

# A circular interpretation of the Euler-Maclaurin formula

Jean-Paul Berrut

Department of Mathematics  
 University of Fribourg  
 CH-1700 Fribourg/Pérolles  
 Switzerland  
 June 2004

**Abstract.-** The present work makes the case for viewing the Euler-Maclaurin formula as an expression for the effect of a jump on the accuracy of Riemann sums on circles and draws some consequences thereof, e.g., when the integrand has several jumps. On the way we give a construction of the Bernoulli polynomials tailored to the proof of the formula and we show how extra jumps may lead to a smaller quadrature error.

**Subject classification:** AMS(MOS) 65D30.

**Key words:** definite integral, Riemann sum, trapezoidal rule, error formula.

## 1. Introduction

In the present work we discuss the approximation of the definite integral

$$I := \int_0^L f(x) dx$$

of a (piecewise) smooth function  $f$  from an equidistant sample of its values by the (*composite*) *trapezoidal rule* [Gau, Häm-Hof, Hen, Kin-Che, Kre, Sch]:

$$T_f(h) := h \left[ \frac{f(0)}{2} + \sum_{k=1}^{N-1} f(kh) + \frac{f(L)}{2} \right], \quad h := \frac{L}{N}, \quad N \in \mathbb{N}.$$

The appraisal of the error  $T_f(h) - I$ , and the basis of one approach to Romberg extrapolation, is the standard *Euler-Maclaurin formula (EMF)* given in the following theorem [Kre]. Throughout this paper,  $f^{(j)}$  will denote the  $j$ -th derivative of  $f$  and  $C^q[a, b]$  the set of all  $q$ -times continuously differentiable functions on  $[a, b]$ .

### Theorem 1 (Euler–Maclaurin formula (EMF) for the trapezoidal rule)

Let  $f \in C^{2m+2}[0, L]$  for some  $m \geq 0$ . Then, for every  $N \in \mathbb{N}$  and with  $h := \frac{L}{N}$ , the error of the trapezoidal rule may be written as

$$T_f(h) - I = a_2 h^2 + a_4 h^4 + \dots + a_{2m} h^{2m} + L \frac{B_{2m+2}}{(2m+2)!} f^{(2m+2)}(\xi) h^{2m+2} \quad (1.1)$$

for some  $\xi \in [0, L]$ , where

$$a_{2j} := \frac{B_{2j}}{(2j)!} [f^{(2j-1)}(L) - f^{(2j-1)}(0)] \quad (1.2)$$

and with  $B_\ell$  denoting the  $\ell$ -th Bernoulli number.

The speed of convergence of  $T_f(h)$  toward  $I$  as  $h \downarrow 0$  is thus determined by the differences between the derivatives of odd orders at the extremities of the interval: in general, i.e., without the special property  $f'(L) = f'(0)$ , one has  $\mathcal{O}(h^2)$ -convergence; every equality of another odd order derivative eliminates a further  $h^{2j}$ -term. The method is therefore especially efficient when  $f$  is  $L$ -periodic and in  $C^{2m+2}(-\infty, \infty)$  [Wei]. Notice that  $\xi$  varies with  $h$  and that the  $h^2$ -behavior of the error may show up only once  $h$  is small enough.

The question we address here is the following: *how do we understand the fact that for  $h$  small enough the integration error almost solely depends on differences in the behavior of the function at the extremities and not on what happens in-between ?*

Our answer is to view the trapezoidal rule as a Riemann sum on a circle. This interpretation considers the values of  $f$  and its derivatives at the extremities as the left and right limits at a jump and explains why they govern the accuracy; it also leads to a generalization of the formula to functions with several jumps.

Note that, when the derivatives at the extremities are known, one may use them in (1.1) to construct quadrature rules with higher orders of convergence. Such rules may also be obtained without knowledge of the derivatives by replacing the latter with divided differences [Lyn2, Rok, Alp].

## 2. Bernoulli polynomials and Bernoulli numbers

The circle interpretation will yield as a by-product a somewhat simpler proof of the EMF. The Bernoulli polynomials (BP) are an essential ingredient of all such proofs (the Bernoulli numbers are the values of the BP at zero). They are usually described at the onset, without

connection to the EMF. We shall instead construct them as recursive integrals of the constant 1 with just the right properties for a self-contained proof.

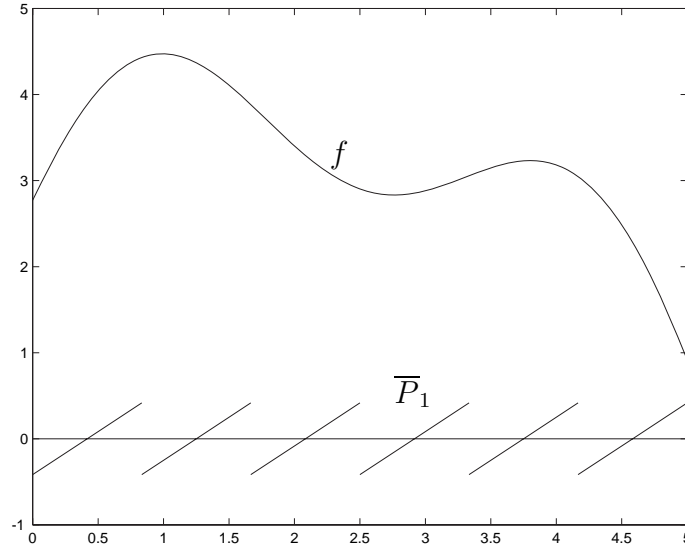
Let us first give a flavour of the latter. The trapezoidal sum is obtained from an integration by parts of  $f(x) = f(x) \cdot 1$  over each subinterval  $[kh, (k+1)h]$ , where the primitive of 1 is the line  $x - c_k$  connecting the points  $(kh, -\frac{h}{2})$  and  $((k+1)h, \frac{h}{2})$ , with  $c_k = (k + \frac{1}{2})h$  being the center of the interval:

$$\begin{aligned} \int_{kh}^{(k+1)h} f(x)dx &= (x - c_k)f(x)\Big|_{kh}^{(k+1)h} - \int_{kh}^{(k+1)h} f'(x)(x - c_k)dx \\ &= \frac{h}{2} [f(kh) + f((k+1)h)] - h \int_{kh}^{(k+1)h} f'(x) \left( \frac{x}{h} - (k + \frac{1}{2}) \right) dx. \end{aligned}$$

This yields

$$I = \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} f(x)dx = T_f(h) - h \int_0^L f'(x) \overline{P}_1\left(\frac{x}{h}\right) dx, \quad (2.1)$$

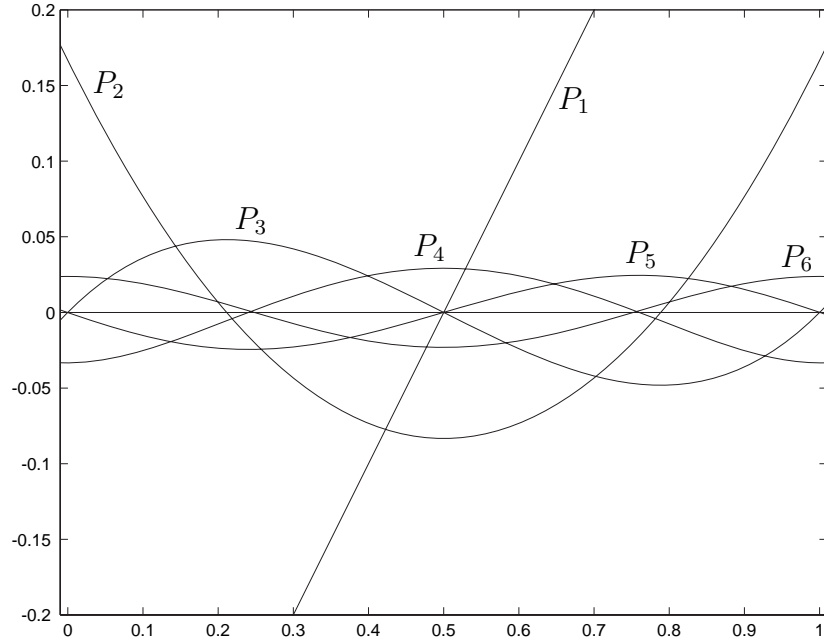
where  $\overline{P}_1$  stands for the  $h$ -periodic continuation of the function given on  $[0, 1]$  by  $P_1(x) = x - \frac{1}{2}$  (see Figure 1). %



**Figure 1.** An example of  $f$  and  $\overline{P}_1$  in equation (2.1) with  $N = 6$

The differences in the derivatives at extremal nodes in the EMF are obtained by recursively applying integration by parts to the last integral of (2.1) on every subinterval separately, thereby differentiating  $f'$  and integrating  $\overline{P}_1$  again and again. *The  $h$ -periodic*

extensions  $\overline{P}_\ell$  of the primitives of  $P_1$  are made continuous at the integer multiples of  $h$ , so that no value at an interior node appears, in contrast with the first integration above. Continuity is achieved by constructing the  $P_\ell$  for  $\ell$  even as even functions with respect to  $\frac{h}{2}$  and the  $P_\ell$  for  $\ell$  odd as odd functions with zero values at the extremities of  $[0, h]$ . (A function  $g$  is even with respect to  $a$  when  $g(a-x) = g(a+x)$ , odd when  $g(a-x) = -g(a+x)$ .) These primitives are the Bernoulli polynomials, which we denote by  $P_\ell$  in order to distinguish them from the Bernoulli numbers  $B_\ell := P_\ell(0)$ .



**Figure 2.** Bernoulli polynomials  $P_1, \dots, P_6$  on  $[0, 1]$

The BP are usually constructed and given on  $[0, 1]$ . There  $P_1(x) = x - \frac{1}{2}$  is odd with respect to  $\frac{1}{2}$ . We obtain two new polynomials simultaneously. Suppose that  $P_{2k-1}$  has been determined and that it is monic (i.e., the coefficient of its term of highest degree equals 1) and odd with respect to  $1/2$ . Then any primitive of the form  $a \int_b^x P_{2k-1}(u) du + c$ , with  $a, b, c \in \mathbb{R}$ , is even with respect to  $1/2$ , and one may choose  $a = 2k$  to have a monic polynomial. We thus consider

$$P_{2k}(x) = P_{2k}^*(x) + B_{2k}, \quad P_{2k}^*(x) := 2k \int_0^x P_{2k-1}(u) du \quad (2.2)$$

for some constant  $B_{2k}$ . Equation (2.2) splits  $P_{2k}$  into  $P_{2k}(0)$  and the term by term integral

of  $P_{2k-1}$ . A further integration from  $1/2$  yields

$$P_{2k+1}(x) = (2k+1) \left[ \int_{1/2}^x P_{2k}^*(u) du + B_{2k}(x - \frac{1}{2}) \right] + B_{2k+1} \quad (2.3)$$

for some constant  $B_{2k+1}$ , which we take as 0 in order to make  $P_{2k+1}$  odd with respect to  $1/2$ . Then requiring  $P_{2k+1}(0) = 0$ , i.e.,

$$B_{2k} = -2 \int_0^{1/2} P_{2k}^*(u) du = - \int_0^1 P_{2k}^*(x) dx,$$

guarantees that  $P_{2k+1}(1)$  vanishes, too, and fully determines  $P_{2k}$  and  $P_{2k+1}$  in (2.2) and (2.3).  $P_\ell$  is called the *Bernoulli polynomial of degree  $\ell$* , the constant  $B_\ell = P_\ell(0)$  the  $\ell$ -th *Bernoulli number*. Tables and graphs of  $P_\ell$  and  $B_\ell$  appear in many books, among them the classical [Abr-Ste].

The parity may be written as

$$P_\ell(1-x) = (-1)^\ell P_\ell(x). \quad (2.4)$$

Although it will not be used in this paper, we notice that this implies a vanishing mean of  $P_\ell$  over the interval  $[0, 1]$ , i.e.,  $\int_0^1 P_\ell(x) dx = 0 \ \forall \ \ell$  (for  $\ell$  even, this is obtained from  $\int_0^1 P_{2k}(x) dx = \int_0^1 P_{2k}^*(x) dx + B_{2k} = 0$ ).  $P_\ell$  is often constructed from  $P_{\ell-1}$  just by requiring that property [Hen, p. 282].

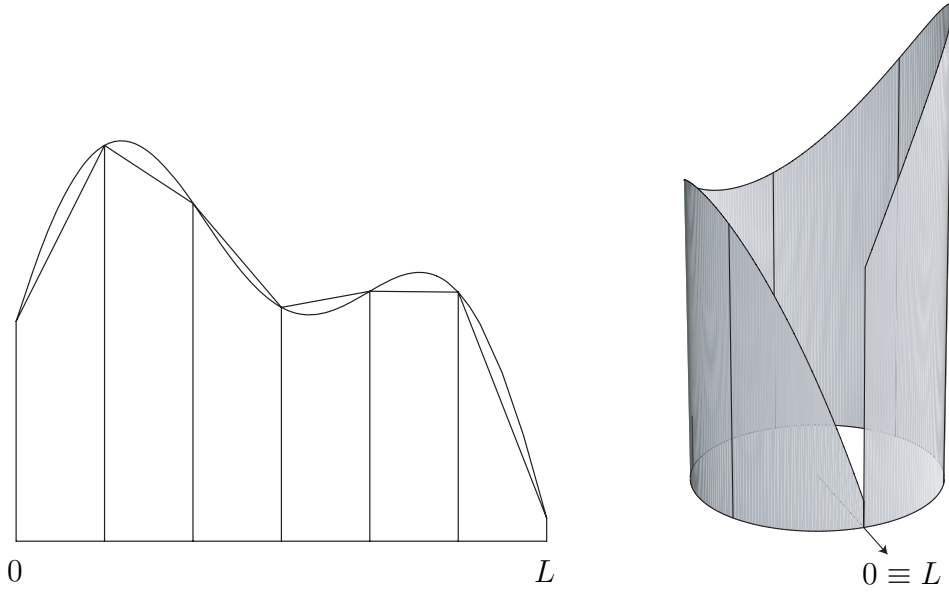
The relation

$$P'_{\ell+1}(x) = (\ell+1)P_\ell(x), \quad (2.5)$$

which follows from (2.2) for  $\ell$  even and from (2.3) and (2.2) for  $\ell$  odd, will be crucial in the development.

### 3. The circle interpretation of the trapezoidal rule

Let us now come to our main point, namely that the trapezoidal rule and the EMF should be interpreted on a circle. For that purpose, think of the interval  $[0, L]$  as being rolled up on the circle  $\mathcal{D}$  of radius  $\frac{L}{2\pi}$  through the application that maps  $x$  onto the point  $\frac{L}{2\pi}(\cos \phi, \sin \phi) \in \mathbb{R}^2$  with polar angle  $\phi = \frac{2\pi}{L}x$ ,  $\phi : [0, L] \mapsto [0, 2\pi]$ , and let  $f$  be correspondingly defined on the circle.  $x$  now also denotes arc length on  $\mathcal{D}$ . This makes the extremities  $x = 0$  and  $x = L$  the same point on  $\mathcal{D}$ , and the values of  $f$  and its derivatives at 0 and  $L$  their left and right limits respectively at that same point, see Figure 3. In fact,  $f(0) = f(0+)$  and  $f(L) = f(0-)$  where, as usual,  $f(x\pm) := \lim_{\epsilon \rightarrow 0+} f(x \pm \epsilon)$ . In the generic case, i.e., when  $f$  and its derivatives are not  $L$ -periodic,  $0 \sim L$  becomes a *point of discontinuity (jump)* of  $f$ .



**Figure 3.** Composite trapezoidal integration and its circular interpretation

The trapezoidal rule is usually introduced as the area under the piecewise linear interpolant of  $f$  between equidistant points. It may however also be seen [Ber] as the area under the — possibly balanced, see [Hen, p. 333] — trigonometric polynomial of minimal degree

$$p(x) = \sum_{n=-[N/2]}^{[N/2]} b_n e^{in\frac{2\pi}{L}x}$$

interpolating the same values. (As usual,  $[x]$  denotes the entire part of  $x$ .) Indeed, in view of  $\int_0^L e^{in\frac{2\pi}{L}x} dx = \frac{L}{2\pi} \int_0^{2\pi} e^{in\phi} d\phi = 0$ , one has

$$\int_0^L p(x) dx = \int_0^L b_0 dx = L \cdot b_0.$$

But the  $b_n$  are the trapezoidal approximations of the Fourier coefficients of  $f$  [Hen, p. 352], so that  $b_0 = \frac{1}{L} T_f(h)$  and  $T_f(h) = \int_0^L p(x) dx$ . *The negative influence of the jump on the accuracy of the interpolating trigonometric polynomial explains why the convergence of the trapezoidal rule hinges on the values of  $f$  and its derivatives at that point.*

After changing  $f(0)$  to

$$f(0) := \frac{1}{2} [f(0) + f(L)], \quad (3.1)$$

the trapezoidal rule becomes a Riemann sum on the circle:

$$T_f(h) = h \sum_{k=0}^{N-1} f(kh) = h \sum_{k=1}^N f(kh).$$

In Theorem 1 the location of the jump coincides with a node. It has been known for some time [Lyn1] that one can prove a similar result for any Riemann sum with equidistant nodes (and evaluation set identical with the partition), i.e., for every rule

$$R_f(h) := h \sum_{k=0}^{N-1} f((k+t)h) = h \sum_{k=1}^N f((k-1+t)h), \quad 0 \leq t < 1 \quad (3.2)$$

(notice that our range for  $t$  differs from that of [Lyn1] and [Ell]). On the circle  $\mathcal{D}$  we define the  $L$ -periodic function

$$\hat{f}(x) := f(x - (1-t)h) = f(x - \hat{t}h)$$

with

$$\hat{t} := 1 - t.$$

Since  $I = \int_0^L \hat{f}(x)dx$ , one may view  $R_f(h)$  for all  $t$  as trapezoidal integration:

$$R_f(h) = T_{\hat{f}}(h).$$

This allows us to start the proof for all  $t$  as in (2.1) and eliminates  $\hat{t}h$  from most of the development. The jump in  $\hat{f}$  is located at  $\hat{t}h$  in the first interval  $[0, h]$ .  $t$  is the relative distance of the jump to the node which follows it,  $\hat{t}$  to that which precedes it. Notice that the circle interpretation automatically defines  $\hat{f}$  on  $[0, \hat{t}h]$ .

#### 4. The EMF on the circle

To prove the generalization of Theorem 1 to  $R_f(h)$  along Elliott's lines in [Ell], we first notice that, in order for the zero values of the periodically extended odd degree Bernoulli polynomials  $\overline{P}_{2k+1}$  to lie at the extremities of the subintervals  $[kh, (k+1)h]$ , we must define  $\overline{P}_\ell$  on  $[kh, (k+1)h]$  as

$$\overline{P}_\ell\left(\frac{x}{h}\right) := P_\ell\left(\frac{x}{h} - k\right)$$

(this is somewhat simpler than the corresponding function in [Ell]). As sketched in §2, if  $\hat{f}'$  is absolutely integrable one can evaluate  $\int \hat{f}(x)dx$  on each of the last  $N-1$  intervals as

$$\int_{kh}^{(k+1)h} \hat{f}(x)dx = h\overline{P}_1\left(\frac{x}{h}\right)\hat{f}(x)\Big|_{kh}^{(k+1)h} - h \int_{kh}^{(k+1)h} \hat{f}'(x)\overline{P}_1\left(\frac{x}{h}\right)dx, \quad k = 1, \dots, N-1. \quad (4.1)$$

If  $t = 0$ , (4.1) holds in the first interval also, whereas for  $t \neq 0$  the jump is to be taken into account as

$$\begin{aligned} \int_0^h \hat{f}(x) dx &= \int_0^{\hat{t}h} \hat{f}(x) dx + \int_{\hat{t}h}^h \hat{f}(x) dx \\ &= h \left[ P_1\left(\frac{x}{h}\right) \hat{f}(x) \Big|_0^{\hat{t}h} + P_1\left(\frac{x}{h}\right) \hat{f}(x) \Big|_{\hat{t}h}^h \right] - h \int_0^h \hat{f}'(x) P_1\left(\frac{x}{h}\right) dx, \end{aligned} \quad (4.2)$$

where we have chosen the continuous function  $P_1$  as the primitive of 1 and used the fact that  $\overline{P}_1 = P_1$  on the first interval. In (4.1),  $\overline{P}_1(\frac{x}{h})$  equals  $\frac{1}{2}$  at every right extremity and  $-\frac{1}{2}$  at every left one and, similarly,  $P_1(0) = -\frac{1}{2}$  and  $P_1(1) = \frac{1}{2}$  in (4.2). The value  $\hat{f}(\hat{t}h)$  at the jump is  $f(L)$  on the left and  $f(0)$  on the right. Summing over all intervals yields, with (2.4) for  $\ell = 1$ ,

$$\begin{aligned} T_{\hat{f}}(h) - \int_0^L \hat{f}(x) dx &= R_f(h) - I = \gamma P_1(t) h [f(L) - f(0)] + h \int_0^L \hat{f}'(x) \overline{P}_1\left(\frac{x}{h}\right) dx, \\ \gamma &= \begin{cases} 0, & t = 0, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.3)$$

The right-hand integral may be recursively evaluated over each subinterval, taking (2.5) into account:

$$\int_{kh}^{(k+1)h} \hat{f}^{(\ell-1)}(x) \overline{P}_{\ell-1}\left(\frac{x}{h}\right) dx = \frac{h}{\ell} \overline{P}_{\ell}\left(\frac{x}{h}\right) \hat{f}^{(\ell-1)}(x) \Big|_{kh}^{(k+1)h} - \frac{h}{\ell} \int_{kh}^{(k+1)h} \hat{f}^{(\ell)}(x) \overline{P}_{\ell}\left(\frac{x}{h}\right) dx.$$

For  $k = 0$  and  $t \neq 0$  the integrated term may be split as in (4.2). As anticipated in §2,  $\overline{P}_{\ell}(\frac{x}{h})$  is continuous, equaling  $P_{\ell}(0)$  at every subinterval extremity  $kh$ ; the sum of the contributions of the integrated terms at the nodes vanishes — it telescopes — only the terms at the jump remain and we have

$$\int_0^L \hat{f}^{(\ell-1)}(x) \overline{P}_{\ell-1}\left(\frac{x}{h}\right) dx = \frac{h}{\ell} P_{\ell}(\hat{t}) [f^{(\ell-1)}(L) - f^{(\ell-1)}(0)] - \frac{h}{\ell} \int_0^L \hat{f}^{(\ell)}(x) \overline{P}_{\ell}\left(\frac{x}{h}\right) dx.$$

When recursively inserting this into (4.3), the factors  $\frac{h}{\ell}$  lead to powers of  $h$  and factorials. It remains to change the variable to  $x + \hat{t}h$  in the last integral to retrieve  $f$ . Considering that, according to (2.4), the negative sign arising in  $P_{\ell}(1 - t)$  and  $\overline{P}_{\ell}(\frac{x}{h} - t)$  compensates that of every new integration by parts, and in view of the  $L$ -periodicity of all integrands, we finally obtain the following formula.



## Theorem 2 (EMF for equispaced Riemann sums)

Let  $f \in C^{q-1}[0, L]$  with  $f^{(q)}$  absolutely integrable on  $[0, L]$ ,  $q \geq 2$ . Let  $R(h)$  be any Riemann sum (3.2) of  $I$  on  $\mathcal{D}$  with  $f$  being  $L$ -periodically extended and defined at the jump 0 as in (3.1). Then the integration error may be written as

$$R_f(h) - I = \gamma a_1 h + \sum_{\ell=2}^q a_\ell h^\ell - \frac{h^q}{q!} \int_0^L f^{(q)}(x) \bar{P}_q(t - \frac{x}{h}) dx \quad (4.4)$$

with  $\gamma$  from (4.3) and

$$a_\ell := \frac{P_\ell(t)}{\ell!} [f^{(\ell-1)}(L) - f^{(\ell-1)}(0)].$$

Formula (4.4) states that the Riemann sum error is  $\mathcal{O}(h)$  unless  $f(L) = f(0)$  or  $t = 0$  or  $t = 1/2$ . With  $t = 0$  one has the trapezoidal rule, in which case  $\gamma = 0$ ,  $P_\ell(0) = 0$  for odd  $\ell > 1$  and  $q = 2m + 2$  yield formula (1.1) after combining the two terms in  $h^{2m+2}$  and applying the integral mean value theorem [Häm-Hof, p. 281; Kin-Che, p. 484]. If  $t = 1/2$ , then  $P_\ell(1/2) = 0$  for odd  $\ell$  leads to a similar formula for the *midpoint rule* [Ell, p. E36; Lie-Lü-Shi, p. 25].

## 5. A generalization of the EMF to functions with several jumps

Formula (4.4) and its proof express that the accuracy of a Riemann sum for a function  $f$  with a jump  $c$ , at which the value is taken to be  $\frac{f(c-) + f(c+)}{2}$ , is determined for  $h$  small enough by the differences of the left and right values of  $f$  and its derivatives at  $c$ . Once  $f$  is looked at on a circle, the fact that the jump originated from joining the extremities of the interval is irrelevant. The coefficients in (4.4) merely depend on the distance  $th$  from  $c$  to the node  $x_k$  that follows it on the circle; if  $t = 0$  or  $t = 1/2$ , i.e., if  $c$  coincides with a node or lies at equal distance of two nodes, then the first term vanishes and the error is generically  $\mathcal{O}(h^2)$ ; otherwise it is  $\mathcal{O}(h)$ .

The same proof naturally delivers a generalization of (4.4) to functions with several jumps. Let  $f$  be *piecewise*  $C^{q-1}[0, L]$ , i.e.,  $(q-1)$ -times continuously differentiable on  $[0, L]$  except at interior jumps, say  $c_1, \dots, c_J$ , at which the limits of  $f$  and its  $q-1$  derivatives exist on both sides. Denote by  $c_0 := 0 \sim L$  the abscissa of the extremities and, as in (3.1), set the value at the jumps as

$$f(c_j) := \frac{f(c_j-) + f(c_j+)}{2}, \quad j = 0, \dots, J. \quad (5.1)$$

Let  $I$  be approximated with a Riemann sum (3.2). For every jump  $c_j$ , determine  $t_j \equiv -\frac{c_j}{h} \bmod 1$ , the location of  $c_j$  with respect to the node that follows it. Obviously,  $t_0 = t$ . Subdividing the interval  $[0, L]$  according to the abscissae  $kh$  and to the jumps  $c_j$  and repeating the proof of §4, we obtain the following formula.

**Theorem 3 (EMF for Riemann sums of piecewise smooth functions)**

Let  $f$  be piecewise  $C^{q-1}[0, L]$  and let  $R_f(h)$  be any Riemann sum (3.2) of  $I$  on  $\mathcal{D}$  with  $f$   $L$ -periodically extended. Let  $c_j$  denote the jumps of  $f$  and define  $f(c_j)$  as in (5.1) and  $t_j \equiv -\frac{c_j}{h} \bmod 1$ ,  $j = 0, \dots, J$ .

If  $f^{(q)}$  is absolutely integrable between every two consecutive  $c_j$  then the integration error may be written as

$$R_f(h) - I = a_1 h + \sum_{\ell=2}^q a_\ell h^\ell - \frac{h^q}{q!} \int_0^L f^{(q)}(x) \sum_{j=0}^J \overline{P}_q(t_j - \frac{x}{h}) dx \quad (5.2)$$

with

$$a_1 := \sum_{j=0}^J \gamma_j P_1(t_j) [f(c_j-) - f(c_j+)], \quad \gamma_j := \begin{cases} 0, & t_j = 0, \\ 1, & t_j \neq 0, \end{cases}$$

and

$$a_\ell := \sum_{j=0}^J \frac{P_\ell(t_j)}{\ell!} [f^{(\ell-1)}(c_j-) - f^{(\ell-1)}(c_j+)]. \quad (5.3)$$

The factors  $\gamma_j$  are a means of expressing that the sum in  $a_1$  is merely to contain the terms corresponding to jumps outside the nodes, where the magnitude of the jumps has an influence. Notice that  $t$ , the parameter that determines the Riemann sum, enters the formula through  $t_0$  only: the jump at the extremities is no different from any other.

Equation (5.2) is a generalization of formula (2.11) in [Lyn1]. By summing functions of compact support one may also derive (5.2) from that formula (2.11); the proof in [Lyn1] is less elementary though, as it involves the Poisson summation formula and the Fourier series of  $\overline{P}_\ell$ .

## 6. Examples

*Example 1.* Since the jump at the extremities is no different from any other, an interior jump will not necessarily slow down the convergence of Riemann sums. As an example, use the trapezoidal rule to integrate

$$f(x) = \cos 60x + r(x)$$

on  $[0, 2]$ , where the function

$$r(x) := \begin{cases} 0, & x < 1, \\ 1 + (x - 1)^{10}, & x \geq 1, \end{cases}$$

is discontinuous in the center of the interval. (The high frequency 60 was chosen to slow down convergence, so that the cancellation in the computation of the orders – see below – is not too severe; the relatively high exponent 10 is necessary for the derivatives at the jump not to vanish.) Table 1 displays the results for increasing *even* numbers of nodes, so that the abscissa 1 of the interior jump is not one of them and  $P_\ell(t_1) = 0$ ,  $\ell$  odd. The first column gives the number of nodes minus 1, the second the calculated orders of convergence while the following columns contain the orders with the first three Romberg steps, using the formula for general  $h_k$  [Sch, p. 133].

**Table 1.** Experimental convergence orders for  $f(x) = \cos 60x + r(x)$  on  $[0, 2]$

$N$	Order estimate	1 <sup>st</sup> Romberg step	2 <sup>nd</sup> Romberg step	3 <sup>d</sup> Romberg step
3	1.94790851695372			
7	3.25984172808322	-1.85921286236476		
15	-0.84025938726864	0.74958411148075	0.63882403712488	
31	1.44769002315812	0.86209983657621	0.81505824722645	0.79504194836318
63	2.49453018431671	4.42574316519551	3.36637170662472	3.20800608098070
127	2.09705023858407	4.63735591276022	6.70951901799688	5.55765720679422
255	2.02294955773993	4.13349341257332	6.71571953436765	8.79650553194619
511	2.00564379774331	4.03456602581793	6.15776307487585	8.74619060349973
1023	2.00140301386587	4.00993186507600	6.04335134386092	8.16998453814465
2047	2.00034998470753	4.00317849384084	6.01227420004377	8.26719047830944

The orders clearly reflect the error as an expression in  $h^2$ , just as without interior jump. (The orders of convergence were estimated the usual way [Boo, p. 23]: assuming that the error behaves asymptotically as

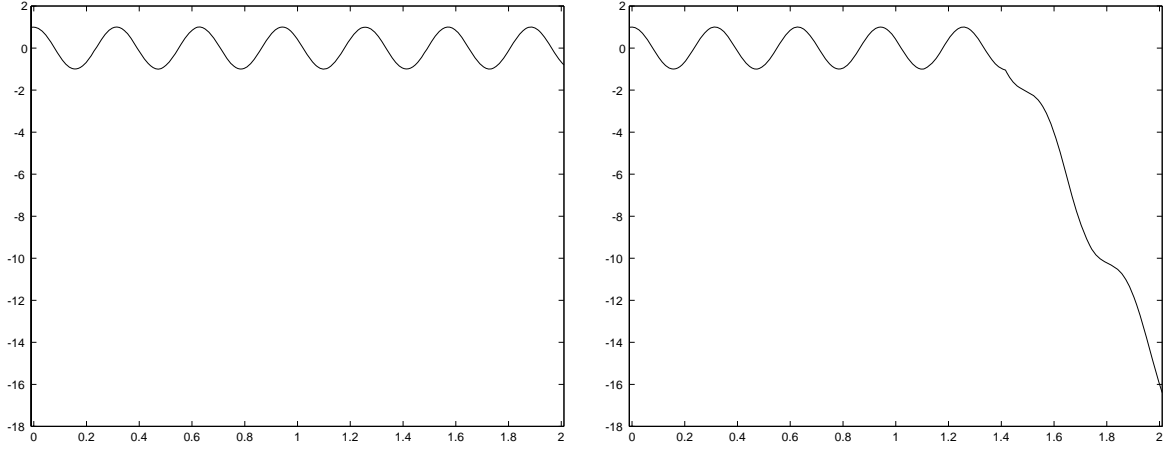
$$e_h \approx C \cdot h^p, \quad (6.1)$$

one divides two of these approximations for  $h$  and  $\hat{h}$  to eliminate the unknown  $C$  and get  $\frac{e_h}{e_{\hat{h}}} \approx (h/\hat{h})^p$  or

$$p \approx \frac{\log(e_h/e_{\hat{h}})}{\log(h/\hat{h})} = \frac{\log(e_h/e_{\hat{h}})}{\log(\hat{N}/N)}. \quad (6.2)$$

Here we have  $N = 2^\ell - 1$  for the first  $\ell \in \mathbb{N}$ .)

*Example 2.* The terms in the sums  $a_j$  may cancel each other and so extra jumps even lead to a smaller error  $R_f(h) - I$ . This is one of those instances in which a numerical method surprises with better results than those to be expected from the classical theory. To be specific, suppose that  $f \in C^\infty[0, L]$  with  $f'(L) \neq f'(0)$ , so that the trapezoidal values  $T_f(h)$  converge toward  $I$  as  $\mathcal{O}(h^2)$ . We will now construct a sequence of examples with a knick for which the rule yields an error proportional to  $h^4$ .



**Figure 4.** The trapezoidal rule has an  $\mathcal{O}(h^2)$ -error for the smooth function  $f(x) = \cos 20x$  (left), an  $\mathcal{O}(h^4)$ -one when subtracting a broken line (right).

For that purpose, we will subtract from  $f$  for given  $h$  a line broken at an abscissa  $s$ ,

$$l(x) = \begin{cases} \alpha^-(x - s), & x \leq s, \\ \alpha^+(x - s), & x \geq s, \end{cases}$$

with constants  $\alpha^-$  and  $\alpha^+$  to be determined, and integrate  $f - l$ . Since  $l$  is continuous,  $a_1$  remains 0.  $a_2 = 0$  requires

$$P_2(0) [(f'(L) - \alpha^+) - (f'(0) - \alpha^-)] + P_2(t_s) [(f'(s-) - \alpha^-) - (f'(s+) - \alpha^+)] = 0,$$

where  $t_s$  denotes the relative distance of  $s$  to the following node. Since  $f'(s-) = f'(s+)$ , the condition on the slopes is

$$\alpha^+ - \alpha^- = \frac{P_2(0)}{P_2(0) - P_2(t_s)} [f'(L) - f'(0)].$$

This choice warrants an error  $\mathcal{O}(h^4)$ , for  $f''(s^+) = f''(s^-)$  and  $[\alpha^-(x-s)]'' = [\alpha^+(x-s)]'' = 0$  eliminate the  $\mathcal{O}(h^3)$ -term.

Table 2 gives the results when integrating  $f - l$  on  $[0, 2]$  for  $f(x) = \cos 20x$  and the arbitrary choices  $s = \sqrt{2}$  and  $\alpha^- = 0$ . The better precision of the trapezoidal rule for the broken functions is obvious. (Estimating the order for  $f - l$  by means of (6.2) is suitable: the constant  $C$  in (6.1) does not depend on  $l$  since in (5.3)  $(f - l)^{(\ell-1)} = f^{(\ell-1)}$  for every  $\ell > 2$ .)

**Table 2.** Integration errors for  $f(x) = \cos 20x$  and  $f(x) - l(x)$  on  $[0, 2]$

$N$	$T_f(h) - I$	Order estimate	$T_{f-l}(h) - \int(f-l)$	Order estimate
1	0.29580628032377		-0.037255565802397	
2	0.53735737296329	-0.86122919806707	1.77921264042887	-5.57763558803431
4	-0.09235918212477	2.54055457830827	0.21810463474162	3.02814661004104
8	-0.16193620023028	-0.81009822808470	-0.08432024601368	1.37106944637715
16	-0.02178182701936	2.89422865592582	-0.00237783846521	5.14815645994509
32	-0.00498221601737	2.12826548975059	-0.00013121887884	4.17960350647318
64	-0.00122071894493	2.02905649855289	-0.00000796966030	4.04131325798995
128	-0.00030368194030	2.00709805593859	-0.00000049461914	4.01012824624056
256	-0.00007582769004	2.00176443581766	-0.00000003085974	4.00252013723091
512	-0.00001895113547	2.00044048362570	-0.00000000192790	4.00062716250152
1024	-0.00000473742237	2.00011008188820	-0.00000000012048	4.00012822550482
2048	-0.00000118433300	2.00002751808361	-0.00000000000753	3.99994591053435

## References

- [Abr-Ste] Abramowitz M., Stegun I. A., *Handbook of Mathematical Functions* (Dover, New York, 1965).
- [Alp] Alpert, B. K., *Sparse Representation of Smooth Linear Operators* (Research Report YALEU/DCS/RR-814, Yale University, 1990).
- [Ber] Berrut J.-P., Fascinante interpolation, *Bull. Soc. Frib. Sc. Nat.* **83** (1994) 3–20.
- [Boo] de Boor C., *A Practical Guide to Splines* (Applied Mathematical Sciences 27, Springer, New York, 1978).

- [Ell] Elliott D., The Euler–Maclaurin formula revisited, *J. Austral. Math. Soc.* **B 40** (Electronic) (1998) E27–E76.
- [Gau] Gautschi W., *Numerical Analysis, an Introduction* (Birkhäuser, Boston, 1997).
- [Häm-Hof] Hämmerlin G., Hoffmann K.-H., *Numerical Mathematics* (Springer-Verlag, New York, 1999).
- [Hen] Henrici P., *Essentials of Numerical Analysis with Pocket Calculator Demonstrations* (Wiley, New York, 1982).
- [Kin-Che] Kincaid D., Cheney W., *Numerical Analysis, Mathematics of Scientific Computing* (Wadsworth, Belmont, 1991).
- [Kre] Kress R., *Numerical Analysis* (Springer-Verlag, New York, 1998).
- [Lie-Lü-Shi] Liem C. B., Lü T., Shih T. M., *The Splitting Extrapolation Method* (World Scientific, Singapore, 1995).
- [Lyn1] Lyness J. N., The calculation of Fourier coefficients by the Möbius inversion of the Poisson summation formula — Part II. Piecewise continuous functions and functions with poles near the interval  $[0, 1]$ , *Math. Comp.* **25** (1971) 59–78.
- [Lyn2] Lyness J. N., Computational techniques based on the Lanczos representation, *Math. Comp.* **28** (1974) 81–123.
- [Rok] Rokhlin V., End-point corrected trapezoidal quadrature rules for singular functions, *Computers Math. Applic.* **20** (1990) 51–62.
- [Sch] Schwarz H. R., *Numerische Mathematik* (4<sup>te</sup> Aufl., Teubner, 1997; english translation of the second edition: *Numerical Analysis. A Comprehensive Introduction*, Wiley, New York, 1989).
- [Wei] Weideman J. A. C., Numerical integration of periodic functions: a few examples, *Am. Math. Mon.* **109** (2000) 21–36.