel. Then M admits an extrinsically homogeneous immersion into $\mathbb{R}^m$.

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Introduction

In Riemannian geometry special geometric structures are often locally characterized by the parallelism of certain tensors. In his thesis, Nomizu [8] showed that a Riemannian manifold (M, g_M) is locally homogeneous, if and only if it admits a metric connection $\tilde{\nabla}$, called canonical connection, such that its torsion, its curvature tensor and the tensor $\Gamma = \nabla - \tilde{\nabla}$, where ∇ denotes the Riemannian connection on M, are $\tilde{\nabla}$-parallel. Even before it was observed by É. Cartan that a Riemannian manifold is locally symmetric, if and only if the Riemannian connection is canonical in this sense.

A technique due to Strübing [13] shows that in the case of complete submanifolds, the parallelism of a certain structure often implies extrinsic geometric properties, which are even global. An analogy to Nomizu's theorem was given by Olmos [10] and (in a more general situation) by Eschenburg [3]. Extrinsic homogeneity of closed submanifolds in $\mathbb{R}^m$ is equivalent to the existence of a metric connection $\tilde{\nabla}$ on the trivial bundle $M \times \mathbb{R}^m \cong TM \oplus \nu M$, such that TM and νM are $\tilde{\nabla}$-parallel subbundles and such that the difference tensor $\Gamma = \partial - \tilde{\nabla}$ with the usual derivative ∂ in $\mathbb{R}^m$ is $\tilde{\nabla}$-parallel. If the normal part of $\tilde{\nabla}$ coincides with the usual normal connection, then, according to Olmos and Sánchez [9], M is essentially an orbit of an s-representation and vice-versa. If moreover the tangent part of the canonical connection is just the Riemannian connection, then M is extrinsically symmetric in $\mathbb{R}^m$. This analogy to É Cartan's characterization of locally symmetric spaces is due to Ferus [4].

Katsuda [5] showed a pinched version of Nomizu's theorem. Another pinching result in the case of compact symmetric spaces was provided by Min-Oo and Ruh [7]. In this paper we show a pinching theorem for extrinsically homogeneous submanifolds of Euclidean space, obtained from the characterization of Olmos and Eschenburg. The technique we use is somehow similar to the one used by Katsuda. We further discuss in detail the case of orbits of s-representations and the case of extrinsically symmetric submanifolds. In the last case the result was shown by the author and can be found in [11].

1. Preliminaries

A real valued function f defined on a bounded domain Ω of some Euclidean space is said to be of class $\mathcal{C}^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in [0,1]$, if it is bounded in the $\mathcal{C}^{k,\alpha}$ Hölder-norm:

$$\|f\|_{k,\alpha} = \sum_{0\le|\beta|\le k} \sup_{x\in\Omega} |\partial_\beta f(x)| + \sum_{|\beta|=k} \sup_{x\neq y} \frac{|\partial_\beta f(x) - \partial_\beta f(y)|}{|x-y|^\alpha}.$$

A tensor on a compact manifold M resp. a mapping between two manifolds is said to be of class $\mathcal{C}^{k,\alpha}$, if there are local coordinates such that in these coordinates its components are of class $\mathcal{C}^{k,\alpha}$. For a compact manifold M we denote by $\mathcal{C}^{k,\alpha}(M,\mathbb{R}^m)$ the Hölder space of $\mathcal{C}^{k,\alpha}$ functions form M to $\mathbb{R}^m$. We have the following embedding theorem for Hölder spaces:

PROPOSITION 1. *Let M be a compact manifold, k_1, k_2 be two positive integers and $0 \le \alpha_1, \alpha_2 \le 1$ such that*

$$k_1 + \alpha_1 > k_2 + \alpha_2.$$

Then the canonical embedding

$$\mathcal{C}^{k_1,\alpha_1}(M,\mathbb{R}^m) \longrightarrow \mathcal{C}^{k_2,\alpha_2}(M,\mathbb{R}^m)$$

is compact, i.e. any bounded sequence in $\mathcal{C}^{k_1,\alpha_1}(M,\mathbb{R}^m)$ has a convergent subsequence in $\mathcal{C}^{k_2,\alpha_2}(M,\mathbb{R}^m)$.

By $\mathcal{M}(\Lambda, d, v, n)$ we denote the class of n-dimensional compact Riemannian manifolds M with bounded sectional curvature $|K| \le \Lambda^2$ and diameter $\mathrm{diam}(M) \le d$ and admitting moreover a lower bound on the volume ($\mathrm{vol}(M) \ge v$). M. Gromov, A. Katsuda, S. Peters and R. Greene and H. Wu (see [12, Appendix]) provided the following convergence result for sequences in $\mathcal{M}(\Lambda, d, v, n)$:

THEOREM 2. *Let* $(M_i, g_i)_{i\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}(\Lambda, d, v, n)$ *and let* $\alpha' \in]0, 1[$ *be fixed. Then there exists a subsequence* $(M_{i_j}, g_{i_j})_{j\in\mathbb{N}}$ *and a smooth manifold* M *equipped with a Riemannian metric* g *of class* $\mathcal{C}^{1,\alpha'}$ *such that the following holds: There is an integer* j_0 *such that for all* $j \geq j_0$ *there are* $\mathcal{C}^\infty$*−diffeomorphisms* $f_{i_j}: M \longrightarrow M_{i_j}$ *such that the sequence of pullback metrics* $(f_{i_j}^* g_{i_j})_{j\in\mathbb{N}}$ *on* M *converges to* g *in the* $\mathcal{C}^{1,\alpha}$ *topology* $(0 < \alpha < \alpha')$.

Let $|\cdot|$ be a norm on $\mathbb{R}^n$ and let $\|\cdot\|_l$ be a norm on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $|Ax| \leq \|A\|_l \cdot |x|$, where $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$; e.g. the operator norm on linear endomorphisms. Then $(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n), \|\cdot\|_l)$ is a Banach space. For further use, we state the following Gronwall-type inequality:

LEMMA 3. *Let* $f, b: [0, T] \longrightarrow \mathbb{R}^n$ *and* $A: [0, T] \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ *be smooth functions. Assume that the functions* $A(t)$ *and* $b(t)$ *are bounded on* $[0, T]$, *i.e.* $\|A(t)\|_l \leq A_0$ *and* $|b(t)| \leq b_0$. *If* $f'(t) = A(t)f(t) + b(t)$, *then*

$$|f(t)| \leq |f(0)|e^{A_0 t} + \frac{b_0}{A_0}(\mathrm{e}^{A_0 t} - 1).$$

2. Extrinsically Homogeneous Submanifolds

Consider a closed (i.e. compact and connected) n-dimensional Riemannian manifold (M^n, g) and an isometric immersion

$$f: (M, g_M) \longrightarrow (\mathbb{R}^m, g_{\mathrm{can}})$$

of M into the m-dimensional Euclidean space with its canonical metric. The immersion f induces a splitting of the trivial bundle $\mathcal{E} = f^*T\mathbb{R}^m \cong T\mathbb{R}^m\,|_{f(M)} \cong M \times \mathbb{R}^m$ over M as a direct sum of the tangent bundle TM and the normal bundle νM, i.e. $\mathcal{E} = TM \oplus \nu M$. The bundle metric g on $\mathcal{E}$ induced by g_{can} splits accordingly $g = g^T \oplus g^\perp$, where $g^T \cong g_M$. Since our considerations are of local nature, we identify always locally vector fields on M with the corresponding vector fields on $f(M)$. In the following, vectors tangent to M will be denoted by capital Roman letters and normal vectors by Greek ones. If X is an element of $\mathcal{E}$, we denote by X^T its tangent and by $X^\perp$ its normal component. Let ∂ denote the canonical derivative in $\mathbb{R}^m$ and ∇ the Riemannian connection on M. The normal bundle νM is equipped with a metric connection $\nabla^\perp$ defined by $\nabla^\perp_X \xi = (\partial_X \xi)^\perp$. The second fundamental form α of f is defined by $\alpha(X, Y) = \partial_X Y - \nabla_X Y$ and the corresponding shape operator A by $A_\xi X = \nabla^\perp_X \xi - \partial_X \xi$. Since ∂ is metric, the second fundamental form and the shape operator are related by $g(A_\xi X, Y) = g^\perp(\alpha(X, Y), \xi)$. Moreover, R denotes the Riemannian curvature tensor, K the sectional curvature and $\mathrm{inj}(M)$ the injectivity radius of M.

The submanifold $f(M)$ is called *extrinsically homogeneous,* if for any pair of points p, q in $f(M)$, there exists an isometry of $\mathbb{R}^m$ mapping p to q while leaving

$f(M)$ invariant. Hence, extrinsically homogeneous submanifolds of $\mathbb{R}^m$ are orbits of subgroups of the isometry group of $\mathbb{R}^m$.

A connection $\tilde{\nabla}$ on $\mathcal{E}$ is called *(extrinsic) canonical connection* (w.r.t. f) if

(1) $\tilde{\nabla}$ is metric;
(2) TM is a $\tilde{\nabla}$-parallel subbundle of $\mathcal{E}$;
(3) The difference tensor $\Gamma = \partial - \tilde{\nabla}$ is $\tilde{\nabla}$-parallel.

By the second property the connection $\tilde{\nabla}$ splits as $\tilde{\nabla} = \tilde{\nabla}^T \oplus \tilde{\nabla}^\perp$ in a connection $\tilde{\nabla}^T$ on the tangent bundle and a connection $\tilde{\nabla}^\perp$ on the normal bundle, both of which are metric. Moreover the second fundamental form and the shape operator are $\tilde{\nabla}$-parallel (cf. [1, p. 204]). Since M is closed, $f(M)$ cannot be totally geodesic in $\mathbb{R}^m$. Thus the second fundamental form α does not vanish. Hence the connection on $\mathcal{E}$ induced by ∂ does not preserve TM and νM and is therefore not canonical.

As an analogy to Nomizu's characterization [8] of abstract Riemannian homogeneous spaces, a result due to Olmos [10] and Eschenburg [3] characterizes the extrinsically homogeneous submanifolds of Euclidean space:

THEOREM 4 ([3, 10]). *A closed submanifold of Euclidean space is extrinsically homogeneous, if and only if it admits a canonical connection.*

3. Almost Canonical Connections

Let (M^n, g_M) be a compact connected n-dimensional Riemannian manifold and let $f:(M, g_M) \longrightarrow (\mathbb{R}^m, g_{\mathrm{can}})$ be an isometric immersion. We denote by $\|T\|_0$ the supremum of the norm of the tensor T with unit vectors as arguments.

Given a connection $\tilde{\nabla}$ on $\mathcal{E}$, we define a tensor $\tilde{\alpha} \in TM^* \otimes TM^* \otimes \nu M$ by $\tilde{\alpha}(X, Y) = (\tilde{\nabla}_X Y)^\perp$ and a tensor $\tilde{A} \in \nu M^* \otimes TM^* \otimes TM$ by $\tilde{A}(\xi, X) = \tilde{A}_\xi X = -(\tilde{\nabla}_X \xi)^T$. Again $\tilde{\nabla}$ induces a tangent connection $\tilde{\nabla}^T$ on M defined by $\tilde{\nabla}^T_X Y = (\tilde{\nabla}_X Y)^T = \tilde{\nabla}_X Y - \tilde{\alpha}(X, Y)$ and a normal connection $\tilde{\nabla}^\perp$ defined by $\tilde{\nabla}^\perp_X \xi = (\tilde{\nabla}_X \xi)^\perp$, where X and Y are tangent vector fields and ξ is a normal vector field on M.

Let $\varepsilon > 0$. A connection $\tilde{\nabla}$ on $\mathcal{E}$ is said to be an *ε-almost canonical connection* (w.r.t. f) if

(1) $\|\tilde{\nabla} g\|_0 < \varepsilon$;
(2) $\|\tilde{\alpha}\|_0 < \varepsilon$, i.e. $\tilde{\nabla}$ almost preserves TM;
(3) $\|\tilde{\nabla}\Gamma\|_0 < \varepsilon$, where $\Gamma = \partial - \tilde{\nabla} \in TM^* \otimes \mathcal{E}^* \otimes \mathcal{E}$ and

$$(\tilde{\nabla}_X \Gamma)(Y, Z) = \tilde{\nabla}_X(\Gamma(Y, Z)) - \Gamma\big(\tilde{\nabla}^T_X Y, Z\big) - \Gamma(Y, \tilde{\nabla}_X Z)$$

for tangent vector fields X, Y and a section Z in $\mathcal{E}$.

Notice that

$$\begin{aligned}\|\tilde\nabla g\|_0 = \|\tilde\nabla g - \partial g\|_0 \ &= \ \|\Gamma g\|_0 \\ &= \sup\{|g(\Gamma_X Y, Z) + g(Y, \Gamma_X Z)|; \\ &\qquad X \in TM,\ Y, Z \in \mathcal{E},\ |X| = |Y| = |Z| = 1\}.\end{aligned} \tag{1}$$

It is sometimes advantageous to consider $\tilde\alpha$ and $\tilde A$ as elements of $\mathrm{Hom}(\mathcal{E}\otimes\mathcal{E}, \mathcal{E})$ as follows:

$$\begin{aligned}\tilde\alpha(X, Y) &= \tilde\alpha(X^T, Y^T), \\ \tilde A_X Y &= \tilde A_{X^\perp} Y^T; \quad X, Y \in \mathcal{E}.\end{aligned}$$

From now on we always assume that $\tilde\nabla$ is an ε-almost canonical connection.

Take two tangent vector fields X, Y and a normal vector field ξ. Since $g(Y, \xi) = 0$, we get $0 = Xg(Y, \xi) = (\tilde\nabla_X g)(Y, \xi) - g(\tilde\nabla_X Y, \xi) - g(Y, \tilde\nabla_X \xi)$. Thus $\tilde\alpha$ and $\tilde A$ are related by

$$g(\tilde\alpha(X, Y), \xi) = g(Y, \tilde A_\xi X) + (\tilde\nabla_X g)(Y, \xi).$$

If $Y = \tilde A_\xi X$, we get $|\tilde A_\xi X| = g(\tilde\alpha(X, \tilde A_\xi X), \xi) - (\tilde\nabla_X g)(\tilde A_\xi X, \xi)$. Hence the following lemma is immediate:

LEMMA 5. $\|\tilde A\|_0 < 2\varepsilon$, *i.e.* $\tilde\nabla$ *almost preserves* νM.

Although the connection $\tilde\nabla^T$ might not be geodesically complete, we get an estimate for the speed of $\tilde\nabla^T$-geodesics on M at least for times smaller than a certain value.

LEMMA 6. *Let* $\gamma\colon [0, T] \longrightarrow M$ *be a* $\tilde\nabla^T$*-geodesic, i.e.* $\tilde\nabla^T_{\gamma'(t)}\gamma'(t) = 0$, *then for* $0 \le t < \frac{2}{3\varepsilon|\gamma'(0)|}$ *we have*

$$\frac{2|\gamma'(0)|}{2 + 3\varepsilon t|\gamma'(0)|} \le |\gamma'(t)| \le \frac{2|\gamma'(0)|}{2 - 3\varepsilon t|\gamma'(0)|}.$$

Proof. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\gamma'(t)|^2 = 2\cdot|\gamma'(t)|\cdot\frac{\mathrm{d}}{\mathrm{d}t}|\gamma'(t)| \ &= \ \frac{\mathrm{d}}{\mathrm{d}t}g(\gamma'(t), \gamma'(t)) \\ &= 2g\big(\partial_{\gamma'(t)}\gamma'(t),\ \gamma'(t)\big) \\ &= \big(\Gamma_{\gamma'(t)}g\big)(\gamma'(t),\ \gamma'(t)) + 2g(\tilde\alpha(\gamma'(t),\ \gamma'(t)), \gamma'(t)),\end{aligned}$$

we get by Formula (1):

$$-\frac{3}{2}\varepsilon \le \frac{\frac{\mathrm{d}}{\mathrm{d}t}|\gamma'(t)|}{|\gamma'(t)|^2} \le \frac{3}{2}\varepsilon.$$

Integration now yields the claim. □

4. Parallel Displacement and Almost Isometries

Let M be a closed submanifold of $\mathbb{R}^m$. Assume that the trivial bundle $\mathcal{E} = M \times \mathbb{R}^m = T\mathbb{R}^m\mid_M = TM \oplus \nu M$ is equipped with an ε-almost canonical connection $\tilde{\nabla}$. Let p and q be two points on M and let $c\colon [0, L] \to M$ be a curve joining p and q. Let $P\colon \mathbb{R}^m \cong \mathcal{E}_p \longrightarrow \mathcal{E}_q \cong \mathbb{R}^m$ denote the linear map given by the $\tilde{\nabla}$-parallel translation along c and let $C = \sup\{|c'(t)|;\ t \in [0, L]\}$.

Notation. In this section we have to deal with quite a lot of estimates. In order to make these estimates easier and the proofs more readable, we introduce the following notation: By k or $k_i,\ i \in \mathbb{N}$ we denote nonvanishing constants depending on ε which converge to a nonvanishing constant if ε tends to 0. The exact value of k and k_i might change from formula to formula.

LEMMA 7. *If X and Y are unit vectors in $\mathbb{R}^m$, then*

$$|\langle P(X), P(Y)\rangle - \langle X, Y\rangle| \leq \varepsilon \cdot k.$$

Proof. Let X and Y be two unit vectors in $\mathbb{R}^m \cong \mathcal{E}_p$ and let $X(t)$ and $Y(t)$ be the vector fields along $c(t)$ obtained by the $\tilde{\nabla}$-parallel translations of X and Y. Then $P(X) = X(L)$ and $P(Y) = Y(L)$. Further

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle X(t), Y(t)\rangle &= (\langle \Gamma_{\bar{c}'}\bar{X}(t),\ \bar{Y}(t)\rangle + \langle \bar{X}(t),\ \Gamma_{\bar{c}'}\bar{Y}(t)\rangle) \cdot |X(t)| \cdot |Y(t)| \cdot |c'(t)| \\ &\leq |X(t)| \cdot |Y(t)| \cdot C \cdot \varepsilon,\end{aligned}$$

where

$$\bar{c}' = \frac{c'(t)}{|c'(t)|},\ \ \bar{X}(t) = \frac{X(t)}{|X(t)|} \quad \text{and} \quad \bar{Y}(t) = \frac{Y(t)}{|Y(t)|}.$$

As in the proof of Lemma 6 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|X(t)|^2 = 2 \cdot |X(t)| \cdot \frac{\mathrm{d}}{\mathrm{d}t}|X(t)| = \frac{\mathrm{d}}{\mathrm{d}t}\langle X(t), X(t)\rangle \leq \varepsilon \cdot C \cdot |X(t)|^2.$$

Thus

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}|X(t)|}{|X(t)|} \leq \frac{1}{2} \cdot \varepsilon \cdot C.$$

Integration yields $|X(t)| \leq \mathrm{e}^{\frac{1}{2}\varepsilon C t}$. Thus $\frac{\mathrm{d}}{\mathrm{d}t}\langle X(t), Y(t)\rangle \leq \varepsilon \cdot C \cdot \mathrm{e}^{\varepsilon C L}$. A second integration shows the statement. □

COROLLARY 8.

$$\|P\|_0 \leq \sqrt{\varepsilon \cdot k + 1}.$$

LEMMA 9. *Let X be a an element of $\mathcal{E}_p$ and let $X(t)$ denote the induced $\tilde{\nabla}$-parallel vector field along c. Then $|X(t)| \leq \mathrm{e}^{\frac{1}{2}\varepsilon C L} \cdot |X^T(0)|$.*

Proof. Analogously to the proof of Lemma 6 we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|X(t)|^2 &= 2\cdot|X(t)|\cdot\frac{\mathrm{d}}{\mathrm{d}t}|X(t)| = \frac{\mathrm{d}}{\mathrm{d}t}g(X(t), X(t))\\ &= 2g(\partial_{c'(t)}X(t),\ X(t)) = (\Gamma_{c'(t)}g)(X(t),\ X(t)).\end{aligned}$$

By Formula (1) we now obtain:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}|X(t)|}{|X(t)|} \le \frac{1}{2}\cdot\varepsilon\cdot C.$$

Integration now yields again the claim. □

LEMMA 10. *Let X be a an element of T_pM and let $X(t)$ and $X^T(t)$ denote the corresponding $\tilde{\nabla}$-parallel and $\tilde{\nabla}^T$-parallel vector fields along c. Then $|X^T(t)| \le e^{\frac{3}{2}\varepsilon CL}\cdot|X^T(0)|$ and $|X(t) - X^T(t)| \le \varepsilon\cdot t\cdot k$.*

Proof. The proofs of these inequalities are similar to the proof of Lemma 9. In the first case we observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|X^T(t)|^2 &= 2\cdot|X^T(t)|\cdot\frac{\mathrm{d}}{\mathrm{d}t}|X^T(t)|\\ &= \frac{\mathrm{d}}{\mathrm{d}t}g(X^T(t), X^T(t)) = 2\cdot g\big(\partial_{c'(t)}X^T(t),\ X^T(t)\big)\\ &= \big(\Gamma_{c'(t)}g\big)(X^T(t),\ X^T(t)) + 2\cdot g(\tilde{\alpha}(c'(t),\ X^T(t)), X^T(t)).\end{aligned}$$

In the second case we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|X(t) - X^T(t)|^2 = {}&\big(\Gamma_{c'(t)}g\big)(X(t) - X^T(t),\ X(t) - X^T(t))\\ &+ 2\cdot g(\tilde{\alpha}(c'(t), X^T(t)),\ X(t) - X^T(t))\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|X(t) - X^T(t)| &\le \frac{1}{2}\cdot\varepsilon\cdot C\cdot(|X(t) - X^T(t)| + 2|X^T(t)|)\\ &\le \frac{1}{2}\cdot\varepsilon\cdot C\cdot(|X(t)| + 3|X^T(t)|) = \varepsilon\cdot k.\end{aligned}$$

Since $X(0) = X^T(0) = X$, the lemma follows now by integration. □

By $\bar{g}$ we denote the unique affine transformation of $\mathbb{R}^m$ satisfying $\bar{g}(p) = q$ and $\bar{g}_*\,|_p = P$.

LEMMA 11. *Let $X^T(t)$ be a $\tilde{\nabla}^T$-parallel vector field in TM along c, then $\|\bar{g}_*\circ\Gamma_{X^T(0)}\circ\bar{g}_*^{-1} - \Gamma_{X^T(L)}\|_0 < \varepsilon\cdot k$.*

Proof. Let Y be a unit vector in $\mathcal{E}_q$ and let $Y(t)$ be the $\tilde{\nabla}$-parallel vector field along c given by $Y(L) = Y$. By $A(t)$ we denote the $\tilde{\nabla}$-parallel vector field along c with $A(0) = \Gamma_{X^T(0)}Y(0)$. Then $A(L) = (\bar{g}_* \circ \Gamma_{X^T(0)} \circ \bar{g}_*^{-1})Y$. Since $\tilde{\nabla}_{c'(t)}(\Gamma_{X^T(t)}Y(t)) = (\tilde{\nabla}_{c'(t)}\Gamma)_{X^T(t)}Y(t)$, we get for $Z(t) = A(t) - \Gamma_{X^T(t)}Y(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|Z(t)|^2 = 2\cdot|Z(t)|\cdot\frac{\mathrm{d}}{\mathrm{d}t}|Z(t)| &= \frac{\mathrm{d}}{\mathrm{d}t}g(Z(t),\ Z(t))\\ &= \left(\Gamma_{c'(t)}g\right)(Z(t),\ Z(t)) + 2g\left(\left(\tilde{\nabla}_{c'(t)}\Gamma\right)_{X^T(t)}Y(t),\ Z(t)\right).\end{aligned}$$

Hence with the estimates of Lemmas 9 and 10 we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|A(t) - \Gamma_{X^T(t)}Y(t)\right| &\le \varepsilon\cdot C\left(\left|A(t) - \Gamma_{X^T(t)}Y(t)\right| + 2|X^T(t)|\cdot|Y(t)|\right)\\ &\le \varepsilon\cdot C(|A(t)| + (\|\Gamma\|_0 + 2)\cdot|X^T(t)|\cdot|Y(t)|)\\ &\le \varepsilon\cdot\kappa.\end{aligned}$$

As $A(0) = \Gamma_{X^T(0)}Y(0)$ this lemma follows by integration. □

PROPOSITION 12. *Assume*[⋆] *that the canonical connection $\tilde{\nabla}$ does not coincide with the connection on $\mathcal{E}$ induced by ∂. Let $X \in T_pM$ be a unit vector and let γ be the $\tilde{\nabla}^T$-geodesic on M with $\gamma'(0) = X$. Consider further the $\tilde{\nabla}^T$-geodesic $\tilde{\gamma}(t)$ on M defined by $\tilde{\gamma}'(0) = X^T(L)$, where $X^T(t)$ is the $\tilde{\nabla}^T$-parallel vector field along c with $X^T(0) = X$. Then for $0 \le t \le T < \frac{2}{3\varepsilon\sqrt{\varepsilon CLe^{\varepsilon CL}+1}}$ we obtain*

$$|(\bar{g}\circ\gamma)(t) - \tilde{\gamma}(t)| < \varepsilon\cdot h_\varepsilon(t)$$

with a function h_ε which does not diverge if ε tends to 0.

Proof. Recall that k and $k_i,\ i \in \mathbb{N}$ denote nonvanishing constants depending on ε which converge to a nonvanishing constant if ε tends to 0. Their exact values might change from formula to formula.

Let $E(t) = \gamma'(t)$ be the tangent vector field of γ. Then $E'(t) = \partial_{\gamma'(t)}E(t) = \partial_{\gamma'(t)}E(t) - \tilde{\nabla}^T_{\gamma'(t)}E(t) = \Gamma_{\gamma'(t)}E(t) + \tilde{\alpha}(\gamma'(t), E(t))$.

Let F be the tangent vector field of the curve $\bar{g}\circ\gamma$, i.e. $F(t) = \bar{g}_*E(t)$. Since $\bar{g}$ is an affine map we obtain

$$\begin{aligned}F'(t) = \partial_{(\bar{g}\circ\gamma)'(t)}F(t) &= \partial_{\bar{g}_*(\gamma'(t))}\bar{g}_*E(t) = \bar{g}_*\left(\partial_{\gamma'(t)}E(t)\right)\\ &= \bar{g}_*\left(\Gamma_{\gamma'(t)}E(t) + \tilde{\alpha}(\gamma'(t), E(t))\right)\\ &= \left(\bar{g}_*\circ\left(\Gamma_{\gamma'(t)} + \tilde{\alpha}(\gamma'(t),\cdot)\right)\circ\bar{g}_*^{-1}\right)F(t).\end{aligned}$$

[⋆]Since we think of ε to be small and since M cannot be totally geodesic, the supremum norm of the second fundamental form is bounded away from 0. Therefore this condition, which especially implies $\|\Gamma\|_0 \neq 0$, is only of technical and not of conceptual nature.

Thus we get the following ordinary differential equation:

$$(F)\begin{cases}(\bar g\circ\gamma)'(t)=F(t);\\ F'(t)=\left(\bar g_*\circ\left(\Gamma_{\gamma'(t)}+\tilde\alpha(\gamma'(t),\cdot)\right)\circ\bar g_*^{-1}\right)F(t);\\ \text{initial conditions } (\bar g\circ\gamma)(0)=q,\ \ F(0)=\bar g_*X.\end{cases}$$

Consider now the tangent vector field G of the $\tilde\nabla^T$-geodesic $\tilde\gamma$. Since G is $\tilde\nabla^T$-parallel, we have:

$$(G)\begin{cases}\tilde\gamma'(t)=G(t);\\ G'(t)=\Gamma_{\tilde\gamma'(t)}G(t)+\tilde\alpha(\tilde\gamma'(t),G(t));\\ \text{initial conditions } \tilde\gamma(0)=q,\ \ G(0)=X^T(L).\end{cases}$$

The difference of (F) and G gives rise to the following differential equation

$$(H)\begin{cases}(\bar g\circ\gamma-\tilde\gamma)'(t)=F(t)-G(t);\\ F'(t)-G'(t)=\Gamma_{\tilde\gamma'(t)}(F(t)-G(t))+Z(t);\\ \text{initial conditions } (\bar g\circ\gamma-\tilde\gamma)(0)=0\\ \text{and } F(0)-G(0)=\bar g_*X-X^T(L),\end{cases}$$

where $Z(t)=\Delta(t)F(t)+\bar g_*\tilde\alpha(\gamma'(t),\ \bar g_*^{-1}F(t))-\tilde\alpha(\tilde\gamma'(t),\ G(t))$ and $\Delta(t)=\bar g_*\circ\Gamma_{\gamma'(t)}\circ\bar g_*^{-1}-\Gamma_{\tilde\gamma'(t)}$.

By Lemma 6 we get

$$\left\|\Gamma_{\tilde\gamma'(t)}\right\|_0\le\frac{2}{2-3\varepsilon T}\|\Gamma\|_0. \tag{2}$$

Let X be a $\tilde\nabla$-parallel section in $\mathcal{E}$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\tilde\gamma'(t)}\right)X(t)=\tilde\nabla_{\tilde\gamma'(t)}\partial_{\tilde\gamma'(t)}X=(\tilde\nabla_{\tilde\gamma'(t)}\Gamma)_{\tilde\gamma'(t)}X$$

and hence $\|\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\tilde\gamma'(t)}\|_0\le|\tilde\gamma'(t)|^2\cdot\|\tilde\nabla\Gamma\|_0$. Using again Lemma 6 yields:

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\tilde\gamma'(t)}\right\|_0\le\varepsilon\cdot k. \tag{3}$$

Observe that $\bar g^{-1}$ is obtained by parallel translation along c in the reverse direction, hence the estimate of Corollary 8 also holds for $\bar g^{-1}$. Together with $\|\frac{\mathrm{d}}{\mathrm{d}t}(\bar g_*\circ\Gamma_{\gamma'(t)}\circ\bar g_*^{-1})\|_0\le\|\bar g_*^{-1}\|_0\cdot\|\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\gamma'(t)}\|_0\cdot\|\bar g_*\|_0$ we get

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar g_*\circ\Gamma_{\gamma'(t)}\circ\bar g_*^{-1}\right)\right\|_0\le\varepsilon\cdot k. \tag{4}$$

Since $\Delta(t) = \int_0^t \Delta'(s)\,\mathrm{d}s + \Delta(0)$, the estimates (3) and (4) together with Lemma 11 provide the following estimate of $\|\Delta(t)\|_0$:

$$\begin{aligned}
\|\Delta(t)\|_0 &\leq \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}s}\Delta(s) \right\|_0 \mathrm{d}s \;+\; \|\Delta(0)\|_0 \\
&= \int_0^t \left(\left\| \frac{\mathrm{d}}{\mathrm{d}s}\Gamma_{\tilde{\gamma}'(s)} \right\|_0 + \left\| \frac{\mathrm{d}}{\mathrm{d}s}\left(\bar{g}_* \circ \Gamma_{\gamma'(s)} \circ \bar{g}_*^{-1}\right) \right\|_0 \right) \mathrm{d}s \,+ \\
&\qquad + \|\Delta(0)\|_0 \\
&\leq \varepsilon \cdot k.
\end{aligned}$$

Moreover by Lemma 6 and Corollary 8 we get $|F(t)| = |\bar{g}_* E(t)| \leq \|\bar{g}_*\|_0 \cdot |E(t)| \leq k$ and hence

$$|\Delta(t)F(t)| \leq \varepsilon \cdot k.$$

Since $|\bar{g}_*\tilde{\alpha}(\gamma'(t),\ \bar{g}_*^{-1}F(t))| = |\bar{g}_*\tilde{\alpha}(\gamma'(t),\ E(t))| \leq \|\bar{g}_*\|_0 \cdot \|\tilde{\alpha}\|_0 \cdot |\gamma'(t)|^2 \leq \varepsilon \cdot k$ and $|\tilde{\alpha}(\tilde{\gamma}'(t), G(t))| \leq \|\tilde{\alpha}\|_0 \cdot |\tilde{\gamma}'(t)|^2 \leq \varepsilon \cdot k$, we obtain the estimate:

$$|Z(t)| \leq \varepsilon \cdot k.$$

Applying to (H) the Gronwall-type inequality of Lemma 3 together with Lemma 10 yields:

$$|F(t) - G(t)| \leq \varepsilon \cdot (k_1 \cdot \mathrm{e}^{k \cdot t} + k_2 \cdot (\mathrm{e}^{k \cdot t} - 1)),$$

where $k = \frac{2}{2 - 3\varepsilon T}\|\Gamma\|_0$. Since $\|\Gamma\|_0 \neq 0$, integration shows the claim:

$$\begin{aligned}
|(\bar{g} \circ \gamma - \tilde{\gamma})(t)| &\leq \int_0^t |F(s) - G(s)|\,\mathrm{d}s \\
&\leq \varepsilon \cdot \left(\frac{k_1}{k} \cdot (\mathrm{e}^{kt} - 1) + \frac{k_2}{k} \cdot (\mathrm{e}^{kt} - kt - 1) \right).
\end{aligned}$$

□

Observation. The conclusion of Proposition 12 holds also for a broken geodesic line γ and the corresponding broken geodesic line $\tilde{\gamma}$.

5. The Main Result

Let $\mathcal{M}_{\mathrm{im}}(\Lambda, d, n, m, \varepsilon)$ be the set of all triples $(M^n,\ f,\ \tilde{\nabla})$ consisting of a closed n-dimensional manifold M^n, an immersion f of M into the m-dimensional Euclidean space $(\mathbb{R}^m, g_{\mathrm{can}})$ and an ε-almost canonical connection $\tilde{\nabla}$ w.r.t. f satisfying:

(1) The diameter of M measured in the pullback metric $f^* g_{\mathrm{can}}$ is bounded form above by d;
(2) $\|\alpha\|_0 < \Lambda$;
(3) $\|\Gamma\|_0 < \Lambda$.

The first two conditions exclude collapsing and the last two bounds together with the definition of an ε-canonical connection provide an estimate for $\|\partial\Gamma\|_0$ in the following way: Let $\Gamma^T = \nabla - \tilde{\nabla}^T$ and take two vectors X and Y in T_pM. Then $\Gamma^T_X Y = \Gamma_X Y - \alpha(X,Y) - \tilde{\alpha}(X,Y)$. Considering a vector $Z \in \mathcal{E}_p$ we get $(\partial_X \Gamma)(X,Y) = (\tilde{\nabla}_X \Gamma)(Y,Z) + \Gamma_X\Gamma_Y Z - \Gamma_Y\Gamma_X Z - \Gamma_{\Gamma^T_X Y} Z$. Thus $\|\partial\Gamma\|_0 < \varepsilon(1+\Lambda) + 2\Lambda^2(1+\varepsilon)$.

Let $(M, f, \tilde{\nabla})$ be an element of $\mathcal{M}_{\rm im}(\Lambda, d, n, m, \varepsilon)$, then $(M, f^*g_{\rm can})$ lies in $\mathcal{M}(\Lambda\sqrt{2}, d, (\Lambda\sqrt{2})^{-n}\mathrm{vol}(S^n, g_0), n)$,[⋆] where (S^n, g_0) denotes the standard unit sphere of dimension n (see [11]).

PROPOSITION 13. *Let Λ, $d > 0$ and let M be a closed manifold of dimension n. Assume that there exist a sequence (f_i) of immersions of M into $\mathbb{R}^m$ and a sequence $(\tilde{\nabla}_i)$ of connections on $\mathcal{E} = M \times \mathbb{R}^m$. If for each positive integer i the triple $(M, f_i, \tilde{\nabla}_i)$ lies in $\mathcal{M}_{\rm im}(\Lambda, d, n, m, \frac{1}{i})$, then there exists an extrinsically homogeneous immersion of M into $\mathbb{R}^m$.*

Proof. Multiplying f_i by a constant, we can assume w.r.g. that $d = 1$. As we have to consider subsequences several times, we do not introduce a special notation in order to keep this proof readable.

Since $(M, f_i^* g_{\rm can})$ is a sequence in $\mathcal{M}(\Lambda\sqrt{2}, 1, (\Lambda\sqrt{2})^{-n}\mathrm{vol}(S^n, g_0), n)$, we can assume by Theorem 2 that, after passing to a subsequence, there is a $\mathcal{C}^{1,\alpha'}$-Riemannian metric g_M on M and diffeomorphisms h_i of M such that the metrics $g_{Mi} := \tilde{f}_i^* g_{\rm can}$, $\tilde{f}_i = f_i \circ h_i$, converge to g_M in the $\mathcal{C}^{1,\alpha}$-topology, $0 < \alpha < \alpha' < 1$. The diameter of (M, g_M) is also bounded by 1. Let $\|\cdot\|_0^{g_{Mi}}$ denote the supremum norm w.r.t. the metric g_{Mi}.

Now we fix a point p_0 on M. By composition with an appropriate translation of $\mathbb{R}^m$ we can assume that $\tilde{f}_i(p_0) = 0$. Since the second fundamental form coincides with the Hessian of the immersion, the $\mathcal{C}^2$-norm of $\tilde{f}_i$ on (M, g_{Mi}) is given by $\|\tilde{f}_i\|_{\mathcal{C}^2((M,g_{Mi}),\mathbb{R}^m)} = \|\tilde{f}_i\|_0^{g_{Mi}} + \|\mathrm{d}\tilde{f}_i\|_0^{g_{Mi}} + \|\alpha_i\|_0^{g_{Mi}}$. Notice that this norm is equivalent to the norm of Proposition 1. Since M is connected and $\tilde{f}_i(p_0) = 0$, the maximum of $\tilde{f}_i$ is not greater than the diameter bound $d = 1$ of M. Thus we get $\|\tilde{f}_i\|_{\mathcal{C}^2((M,g_{Mi}),\mathbb{R}^m)} \leq 2 + \Lambda$. The norms $\|\cdot\|_0^{g_i}$ converge to $\|\cdot\|_0$, the supremum norm corresponding g. Hence the sequence $(\tilde{f}_i)$ is bounded in $\mathcal{C}^2((M, g_M), \mathbb{R}^m)$ as well for $i > i_0$. By the embedding theorem for Hölder spaces (see Proposition 1) there exist a function $f \in \mathcal{C}^{1,\beta}((M, g_M), \mathbb{R}^m)$, $0 < \beta < 1$ and a subsequence of $(\tilde{f}_i)$ converging to f in $\mathcal{C}^{1,\beta}((M, g_M), \mathbb{R}^m)$. From now on we restrict our attention to this subsequence $(M, \tilde{f}_i, \tilde{\nabla}_i)$. Since $\tilde{f}_i$ converges to f in the $\mathcal{C}^1$-topology, f is an isometric immersion of (M, g_M) into $\mathbb{R}^m$. Thus the metrics g_i on $\mathcal{E}$ induced by $\tilde{f}_i$ converge uniformly to the $\mathcal{C}^0$ metric g on $\mathcal{E}$ given by f. Notice that g_{Mi} and g_M coincide with the tangent parts of g_i and g. By $\|\cdot\|_0^{g_i}$ we denote the supremum norm w.r.t. g_i. The supremum norm corresponding to g is simply denoted by $\|\cdot\|_0$. Moreover the tangent and normal bundles of $\tilde{f}_i$, denoted by T_iM and ν_iM converge

[⋆]The definition is stated in Section 1.

as subbundles of $\mathcal{E}$ to the tangent and normal bundle of f, denoted by TM and νM, and the corresponding projections converge w.r.t. $\|\cdot\|_0$.

By assumption $\|\Gamma_i\|_0^{g_i} < \Lambda$ and $\|\partial\Gamma_i\|_0^{g_i} < \frac{1+\Lambda}{i} + 2\Lambda^2(1+\frac{1}{i}) < 1+\Lambda+4\Lambda^2$. Hence, since $\|\cdot\|_0^{g_i}$ converges to $\|\cdot\|_0$, the tensors Γ_i and $\partial\Gamma_i$ admit for great i a bound in the $\|\cdot\|_0$-norm not depending on i. Thus for great i the sequence (Γ_i) is $\mathcal{C}^0$-bounded and equicontinuous. By the Arzelà-Ascoli theorem there exists a continuous tensor field Γ on $\mathcal{E}$ and a subsequence of (Γ_i) converging uniformly to Γ. We observe that the connections $\tilde{\nabla}_i = \partial - \Gamma_i$ on $\mathcal{E}$ converge to the continuous connection $\tilde{\nabla} := \partial - \Gamma$.

We now prove that $f(M)$ is extrinsically homogeneous. Let p and q be two points on M and let $c : [0, L] \to M$ be a curve joining p and q. Let $\bar{g}_i$ be the unique affine transformation of $\mathbb{R}^m$ mapping $\tilde{f}_i(p)$ to $\tilde{f}_i(q)$ whose derivative P_i at $\tilde{f}_i(p)$ coincides with the $\tilde{\nabla}_i$-parallel translation along c. By Corollary 8 $\|P_i\|_0$ is bounded by $\sqrt{\frac{CL}{i}e^{\frac{CL}{i}} + 1} \leq \sqrt{CLe^{CL} + 1}$. Thus there exists a linear map P of $\mathbb{R}^m$ and a subsequence (P_i) converging uniformly to P. Notice that by Lemma 7 P is an isometry of $\mathbb{R}^m$ and that P coincides with the $\tilde{\nabla}$-parallel translation along c.* The corresponding subsequence of affine maps $(\bar{g}_i)$ converges to the isometry $\bar{g}$ of $\mathbb{R}^m$ mapping $f(p)$ to $f(q)$ whose derivative at $f(p)$ is given by P. Let X be a unit tangent vector at p w.r.t. g_M and let X_i be the unit vector w.r.t. g_{M_i} obtained by rescaling X. Let γ_i denote the $\tilde{\nabla}_i^T$-geodesic defined by $\gamma_i'(0) = X_i$ and let $\tilde{\gamma}_i$ denote the $\tilde{\nabla}_i^T$-geodesic defined by $\tilde{\gamma}_i'(0) = X_i^T(L)$, where $X_i^T(t)$ is the $\tilde{\nabla}_i^T$-parallel tangent vector field along c defined by X_i. Since $\tilde{\nabla}_i^T$ and $\bar{g}_i$ converge uniformly to $\tilde{\nabla}^T$ and $\bar{g}$, the geodesics γ_i and $\tilde{\gamma}_i$ converge pointwise to the $\tilde{\nabla}^T$-geodesics γ and $\tilde{\gamma}$ defined by $\gamma'(0) = X$ and $\tilde{\gamma}'(0) = X(L) = \bar{g}_*X$. Proposition 12 shows that

$$|(\bar{g} \circ f \circ \gamma)(t) - f \circ \tilde{\gamma}(t)| = \lim_{i\to\infty} |(\bar{g}_i \circ \tilde{f}_i \circ \gamma_i)(t) - \tilde{f}_i \circ \tilde{\gamma}_i(t)| = 0.$$

Recall that Proposition 12 can easily be generalized to broken geodesic lines. Since by connectedness and compactness, any two points of M can be joint by a broken geodesic line, $\bar{g}$ leaves $f(M)$ invariant. Thus $f(M)$ is extrinsically homogeneous. □

Although our proof guarantees a priori only that the limit immersion f is of class $\mathcal{C}^{1,\beta}$, the image $f(M)$ is a smooth submanifold of $\mathbb{R}^m$, since $f(M)$ is extrinsically homogeneous.

THEOREM 14. *There exists a constant $\varepsilon > 0$ depending on Λ, n, m and d with the following property:*
If a triple $(M, f, \tilde{\nabla})$ lies in $\mathcal{M}_{\mathrm{im}}(\Lambda, d, n, m, \varepsilon)$, then M can be immersed into $\mathbb{R}^m$ as extrinsically homogeneous submanifold.

*This fact will be important in the proof of Proposition 16.

To avoid rescaling (blowing up), ε depends on d. To get rid of this dependence, one might replace the condition $\|\tilde{\nabla}\Gamma\|_0 < \varepsilon$, as stated in the definition of an ε-almost canonical connection, by the rescaling invariant condition $\|\tilde{\nabla}\Gamma\|_0 \cdot d^3 < \varepsilon$.

Proof. W.r.g. let $d = 1$. Assume by contradiction that for each positive integer i, there exists a triple $(M_i, f_i, \tilde{\nabla}_i)$ in $\mathcal{M}_{\mathrm{im}}(\Lambda, 1, n, m, \frac{1}{i})$ such that M_i does not admit an immersion into $\mathbb{R}^m$ as extrinsically homogeneous submanifold. Since the sequence $(M_i, f_i^* g_{\mathrm{can}})$ is contained in $\mathcal{M}(\Lambda\sqrt{2}, 1, (\Lambda\sqrt{2})^{-n}\mathrm{vol}(S^n, g_0), n)$, Theorem 2 implies that, after passing to a subsequence, there are a smooth manifold M and diffeomorphisms $h_i : M \to M_i$. Now we get a new sequence $(M, f_i \circ h_i, \tilde{\nabla}_i)$ satisfying the assumptions of Proposition 13. Thus M admits an extrinsically homogeneous immersion into $\mathbb{R}^m$, a contradiction. □

The following example shows that the condition $\|\Gamma\|_0 < \Lambda$ is necessary: Consider the Grassmann manifold $\mathrm{G/K}$ of all oriented n-dimensional linear subspaces in $\mathbb{R}^{2n}$, where $\mathrm{G} = \mathrm{O}(2n)$ and $\mathrm{K} = \mathrm{O}(n) \times \mathrm{O}(n)$. The corresponding Cartan decomposition is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $M \cong \mathrm{SO}(n)$ be a component of the orbit of $\xi = \left(\begin{smallmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{smallmatrix}\right) \in \mathfrak{p}$ under the isotropy representation of $\mathrm{G/K}$. The isotropy group H of ξ is now the diagonal in $\mathrm{O}(n) \times \mathrm{O}(n)$. As in the abstract case, an (extrinsic) canonical connection corresponds to a reductive decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ (cf. [1, Section 7.1]). For $\lambda \neq -1$ any of the following complements $\mathfrak{m}_\lambda$ of $\mathfrak{h}$ give rise to a reductive decomposition: $\mathfrak{m}_\lambda = \{\left(\begin{smallmatrix} X & 0 \\ 0 & -\lambda X^T \end{smallmatrix}\right);\ X \in \mathfrak{o}(n)\}$. Thus for any sequence λ_i converging to -1 with $\lambda_i \neq -1$ we get a sequence of (extrinsic) canonical connections $\mathfrak{m}_{\lambda_i}$, which does not converge to a connection at all. Notice that the normal part of the canonical connection given by $\mathfrak{m}_1$ coincides with the usual normal connection (see Section 6).

6. Examples

6.1. ORBITS OF s-REPRESENTATIONS

An s-representation is the isotropy representation of a semisimple symmetric space S. Assumed that S is irreducible, all non-vanishing orbits of the isotropy representation of S are full submanifolds, i.e. they are not contained in a proper affine subspace, since in the case of symmetric spaces the isotropy representation and the holonomy representation coincide. If S is of noncompact type, then each unit tangent vector X of S defines a point $X(\infty)$ in the spherical boundary at infinity $S(\infty)$ of S, such that the unit sphere of a given tangent space can be identified with $S(\infty)$. The orbit of X under the action of the isotropy representation at the foot point of X coincides under the above identification with the orbit of $X(\infty)$ under the usual action of the isometry group of S on $S(\infty)$. Thus orbits of s-representations (as submanifolds of the corresponding tangent space seen as $\mathbb{R}^n$) can be seen as standard imbeddings of real flag manifolds (also known as R-spaces) into Euclidean space as considered by Kobayashi and Takeuchi [6]. A detailed description of such orbits can be found in [1, p. 46–52].

Let $\nabla^{\perp}$ be the usual normal connection on νM. We say that $f(M)$ has *extrinsically homogeneous normal holonomy bundle,* if for any points p and q on M and any curve c on M joining p and q, there exists an isometry $\bar{g}$ of $\mathbb{R}^m$ mapping $f(p)$ to $f(q)$, leaving $f(M)$ invariant and such that the mapping $\bar{g}_* |_{\nu_p M} \colon \nu_p M \longrightarrow \nu_q M$ coincides with the $\nabla^{\perp}$-parallel transport along c.

Olmos and Sánchez [9] (see also [1, p. 164 and p. 211]) gave the following characterization of orbits of s-representations:

THEOREM 15 ([9]). *Let M be a full closed submanifold of $\mathbb{R}^m$. Then the following assertions are equivalent:*

(1) *M admits a canonical connection $\tilde{\nabla}$ whose normal part coincides with the usual normal connection on $\mathcal{E}$, i.e. $\tilde{\nabla}^{\perp} = \nabla^{\perp}$.*
(2) *M is the orbit of an s-representation;*
(3) *M has extrinsically homogeneous normal holonomy bundle.*

In this case $\tilde{\nabla}\Gamma = 0$ is equivalent to the following two conditions: $\tilde{\nabla}\alpha = 0$ and $\tilde{\nabla}^T(\nabla - \tilde{\nabla}^T) = 0$, where ∇ is the Riemannian connection on M.

This in mind, we restrict our attention to the tangent part of the canonical connection. In analogy to Proposition 13 we get:

PROPOSITION 16. *Let M be a closed manifold of dimension n and let $\Lambda, d > 0$. Assume that there exist a sequence (f_i) of immersions of M into $\mathbb{R}^m$ and a sequence $(\tilde{\nabla}_i)$ of connections on $\mathcal{E} = M \times \mathbb{R}^m$ satisfying $\tilde{\nabla}_i^{\perp} = \nabla_i^{\perp}$, where $\nabla_i^{\perp}$ is the usual normal connection on $\mathcal{E}$ given by f_i. If for each positive integer i the triple $(M, f_i, \tilde{\nabla}_i)$ lies in $\mathcal{M}_{\mathrm{im}}(\Lambda, d, n, m, \frac{1}{i})$, then M can be immersed into an affine subspace of $\mathbb{R}^m$ as orbit of an s-representation.*

Proof. The proof of Proposition 13 can essentially be copied and we refer to this proof for the chosen notations. The only delicate point is that the limit immersion f constructed in the proof of Proposition 13 is a priori only of class $\mathcal{C}^{1,\beta}$ and might not give rise to a normal connection $\nabla^{\perp}$. But we show that in the case at hand f is of class $\mathcal{C}^{2,\beta}$. Let Y and Z be $\tilde{\nabla}_i^T$-parallel tangent vector fields, then $(\partial_X \alpha_i)(Y, Z) = (\tilde{\nabla}_X \alpha_i)(Y, Z) - \alpha_i(\Gamma_{iX} Y, Z) - \alpha(Y, \Gamma_{iX} Z) - A_i(\alpha_i(Y, Z), X)$, where α_i and A_i are the second fundamental form and the shape operator of $\tilde{f}_i$ and $\Gamma_i = \nabla_i - \tilde{\nabla}_i^T$. Thus $\|\partial\alpha_i\|_0^{g_i} \le \|\tilde{\nabla}_i \alpha_i\|_0^{g_i} + 2\|\alpha_i\|_0^{g_i} \|\Gamma_i\|_0^{g_i} + (\|\alpha_i\|_0^{g_i})^2$. Again we can assume that $0 \in \tilde{f}_i(M)$. Recall that the second fundamental form coincides with the Hessian of the immersion. Thus the $\mathcal{C}^3$-norm of $\tilde{f}_i$ on (M, g_{M_i}) is given by $\|\tilde{f}_i\|_{\mathcal{C}^3((M, g_{M_i}), \mathbb{R}^m)} = \|\tilde{f}_i\|_0^{g_{M_i}} + \|\mathrm{d}\tilde{f}_i\|_0^{g_{M_i}} + \|\alpha_i\|_0^{g_{M_i}} + \|\partial\alpha_i\|_0^{g_{M_i}} \le 2 + \frac{1}{i}\varepsilon + \Lambda + 3\Lambda^2$. The norms $\|\cdot\|_0^{g_{M_i}}$ converge to $\|\cdot\|_0$, the supremum norm corresponding g_M. Hence the sequence $(\tilde{f}_i)$ is bounded in $\mathcal{C}^3((M, g_M), \mathbb{R}^m)$ as well for big i. By the embedding theorem for Hölder spaces (see Proposition 1) there exist a function $f \in \mathcal{C}^{2,\beta}((M, g_M), \mathbb{R}^m)$, $0 < \beta < 1$ and a subsequence of $(\tilde{f}_i)$ converging to f in $\mathcal{C}^{2,\beta}((M, g_M), \mathbb{R}^m)$. This shows that f gives rise to a normal connection $\nabla^\perp$. Moreover the proof of Proposition 13 now shows that $f(M)$ has extrinsically homogeneous normal holonomy bundle. Reduction of the codimension eventually yields the claim. □

Using the same arguments as in the proof of Theorem 14, we get as a consequence of the above proposition the following pinching result for orbits of s-representations.

THEOREM 17. *There exists a constant $\varepsilon > 0$ depending on d, Λ, n and m with the following property:*

If a triple $(M, f, \tilde{\nabla})$ with $\tilde{\nabla}^\perp = \nabla^\perp$ lies in $\mathcal{M}_{\mathrm{im}}(\Lambda, d, n, m, \varepsilon)$, then M can be immersed into an affine subspace of $\mathbb{R}^m$ as an orbit of an s-representation.

6.2. EXTRINSICALLY SYMMETRIC SUBMANIFOLDS

A submanifold of a Euclidean space is called *extrinsically symmetric,* if it is invariant under the reflections at each of their normal spaces. Ferus [4] has classified and characterized these submanifolds as follows:

- The connected extrinsically symmetric submanifolds of Euclidean space are products of closed extrinsically symmetric submanifolds with totally geodesic ones.
- The full closed extrinsically symmetric submanifolds of Euclidean space are exactly the symmetric orbits of s-representations.
- A closed submanifold of Euclidean space is extrinsically symmetric if and only if its second fundamental form is parallel, i.e. $\nabla^\perp \alpha = 0$.

The pinching theorem for closed extrinsically symmetric submanifolds of Euclidean space resulting from this characterization can be be found in [11].

THEOREM 18 ([11]). *There exists a constant $\varepsilon > 0$ depending on d, Λ, n and m with the following property:*

If a triple $(M, f, \nabla \oplus \nabla^\perp)$ lies in $\mathcal{M}_{\mathrm{im}}(\Lambda, d, n, m, \varepsilon)$, then M can be immersed into $\mathbb{R}^m$ as an extrinsically symmetric submanifold.

As the characterization of Ferus does not use a supplementary connection any more, this pinching result does not assume the existence of a supplementary ε-almost canonical connection. Instead we assume that $\nabla \oplus \nabla^\perp$ is ε-almost canonical, which only means that $\|\nabla^\perp \alpha\|_0 < \varepsilon$. Moreover in this case we do not need to construct the extrinsic isometries by a limit process, as they are given by the reflections at the normal spaces. Notice also that the condition $\|\Gamma\|_0 < \Lambda$ is now redundant since $\|\alpha\|_0 < \Lambda$.

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