

1 Paradoxes in the logistic equation?

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1 **Abstract** : Levins and Ginzburg raised two paradoxes concerning the logistic equation of
2 population growth. None of them received a satisfactory answer within the frame of the
3 original equation. Here, we propose solutions to both paradoxes.

4 *keywords* : logistic model, population dynamic, population regulation, Verhulst equation.

5 The logistic equation of population growth occupies a unique and fascinating
6 position in the development of ecological thinking. Proposed in the first half of the
7 nineteenth century by the Belgian mathematician Pierre-François Verhulst (1838) as a
8 potential solution to the dilemma of Malthusian exponential growth, it was rediscovered
9 and imposed to biologists as a simple model of population self-regulation in the early
10 twentieth century by the American biologist Raymond Pearl and his disciples (Kingsland
11 1985).

12 The logistic equation has since inspired and stimulated much ecological work,
13 including modeling, experimental and field research. Nowadays, the concept of carrying
14 capacity can hardly be dissociated from the model. It has also permeated evolutionary
15 biology in the 1950's and the concept of r/K selection was named after its parameters in
16 the 1970's. Finally, its remarkable fitting abilities to early datasets have also contributed to
17 the popularity of the model (Pearl 1925; Feller 1940; Hutchinson 1978). The logistic
18 equation can be found in many biology textbooks and belongs to basic education in
19 ecology. Independently of the status that one gives to this model, it has been and remains a
20 cornerstone of empirical and theoretical ecology.

21 The logistic equation however also elicited strong attacks and vigorous criticisms, so
22 that ecologists have developed a love-hate relationship with the logistic equation (Olson
23 1992). Many legitimate criticisms address the phenomenological foundations of the model
24 and its adequacy to describe the real world. They include for example the difficulty in the

1 interpretation of its parameters (e.g., Olson 1992; Miller et al. 2002), or its unrealistic form
2 of density dependence (e.g., Getz 1996; McCarthy 1997; Courchamp et al. 1999; Etienne
3 2002; Kent et al. 2003). Other criticisms have been raised with regard to internal
4 contradictions. Named hereafter paradoxes, these apparent contradictions challenge the
5 consistency of the model itself, and we shall focus here on this issue. We believe that
6 clarifying such points might help understanding the significance and limitations of this
7 historically and educationally important model. Our aim is to discuss two paradoxes raised
8 respectively by Levins (Hutchinson 1978) and Ginzburg (1992), and we show how they
9 can be solved within the framework of the original formulation of the model.

10 **Levins' paradox**

11 The usual "ecological" form of the logistic equation is:

$$12 \quad \frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right). \quad (1)$$

13 The first paradox, raised by Levins, has been exposed in Hutchinson (1978, page 4) as
14 follows: “ r (...) be negative, setting $N > K$, leads to an impossible result, for dN/dt is
15 positive and increases indefinitely with N ”. Indeed, it can be shown that infinity is even
16 reached in a finite time and, to avoid such a situation, Levins suggested that r should be
17 constrained to non-negative values. If so the paradox disappears, but at the cost of
18 excluding biologically meaningful situations, namely declining populations. Obviously, r
19 has to be negative in sink environments (Hairston et al. 1970; Pulliam 1988). Actually, a
20 more sensible resolution emerges from Verhulst's original formulation (1838):

$$21 \quad \frac{dN}{dt} = aN - bN^2, \quad (2)$$

22 where a is the Malthusian growth rate and, as such, has to admit both signs. In the second
23 term added to avoid infinite growth, b is a friction coefficient and therefore strictly

1 positive. Eq. (1) is derived from Eq. (2) by writing the latter as:

$$2 \quad \frac{dN}{dt} = aN \left(1 - \frac{N}{a/b} \right), \quad (2')$$

3 and identifying $r = a$ and $K = a/b$. These relations are well known, and can be found in
4 several textbooks. However, a simple but important consequence that, to our knowledge,
5 has never been pointed out is that r and K must have the same sign since $b = r/K$ is
6 positive. This constraint rules out Levin's case ($r < 0$ and $K > 0$) and therefore solves the
7 paradox without limiting the sign of r . However, with $r < 0$, K is also negative and cannot
8 be interpreted as a "carrying capacity"; it is easy to see that, if $r < 0$, then all the solutions
9 with $N(0) \geq 0$ converge to 0, as expected in sink environments. We see that this leads to a
10 difficulty in the interpretation of the parameter $K = a/b$ as a "carrying capacity". A way to
11 avoid this problem is to redefine the carrying capacity as K_∞ :

$$12 \quad K_\infty = \lim_{t \rightarrow \infty} N(t), \text{ for } N(0) > 0. \quad (3)$$

13 As a consequence, in the logistic case we have :

$$14 \quad K_\infty = \begin{cases} a/b = K, & \text{if } a > 0, \\ 0, & \text{if } a \leq 0. \end{cases} \quad (4)$$

15 It is worth emphasizing that, while a and b are two independent parameters in Verhulst's
16 equation (Eq. (2)), K depends on r in the usual ecological form (Eq. (1)). To our
17 knowledge, the first apparition of the ecological form goes back to Lotka (1925), with a
18 typo in the formula.

19 At this stage, it is helpful to distinguish between two versions of the logistic model.
20 We call "constrained" logistic the model derived from Verhulst approach, i.e., with $r = a$,
21 $K = a/b$, and b strictly positive. In the "free" logistic model, r and K are considered
22 independent. A classical derivation (e.g., Begon et al. 1996) of the logistic equation
23 assumes a linear decrease of the per capita growth rate (see Fig. 1a). In this approach, r and

1 K are implicitly assumed to be both positive and independent. Levins' paradox appears in
2 the free model with r negative (Fig. 1b), but not in the constrained model (Fig. 1c). It is
3 worth noting that Levin's configuration (Fig. 1b) leads to a positive density-dependence.

4 **Ginzburg's paradox**

5 Ginzburg (1992) considered incorporating additional mortality into the logistic
6 equation without changing any other aspects of the environment, and proposed two ways to
7 do so. Firstly:

$$8 \quad \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - \mu N, \quad (5)$$

9 where μN ($\mu > 0$) is the additional mortality term. The new equilibrium density is $K(r - \mu) / r$
10 ($\mu < r$). Ginzburg rejected this approach because it “disagrees with our intuition about
11 unchanging equilibrium”. He claimed that, as the resources do not change, the population
12 with higher mortality might attain the same K , but more slowly. This point is obviously
13 disputable. Like all population equations, the logistic is a conservation or balance equation,
14 i.e., the size variation over a time interval is the difference between input and output over
15 that interval. In any case, it is admissible to act on such an equation by adding a term
16 reflecting increased mortality, e.g., by harvesting a constant fraction of the population per
17 unit of time. If one's intuition is puzzled by the consequences of such a change, one must
18 either adapt or reject the whole model, i.e., the logistic itself. Moreover, our own intuition
19 is not hurt by the decrease of the equilibrium density: with additional mortality, individuals
20 are continuously removed from the population, and thus the population might never reach
21 K again. Such a behaviour is ultimately dictated by the model, and not the intuition.

22 Secondly, Ginzburg proposed to introduce additional mortality in r only:

$$23 \quad \frac{dN}{dt} = (r - \mu)N \left(1 - \frac{N}{K}\right) = rN \left(1 - \frac{N}{K}\right) - \mu N \left(1 - \frac{N}{K}\right), \quad (6)$$

1 which does not change K . But Ginzburg stated that the additional term $-\mu N(1 - N/K)$
2 cannot represent mortality since it is positive for $N > K$ (note that it is valid only for $K > 0$
3 i.e. $\mu < r$).

4 This paradoxical situation fostered a lively debate on equations of population growth
5 in the issues of September 1992 and February 1993 of TREE. Most of these answers were
6 based on models different from the logistic one and we will not tackle them, because our
7 aim is to solve the paradox internally to the logistic equation. Only Young (1992) and
8 Berryman (1992) replied within the framework of the logistic, but their answers are
9 unsatisfactory. Indeed, Young's and Ginzburg's second approach (Eq. (6)) are
10 mathematically equivalent (set $p = 1 - \mu/r$ in Eq. (7) of Young 1992), and therefore deserve
11 the same analysis given below. With regard to Berryman, although we share his
12 conclusions (Berryman 1992), we cannot adopt his line of reasoning. He tackled the
13 problem by decomposing the model into two density dependent linear birth and death rates
14 leading to the following form of the logistic equation: $dN/Ndt = (n - \nu N) - (s + \sigma N)$; n and s
15 are the intrinsic birth and death rates, ν and σ the density dependent birth and death
16 coefficients. Such a derivation can be found in contemporary textbooks of ecology (e.g.,
17 Gotelli 2001) and was already proposed in Kostitzin (1937, pp. 15). A first difficulty is that
18 the number of parameters is different in Ginzburg's and Berryman's equations, leading to
19 problems with their identification. A more fundamental problem lies in the linear birth
20 term with a negative slope, since it becomes negative for large values of N . This is
21 conceptually unsatisfactory since it requires a limitation in the values of N , and therefore
22 provides a model different from Verhulst's logistic equation. In our opinion, one should
23 abandon this approach and rather stick to Verhulst's derivation, which does not contain this
24 awkward feature.

1 Again, returning to Verhulst framework (Eq. (2)) solves Ginzburg's paradox. In the
 2 constrained logistic model, the only way of maintaining K fixed and respecting the
 3 equalities $r = a$ and $K = a/b$ is to act on b . If a becomes $a - \mu$ ($0 < \mu < a$ to preserve $K > 0$)
 4 then b must be replaced by $b - \delta$ ($0 < \delta < b$), so that $K = a/b = (a - \mu) / (b - \delta)$. Solving for δ ,
 5 we get $\delta = \mu b / a$, and Eq. (2) becomes:

$$6 \quad \frac{dN}{dt} = (a - \mu)N - (b - \delta)N^2 = aN - bN^2 - \mu N + \frac{\mu b}{a}N^2, \quad (7)$$

7 It is now apparent that the paradoxical mortality term of Eq. (6) (i.e., $-\mu N(1 - N/K)$) is
 8 composed of the losses due to increased mortality $-\mu N$ and of $\frac{\mu b}{a}N^2$, a correction of the
 9 second order friction term. The first term is always negative and the second always

10 positive since K , and hence a , are positive. Clearly, $\frac{\mu b}{a}N^2 = \left(\frac{N}{K}\right)\mu N$ is larger than μN

11 when $N > K$, and the total contribution is positive. This means that the gain obtained by the
 12 reduction in the quadratic friction term overcomes the loss due to the linear mortality term.

13 There is no paradox in this, and we want to illustrate it with an example. Let us consider an
 14 animal population with cannibalism (e.g. a hypothetical spider population): let a denote the
 15 difference between intrinsic birth and death (cannibalism excluded) rates; the brake bN^2 is

16 due only to cannibalism, assumed to be proportional to the number of meetings per unit
 17 time; a predator is added to the system, and removes μN spiders per unit time. If one

18 decrees that the equilibrium density must remain constant, then the only choice (except

19 increasing trivially a) is to act on b . The intensity of cannibalism must be weakened, and the

20 number of individuals saved in this way per unit time is δN^2 ($\delta = \frac{\mu b}{a}$). For large values of

21 N ($N > K$), the number δN^2 of individuals saved from cannibalism overcomes the number

22 μN of individuals killed by the additional predator. Clearly, Ginzburg's paradoxical term

1 does not reflect mortality only. Again, returning to the historical formulation of Verhulst
2 puts a constraint on parameters r and K , which also solves Ginzburg's paradox.

3 **Discussion**

4 We would like to stress some important issues raised in this paper. Firstly, we
5 showed that two paradoxes or apparent contradictions appear only in the free logistic
6 model, but disappear in the constrained one. Other "paradoxes" can be produced, e.g. by
7 increasing r without changing K , or by decreasing the carrying capacity with $r < 0$ and
8 $K > 0$. In both cases, the additional terms have the "wrong" sign. Again, both "paradoxes"
9 do not persist in the constraint model. Secondly, it is important to be aware that the choice
10 of the derivation is not innocuous. The approach proposed in Kostitzin (1937) and used in
11 Berryman's reply (1992) is an example. In the same vein, following Lotka's (1924)
12 derivation based on Taylor's expansion leads to the same difficulty because there is no
13 constraints on the quadratic term. Thirdly, we believe that the logistic equation should be
14 considered as an abstract object conveying the message that a population may exhibit self-
15 regulation via some implicit density-dependent mechanism modeled in the friction term.
16 The phenomenological identification of the parameters requires additional knowledge in
17 each particular situation. Indeed, this implies describing how the model can be
18 implemented in the real world, i.e. how energy or matter is used, enters and leaves the
19 system (Saucy & Gabriel 1999). We will address this challenging question in a
20 forthcoming paper. Finally, although our goal is neither to rescue nor to dismiss the model,
21 we observe that the logistic equation is of historical importance and of educational
22 significance. For this reason we think valuable to clarify its conceptual framework and its
23 limitations. We hope that solving these two paradoxes will help reaching this goal.

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1 **Figure legend**

2 **Fig. 1.** Per capita growth rate as a function of population density N for the logistic model.

3 **a)** Usual configuration with r and K positive; K_{∞} as a new definition of the carrying

4 capacity (see Eq. 4). **b)** Levins' situation leading to an inverse density dependence. **c)**

5 Constrained model with r negative.

