



Institut de Physique Théorique  
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# **Modelling markets dynamics : Minority games and beyond**

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**Damien Challet**

de Pleujouse (JU)

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- Prof. Yi-Cheng Zhang, directeur de thèse
- Dr Michel Dacorogna, Zürich Re, Suisse, examinateur
- Dr Matteo Marsili, SISSA Trieste, Italie, examinateur

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Le directeur de thèse

Le doyen

Prof. Yi-Cheng Zhang

Prof. Beat Hirsbrunner

# Résumé

A partir du problème du bar d'El Farol, nous construisons le jeu de la minorité, un modèle extrêmement simple de compétition entre des agents adaptatifs, et étudions ses propriétés. Il en ressort que ce modèle présente une transition de phase avec brisure de symétrie. Un formalisme adéquat est introduit, qui montre le lien entre le jeu de la minorité et les verres de spin. Il permet de montrer que ce type de modèles possède généralement une fonction de Lyapunov, l'information disponible. Des méthodes mathématiques de la physique statistique des systèmes désordonnés sont appliquées pour trouver la solution exacte du jeu de la minorité. De plus, l'impact que les joueurs ont sur le jeu se révèle d'une importance fondamentale : si ceux-ci en tiennent compte, ils sont capables de minimiser leurs pertes.

Diverses extensions et modifications de ce modèle sont considérées. Par exemple, la présence d'un processus évolutif de type darwiniste est globalement bénéfique pour le système, qui reproduit la loi de durée de vie des espèces biologiques.

Un lien formel est établi entre le jeu de la minorité et les marchés financiers, ce qui permet d'étudier le rôle des différents types d'acteurs dans ces marchés. Par exemple, nous montrons que les spéculateurs ne sont pas des parasites, mais au contraire qu'ils permettent à ceux qui utilisent la bourse dans un autre but que la pure spéculation d'obtenir un prix plus juste, dont ils réduisent les fluctuations.

Enfin, le jeu de la minorité est modifié pour modéliser de façon plus réaliste les marchés financiers : le prix ainsi obtenu a des propriétés statistiques non triviales.



# Abstract

Starting from the El Farol's bar problem, we introduce the minority game, an extremely simple model of competition between adaptive agents. We study its properties, and find that it displays a phase transition with symmetry breaking. An adequate mathematical formalism is introduced and reveals the spin glass nature of the model. We prove that available information is a Lyapunov function for this model. Using mathematical tools of disordered systems' statistical mechanics, we obtain the exact solution of the minority game. In addition, the market impact is shown to be crucially relevant : if the players account for it, they are able to minimize their losses.

Several extensions and modifications of the game are studied. In particular, the system globally benefits from the presence of a Darwinist process, and reproduces the biological species life time distribution.

The minority game is formally linked to the financial markets. This allows the study of the role of several kinds of agents. For instance, we show that speculators are not parasites, but play an important role in financial markets : they reduce the price's fluctuations and allow people who use the markets for other reasons than pure speculation to obtain fair prices.

Finally, the minority game is extended further in order to model more realistically markets dynamics. A price is obtained, which has non-trivial statistical properties.



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# Introduction

The perception of the very structure of nature changed tremendously at the beginning of the twentieth century. General Relativity revised the concepts of space and time, and Quantum Mechanics those of the structure of matter. Both theories concern domains outside daily experience. They are so fascinating and so rich that the major trend in physics has been to explore as much as possible their implications.

Although fundamental Physics is nothing less than fundamental, it is not the whole subject. There was a niche for a new trend in Physics, *Econophysics*, which is still very vaguely defined, but whose aim is to promote new<sup>1</sup> fields of research such as economy or financial markets, and rediscovering daily life, which, far from becoming a monotone and sterile land for physicists, remains for them an inexhaustible source of questions and inspiration.

Econophysicists expect that the application to other fields of concepts and statistical methods borrowed from physics will yield fruitful results<sup>2</sup>. In the case of financial markets, one can consider the price as a fluctuating macroscopic quantity resulting from interactions between agents, seen as particles. This is of course an imperfect analogy, for instance because agents' behavior is probably not as simple as that of electrons. But statistical physics shows that the properties of macroscopic quantities do not crucially depend on all the details of the interaction between simple elements. Some physicists' Holy Grail is to directly apply those methods to economy and financial markets [1]. Unfortunately (?) human behavior is more subtle than that of electrons : whereas the latter have permanent properties, like mass, charge and spin, which can be erected as truths, there is nothing like that regarding human behavior, except the fact that there is nothing like that. Apart from being a joke, this assertion aims to show that the permanent properties of human behavior are probably not quite similar to those physicists are used

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<sup>1</sup>for physicists, of course.

<sup>2</sup>A good example of Physicists' concepts (diffusion) and methods applied to market prices can be found in [2].

to.

There are two main fields of study of financial markets. The first one consists in characterizing the statistical properties of prices (see for instance [2]) —this is now possible due to the large amount of available digital data. On the other hand, physicists as well as economists are looking for models reproducing the properties of prices [3, 4]. More precisely, they try to determine which are the essential ingredients to obtain those properties. There are two main approaches. One is to describe the price with a stochastic differential equation [4]. The other possibility is *agent based* financial markets [3], where the behavior of agents is modeled, and a price defined. Economists usually study very refined models [3], which are often too refined to be easily understood, whereas physicists most of the time propose models being too simple to be deeply relevant for financial markets.

The minority game [5] is a compromise between the too complicated models of economists and the too simple models of physicists, probably because it is physicists' simplification of an economist's model. It shows, maybe in the simplest way, the interplay between the behavior of agents and the macroscopic properties they create. Thanks to its extreme simplicity, it is mostly exactly solvable by tools of statistical physics. Therefore, it is hopefully a good example of how physicists can contribute to the understanding of financial markets. In addition, the minority game is also relevant in areas such as biology and sociology.

# Chapter 1

## Financial markets

### 1.1 How do financial markets work

A market place is basically designed for exchanging goods at fair prices — in financial markets, goods are for instance shares of an asset. One condition to obtain fair prices is that several people are wishing to sell a given good at the same time that several other people are willing to buy the same good, so that a kind of dynamical auction process can take place : prospective sellers publish at what price there are willing to sell which quantity of goods, and potential buyers publish their wishes (volume, price), giving rise to a whole distribution of asked prices and bid prices (see Fig. 1.1). Actually, agents (sellers or buyers) place bid or ask orders on a book, called *order book*. Agents are free to change their orders at any time.

Whenever some sellers and some buyers agree on a price, a given quantity of shares — called *volume* — is exchanged at that price. Most of the time however, no transaction takes place, because the highest price a potential buyer is willing to pay is lower than the lowest price at which a seller would accept to exchange his good. The difference between these two prices is called the *spread*.

In most markets, there are persons who have a special function : the *market makers*. In some markets such as the New York Stock Exchange (NYSE)<sup>1</sup>, there is one market maker per stock, so that all orders are centralized. In others markets such as the NASDAQ<sup>2</sup>, called over the counter market (OTC), there is a variable number of market makers per stock. Market makers are in some sense the oil of the markets : they smooth the dynamics of the market by their actions. For instance, when there are few people interested in trad-

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<sup>1</sup>[www.nyse.com](http://www.nyse.com)

<sup>2</sup>[www.nasdaq.com](http://www.nasdaq.com)

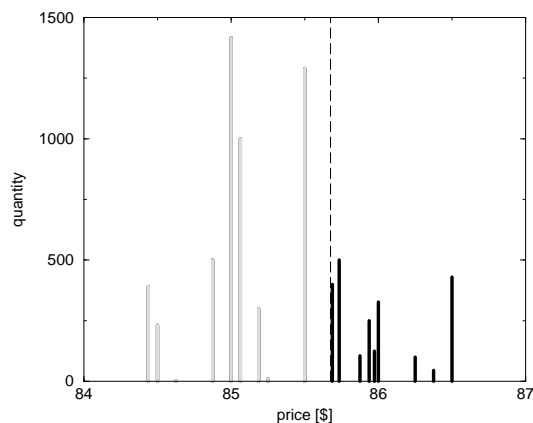


Figure 1.1: Order book of the stock price of Sun Microsystems on Island ECN (January, 24th 2000, at 09:47:57). The bid orders are plotted in gray and the ask orders in black. The last price paid is  $85 \frac{5}{8}$  \$ (dashed line).

ing a given stock, without market makers, the spread would often be very large, and the price would vary with jumps : this is a bad situation, where one may not obtain a fair price. In such situations, a market maker will reduce the spread by placing his own orders on the order book. By doing this, he takes risks ; however, market makers are allowed to maintain a spread even in much less risky situations. In this case, the spread is the price people willing to trade pay for the services market makers offer.

As everybody knows, profit making is possible by buying a good at a low price and selling it at a higher price ; this is called taking a *long position* and closing it. A less known possibility to make profit is to sell a good without possessing it at a high price and to buy it afterwards at a lower price (*short selling*) ; this is called a *short position*. Since markets prices grow on average, some traders are reluctant to take short positions.

Consequently, there are two opposite motivations for buying an asset, and two opposite motivations for selling it.

## 1.2 Predictability

A major issue concerning financial markets is their predictability. Indeed, if financial markets are predictable, how can they remain predictable, since so many smart people try to make money by forecasting future price moves. Reversely, if financial markets are not predictable, how can so many people make money ?

The story began with Bachelier back in the 1900's : for him, stock prices



follow a random walk, thus financial markets are *zero sum games*, that is, the gain of a speculator is zero on average [6]. Later, this theory has been developed and is known as the *Efficient Market Hypothesis* (EMH) [7] : a market is efficient if there are no *arbitrages*, i.e. possibilities of predictable gains. A market is efficient if three conditions are met :

1. All agents are rational and maximize their gain. They also know that all other agents are rational and gain maximizers.
2. Information is easily and immediately available.
3. There are no transaction costs.

The first assumption is very common in Economics. If it is true, all the sophisticated tools and concepts of Game Theory can be applied. However, it is more and more debated (see below) [13, 14]. In the second assumption, the term “information” is not precisely defined : there are many kinds of information, so that EMH is often found under three forms :

**Weak** All the public information on past prices and volumes is reflected on the current price at every time.

**Intermediate** All public information of any kind (earnings, political news, ...) is included in the price at every time.

**Strong** All kinds of information, including secret information are fully reflected on the price at every time.

Empirical evidences show that the strong form is too strong and incorrect [8] ; it is also not very well defined. The intermediate form is under discussion amongst economists and practitioners, whereas the weak form is often believed to be true by many economists. However, there are many facts that contradict even the weakest form of EMH. The best-known and persistent one is the *January effect* : during this month, stocks generate abnormally high returns. In addition, empirical studies [9, 12] show that there are systematic correlations in most financial markets<sup>3</sup>. Economists call such facts *small anomalies* without significance. But these anomalies are quite frequent<sup>4</sup> (see [8]).

It is worth reviewing some methods used by practitioners, and which form of EMH they rely on. The *Portfolio Theory* optimizes the composition of a

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<sup>3</sup>However, transaction costs may be an obstacle to exploit them.

<sup>4</sup>We let the reader to judge if they are significant.

portfolio ; its standard form is based on the assumption that the market are efficient<sup>5</sup>. On the other hand, *Technical Analysis* studies time series of old prices in order to find patterns like trends, cycles, and other regularities, and predicts future prices. This yields gains if the weak form of EMH is not verified. Furthermore, *Fundamentals Analysis* tries to find differences between the actual value of an asset and its “intrinsic” value . This implies the break of the intermediate form. All these techniques are used by practitioners, but they are recommended in three different market regimes, thus it does not make sense to use them simultaneously. However, they coexist and are all used. This shows at best that the efficiency issue is not simple and that nobody has a definitive answer. Even worse, nobody agrees on how to measure efficiency.

### 1.3 Fluctuations

Another very important quantity in financial markets is prices’ fluctuations. It is known that the fluctuations are *clustered* in time, that is, if a stock price has large fluctuations at a given time, it is very probable that these large fluctuations will last a while, and then decrease. Fluctuations are synonyms of risk for investors, since they are related to the uncertainty of a price.

### 1.4 Rationality–Bounded Rationality–Induction

The most important hypothesis of EMH is that all agents are rational gain maximizers. This supposes that they have access to all the information they need in order to *deduce* what is the best decision to take, and is mostly unrealistic.

First, the instantaneous access to all valuable information may not be not achieved in most cases, even with the help of computers, so that one often has to deal with uncertain or incomplete information.

Nevertheless, suppose that information is not an issue. The major problem is the assumption of rationality and deduction. In practice, humans cannot cope with too much information (complete or not) : there have *bounded rationality* [13, 14]. Information processing capabilities supposed by a rational behavior are often well beyond human limits [14]. Therefore, humans do not look for *optimal* choices, but *satisfactory* choices [13]. Instead of being deductive, humans are rather *inductive*, that is, they constantly build new hypothesis, test them, discard those who turn out to be wrong, and trust

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<sup>5</sup>Or marginally efficient [10, 11]

those that appear to be correct [14]. For instance, pattern recognition is an inductive task, while criminal investigations *à la* Colombo is a deductive task. When a situation is either too complicated or requires a too fast decision for their deduction capabilities, humans switch to induction.

Suppose nevertheless that ONE person only has a gift for deduction. He cannot ever take a rational decision, because other people do not behave rationally. Therefore, most problems in Economics are ill posed, and bounded rationality, like induction, is more suited than rationality for solving them.

## 1.5 The El Farol's Bar Problem

In order to illustrate ill posed problems in Economics, W. Brian Arthur introduced his famous El Farol's bar problem. Going to this bar is not enjoyable if more than sixty people — the comfort level — are simultaneously present. Suppose that there are more than sixty people wishing to go to the bar, and that they do not know each other. Some relevant information about the system is the past attendance, but it may be not enough to take a rational decision ; in addition, since agents do not know each other, they cannot be sure that other agents will be rational. Note that heterogeneity amongst customers is mandatory : if all potential customers analyze the situation in the same way, all will lose.

Induction can be used : customers have their own mental schemes, or behavior rules (strategies), which they use as attendance predicting devices. They keep track of the performance of the latter and use the one with the highest score. If the device they use predicts an attendance lower than the comfort level, they decide to go to the bar, and reversely. Surprisingly, this model is able to reproduce an attendance fluctuating around sixty using Arthur's behavior rules<sup>6</sup> [14], which is a kind of equilibrium.

This model raised a lot of interest in the Economics community. Many authors tried to understand the model and published papers with evocative titles such as "Clarifying the El Farol's problem" [15] ; some authors claim to have found rigorous results [16]. One reason of the encountered difficulties is that people do not interact directly among themselves, but through a common closed environment which they create themselves.

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<sup>6</sup>For instance, the predicted next attendance is the same as last week's, a mirror image around 50 of last week's, a constant one, a (rounded) average of the last four weeks, etc ...



# Chapter 2

## The Minority Game

Several years after its introduction, the El Farol's bar problem remained mysterious. In particular, it had been realized that standard tools of Game Theory were of little use for the strategies Arthur defined. Casti beg for a better formulation and appropriate mathematical tools for what he called *the most important problem in complex systems theory* [18]. What was missing was a good parameterization of the behavior rules. Indeed, there is no systematic way of deriving rules given by Arthur (who gave them only as examples). Only recently (up to our knowledge) this was done by economists [19], who curiously did not undertake a systematic study of Arthur's model ; they failed to reproduce Arthur's results, and concluded that this model is "suspect". This does not reflect a better understanding of why Arthur's agents seems to reach an equilibrium.

### 2.1 From El Farol to MG

When simplifying the El Farol problem, one has to keep its two key features, namely

- **Frustration** : not all persons can win at the same time, even if they are willing to. This makes the model non trivial, and keeps the competition going.
- **Inductive agents** : agents have strategies and strategies have scores which evolve in time. This makes agents adaptive.

Hundred agents competing for sixty seats is a rather specific definition<sup>1</sup>. A more general definition is  $N$  agents and  $L$  seats. If agents behave in such a

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<sup>1</sup>Which, again, was only though as an example by Arthur.

way that the attendance fluctuate around  $L$ , they reach a kind of equilibrium. But the fluctuations around the comfort capacity are in themselves very interesting : the larger they are, the less optimally agents behave. So the average attendance determines the overall capacity of agents to learn a given comfort level while the fluctuations around the comfort level measure the strength of cooperation. Generally speaking, the fluctuations are a direct consequence of competition.

Symmetry is a concept of particular importance for physicists. In the El Farol's bar problem, both decisions are not *a priori* equivalent, hence, the problem is not symmetric. The idea is to first study and understand the symmetric problem, and then to generalize this knowledge to asymmetric situations. Building the symmetric El Farol's problem is obtained by setting a comfort capacity of  $L = N/2$ . Agents wins if their choice is that of *minority*.

## 2.2 No information — Nash equilibria

If a minority game is played once, the Game Theory allows the study of the best choices to be taken for agents. When all agents behave in such a way that if they deviate from their current behavior they win less, a Nash equilibrium is attained [17]. For a minority game, the only rational behavior consists of taking random decisions. Of course, this is not very satisfactory and one has to give information to agents in order to obtain a non trivial behavior.

## 2.3 Information

If the game is repeated, at least one agent may think that past attendance is of valuable information, and that next attendance depend on previous ones<sup>2</sup>. That is how people behave in the El Farol's bar problem : Arthur's behavior rules predicts an attendance  $A(t+1) \in \mathcal{N} = \{0, \dots, N\}$  from the knowledge of the attendance of past days  $\{A(t), A(t-1), \dots\} \in \mathcal{N}^t$ . These are mathematical functions from an exponentially growing set into a constant set ; with such a setup, there are  $N^{N^t}$  such functions. One can remove the time dependence by supposing that agents do not remember or take into account attendance older than  $M$  time steps ( $M$  stands for memory), so that there remain  $N^{N^M}$  possible behavior rules. From agents' point of view, the only relevant prediction is whether the bar will be crowded or not, that is,

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<sup>2</sup>If he acts thinking so, then next attendance will indeed depend on previous attendance, just because he thinks so.

there only are interested in a binary prediction. There are  $2^{N^M}$  such behavior rules, which is still a very large number.

The last step leading to the Minority Game is to simplifying the information given to agents : indeed, it is responsible to the factor  $N^M$  in the number of behavior rules. The key observation is that it does not make much sense to base binary decisions on highly detailed information. From now, the common information given to all agents will be the  $M$  last correct decisions, i.e. the last  $M$  decisions of the minority. There are consequently  $2^M$  possible pieces of information, and  $2^{2^M}$  different behavior rules.

## 2.4 The Minority Game

In this section the Minority Game (MG) [20] is defined *ab initio* : at a given time, some people have two choices, say,  $A$  or  $B$  ; they take their decisions simultaneously, without any kind of communication between them ; those who happen to be in minority win. This is exactly the inverse of a democratic vote. Indeed, in a minority game, it is not the interest of any agent to behave in the same way than the herd and there is no prior communication. The minority mechanism is a way to implement competition for a scarce resource (the maximum number of winners is limited). Agents are faced to two main types of information :

- **Public information** : for instance, the “state of the world”. The most basic global information about the system is which choices were successful in the past. The precise attendance is also a valuable source of information, although maybe too detailed.
- **Private information** : Adaptive agents are able to trace how well they perform and to modify their behavior when needed.

Adaptive agents have global information processing devices<sup>3</sup> that transform the global information into a behavior. Mathematically speaking, such a device is function  $a$  from the set of all possible global pieces information  $I$  into the set of available choices  $C$ . Thus the choice of the kind of global information is crucial for the model. For the sake of greatest simplicity possible, the agents are given only the set of the right choices. It does not make sense to give access at one time to all previous right choices, thus an arbitrary limit is given : only the last  $M$  last winning choices are accessible. With this particular choice, the global information can be encoded by a string of  $M$  bits,

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<sup>3</sup>In the following, we shall call it simply strategies.

0 standing for “ $A$  was winning” and 1 for “ $B$  was the right choice” (or the reverse), and there are  $P = 2^M$  possible information ; furthermore, a strategy is simply a lookup table with  $P$  rows (see table below for an example)

signal	choice
000	1
001	0
010	0
011	1
100	1
101	0
110	1
111	0

Such a function can be thought as a “theory of the world” : to every “state of the world” corresponds a fixed choice, i.e. an *a priori belief*. At this point, there are several ways of implementing adaptation

- **One variable strategy** Each agent gets one randomly drawn strategy at the beginning of the game and is able to modify it during the game.
- **Several constant strategies** Each agents gets a set of strategies, keep track of their performance and use at each time step the one that seems to be the most adequate.

The first possibility belongs to the class of genetic algorithms and may appear the most natural, but it requires *a priori*  $P$  registers in order to keep track of the performance of the strategy for each possible information, while the second one only needs one register per strategy, therefore it is more economical to give  $S$  strategies per agent.

Finally the way the agents perceive the success of their strategies is fixed in the following way : after each time steps, agents add a point to the score of each of her strategies that have predicted correctly what action to take, and subtract one point to the others. This setup is maybe conceptually the simplest payoff possible and entails that agents do not have any way to know the precise outcome of the games.

## 2.5 Algorithm

The minority game is a closed dynamical system, hence, any formal description has to be self-consistent, looking maybe complicated.



$N$  agents play at each time step  $t$  a minority game (MG). They base their decisions on the common pieces of information  $\mu(t) \in P$  where  $P$  is the set of all possible piece of information. In order to react to the latter, they have  $S$  strategies (behavior rules)  $a_{i,s}$ ,  $i = 1, \dots, N$ ,  $s = 1, \dots, S$ , which are a function from the set  $P$  into the set of possible actions ; here, the possible actions are  $-1$  and  $+1$ , hence  $a_{i,s}^\mu = -1$  or  $+1$ . Each strategy  $a_{i,s}$  is given a score  $U_{i,s}(t)$  which aims at reflecting their perceived success rate. At time  $t$ , agent use the strategy  $s_i(t)$  according to

$$s_i(t) = \arg \max_{s=1, \dots, S} U_{i,s}(t) \quad (2.1)$$

that is, the one with the highest score, and they take the decision given by this strategy, namely  $a_i(t) = a_{i,s_i(t)}^\mu$ . The decisions of all agents are aggregated into the global quantity

$$A(t) = \sum_i a_i(t) \quad (2.2)$$

The history  $\mu(t)$  is updated into

$$\mu(t+1) = [2\mu + (1 + \text{sgn}A(t))/2] \text{ MOD } P \quad (2.3)$$

and the reward

$$-a_i(t)\text{sgn}[A(t)] = \pm 1 \quad (2.4)$$

is given to agents  $i = 1, \dots, N$ . In turn, the latter update the score of their strategies according to

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}\text{sgn}[A(t)] \quad (2.5)$$

Eqs (2.1), (2.2), (2.3) and (2.5) completely define the model, whose dynamical quantities are  $s_i$ ,  $U_{i,s}$  and  $\mu$ . Note that Eq. (2.4) encode the minority mechanism ; any odd<sup>4</sup> function encodes the minority mechanism as well. In the following, we shall generalize this equation to more general payoff function  $G(x)$ , and shall focus on  $G(x) = x$ , for practical reasons.

## 2.6 Macroscopic properties

Since the MG has only three parameters  $N$ ,  $M$  and  $S$ , it is very suitable for systematic study and analysis. In principle, any macroscopic scalar quantity only depends on these three parameters.

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<sup>4</sup>In principle, any function  $G$  such that  $xG(x) > 0$  would be acceptable ; here we consider only odd functions for symmetry reasons.

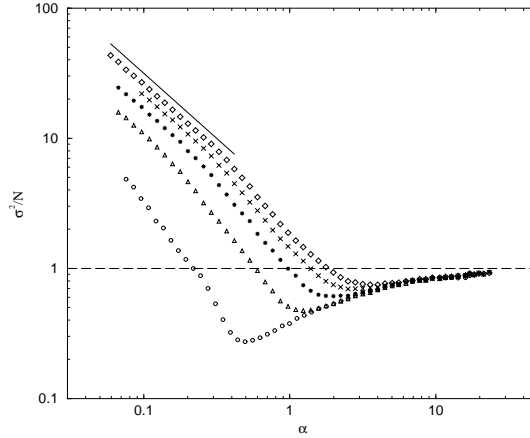


Figure 2.1: Normalized fluctuations versus the control parameter  $\alpha = 2^M/N$  for  $S = 2$  (circles),  $S = 3$  (triangles),  $S = 4$  (stars),  $S = 5$  (x) and  $S = 6$  (diamonds) ( $M=8$ , average over 50 samples). The straight line is  $\propto 2^M/N$ .

### 2.6.1 Cooperation

One striking property of this model is the fact that agents cooperate. A relevant measure of cooperation is provided by the fluctuations of attendance  $A(t)$

$$\sigma^2 = \langle A^2 \rangle \quad (2.6)$$

where the average is taken over the time. Agents taking random decisions would produce fluctuations equal to  $N$ , so that agents cooperate if they manage to produce fluctuations lower than  $N$ . Figure 2.1 shows normalized fluctuations measured at fixed  $M$  and  $S$  versus  $\alpha = 2^M/N$  for various  $S$  [21, 22]. For fixed  $S$ , one can distinguish *grosso modo* three regions :

- **Small  $\alpha$**  : Agents display a herding behavior and produces non-Gaussian fluctuations  $\sigma^2 \sim N^2$ . This region has been called *crowded* region since it is reach by keeping  $M$  constant and increasing  $N$ .
- **Intermediate  $\alpha$**  : Best coordination is achieved.
- **Large  $\alpha$**  : Coordination slowly disappears and the variance of the outcome tends to the one that would be produced by agents taking random decisions.

Savit *et al.* noted that if  $S = 2$ ,  $\sigma^2/N$  versus  $2^M/N$  collapse on the same curve for any  $M$  and  $N$  [21]. This is true for any  $S$  at fixed  $S$  [22], and is a

very interesting observation : it implies that  $2^M/N$  is the *control parameter* of the MG.

When  $S$  is varied, the crowded region moves to the right, whereas  $\sigma^2/N$  for  $N \ll 2^M$  seems to collapse on roughly the same curve. The minimum of  $\sigma^2/N$  is less and less pronounced when  $S$  is larger, and its position is linearly dependent on  $S$ . This suggest the existence of a control parameter that makes fluctuations have their minimum at the same abscissa and collapse on the same curve  $\propto N/P$  for small  $\alpha$ .

At first look, emergence of cooperation may be strange, since agents are selfish and faced to a competitive situation. It took almost two years before this could be explained [24]. But agents behave very badly for small  $2^M/N$ , and this indicates that cooperation does not arise in all circumstances.

### 2.6.2 Predictability – Phase transition

The next major important quantity is the predictability of the next minority choice, particularly with respect to the controversy about financial markets and EMH. In the MG, there are three kinds of information. The first one is the histories, which are public common pieces of information encoding the previous  $M$  last minority choices. Another kind of information agents receive is the rewards. For them, it is the only way of learning something about the game they are playing. When the game is repeated, agents accumulate information about the system, and encode them into the scores of their strategies. The latter therefore contain valuable information about the performance of each of their strategies and, in principle, about the system. In principle means “if the next minority choice is predictable”.

Because of the symmetric design of the MG,  $\langle A \rangle = 0$ , hence, on average, at first sight the sign  $\chi(t+1)$  of the outcome  $A(t+1)$  is not predictable. But agents behave conditionally : they know at each time step the history of the game ; as a consequence, one has to consider the time average of  $A(t+1)$  conditional to  $\mu(t) = \nu$ . The next minority choice is predictable if the conditional average  $\langle A|\nu \rangle \neq 0$ . A measure of the predictability is

$$\Theta = \lim_{T \rightarrow \infty} \sum_{\nu=1}^P \rho^\nu \left[ \frac{1}{T} \sum_{t=1}^T \chi(t) \delta_{\nu, \mu(t)} \right]^2 = \sum_{\nu=1}^P \rho^\nu \langle \chi|\nu \rangle^2 = \overline{\langle \chi \rangle^2} \quad (2.7)$$

where  $\rho^\nu = T^\nu/T = 1/T \sum_{t=1}^T \delta_{\nu, \mu(t)}$  is the frequency of history  $\nu$ . Figure 2.2 shows that for small  $\alpha$ ,  $\Theta = 0$ . At the point where the fluctuations are minimal ( $\alpha_c \simeq 0.3$ ),  $\Theta$  begins to differ from zero and then monotonically grows [21, 24]. Therefore, for  $\alpha > \alpha_c$ , the outcome is predictable. This predictability is related to asymmetries in the outcomes. In language of sta-

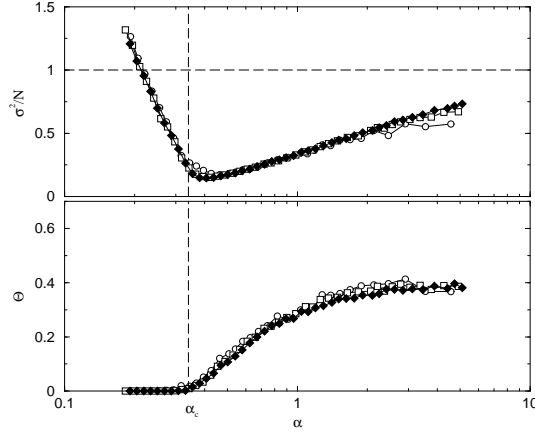


Figure 2.2: Above : Normalized fluctuations versus  $\alpha = 2^M/N$  ; below : order parameter  $\Theta$  ; ( $M = 5$  (circles),  $M = 6$  (squares),  $M = 8$  (diamonds) (300  $P$  iterations, average over 200 samples))

tistical mechanics, MG undergoes a phase transition with symmetry breaking as  $\alpha$  is varied, and  $\Theta$  is an order parameter of the system. Indeed, for  $S = 2$  and  $\alpha < \alpha_c = 0.3374..$ , the outcome is symmetric and this phase is called *symmetric* ; on the other hand,  $\Theta \neq 0$  for  $\alpha > \alpha_c$  : this is the *asymmetric phase*.

An asymmetry means that the outcome is probabilistically predictable, thus one expects that adaptive agents could exploit the asymmetry, reduce it and finally completely remove it, making the game *efficient* for agents, as supposed by the EMH. This is achieved if there are enough agents ( $N > N_c = P/\alpha_c$ ). Indeed, even if the number of agents is fixed in the MG, the presence of available information would attract more agents until information disappears. Therefore, at the simple level of the MG, one apparently recovers the weak form of EMH. However, the MG is never efficient, except at the critical point (see next subsection).

### 2.6.3 Persistence – anti-persistence

The order parameter  $\Theta$  introduced above is not sufficient to describe the statistical properties of the canonical MG. Indeed, although it gives a good overall description of the asymmetric phase, it tells nothing about the dynamics of the symmetric phase. Another kind of information is needed. The right one is given by the correlation between the outcome at time  $t$  and that at time  $t + \delta t$  *conditional to the history* [24]

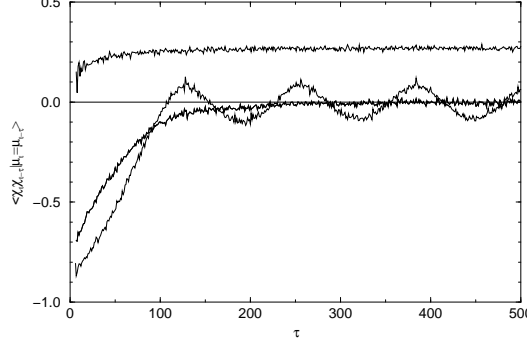


Figure 2.3: Temporal correlation of  $\chi_t$  on the same history,  $\langle \chi_t \chi_{t+\tau} | \mu_t = \mu_{t+\tau} \rangle$ , averaged over all histories versus  $\tau$  ( $10^6$  iterations,  $M = 6$ ,  $\alpha = 0.5, 0.22, 0.1$ )

$$\overline{\langle \chi(t) \chi(t + \delta t) \rangle} = \sum_{\nu=1}^P \rho^\nu \langle \chi(t) \chi(t + \delta t) | \mu(t) = \mu(t + \delta t) = \nu \rangle \quad (2.8)$$

As reported by Fig 2.3, the system behaves very differently between the different phases. In the symmetric phase, the system behaves in a *periodic* and *anti-persistent* way<sup>5</sup>. The period is  $2P$ , meaning that agents tend to do the opposite of what they did when the history was the same.

The asymmetric phase is characterized by *persistence* : agents tend to repeat their last action. The critical point is the only point where the system has no long time correlations, in other words the MG is arbitrage free only at the critical point. Therefore, the game *seems* efficient to agents in the symmetric phase, because they cannot detect anti-persistence with their simple strategies, but actually is not efficient. A better equipped agent outsmarts agents who cannot profit from anti-persistence (see chapter 5). These findings are confirmed, although under another form, by ref. [25], in which the mutual information in the minority sign  $\chi(t)$  is studied.

#### 2.6.4 Unused strategies—freezing

In the asymmetric phase, since the outcome is predictable, there is *a priori*<sup>6</sup> a best strategy  $a_{\text{best}}^\mu = -\text{sgn} \langle \chi | \mu \rangle$ , and one expects agents to use preferably strategies that are the most anti-correlated with  $\langle \chi | \mu \rangle$ . In other words, in the stationary state, agents use only  $n$  strategies amongst their  $S$  strategies.

<sup>5</sup>The behavior of each agent ( $s_i(t)$ ) is also anti-persistent in this phase.

<sup>6</sup>i.e. for an agent standing outside of the game.

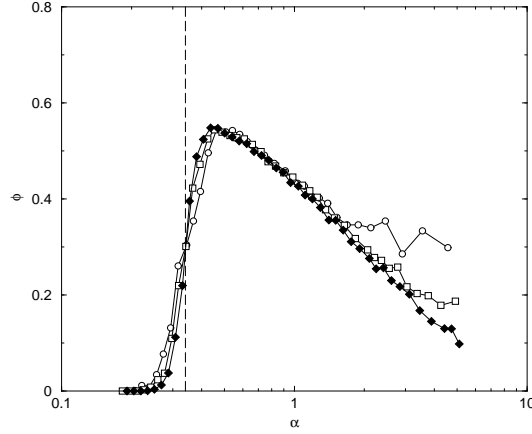


Figure 2.4: Fraction of frozen agents versus  $\alpha$  for various system sizes ( $M = 5$  (circles),  $M = 6$  (squares) and  $M = 7$  (diamonds) (average over 200 samples). It allows to locate rather precisely the critical point.

For instance, for  $S = 2$ , some agents always use the same strategy : there are *frozen*. The fraction of frozen agents  $\phi$  is reported in Fig. 2.4 : it is maximal at the critical point and exponentially decreases when  $\alpha$  increases. Below the critical point, due to finite size effect, some agents are also frozen. This gives a powerful tool to locate precisely the critical point. Indeed, on figure 2.4  $\phi$  plotted for various  $M$  crosses at one point, located at  $\alpha \simeq 0.34$ , which is indeed a good approximation to the exact value ( $\alpha_c = 0.3374\dots$ ).

For  $S > 2$ , one measures  $1 - n(\alpha)/S$ , and also finds the same behavior as that of  $\phi$  for  $S = 2$ .

## 2.7 Geometrical interpretation of the cooperation

At this stage of the study of MG, it is often very hard to understand the rich behavior of the model. In particular, two questions are important :

- How can cooperation arise since an agent do not know anything about other agents, and nothing at all about the game itself ?
- Why  $P/N$  is the control parameter ?

Before exposing the exact analytical solution, it is useful to acquire its own intuitive understanding of the (non)-cooperative behavior of agents. A geometric approach is well suited for this task.

Strategies can be seen as vectors of  $2^M$  components, belonging to  $\mathcal{H}_P$ , the hypercube of dimension  $P$ . Even if it is very hard to have a proper representation of a hypercube of dimension  $P$ , there is at least one notion that is commonly intuitively understood : the distance. In hypercubes, the appropriate distance is the Hamming distance : if  $a$  and  $b$  belong to  $\mathcal{H}_P$ , their hamming distance is defined as

$$d(a, b) = \frac{1}{4P} \sum_{\mu=1}^P (a^\mu - b^\mu)^2 \quad (2.9)$$

and is the fraction of components that differ between  $a$  and  $b$ . There are three interesting special cases :

- if  $d(a, b) = 0$ ,  $a$  is *fully correlated* with  $b$ , i.e.  $a = b$ ,
- if  $d(a, b) = 1/2$ ,  $a$  is *uncorrelated* with  $b$ , and
- if  $d(a, b) = 1$ ,  $a$  is *anti-correlated* with  $b$ , i.e.  $a = -b$ .

Note that  $d(a, b)$  is also the probability that  $a^\mu = b^\mu$  for a random  $\mu$ . Therefore, the gain of an agent is directly related to the average distance from other agents she<sup>7</sup> can achieve. Since large fluctuations are synonyms of resource waste, or large losses for agents, there must be a relationship between the fluctuations and the distance. Actually, there is a complete equivalence between these two quantities. Indeed, the average distance between agents is given by

$$\langle d \rangle = \frac{1}{N(N-1)} \sum_{i \neq j} \langle d(a_i, a_j) \rangle, \quad (2.10)$$

while the fluctuations can be written as a sum of random outcome plus correlations among agents

$$\sigma^2 = N + \sum_{i \neq j} \langle a_i a_j \rangle. \quad (2.11)$$

Combining both equations yields to

$$\frac{\sigma^2}{N} = 1 - 2(N-1) \left[ \langle d \rangle - \frac{1}{2} \right] \quad (2.12)$$

This equation shows that the fluctuations and the distance are equivalent. Hence, one can interpret the (non)-cooperative behavior as a geometrical property. In general, it is not possible to calculate explicitly the average distance [26].

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<sup>7</sup>We consider agent alternatively as a masculine and feminine word.

## 2.8 The Reduced Set of Strategies (RSS)

The previous subsection gives a first way to better understand the nature of the fluctuations in term of the average distance. The problem is now to find a way to evaluate the average distance. One answer comes from the Reduced Set of Strategies (RSS). Suppose that  $M$  is not too small<sup>8</sup> and consider one strategy  $a$  ; change one bit of  $a$  and call the result  $b$ . Obviously  $a$  and  $b$  are almost the same. This raises the following question : how much strategies are really different ?

Of course there is a strategy that is definitively very different from  $a$  : its inverse  $-a$ , but in between there are many other strategies. We give an arbitrary criterion : two strategies  $a$  and  $b$  are said “significantly different” if  $d(a, b) \geq 1/2$ . Going even further, one can introduce the RSS notion : a RSS in hypercube  $\mathcal{H}_P$  is a maximal<sup>9</sup> set  $V_P$  whose all strategies are pairwise *either uncorrelated or anti-correlated*<sup>10</sup>. The central property of a RSS is :

- if  $a$  and  $b$  are uncorrelated,  $d(a, b) = 1/2$  by definition, and  $d(a, -b) = d(-a, b) = d(-a, -b) = 1/2$

As a consequence, a RSS is the union of a set  $U_P$  of pairwise uncorrelated strategies and of the set  $\overline{U}_P$  of the opposite strategies, which are also pairwise uncorrelated according to the central property of RSS.

### 2.8.1 How to build a RSS

Since a RSS  $V_P$  is actually the union of  $U_P$  whose all elements are mutually uncorrelated and of  $\overline{U}_P$ , the set that contains the inverses of all elements of  $U_P$ , it suffices to build  $U_P$ .

Let us describe in details the structure of  $U_P$ . First, it is easy to see that for fixed  $P$ ,  $U_P$  is not unique : for instance, take  $P = 1$  : there are two possible  $U_P$ , namely  $U_1 = \{0\}$  and  $U'_1 = \{1\}$ , that makes only one RSS. In order to show that  $V_P$  is not unique, one has to go to  $P = 2$ , where there are four  $U_P$ , and two possible  $V_P$ .

There is a very simple method for building  $U_{2P}$  from  $U_P$ . If  $a$  and  $b \in U_P$ , one can define the direct product  $c = a \otimes b$  as

$$c^\mu = \begin{cases} a^\mu & \text{if } 1 \leq \mu \leq P \\ b^{\mu-P} & \text{if } P < \mu \leq 2P. \end{cases} \quad (2.13)$$

---

<sup>8</sup> $M \geq 5$  is large enough

<sup>9</sup>Maximal means that it cannot contain more elements having those properties.

<sup>10</sup>Of course for fixed  $P$  there are many RSS, as we shall demonstrate.



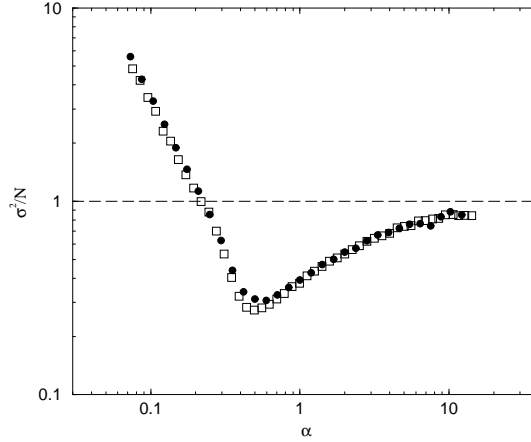


Figure 2.5: Normalized fluctuations versus the control parameter  $\alpha = 2^M/N$  for the RSS (circles) and the whole set of strategies (squares) ( $M = 8, S = 2$ )

that is,  $c$  is made by appending the components of  $b$  to those of  $a$ . It is easy to convince oneself that the elements of  $U_{2P}$  obtained by the algorithm

- for all  $a$  in  $U_P$ , put  $a \otimes a$  and  $a \otimes (-a)$  in  $U_{2P}$

are mutually uncorrelated if all elements of  $U_P$  are mutually uncorrelated. Finally, the RSS  $V_P$  is simply  $U_P \cup \overline{U_P}$ .

There are generally many RSS, but given one strategy, there is only one RSS that contains it. Therefore, the total number of RSS for a given  $M$  is given by  $2^P/(2P)$ . The method described above gives the RSS that contains the strategy  $a^\mu = 1$  for all  $\mu \in P$ , but it is easy to build the RSS  $V'_P$  that contains any strategy, say,  $c$ : for all  $a \in V_P$ ,  $a'$  such that  $a'^\mu = a^\mu c^\mu$  belongs to  $V'_P$ . It can be shown that the construction method given above yields to maximal size RSS ( $2^{M+1}$ ) [22].

### 2.8.2 RSS in action

The best way to test the validity of the RSS concept is to carry numerical simulations. Fig 2.5 illustrates that the macroscopic behavior of the MG is not changed if the agents are forced to draw their strategies from a RSS<sup>11</sup>.

RSS gives a very intuitive (but wrong) interpretation of the control parameter, which was numerically found to be  $\alpha = P/N$ . Indeed, since there are  $2P$  elements in  $V_P$ ,  $SN/(2P)$  is the fraction of the strategy set that

<sup>11</sup>Finite size effects are more pronounced when agents draw their strategies from the RSS.

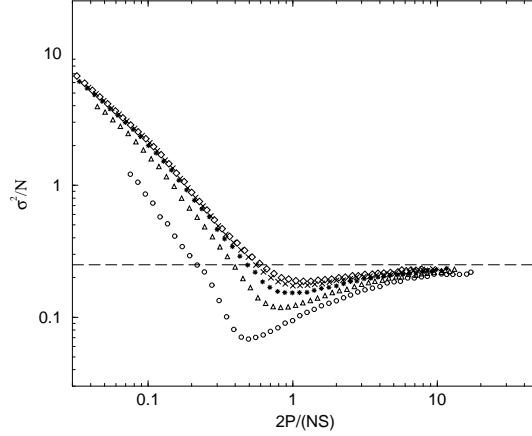


Figure 2.6: Normalized fluctuations versus the RSS control parameter  $\alpha_{\text{RSS}} = 2^{M+1}/(NS)$  for  $S = 2$  (circles),  $S = 3$  (triangles),  $S = 4$  (stars),  $S = 5$  (x) and  $S = 6$  (diamonds)

is sampled by the agents, or more generally the average number of time a strategy has been drawn by agents, thus RSS predicts that the RSS control parameter is<sup>12</sup>  $\alpha_{\text{RSS}} = 2P/(NS)$ . Figure 2.6 shows that it is valid for small  $\alpha$  and large  $S$ . Hence, it is intuitive, but not completely satisfactory.

### 2.8.3 RSS : analytical results

RSS is a valuable tool for an intuitive analytical understanding of cooperation and herding effects in the MG. If all agents draw their strategies in a RSS, the average distance is simply

$$\langle d \rangle = \frac{1}{2} + \frac{\langle n_a \rangle - \langle n_s \rangle}{2} \quad (2.14)$$

where  $\langle n_s \rangle$  is the averaged fraction of agents using the same strategy and  $\langle n_a \rangle$  is the averaged fraction of agents using anti-correlated strategies. Johnson *et al.* [27] call these quantities *crowds*, respectively *anti-crowds*. Combining Eqs (2.12) with (2.14) yields

$$\sigma^2/N = 1 + (N-1)(\langle n_s \rangle - \langle n_a \rangle) \quad (2.15)$$

Eq. (2.15) shows that agents do not interact with others unless they are correlated with others agents. It confirms the intuitive feeling that agents using the same strategy are responsible for the herding effect, that is, the

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<sup>12</sup>Remember that this makes sense only for the symmetric phase

huge fluctuations, and anti-correlated agents reduce the fluctuations, thus are the cause of the cooperation. For small  $\alpha$ ,  $\langle n_s \rangle$  dominates  $\langle n_a \rangle$ , and reversely for  $\alpha > O(1)$ .

The RSS concept allowed to find that  $\sigma^2/N \propto 1/\alpha$  for large  $\alpha$  [20], and finally Johnson *et al.* [27] found an approximated expression for  $\sigma^2/N$  by the following reasoning : at each time step, strategies can be sorted by their score. By finding how many of agents whose best strategy is the  $k$ -th on the list, they could find a reasonable approximation to  $\sigma^2$ , namely  $\sigma^2/N = 1 - 1/(2\alpha)$  for large  $\alpha$ , and  $\sigma^2/N = 1/(24\alpha)[1 - 1/(4P^2)]$  for small  $\alpha$ . Note however that this approach only concerns the fluctuations (up to now), which are not the only relevant quantity of the MG. However, it is highly appreciable and helps to build one's own understanding of MG's behavior.

## 2.9 MG as a spin glass

As first noted by Savit *et al.* [21],  $\sigma^2/N$  depends on the precise realization of the MG, that is, of the composition of each agent's set of strategies. This behavior is analogous to that of spin glasses. And indeed, a microscopic mathematical formulation leads to expressions for macroscopic quantities that are similar to spin glass Hamiltonians [24]. The analogy is more striking for the  $S = 2$  case, because the formalism is somewhat lighter. Therefore, in the following, we shall always begin with this case, and afterwards generalize the discussions to any  $S$ . For mathematical reasons, we will mainly consider the linear payoff function  $G(x) = x$ <sup>13</sup>. Note that with this choice, the fluctuations are simply the total losses of agents per time step :

$$\sigma^2 = \overline{\langle A^2 \rangle} = \sum_{i=1}^N \overline{\langle a_i A \rangle} = - \sum_{i=1}^N g_i \quad (2.16)$$

where by definition  $g_i = -\overline{\langle a_i A \rangle}$  is the gain rate of agent  $i$ . For  $S = 2$ , in addition to the symmetry between the two choices<sup>14</sup>, one can exploit the symmetry between the two strategies that each agent possesses. Instead of  $s_i = 1, 2$ , one relabels the strategies by  $s_i = -1, 1$ , or with a more graphical notation,  $s_i = \downarrow, \uparrow$ . This choice underlines at best the spin nature of the model. In addition since the strategies are independently drawn, given a history, with probability 1/2, both strategies of an agent stipulate the same decision, thus one can separate the constant part  $\omega_i^\mu$  of agent  $i$  decisions from their variable component  $\xi_i^\mu$ , so that

<sup>13</sup>The overall structure of the MG is not changed by particular choices of payoff function.

<sup>14</sup>Which is already exploited by labelling them  $-1$  and  $+1$

$$a_{i,s}^\mu = \omega_i^\mu + \xi_i^\mu s \quad (2.17)$$

where

$$\omega_i^\mu = \frac{1}{2} (a_{i,\uparrow} + a_{i,\downarrow}) \quad \text{and} \quad \xi_i^\mu = \frac{1}{2} (a_{i,\uparrow} - a_{i,\downarrow}) \quad (2.18)$$

Note that  $\omega_i^\mu$ , the constant part, and  $\xi_i^\mu$ , the variable part, are not independent : only one is different from zero given a history  $\mu$ . The symmetry between the two strategies is further exploited by introducing

$$\Delta_i(t) = U_{i,\uparrow}(t) - U_{i,\downarrow}(t) \quad (2.19)$$

At time  $t$ , agent  $i$  use the strategy  $s_i(t)$  given by

$$s_i(t) = \text{sgn} \Delta_i(t) \quad (2.20)$$

The constant decisions of all agents have an influence on  $A$  : it also consists of a constant part and a variable one

$$A^{\mu(t)}(t) = \sum_i^N [\omega_i^{\mu(t)} + \xi_i^{\mu(t)} s_i(t)] = \Omega^{\mu(t)} + \sum_i^N \xi_i^{\mu(t)} s_i(t) \quad (2.21)$$

This equation already gives the key element which is at the origin of the behavior the model. The constant part  $\Omega^\mu$  of  $A$  is a bias, and induces an *a priori asymmetry* in the outcome. When the model is in the symmetric phase, agents are able to remove this asymmetry. In the asymmetric phase, this is no more possible for them, and they reach a stationary state where they cooperate. One concludes that cooperation itself is due to symmetry breaking itself due to  $\Omega^\mu$ , that is, to the fact that for each history  $\mu$ , there are agents who always take the same decision. Note that in the asymmetric phase,  $\langle A | \mu \rangle$  is completely correlated with  $\Omega^\mu$ . The order parameter  $\Theta$  is a measure of the asymmetry induced by the  $\Omega$ s, which is convenient for a  $\text{sgn}$  payoff function, but for a linear payoff function, one prefers to use

$$H = \sum_{\mu=1}^P \rho^\mu \langle A \rangle^2 = \overline{\langle A \rangle^2} \quad (2.22)$$

Such functions measure the presence of available information, which can be detected by agents' strategies and accordingly exploited by them. With this notation, one can explicitly expand  $\sigma^2$  and  $H$

$$\sigma^2 = \overline{\Omega^2} + \sum_{i=1}^N \overline{\xi_i \Omega} \langle s_i \rangle + \sum_{i,j=1}^N \overline{\xi_i \xi_j} \langle s_i s_j \rangle \quad (2.23)$$

$$H = \overline{\Omega^2} + \sum_{i=1}^N \overline{\xi_i \Omega} \langle s_i \rangle + \sum_{i,j=1}^N \overline{\xi_i \xi_j} \langle s_i \rangle \langle s_j \rangle = \overline{\langle A \rangle^2} \quad (2.24)$$

Defining

$$h_i = \overline{\Omega \xi_i} \quad \text{and} \quad J_{i,j} = \overline{\xi_i \xi_j} \quad (2.25)$$

one obtains

$$\sigma^2 = \overline{\Omega^2} + \sum_{i=1}^N h_i \langle s_i \rangle + \sum_{i,j=1}^N J_{i,j} \langle s_i s_j \rangle \quad (2.26)$$

$$H = \overline{\Omega^2} + \sum_{i=1}^N h_i \langle s_i \rangle + \sum_{i,j=1}^N J_{i,j} \langle s_i \rangle \langle s_j \rangle \quad (2.27)$$

The field  $h_i$  measures the difference of correlation of the two strategies with  $\Omega^\mu$  whereas the coupling  $J_{i,j}$  accounts for the interaction between agents as well as for agents self-interaction ( $J_{i,i}$ ). The structure of the couplings (2.25) is reminiscent of neural networks models [28] where  $\xi_i^\mu$  play the role of memory patterns. This similarity confirms the conclusion of refs. [21, 23, 22] that the relevant parameter is the ratio  $\alpha = P/N$  between the number of patterns and the number of spins.

Both  $\sigma^2$  and  $H$  look like disordered Hamiltonians where the  $\xi_i$  and  $\omega_i$  — the strategies — play the role of the quenched disorder, while the spins  $s_i$  are dynamical variables. The constant part of  $A^\mu$  acts like a magnetic field. When it is too strong with respect to agents' ability to exploit and remove it, there is residual magnetic field with which agents try to be anti-aligned. When agents removed information, the effective magnetic field is zero : there is no more a privileged direction in the answer space  $\{-1, 1\}^P$ .

Note that  $H$  and  $\sigma^2$  only differ by the diagonal term of the last summation

$$\frac{\sigma^2}{N} = \frac{H}{N} + \frac{1}{N} \sum_i J_{i,i} (1 - \langle s_i \rangle^2) \simeq \frac{H}{N} + \frac{1}{2} (1 - Q) \quad (2.28)$$

where  $Q = \sum_i \langle s_i \rangle^2 / N$  is the *self overlap*, which is related to the way agents use their strategies : if all agent are frozen,  $Q = 1$ , whereas  $Q = 0$  means that all agents use their two strategies with the same frequency. The approximation made above holds for large system sizes.

For  $S > 2$ , the decomposition of Eq (2.17) cannot be done, but the above discussion is still valid. In particular, with probability  $1/2^{S-1}$  all strategies of an agent give the same decision, thus the total outcome is still a sum of constant and variable parts, the constant part being smaller, explaining why the phase transition occurs at higher  $\alpha$  and why the cooperation is less and less intense as  $S$  is increased.

All quantities have to be expressed by vectors. For instance, strategies' set of agent  $i$  is labelled by  $\vec{a}_i$ , and the strategy used by  $\vec{s}_i(t) \in \{0,1\}^S$  (note that only one component is different from zero at any time), so that agent  $i$  takes the decision  $\vec{s}_i(t) \cdot \vec{a}_i^\mu$ . The time average can also be defined  $\langle \vec{s}_i \rangle = (\langle s_i^k \rangle)_k$ . With this notation,

$$\langle a_i^\mu \rangle = \langle \vec{s}_i \rangle \cdot \vec{a}_i \quad \implies \quad \langle A | \mu \rangle = \sum_i \langle \vec{s}_i \rangle \cdot \vec{a}_i^\mu \quad (2.29)$$

and

$$H = \overline{\langle A \rangle^2} = \sum_\mu \rho^\mu (\langle \vec{s}_i \rangle \cdot \vec{a}_i^\mu)^2 \quad (2.30)$$

The fluctuations are, by definition,

$$\sigma^2 = \overline{\langle A^2 \rangle} = \sum_\mu \rho^\mu \left\langle \left( \sum_i a_{s_i,i}^\mu \right)^2 \right\rangle \quad (2.31)$$

For this case we adopt a slightly different definition of the self-overlap

$$G = \frac{1}{N} \sum_i |\langle \vec{s}_i \rangle|^2 = \frac{1}{N} \sum_i \sum_{k=1}^S \langle s_i^k \rangle^2 \quad (2.32)$$

If all agents play pure strategies  $G = 1$  whereas  $G = 1/S$  if  $\langle s_i^k \rangle = 1/S \forall i, k$ . Therefore  $1/G$  is a measure of the “effective” number of strategies that agents play on average. Eq. (2.28) becomes

$$\frac{\sigma^2}{N} = \frac{H}{N} + 1 - G - \frac{1}{N} \sum_i \sum_{k,k' \neq k} \langle s_i^k \rangle \langle s_i^{k'} \rangle \overline{a_{i,s} a_{i,s'}} \cong \frac{H}{N} + 1 - G \quad (2.33)$$

where in the last relation we neglected terms which vanish in the limit  $N \rightarrow \infty$  (because  $\overline{a_{i,s} a_{i,s'}} \sim P^{-1/2}$  for  $s \neq s'$ ). Eq. (2.33) means that the loss of agents come either from the asymmetry  $H$  which they produce or from the stochastic fluctuations of their choices. Indeed if agents play pure strategies,  $G = 1$  and the last term vanishes. Put differently, the stochastic fluctuations  $\sigma^2$  of the market – or volatility – has a systematic contribution  $H$  arising from unexploited asymmetries and a stochastic one  $1 - G$ , which is generated by stochastic choice of agents.

## 2.10 Exponential learning

The original formulation of the MG was aimed at being the most simple possible, and indeed the MG is very easy to introduce with simple words.

But simplicity is a relative notion : mathematically speaking, it is defined with discontinuous functions for the payoff ( $\text{sgn}(x)$ ) and the used strategies ( $\arg \max$ ). While the first difficulty is easily overcome by taking a linear payoff function, the second one is still to be fixed in order to describe further mathematically the MG. The idea is to replace the use-the-best-strategy rule by a probabilistic use of strategies, where at time  $t$ , each strategy  $s$  of agent  $i$  is played with probability  $\pi_{i,s}(t) \in \Delta^N$ , so that each agent  $i$  behavior at time  $t$  is characterized by the vector<sup>15</sup>  $\vec{\pi}_i$  of components  $\pi_{i,s}$ . Inductive agents as defined in the MG use their strategies according to a reinforcing scheme [30], that is, more rewarding strategies are more likely to be used. By definition of probabilities

$$\pi_{i,s}(t) = \frac{f(U_{i,s}(t))}{\sum_{s'=1}^N f(U_{i,s'}(t))} \quad (2.34)$$

where  $f(x)$  is a monotonously growing function of  $x$ . The most simple choice is to consider  $f(x) = x$ , but is not convenient, since it requires that all the payoffs are positive. Another one very handy choice is  $f(x) = \exp(\Gamma x)$ <sup>16</sup> ; this particular choice is familiar both to physicists and economists : the former call it Boltzmann distribution where  $\Gamma$  is the inverse of a temperature, while the latter know it as the Logit model, or exponential learning [31]. The use-the-best-strategy is recovered in the  $\Gamma \rightarrow \infty$  limit.

In principle, one should avoid introducing new parameters in the MG if one aims to study it as originally defined, since in most cases it will also introduce a supplementary source of confusion. However, as we shall show in the next sections, this probabilistic formulation does not change the stationary state of the MG, when it exists, that is, the asymptotic behavior of the MG in the asymmetric phase does not depend on  $\Gamma$ . In the symmetric phase, however, it does change the behavior of the MG.

For  $S = 2$ , only one  $\pi_{i,s}$  is needed and it is handfull to let

$$m_i(t) = \pi_{i,\uparrow}(t) - \pi_{i,\downarrow}(t) = \tanh[\Gamma(U_{i,\uparrow}(t) - U_{i,\downarrow}(t))] = \tanh[\Gamma\Delta_i(t)] \quad (2.35)$$

so that  $m_i \in [-1, 1]$  is the time average of  $s_i$  in the stationary state ; it can be thought as a soft spin, i.e. as a spin that can take continuous values.

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<sup>15</sup>also called mixed strategies profile in the language of Game Theory.

<sup>16</sup>This probabilistic choice has been first considered in the MG context in ref. [29].

## 2.11 Continuous time limit — “thermodynamic” limit

In the following, we shall restrict to the case of linear payoff, and discuss the general case in chapter 3.1. The continuous time should be a variable independent from the size of the systems, therefore one has to find what is the intrinsic time scale of the discrete MG, if there is any.

A direct clue of the existence of a time scale is given by the anti-persistent behavior of the MG in the symmetric phase : the time correlation function of the minority sign conditional to a history oscillates with a  $2P$  period, thus there is a time scale in MG, proportional to  $P$ . Another argument is to observe that the behavior of agents depends on the scores of their strategies ; the scores  $U_{i,s}$  are sums of sub-scores  $U_{i,s}^\mu$  which all have roughly the same importance, therefore, the number of time steps needed to significantly change  $U_{i,s}$  is proportional to  $P$ , in all phases. Hence, the proper time is the rescaled time

$$\tau = t/P, \quad (2.36)$$

which becomes continuous in the  $P \rightarrow \infty$  limit ; equivalently, achieving a given proper time difference  $\delta\tau$  requires  $\tau P$  discrete time steps, that eventually diverges in the  $P \rightarrow \infty$  limit. This implies that the continuous time limit is only valid if the system is in a stationary state, that is, if its state do not change significantly during  $\delta\tau P$  time steps. As it turns out at first sight, it is only the case in the asymmetric phase<sup>17</sup>.

Since  $\alpha = P/N$  is the control parameter of the system, the “thermodynamic” limit consists of taking  $P, N \rightarrow \infty$  while keeping their ratio constant and equal to  $\alpha$  ; this is noted by  $\limth$ . All quantities have to remain finite in the thermodynamic limit. The normalized fluctuations are noted by

$$\sigma_c^2 = \langle (A_c)^2 \rangle = \limth \frac{\sigma^2}{N} = \limth \frac{\langle A^2 \rangle}{N} \quad (2.37)$$

Strategies scores have to be normalized by a factor  $1/P$  (see below), yielding

$$U_{i,s}^c(\tau) = \limth \frac{U_{i,s}(t)}{P} \quad (2.38)$$

All dynamical equations can now be rewritten in the continuous time formulation. The proper continuous time limit of  $U_{i,s}^c$  is the central part of the reformulation

$$\frac{U_{i,s}^c(\tau + \delta\tau) - U_{i,s}^c(\tau)}{\delta\tau} = \limth - \frac{1}{\delta\tau P} \sum_{t=\tau P}^{(\tau+\delta\tau)P-1} a_{i,s}^{\mu(t)} A^{\mu(t)}(t) \quad (2.39)$$

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<sup>17</sup>In fact, it is also true for the symmetric phase with  $\Gamma \ll 1$  (see below).



By the law of large numbers, since the fluctuations of the discrete scores are Gaussian, they vanishes in the thermodynamic limit [35], thus,

$$\dot{U}_{i,s}^c(\tau) = -\overline{a_{i,s}\langle A \rangle} \quad (2.40)$$

This equations shows why the rescaling factor of the scores is  $P$  : in the asymmetric phase, the sum over the histories gives a factor of order  $\sqrt{P}$ , while the one over the agents a factor of order  $\sqrt{N}$ .

Note that the average over the histories  $\mu$  is now an integration. The dynamics of  $S = 2$  agents is given by

$$\dot{m}_i = -\Gamma(1 - m_i^2)\Delta_i \quad (2.41)$$

where  $\Delta_i^c = 2\overline{\xi_i\langle A \rangle}$ . The calculus for any  $S$  is easy and gives [35]

$$\frac{d\pi_{i,s}}{d\tau} = -\Gamma_i\pi_{i,s} \left[ \overline{a_{i,s}\langle A \rangle} - \overline{\langle a_i \rangle \langle A \rangle} \right] \quad (2.42)$$

This equation simply states that the probability of use of a strategy increases if it yields a higher payoff than the payoff averaged over all the agent's strategies. In the following, the  $c$  indices will be dropped, but shall remain present in the mind of the reader when needed. In addition, we shall note integrations by summations, for the sake of simplicity.

## 2.12 Minimization of available information

The MG with linear payoff admits as Lyapunov function available information  $H$ . Indeed for  $S = 2$ , one has [32]

$$\frac{dH}{d\tau} = \sum_i \frac{\partial H}{\partial m_i} \dot{m}_i = - \sum_i \Gamma(1 - m_i^2) \Delta_i^2 < 0 \quad (2.43)$$

since  $\partial H / \partial m_i = \Delta_i$ . For any  $S$ , the derivation is a bit lengthier, and after some algebra one can write [35]

$$\frac{dH}{d\tau} = \sum_{i,s} \frac{\partial H}{\partial \pi_{i,s}} \cdot \frac{d\pi_{i,s}}{d\tau} = -2 \sum_i \Gamma_i \sum_{s=1}^S \pi_{i,s} \left( \frac{dU_{i,s}}{d\tau} - \vec{\pi}_i \cdot \frac{d\vec{U}_i}{d\tau} \right)^2 < 0. \quad (2.44)$$

Eqs (2.43) and (2.44) are a milestone in the understanding of MG. In addition, they imply that in the stationary state ( $dH/d\tau = 0$ ) each of the strategies played by agent  $i$  – those with  $\pi_{i,s} > 0$  – has the same perceived success rate  $\frac{dU_{i,s}}{d\tau}$  in the long run (see below for a discussion of this point). Finally, note that the constant  $\Gamma$  is a learning rate that can be incorporated into the proper time. Therefore, the stationary state, when it exists, does not depend on its precise value [32, 35, 33].

## 2.13 Algebraic interpretation of the phase transition

As shown in the previous section, agents try to minimize available information  $H$  [32, 35], and can actually cancel it when  $\alpha < \alpha_c$ . One can algebraically explain why this occurs, that is, why depending on  $\alpha$  they can or cannot remove the information. In addition, the algebraic approach gives another rigorous proof that  $\alpha$  is the control parameter for the MG.

Since  $H$  is a sum of  $P$  non negative averages  $\overline{\langle A \rangle^2}$ ,  $H = 0$  only if all averages are zero, namely  $\overline{\langle A \rangle} = 0 \ \forall \mu$ , or equivalently

$$\sum_{i=1}^N \xi_i^\mu \langle s_i \rangle = -\Omega^\mu \quad \forall \mu \quad (2.45)$$

These are  $P$  linear equations in  $N$  variables. However the  $N$  variables  $m_i = \langle s_i \rangle$  are restricted to the  $[-1, 1]$  interval. Above  $\alpha_c$  there are  $N\phi$  variables which are frozen at the boundary of this interval ( $m_i = \pm 1$ ). Therefore there are  $(1 - \phi)N$  free variables only. As shown in refs. [32, 35], the point  $\alpha_c$  marks the transition below which the system of equations (2.45) becomes degenerate, i.e. when there are more variables than equations. Exactly at  $\alpha_c$  the number of free variables  $(1 - \phi)N$  exactly matches the number of equations  $P$ . Dividing this equation by  $N$  gives an equation for  $\alpha_c$ ,

$$\alpha_c = 1 - \phi \quad (2.46)$$

which is indeed confirmed numerically to a high accuracy.

When  $\alpha < \alpha_c$ , there are much more free variables ( $N$  indeed) than equations : the solutions of Eqs. (2.45) then belong to a subspace of dimension  $N - P$ . This allows the anti-persistent behavior to take place, because the system is free to move on this subspace.

This means that if agents draw their strategies in such a way that they have no bias ( $\Omega^\mu = 0$ ), the linear system of equation is then homogeneous and the solution  $\langle s_i \rangle = 0$  always exists for all  $i$ 's, implying that the system is always in the symmetric phase, and there is no phase transition. In particular, if  $\alpha > 1$ , the trivial solution is unique, hence  $\sigma^2/N = 1$ . When  $\alpha < 1$ , a subspace of solutions of dimension  $N - P$  arises, and the anti-persistent behavior also takes place. These predictions are fully confirmed by figure 2.7<sup>18</sup>.

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<sup>18</sup>Actually,  $\sigma^2/N$  appears to be slightly lower than 1 for  $\alpha \sim 1$  : this is a finite size effect.

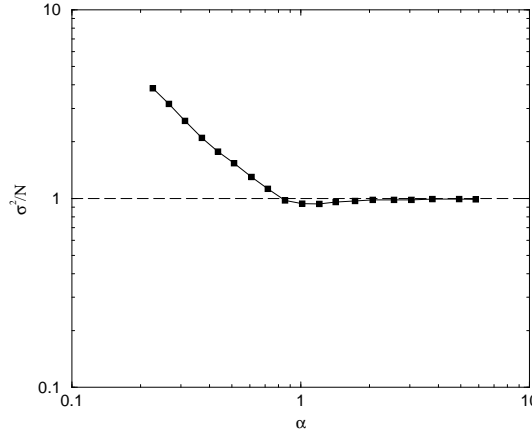


Figure 2.7: Normalized fluctuations versus  $\alpha$  ( $M = 6$ ,  $S = 2$ , average over 50 samples).

This argument easily generalizes to  $S > 2$  strategies[35] (see also appendix 8.2). If agents use, on average,  $n(S)$  strategies (and  $S - n(S)$  are never used) the number of free variables is  $Nn(S)$ . There are  $P$  plus  $N$  equations which these have to satisfy, where the latter  $N$  comes from the normalization condition on the frequency with which each strategy is used. At the critical point, these two numbers are equal, and we find

$$n_c(S) = \alpha_c(S) + 1. \quad (2.47)$$

At the critical point nearly one half of the strategies yield positive virtual gain and are used, whereas the others are not used. From this we find

$$\alpha_c(S) \cong \alpha_c(2) + \frac{S}{2} - 1 \quad (2.48)$$

This shows that actually  $\alpha_c$  grows linearly with  $S$ , but in a slightly less simple way than the prediction given by the RSS ( $\alpha_c(S) \propto S$ ).

## 2.14 Microscopic behavior

The mathematical formulation derived above allow us to explore analytically the microscopic behavior of agents ; for instance, it gives a general explanation of freezing, and of how the gain of an agent depends on his set of strategies as well as on his behavior.

### 2.14.1 Asymmetric phase

#### Unused strategies—Freezing

Numerical simulations showed that for large system sizes, some agents only use a subset of their strategies in the asymmetric phase. In particular, for  $S = 2$ , such agents use one strategy — they are frozen [24].

In the asymmetric phase, an agent can detect and exploit available information if one of her's strategies is more correlated with  $\langle A|\mu \rangle$  than the other. More precisely, we observe that if  $v_i \equiv \langle \Delta_{i,t+1} - \Delta_{i,t} \rangle / 2 \neq 0$  then  $\Delta_{i,t} \simeq v_i t$  grows linearly with time, and the agent's spin will always take the value  $s_i = \text{sign } v_i$ . We find

$$v_i = -\langle \xi_i^{\mu t} A(t) \rangle = -h_i - \sum_{j=1}^N J_{i,j} \langle s_j \rangle \quad (2.49)$$

Note that this is consistent with the dynamical equations in continuous time (2.35) : the stationary state corresponds to  $\dot{m}_i = 0$  for all  $i$ , there are only two possibilities

- $\Delta_i^c = 0$ , so to say, both strategies of agent  $i$  yield the same average score in the long run and she uses both of them,
- $\Delta_i^c \neq 0$ , which implies that  $m_i^2 = 1$ , that is, agent  $i$  is frozen.

It is instructive to consider first the case where other agents than agent  $i$  choose by coin tossing (i.e.  $\langle s_j \rangle = 0$  for  $j \neq i$ ) so that  $v_i = -h_i - J_{i,i} \langle s_i \rangle$ . If  $v_i \neq 0$  then

$$s_i = \text{sign } v_i = -\text{sign } (h_i + J_{i,i} \langle s_i \rangle). \quad (2.50)$$

But this last equation has a solution only if

$$|h_i| > J_{i,i} \quad (2.51)$$

whereas otherwise  $|\langle s_i \rangle| < 1$  and  $v_i = 0$ . For large system sizes,  $J_{i,i} \simeq 1/2$  and that  $h_i$  can be approximated by a Gaussian variable with zero average and variance  $(4\alpha)^{-1}$ . This means that  $|h_i| \ll J_{i,i}$  for  $\alpha \gg 1$ , which implies that most agents have  $\langle s_i \rangle \approx 0$  in this limit and we can indeed neglect agent-agent interaction. This allows to compute the probability for an agent to be frozen

$$\phi = P\{|h_i| > J_{i,i}\} \propto e^{-\alpha/2}, \quad (2.52)$$

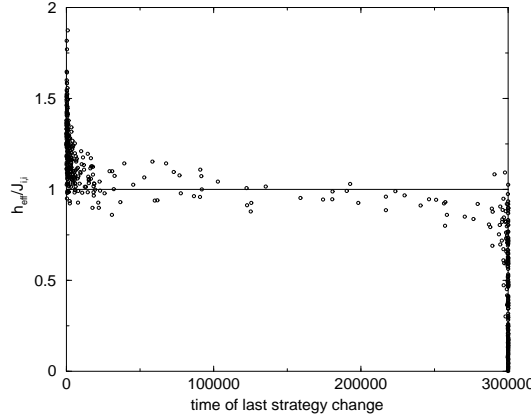


Figure 2.8: Strength of effective field  $h_i$  relative to self impact  $J_{i,i}$  versus the time of last change of strategies for all agents ( $M = 8$ ,  $N = 511$ ,  $S = 2$ ).

for  $\alpha \gg 1$ . Numerical simulations show that  $\phi \propto e^{-(0.37 \pm 0.02)\alpha}$  indeed decays exponentially. As  $\alpha \rightarrow \infty$ , the random agents limit is reached because  $\langle s_i \rangle \rightarrow 0$  for all  $i$  and  $\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle$  for  $i \neq j$ . By Eq. (2.18) we find

$$\sigma^2 = \sum_{\mu} (\Omega^{\mu})^2 / P + \sum_i J_{i,i} \simeq N. \quad (2.53)$$

The same argument applies in general, with the difference that the “bare” field  $h_i$  must be replaced by the “effective” field  $\tilde{h}_i = h_i + \sum_{j \neq i} J_{i,j} \langle s_j \rangle$ . In order for agent  $i$  to get frozen, her effective field  $\tilde{h}_i$  must overcome the self interaction  $J_{i,i}$ , i.e.

$$|\tilde{h}_i| > J_{i,i} \simeq 1/2. \quad (2.54)$$

If this condition is met,  $s_i = -\text{sign } \tilde{h}_i$ . Fig. 2.8 reports that the criterion given by Eq (2.54) is indeed correct.

In addition, a frozen agent, on average, receives a larger payoff than an unfrozen agent (see next section). Loosely speaking, one can say that a frozen agent has a *good* and a *bad* strategy and the good one remains better than the bad one even when she actually uses it. On the contrary, unfrozen agents have two strategies each of which seems better than the other when it is not adopted. In this sense, symmetry breaking in the outcome induces a sort of breakdown in the *a priori* equivalence of agents’ strategies.

This discussion is easily generalized to  $S > 2$ ; in the asymmetric phase, some agents only use  $n < S$  strategies. Eq (2.34) implies that in the stationary state, all used strategies yield the same average score.

We focus on one agent, say  $i$ , and assume that others play their strategies according to some stationary probability distribution  $\vec{\pi}_{-i}$ . If  $\mu(t)$  is drawn

randomly from  $\rho^\mu$  we can consider the decisions of all other agents  $A_{-i}^{\mu(t)}(t) = \sum_{j \neq i} a_{j,s_j(t)}^\mu(t)$  as a stationary process<sup>19</sup>. Since we deal with one agent, we shall drop the subscript  $i$ . In the long run the perceived performance of strategy  $s$  is

$$\begin{aligned} \langle U_s(t+1) - U_s(t) \rangle = \langle \Delta U_s \rangle &= -\overline{a_s \langle A_{-i} \rangle} - \vec{\pi} \cdot \vec{a} \overline{a_s} \\ &\cong -\overline{a_s \langle A_{-i} \rangle} - \pi_s \end{aligned} \quad (2.55)$$

where the approximation in Eq. (2.55) holds for  $P \gg 1$  since  $\overline{a_{s'} a_s} \sim 1/\sqrt{P}$  for  $s' \neq s$ . Because of Eq. (2.34), strategies can either

i) have  $\pi_s > 0$  and  $\langle \Delta U_s \rangle = v$  independent of  $s$  or

ii) have  $\pi_s = 0$  and  $\langle \Delta U_s \rangle < v$ .

This can be understood by a rather simple argument : Imagine that strategy 1 has  $\langle \Delta U_1 \rangle > v$ . Then by the very learning dynamics, the agent shall use strategy 1 more frequently than others and hence  $\pi_1$  shall increase. Because of the last term in Eq. (2.55) that will decrease the perceived performance  $\langle \Delta U_1 \rangle$  of that strategy. On the other hand, if  $\langle \Delta U_1 \rangle < v$  the agent shall use it less frequently, hence its  $\langle \Delta U_1 \rangle$  shall increase. If  $\langle \Delta U_1 \rangle < v$  even when  $\pi_1 \rightarrow 0$  then the agents will never play that strategy, i.e.  $\pi_1 = 0$ .

Let  $n \leq S$  be the number of strategies with  $\pi_s > 0$  and let these be labeled by  $s = 1, \dots, n$ , whereas  $\pi_k = 0$  for  $k > n$ . Taking the sum of Eq. (2.55) on  $s = 1, \dots, n$  we find

$$v = -\frac{1}{n} \sum_{s=1}^n \overline{a_s \langle A_{-i} \rangle} - \frac{1}{n}$$

where  $n$  is fixed by the condition  $v > -\overline{a_k \langle A_{-i} \rangle}$  for all  $k > n$ . Clearly  $-\overline{a_s \langle A_{-i} \rangle} > v > -\overline{a_k \langle A_{-i} \rangle}$  for any  $s \leq n$  and  $k > n$  hence the  $n$  strategies which the agent uses are the  $n$  more efficient ones. Then Eq. (2.55) becomes

$$\pi_s = \frac{1}{n} + \left( \frac{1}{n} \sum_{s'=1}^n \overline{a_{s'} \langle A_{-i} \rangle} - \overline{a_s \langle A_{-i} \rangle} \right),$$

meaning that strategies with a larger  $-\overline{a_s \langle A_{-i} \rangle}$  are played more frequently.

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<sup>19</sup>Leading to an effective external action, similar in essence to the effective external field  $\tilde{h}_i$  for  $S = 2$ .

### Gain

In this section we show how the behavior and the gain of each agent depends on her microscopic constitution and on the asymmetry of the outcome  $A(t)$  in the asymmetric phase. Let us denote the gain of agent  $i$  by  $g_i$ ; by definition,

$$g_i = -\overline{\langle A a_i \rangle}. \quad (2.56)$$

In the asymmetric phase, since the stationary state is of mean field nature,  $\langle s_i s_j \rangle = m_i m_j$  for  $S = 2$ . Consequently, by expanding Eq. (2.56) one obtains

$$\begin{aligned} g_i &= -\overline{\langle A \rangle \omega_i} - \overline{\langle A s_i \rangle \xi_i} \\ &= -\overline{\langle A \rangle \omega_i} - \overline{\langle A \rangle \xi_i m_i} - \overline{\xi_i^2} (1 - m_i^2). \end{aligned} \quad (2.57)$$

The stationary behavior of agent  $i$  is described by  $v_i = -\overline{\langle A \rangle \xi_i}$ . If an agent is non frozen,  $v_i = 0$ , while  $m_i = -\text{sgn} v_i$  otherwise, hence the gain of a generic agent  $i$  is

$$g_i = -\overline{\langle A \rangle \omega_i} + |\overline{\langle A \rangle \xi_i}| - \overline{\xi_i^2} (1 - m_i^2). \quad (2.58)$$

Note that the second term of the above equation vanishes for a non frozen agent  $j$  and therefore

$$g_j = -\overline{\langle A \rangle \omega_j} - \overline{\xi_j^2} (1 - m_j^2) \quad \text{non frozen.} \quad (2.59)$$

On the other hand, the third term of Eq. (2.58) vanishes if agent  $k$  is frozen :

$$g_k = -\overline{\langle A \rangle \omega_k} + |\overline{\langle A \rangle \xi_k}| \quad \text{frozen.} \quad (2.60)$$

In Eqs (2.59) and (2.60), the gain of each agent is expressed as her internal constitution, allowing us to interpret what does the gain of a general agent depends on. In both equations, the first term  $-\overline{\langle A \rangle \omega_i}$ , which represents how much the agents lose due to their bias, is on average negative, due to the impact this bias has on the market. The second term in Eq (2.59) is always negative, and represents the losses due to the switching between strategies, which arises from the neglect of market impact.

By contrast, the term  $\overline{\xi_k^2} (1 - m_k^2)$  disappears for a frozen agent because  $m_k^2 = 1$ . It is replaced by  $|\overline{\langle A \rangle \xi_k}|$  which is always positive and which measures how well agent  $k$  exploits available information. Therefore, on average, the frozen agents gain more than the non frozen ones. This is clearly illustrated by figure 2.9 which also shows that Eqs. (2.59) and (2.60) are exact.

For  $S > 2$ , the same discussion can be handled. The average payoff  $g = -\vec{\pi} \cdot \vec{a} \overline{\langle A_{-i} \rangle} - 1$  delivered is

$$g = -\frac{1}{n} \sum_{s=1}^n \overline{a_s \langle A_{-i} \rangle} + \sum_{s=1}^n \left( \overline{a_s \langle A_{-i} \rangle} - \frac{1}{n} \sum_{s'=1}^n \overline{a_{s'} \langle A_{-i} \rangle} \right)^2 - 1 \quad (2.61)$$

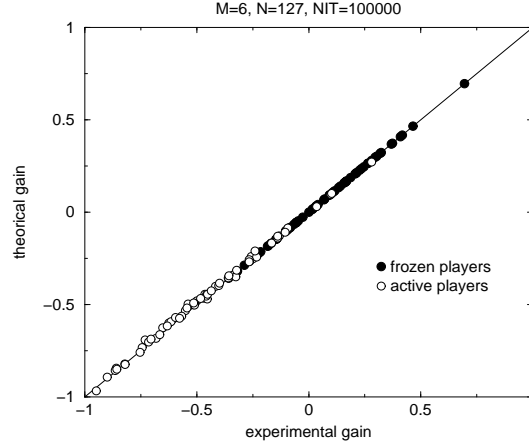


Figure 2.9: Theoretical gain versus experimental gain showing that the frozen agents gain more than the active ones ( $\alpha = 0.5$ ,  $M = 6$ )

With some more algebra, it is easy to check that  $g$  increases as  $n$  decreases : the less strategies she uses, the more she gains on average.

### 2.14.2 Symmetric phase

In the symmetric phase, the learning parameter  $\Gamma$  has an influence on the final state. It is known that for large  $\Gamma$ , a stationary state is attained in all phases [32, 35], and that the behavior of the original MG is recovered in the  $\Gamma \rightarrow \infty$  limit. This subsection is devoted to discuss the dependence of the behavior on the MG on  $\Gamma$  for the  $S = 2$  case [36], even if the discussion can be easily extended to any  $S$ .

#### Learning rate

When agents have no memory ( $P = 1$ ), or equivalently no information, there are two different strategies, and agents again draw their strategies from this set of strategies.

Suppose that they have no prior beliefs:  $\Delta_i(0) = 0$ . Hence  $\Delta_i(t) \equiv y(t)/\Gamma$  is the same for all agents. Consider now  $y(t) = \Gamma \Delta_i(t)$  : it satisfies the equation

$$\begin{aligned} y(t+1) &= y(t) - \frac{1}{N} \sum_{i=1}^N a_i(t) \\ &\simeq y(t) - \Gamma \tanh[y(t)] + O(N^{-1/2}) \end{aligned}$$



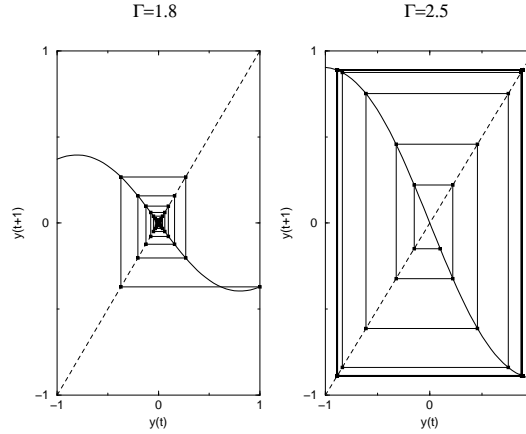


Figure 2.10: Graphical iteration of the map  $y(t)$  for  $\Gamma = 1.8 < \Gamma_c$  and  $\Gamma = 2.5 > \Gamma_c$

Neglecting the last term, which is legitimate for large  $N$ , we find a dynamical system. The point  $y^0 = 0$  is stationary, but it is easy to see that it is only stable for  $\Gamma < \Gamma_c = 2$ . For  $\Gamma > 2$ , a cycle of period 2 arises, as shown in figure 2.10. This has dramatic effects on the optimality of the system. Indeed, let  $\pm y^*$  be the two values of  $y(t)$  in this cycle. Because of the symmetry  $y(t+1) = -y(t) = \pm y^*$  we still have  $\langle A \rangle = 0$ . On the other hand  $\sigma^2 = N^2 y^{*2}$  is of order  $N^2$ , which is even worse than the symmetric Nash equilibrium where  $\sigma^2 = N$ .

This transition from a state where  $\sigma^2 \propto N$  to a state with  $\sigma^2 \propto N^2$  is generic in the minority game.

### Learning rate in the MG

The simple approach followed above can be generalized to the full minority game and it allows to derive the critical learning rate  $\Gamma_c(\alpha)$  as a function of the parameter  $\alpha$  of the MG.

As before, the performance of agents may be worse if they are too reactive. This is shown numerically in figure 4.4. The effect is exactly the same as that discussed previously, in the absence of information ( $P = 1$ ): As the learning rate  $\Gamma$  increases, the stationary solution  $\Delta_i^*$  loses its stability and a bifurcation to a complex dynamics occurs. This is only possible in the low  $\alpha$  phase, where the stationary state is degenerate and the system can attain  $\langle A | \mu \rangle = 0$  by hopping between different states<sup>20</sup>. The plot of  $A(t+1)/N$  vs

<sup>20</sup>In the asymmetric phase  $\alpha > \alpha_c$  the stationary state is unique and this effect is not possible.

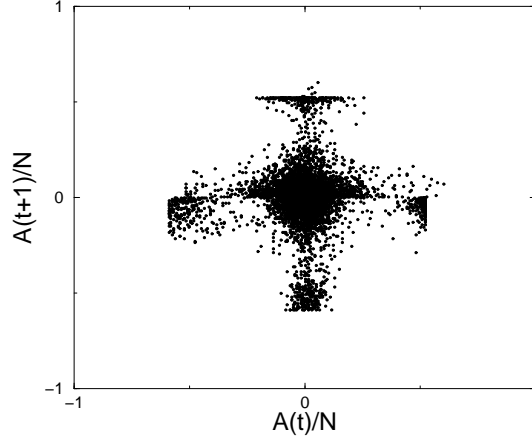


Figure 2.11: Plot of  $A(t+1)/N$  vs  $A(t)/N$  for the MG with  $N = 301$  agents,  $\Gamma = \infty$  and  $P = 16$  ( $\alpha = 0.053\dots$ ).

$A(t)/N$ , in figure 2.11 shows that indeed wild fluctuations occur in one time step: a finite fraction of agents change their mind at each time step. This is what causes, for fixed  $P$ , the cross over from the linear regime  $\sigma^2 \sim N$  to a quadratic dependence  $\sigma^2 \sim N^2$ .

Clearly the continuum time limit, on which our analysis rests, breaks down. Still one can compute the critical learning rate  $\Gamma_c(\alpha)$  which marks the onset of complex dynamics. Let us focus attention on one value of  $\mu = 1$  and on the learning model of Eq. (2.40). Let us define the sequence of times  $t_k$  such that  $\mu(t_k) = 1$  for the  $k^{\text{th}}$  time. Hence we define  $y_i(k) = \Gamma \Delta_i(t_k)$ , which satisfies

$$y_i(k+1) = y_i(k) - \frac{\Gamma}{P} \sum_{t=t_k}^{t_{k+1}-1} A(t) \xi_i^{\mu(t)} \quad (2.62)$$

When  $N \gg 1$ , the sum involves  $\sim P = \alpha N \gg 1$  terms and we may estimate it by the law of large numbers. Let  $y_i^*$  be the solution of Eq. (2.62), then we can set  $y_i(k) = y_i^* + \delta y_i(k)$  and study the linear stability of this solution. With the notation  $\overline{R} = \sum_{\mu} R^{\mu}/P$ , we find

$$\delta y_i(k+1) = \sum_{j=1}^N T_{i,j} \delta y_j(k), \quad T_{i,j} = \delta_{i,j} - \alpha \Gamma \overline{\xi_i \xi_j} (1 - m_j^2) \quad (2.63)$$

where  $m_j = \tanh(y_j^*)$ . The solution  $y_i^*$  is stable if the eigenvalues of  $T_{i,j}$  are all smaller than 1 in absolute value. As  $\Gamma$  increases, the smallest eigenvalue of  $T_{i,j}$  becomes smaller than  $-1$ . Thanks to the results in ref. [37], we have an analytic expression for this eigenvalue, which is  $\lambda_+ = 1 - \Gamma(1 + \alpha^{1/2})^2(1 - Q)/2$ .

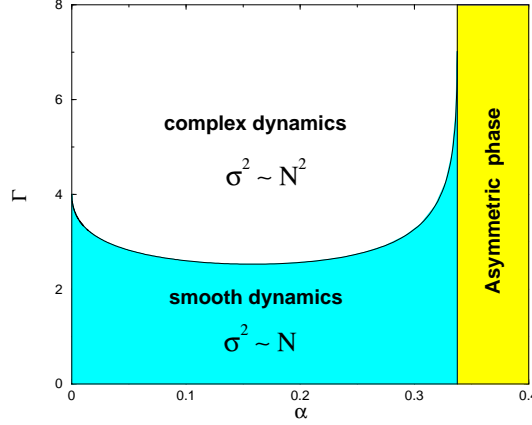


Figure 2.12: Phase diagram of the MG ( $\eta_i = 0$ ) in the  $(\alpha, \Gamma)$  plane.

The stability condition  $\lambda_+ > -1$  then turns into

$$\Gamma < \Gamma_c(\alpha) \equiv \frac{4}{[1 - Q(\alpha)](1 + \sqrt{\alpha})^2}, \quad Q(\alpha) = \frac{1}{N} \sum_{i=1}^N m_i^2 \quad (2.64)$$

which is our desired result. The function  $Q(\alpha)$  is known exactly from the analytic solution [32, 35]. This yields a phase diagram in the  $(\alpha, \Gamma)$  plane which is shown in figure 2.12. For  $\alpha \rightarrow 0$  we find<sup>21</sup>  $\Gamma_c \rightarrow 4$ . As  $\alpha \rightarrow \alpha_c$ ,  $\Gamma_c$  converges to a finite value ( $\simeq 7.0273\dots$ ) with infinite slope. In the asymmetric phase ( $\alpha > \alpha_c$ ) the dynamics is always smooth, hence  $\Gamma_c(\alpha) = \infty$ .

## 2.15 Exact solution

The exact solution exposed in this section only applies to MG having a stationary state, that is, MG in the asymmetric phase, and MG in the symmetric phase with  $\Gamma \ll 1$ . For these MG, the stationary state is obtained after the minimization of  $H$ , a quadratic form of its variables  $\{p_{i,s}\}$ , on which there are  $N$  conditions of normalization. This kind of problem is called quadratic programming and no simple general solution is known. However, in the thermodynamic limit, Statistical Physics gives exact results. The method is the following : one considers  $H$  as a Hamiltonian and tries to find its ground

<sup>21</sup>This differs from our previous result  $\Gamma_c = 2$  without information, because with  $P = 1$  in the MG half of the population has  $a_{+,i} = a_{-,i}$  two equal strategies. This reduces by a factor 2 the effective number of adaptive agents, and accordingly  $\Gamma_c$  takes a factor 2.

state. Statistical physics tells us that if one finds its partition function

$$Z(\beta) = \text{Tr} e^{-\beta H}, \quad (2.65)$$

one also knows the free energy  $F = 1/\beta \ln Z$ , which gives the ground state energy in the zero temperature  $\beta \rightarrow \infty$  limit. The problem here is that one does not know how to calculate exactly  $Z$  for each realization of the quenched disorder – strategies. Thermodynamic quantities are independent from the precise realization of the disorder in the thermodynamic limit, thus the goal is now to compute the averaged free energy over the disorder in the thermodynamic limit, that is, to compute  $\langle \ln Z \rangle$ . Unfortunately, such kind of average is exceedingly — when not impossible — difficult to calculate. One can use the replica trick, which consists of exploiting the identity

$$\langle \ln x \rangle = \lim_{n \rightarrow 0} \frac{\langle x^n \rangle - 1}{n} \quad (2.66)$$

with  $x = Z(\beta)$ . Physically, averaging  $Z^n$  over the disorder is equivalent to taking  $n$  replica of the same system, i.e.  $n$  same realizations of the disorder, and  $N$  spins for each realizations, hence the name. Of course the  $n \rightarrow 0$  limit is to be considered as formal only<sup>22</sup>.

Here, we expose the method for doing the replica calculus for the basic MG with  $S = 2$ . In the following, for each exactly solvable extension of the MG, the whole calculus has to be redone, but the method remains overall the same.

### 2.15.1 Solution for uniformly sampled histories

As originally defined, the MG is a closed system, where the pieces of information are dynamically created by agents, so that the MG is a closed dynamical system, implying that the solution has to be self-consistent. Consequently, the idea of Cavagna [38] — replacing the dynamical creation of histories by an external random process — was thought as a great simplification of the problem and as the door to an analytical description. It is very interesting since the overall structure of the MG apparently only weakly depends on whether the histories are real or completely random (see Fig. 2.13). However as it is showed hereafter, Cavagna’s claim that “all quantities of the minority game are completely independent from the memory of agents” is incorrect in most cases. Indeed, as it appears from the exact solution, all the quantities

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<sup>22</sup>Historically, this limit is the source of very animated discussions between physicists and also between physicists and mathematicians. The replica trick gives correct results in the case of the MG, but there are many other cases for which this question is still open.

depend on the distribution function of  $\rho^\mu$ , the frequencies of appearance of the history  $\mu$ . Hereafter follows the solution for random histories ( $\rho^\mu = 1/P$ ) [32].

The first step is to average  $Z^n$  over the strategies' disorder, that is, to obtain  $\langle Z^n(\beta) \rangle$ . Each replica  $b$  has its own set of dynamical variables  $s_{i,b}$ .

$$\langle Z^n(\beta) \rangle_a = \text{Tr}_s \prod_{\mu,c} \langle e^{-\rho^\mu \beta (A_b^\mu)^2} \rangle_a \quad (2.67)$$

where the average is over the disorder variables  $a_{i,s}^\mu$  and  $\text{Tr}_s$  is the trace on the variables  $s_{i,c}$  for all  $i$  and  $b$ . With the random history process,  $\rho^\mu = 0$  for all  $\mu$ .

Following standard procedures [53, 52], we introduce a Gaussian variable  $z_b^\mu$  so that we can linearize the exponent in Eq. (2.67).

$$\langle Z^n(\beta) \rangle_a = \text{Tr}_s \prod_{\mu} \langle e^{-i\sqrt{\frac{2\beta}{P}} \sum_b z_b^\mu A_b^\mu} \rangle_{a,z} \quad (2.68)$$

This allows us to carry out the averages over  $a$ 's explicitly

$$\langle Z^n(\beta) \rangle_a \simeq \prod_{j=1}^N \prod_{\mu} e^{-\frac{\beta}{2\alpha} \sum_{b,d} z_b^\mu d_b^\mu [1 + \frac{1}{N} \sum_i s_{i,b} s_{i,d}]} \quad (2.69)$$

to leading order in  $N$ . Then we introduce new variables  $Q_{b,d}$  and  $r_{b,d}$  with the identity

$$\begin{aligned} 1 &= \int dQ_{b,d} \delta \left( Q_{b,d} - \frac{1}{N} \sum_i s_{i,b} s_{i,d} \right) \\ &\propto \int dr_{b,d} dQ_{b,d} e^{-\frac{\alpha\beta^2}{2} r_{b,d} (N Q_{b,d} - \sum_i s_{i,b} s_{i,d})} \end{aligned}$$

for all  $b \geq d$ , which allow us to write the partition function as :

$$\langle Z^n(\beta) \rangle = \int d\hat{Q} d\hat{r} e^{-Nn\beta f(\hat{Q}, \hat{r})}$$

with

$$\begin{aligned} f(\hat{Q}, \hat{r}) &= \frac{\alpha}{2n\beta} \text{Tr} \log \hat{T} + \frac{\alpha\beta}{2n} \sum_{b \leq d} r_{b,d} Q_{b,d} \\ &- \frac{1}{n\beta} \log \left[ \text{Tr}_s e^{\frac{\alpha\beta^2}{2} \sum_{b \leq d} r_{b,d} s_b s_d} \right]. \end{aligned} \quad (2.70)$$

The matrix  $\hat{T}$  is given by

$$T_{b,d} = \delta_{b,d} + \frac{\beta}{\alpha} [1 + Q_{b,d}].$$

With the replica symmetric ansatz

$$Q_{b,d} = q + (Q - q)\delta_{b,d}, \quad r_{b,d} = 2r + (R - 2r)\delta_{b,d}$$

the matrix  $\hat{T}$  has  $n - 1$  degenerated eigenvalues  $\lambda_0 = 1 + \frac{\beta(1-q)}{\alpha}$  and one eigenvalue equal to  $\lambda_1 = n \frac{\beta(1+q)}{\alpha} + 1 + \frac{\beta(1-q)}{\alpha}$  therefore, after standard algebra,

$$\begin{aligned} f(q, r) &= \frac{\alpha}{2\beta} \log \left[ 1 + \frac{\beta(Q - q)}{\alpha} \right] \\ &+ \frac{\alpha[1 + q]}{2\alpha + \beta(Q - q)} + \frac{\alpha\beta}{2}(RQ - rq) \\ &- \frac{1}{\beta} \langle \log \int_{-1}^1 ds e^{-\beta V_z(s)} \rangle \end{aligned} \quad (2.71)$$

where we found it convenient to define the “potential”

$$V_z(s) = -\frac{\alpha\beta(R - r)}{2}s^2 - \sqrt{\alpha r} z s \quad (2.72)$$

so that the last term of  $f$  looks like the free energy of a particle in the interval  $[-1, 1]$  with potential  $V_z(s)$  where  $z$  plays the role of disorder.

The saddle point equations are given by :

$$\frac{\partial f}{\partial q} = 0 \quad \Rightarrow \quad r = \frac{1 + q}{[\alpha + \beta(Q - q)]^2} \quad (2.73)$$

$$\frac{\partial f}{\partial Q} = 0 \quad \Rightarrow \quad \beta(R - r) = -\frac{1}{\alpha + \beta(Q - q)} \quad (2.74)$$

$$\frac{\partial f}{\partial R} = 0 \quad \Rightarrow \quad Q = \langle \langle s^2 \rangle_s \rangle_z \quad (2.75)$$

$$\frac{\partial f}{\partial r} = 0 \quad \Rightarrow \quad \beta(Q - q) = \frac{\langle \langle sz \rangle_s \rangle_z}{\sqrt{\alpha r}} \quad (2.76)$$

where  $\langle \cdot \rangle_s$  stands for a thermal average over the above mentioned one particle system. This average is actually over the distribution function of the  $m_i$ .

In the limit  $\beta \rightarrow 0$  we can look for a solution with  $q \rightarrow Q$  and  $r \rightarrow R$ . It is convenient to define

$$\chi = \frac{\beta(Q - q)}{\alpha}, \quad \text{and} \quad \zeta = -\sqrt{\frac{\alpha}{r}}\beta(R - r) \quad (2.77)$$

and to require that they stay finite in the limit  $\beta \rightarrow \infty$ . The averages are easily evaluated since, in this case, they are dominated by the minimum of the potential  $V_z(s) = \sqrt{\alpha r}(\zeta s^2/2 - zs)$  for  $s \in [-1, 1]$ . The minimum is at

$s = -1$  for  $z \leq -\zeta$  and at  $s = +1$  for  $z \geq \zeta$ . For  $-\zeta < z < \zeta$  the minimum is at  $s = z/\zeta$ . With this we find

$$\langle\langle sz \rangle\rangle = \frac{1}{\zeta} \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) \quad (2.78)$$

and

$$\langle\langle s^2 \rangle\rangle = Q = 1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} - \left(1 - \frac{1}{\zeta^2}\right) \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) \quad (2.79)$$

The pdf of the  $m_i$  is

$$\mathcal{P}(m) = \frac{\phi(z)}{2} [\delta(m-1) + \delta(m+1)] + \frac{z}{\sqrt{2\pi}} e^{-(zm)^2/2} \quad (2.80)$$

with  $z = \sqrt{\alpha/(1+Q)}$  ( $Q$  taking its saddle point value) and where  $\phi(z) = 1 - \operatorname{Erf}(z/\sqrt{2})$  is the fraction of frozen agents.

With some more algebra, one easily finds :

$$\chi = \left[ \alpha / \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) - 1 \right]^{-1} \quad (2.81)$$

Finally  $\zeta$  is fixed as a function of  $\alpha$  by the equation

$$\sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} + \left(1 - \frac{1}{\zeta^2}\right) \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) + \frac{\alpha}{\zeta^2} = 2 \quad (2.82)$$

Eq. (2.81) means that  $\chi$  diverges when  $\alpha \rightarrow \alpha_c^+$ , which then implies that at the critical point

$$\operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) = \alpha = \alpha_c. \quad (2.83)$$

This back in the other saddle point equations, yields the following equation for  $\zeta = \zeta_c$  :

$$\sqrt{\frac{2}{\pi}} \frac{e^{-\zeta_c^2/2}}{\zeta_c} + \operatorname{erf} \left( \frac{\zeta_c}{\sqrt{2}} \right) = 2. \quad (2.84)$$

The free energy, at the saddle point, for  $\beta \rightarrow \infty$ , is equal to  $H_c$ , available information, thus

$$H_c = \frac{1+Q}{2(1+\chi)^2} \quad (2.85)$$

where  $Q_c$  and  $\chi$  take their saddle point values (Eqs. (2.79) and (2.81)). The fluctuations are simply given by

$$\sigma_c^2 = H_c + \frac{1}{2}(1-Q) \quad (2.86)$$

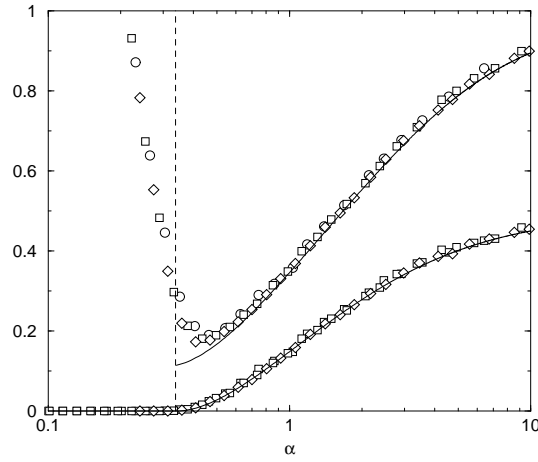


Figure 2.13: Normalized fluctuations (top) and available information (below) versus the control parameter  $\alpha$  ( $S = 2$ ,  $M = 6$  (circles),  $M = 7$  (squares) and  $M = 8$  (diamonds),  $300P$  iterations, average over 100 samples). The continuous lines are theoretical predictions from the exact solution.

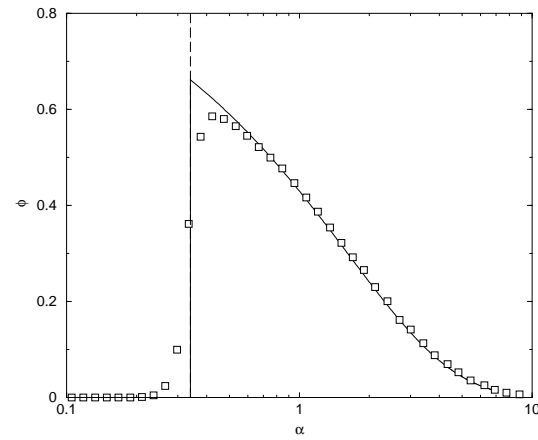


Figure 2.14: Fraction of frozen players from numerical simulation ( $M = 8$ ,  $300P$  iterations, average over 200 samples). The continuous line is the theoretical prediction from the exact solution.



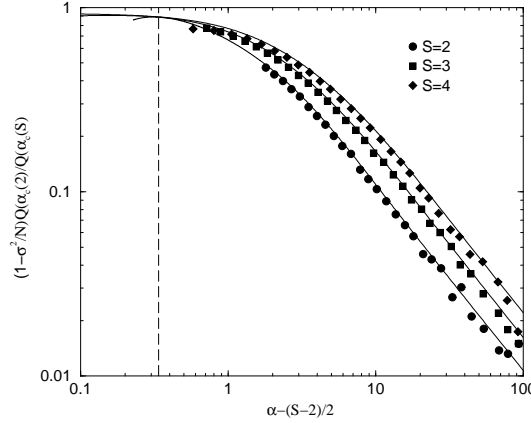


Figure 2.15: Scaled normalized fluctuations versus  $\alpha(S) + 1 - S/2$ , showing that  $\sigma^2/N \rightarrow 1 - 1/\alpha$  when  $\alpha \gg 1$  for  $S = 2$  (circles),  $S = 3$  (squares) and  $S = 4$  (diamonds) ( $M = 10$ ,  $300P$  iterations, average over 100 samples). The continuous lines are theoretical predictions from the exact solution.

Fig. 2.13 and 2.14 report that the theoretical predictions fully agree with these equations.

The exact solution for stationary states can be extended to any  $S$  (see appendix 8.2), and also yields excellent results. Note that for all  $S$ ,  $\sigma^2/N \simeq 1 - 1/\alpha$  for  $\alpha \gg 1$  (see Fig. 2.15).

### 2.15.2 Solution for real histories

The dynamical process on histories does matter. Indeed, by doing careful numerical simulations, one clearly sees that in the asymmetric phase,  $\sigma^2/N$  and  $H/N$  do depend on the nature of histories. Therefore, the solution found in the previous subsection has to be generalized.

The first step is to characterize the properties of the dynamics of real histories, which amounts to study randomly biased diffusion on De Bruijn graphs. This gives a relationship between the strength of the biases and the typical deviation  $\delta\rho^\mu = \rho^\mu - 1/P$  from the uniform distribution they cause. Then we move to the MG and quantify the bias which agents induce on the dynamics of  $\mu$  in the asymmetric phase. Using a simple parametrization of  $\rho^\mu$  which is inferred from numerical data, the calculus of the previous subsection can be generalized to the real histories case. This leads to a self-consistent equation between the asymmetry of the game and the diffusion bias, which can be solved [39].

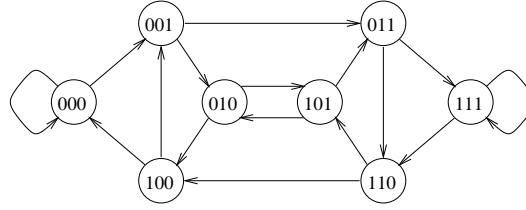


Figure 2.16: De Bruijn graph of order 3

### De Bruijn graphs

A history  $\mu(t)$  can be encoded by a binary sequence of length  $M$  which consists of  $M$  ordered elements  $\{b(t-M), \dots, b(t-1)\}$  where  $b$  is a letter belonging to the alphabet  $\{0, 1\}$ . The next history  $\mu(t+1)$  is obtained by adding  $b(t)$  to the right of  $\mu(t)$  and erasing  $b(t-M)$ . Thus, for a given  $\mu(t)$ , there are two possible  $\mu(t+1)$ , which we call “next neighbours”. This updating rule defines the De Bruijn graph [40] of order  $M$  (see Fig. 2.16 for an example).

Let  $G$  be the  $P \times P$  adjacency matrix of the De Bruijn graph of order  $m$ . if we adopt the convention that its elements are labelled by the decimal value of the binary strings, that is,  $\mu = 0, \dots, P-1$ ,

$$G_{\mu,\nu} = \delta_{[2\mu\%P],\nu} + \delta_{[2\mu\%P]+1,\nu} \quad (2.87)$$

where  $A\%B$  stands for the remainder of the division of  $B$  by  $A$  and  $\delta_{i,j}$  is the Kronecker symbol. The adjacency matrix for  $M = 3$  is :

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (2.88)$$

### Unbiased diffusion

The unbiased diffusion is defined as follows : a particle moves on the directed De Bruijn graph  $G$  and at each time step  $t$ , it jumps with equal probability to one of the next neighbours of the vertex it stands on at this time. Thus the transition probabilities matrix is  $W_0 = G/2$ . In the long run, the fraction of

time spent on vertex  $\nu$  is given by  $[(W_0)^\infty]_{0,\nu}$ . It can be seen (see appendix) that

$$[(W_0)^k]_{\mu,\nu} = \frac{1}{2^k} \sum_{n=0}^{2^k-1} \delta_{[2^k \mu \% P] + n, \nu}. \quad (2.89)$$

In particular,  $(W_0^{M+k})_{\mu,\nu} = \frac{1}{P}$  for all  $k \geq 0$ , that is, all strings  $\mu$  are visited with the same frequency  $\rho^\mu = 1/P$ .

In order to have a intuitive feeling of those graphs, we write them for  $M = 3$  :

$$W_0^2 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad W_0^3 = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (2.90)$$

### Randomly biased diffusion

The perturbations are introduced by adding a term to the transition probabilities matrix  $W_\epsilon = W_0 + \epsilon W_1$  where  $\epsilon$  quantifies the asymmetry and  $W_1$  contains the disorder  $\xi$

$$(W_1)_{\mu,\nu} = (-1)^\nu \xi_\mu (W_0)_{\mu,\nu} \quad (2.91)$$

where the  $\xi$  are iid from the pdf  $P(\xi) = 1/2 \delta(\xi - 1) + 1/2 \delta(\xi + 1)$  and the  $(-1)^\nu$  comes from the normalization of the perturbed probabilities. We are looking for the stationary transition probabilities, i.e.,  $W_\epsilon^\infty$  such that  $W_\epsilon^\infty = \lim_{k \rightarrow \infty} (W_\epsilon)^k$ . It exists since  $W_\epsilon$  is a bounded operator. Its formal series expansion in  $\epsilon$  is noted by  $W_\epsilon^\infty = \sum_{k \geq 0} \epsilon^k W_k^\infty$  where  $W_0^\infty$  is a matrix whose all coefficients are equal to  $1/P$  (see above). The relationship  $W_\epsilon^\infty = W_\epsilon^\infty W_\epsilon$  provides the recurrence

$$W_k^\infty = W_k^\infty W_0 + W_{k-1}^\infty W_1 \quad (2.92)$$

Since  $W_k^M W_0^\infty = 0$ , we iterate  $M - 1$  times this equation by replacing  $W_k^\infty$  with  $W_k^\infty W_0 + W_{k-1}^\infty W_1$  in the r.h.s., yielding to

$$W_k^\infty = W_{k-1}^\infty W_1 V = W_0^\infty [W_1 V]^k, \quad (2.93)$$

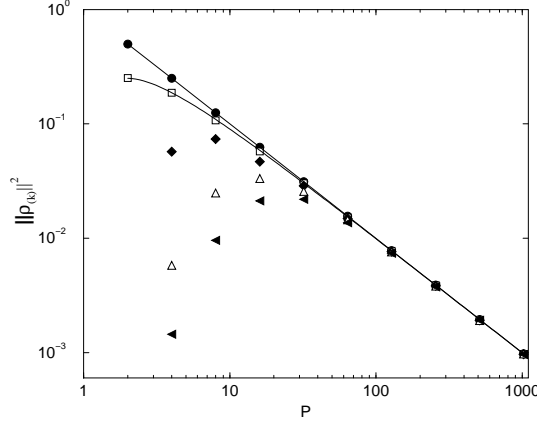


Figure 2.17: Squared norms of  $\rho(k)$  for  $k=0, \dots, 4$  (circles, squares, diamonds, triangles up, full triangles)(average over 500 samples). They decrease as  $1/P$  for large  $P$ . The continuous lines are exact theoretical predictions.

where  $V = \sum_{c=0}^{M-1} (W_0)^c$ . At this point, it is useful to remark that multiplying a matrix on the left by  $W_0^\infty$  is equivalent to averaging its columns :

$$(W_0^\infty A)_{\mu,\nu} = \sum_{a=0}^{P-1} (W_0^\infty)_{\mu,a} A_{a,\nu} = \frac{1}{P} \sum_{a=0}^{P-1} A_{a,\nu} = \text{average of } A\text{'s } \nu\text{-th column} \quad (2.94)$$

thus the matrices  $W_k^\infty$  consist of averages of columns of  $(W_1 V)^k$ . Therefore,  $(W_k^\infty)_{\mu,\nu}$  is the  $k$ -th order correction to the frequency of vertex  $\nu$ , that will be called  $\rho_{(k)}^\nu$  in the following. Note that  $\langle \rho_{(k)}^\nu \rangle_\xi = 0$  for all  $k \geq 1$ . The square root of the second moment of  $\rho_{(k)}^\nu$  averaged over the disorder gives an indication of the typical value of  $\rho_{(k)}^\nu$ . In appendix 2.15.2 we obtain the approximation

$$\langle ||\rho_{(k)}||^2 \rangle_\xi \sim \frac{(1 - 1/P)^k}{P}. \quad (2.95)$$

which is exact for the first order perturbation. Therefore  $\rho_{(k)}^\nu$  is of the same order as the unperturbed  $\rho_{(0)}^\nu$ , thus it cannot be neglected. Fig 2.17 shows that the behavior predicted by Eq (2.95) is indeed correct for large  $P$ .

Finally, one can estimate the second moment of  $\rho^\nu$ . If one supposes that the perturbations at different orders are independent, one obtains

$$\Delta \rho^2 = \frac{1}{P} \sum_{\nu=0}^{P-1} [\langle [\rho^\nu]^2 \rangle_\xi - \langle \rho^\nu \rangle_\xi^2] \simeq \frac{1}{P^2} \left[ \frac{1}{1 - (1 - 1/P)\epsilon^2} - 1 \right] \simeq \frac{1}{P^2} \frac{\epsilon^2}{1 - \epsilon^2} \quad (2.96)$$

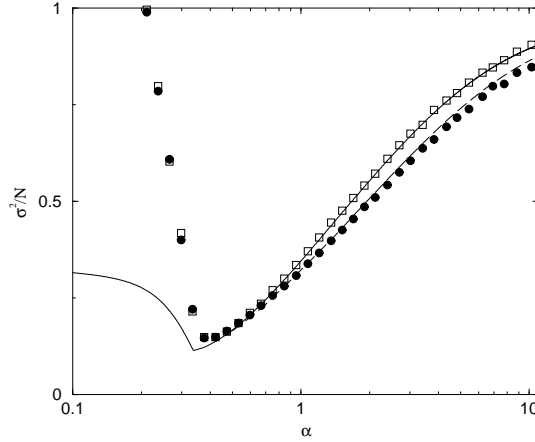


Figure 2.18: Comparison between the fluctuations of MG with uniformly sampled (squares) and real histories (full circles). In the symmetric phase, there are equal whereas they differ significantly in the asymmetric phase. Dashed and continuous lines correspond to theoretical predictions ; they overlap in the symmetric phase ( $M = 8$ ,  $S = 2$ ,  $300P$  iterations, average over 200 samples)

### Application to MG

Before doing any analytic calculations, it is worth looking at Figures 2.18 and 2.19 which clearly show that Cavagna's assertion is right as long as the system is in the symmetric phase. Indeed, if  $\langle A^\mu \rangle = 0$ , the transition probabilities from  $\mu$  to its next neighbours are unbiased, that is  $\epsilon^\mu = 0$  ; therefore in the symmetric phase, where  $\langle A^\mu \rangle = 0$  for all  $\mu$ , the frequencies of visit are uniform  $\rho^\mu = 1/P$ . Accordingly, numerical simulations show that these quantities collapse on the same curve.

As  $\alpha$  increases, the critical point is crossed, and  $\langle A^\mu \rangle \neq 0$  for some  $\mu$ . The dynamics of the history is biased on all such histories and consequently all macroscopic quantities are significantly different : both  $\sigma^2/N$  and  $H/N$  are lower for real histories than for uniformly sampled histories. This can be understood by the facts that  $\sigma^2/N$  and  $H$  are increasing functions of  $\alpha$  and that the biases on the De Bruijn graph of histories reduce the effective number of histories, that can be defined as  $2^{-\overline{\log_2 \rho}}$  : in other words, effective  $\alpha$  of MG with real histories is smaller than that of MG with uniform histories. This explanation is indeed confirmed by Fig 2.20 ; this shows the fraction of frozen agents<sup>23</sup>  $\phi$  which is a decreasing function of  $\alpha$  in the asymmetric phase. As expected from the above argument,  $\phi$  of MG with real histories is

<sup>23</sup>See [24] : they are agents that stop to be adaptive.

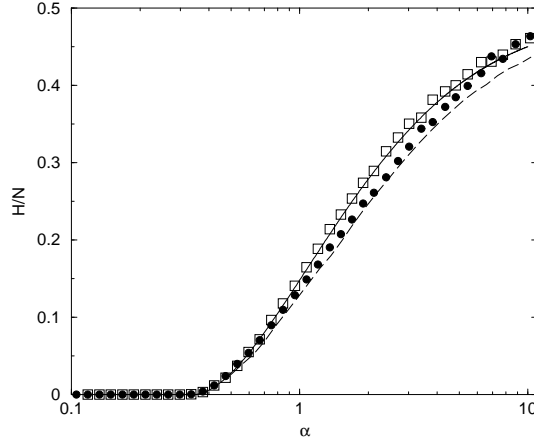


Figure 2.19: Comparison between available information of MG with uniformly sampled (squares) and real histories (circles). Dashed and continuous lines correspond to theoretical predictions ( $M = 8$ ,  $S = 2$ ,  $300P$  iterations, average over 200 samples)

larger than that of MG with uniformly sampled histories.

The bias  $\epsilon^\mu$  on a particular history can be estimated for large  $N$  : in this limit  $A^\mu$  is a Gaussian variable with average  $\langle A^\mu \rangle$  and variance  $\langle (A^\mu)^2 \rangle - \langle A^\mu \rangle^2$ , leading to

$$\epsilon^\mu = \langle \text{sgn}(A^\mu) \rangle \simeq \epsilon_{th}^\mu = \text{erf} \left( \sqrt{\frac{\langle A^\mu \rangle^2}{2[\langle (A^\mu)^2 \rangle - \langle A^\mu \rangle^2]}} \right). \quad (2.97)$$

Fig 2.21 confirms the validity of Eq. (2.97). The figure also shows that the  $\epsilon^\mu$  are unevenly distributed : they are not equal even if the system is deep in the asymmetric phase ( $\alpha \simeq 8.5$  in this figure). Indeed, as a function of  $\mu$ ,  $\langle A^\mu \rangle$  is a random variable with average 0 and variance  $H$  which is an increasing function of  $\alpha$ . Since we studied diffusion of perturbed graphs with only one parameter  $\epsilon$ , we have to map all  $\epsilon^\mu$  onto a scalar quantity, so that we define  $\epsilon$  as the non weighted average<sup>24</sup> of  $\epsilon^\mu$  over the histories. For large  $P$ ,  $\epsilon$  can be approximated by

$$\bar{\epsilon}_{th} = 2 \int_0^\infty dA \frac{e^{-\frac{A^2}{2H}}}{\sqrt{2\pi H}} \text{erf} \left( \frac{A}{\sqrt{2(\sigma^2 - H)}} \right). \quad (2.98)$$

<sup>24</sup>This is clearly an important assumption, but the diffusion on De Bruijn graphs with one  $\epsilon^\mu$  per site leads to a much greater complexity. As it appears on Fig 2.18, 2.19 and 2.24, this assumption is not unrealistic.

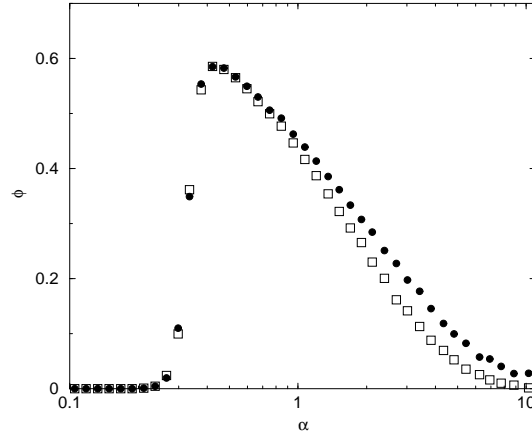


Figure 2.20: Comparison between the fraction of frozen agents in MG with uniformly sampled (squares) and real histories (circles). In the symmetric phase, there are equal whereas they differ significantly in the asymmetric phase ( $M = 8$ ,  $S = 2$ ,  $300P$  iterations, average over 200 samples)

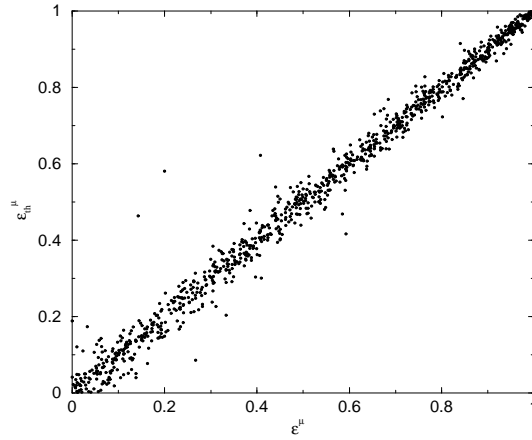


Figure 2.21:  $\epsilon_{th}^\mu$  of Eq (2.97) vs real  $\epsilon^\mu$  ( $M = 10$ ,  $N = 121$ ,  $S = 2$ ,  $1000P$  iterations)

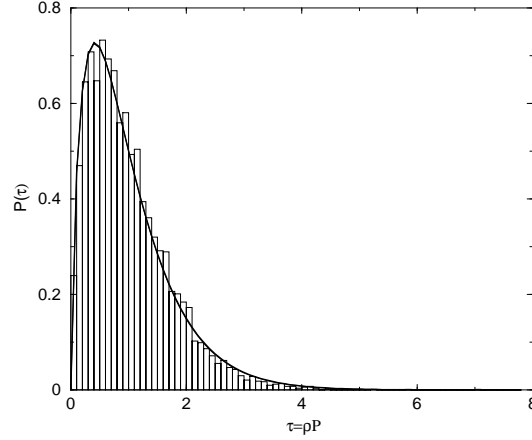


Figure 2.22: Distribution of the frequency of visit of the histories in the minority game. The continuous line is the best fit for a pdf given by Eq (2.99) ( $M = 13$ ,  $N = 801$ ,  $S = 2$ ,  $400P$  iterations)

Here both  $H$  and  $\sigma^2$  are computed analytically by modifying the replica calculus (see below). They depend however on the distribution  $\rho^\mu$ . In order to make Eq. (2.98) a self-consistent equation for  $\bar{\epsilon}_{th}$ , we need to parameterize the distribution of  $\rho^\mu$  by  $\bar{\epsilon}_{th}$  itself.

We could not find *ab initio* the analytic form of the pdf of  $\{\rho^\mu\}$ , but Fig 2.22 shows that

$$P(\tau) \simeq \frac{(\lambda + 1)^{\lambda+1}}{\Gamma(\lambda + 1)} \tau^\lambda e^{-(\lambda+1)\tau} \quad (2.99)$$

is a very good approximation for the pdf of  $\rho = \tau/P$ . The parameter  $\lambda$  is easily connected to  $\bar{\epsilon}_{th}$  :

$$\langle \tau^2 \rangle - \langle \tau \rangle^2 = \frac{1}{1 + \lambda} = P^2 \Delta \rho^2 \simeq \frac{\bar{\epsilon}_{th}^2}{1 - \bar{\epsilon}_{th}^2} \quad (2.100)$$

where we used Eq. (10). This gives  $\lambda \simeq (1 - 2\bar{\epsilon}_{th}^2)/\bar{\epsilon}_{th}^2$ . Note that this approximation requires  $\bar{\epsilon}_{th} < 1/\sqrt{2}$ .

This turns Eq. (2.98) into an equation for  $\bar{\epsilon}_{th}$ , and the theory is self consistent. Figure 2.23 reports measured  $\epsilon$  and its approximation  $\bar{\epsilon}_{th}$ . What clearly appears from this figure is that  $\epsilon$  is far from being negligible, and that  $\bar{\epsilon}_{th}$  is a quite good approximation to  $\epsilon$ .

We can also check the validity of Eq. (2.96) against the self-consistent theory. Fig 2.24 shows that Eq. (2.96) is in good agreement with numerical simulations as long as all histories are visited. In addition the approximation



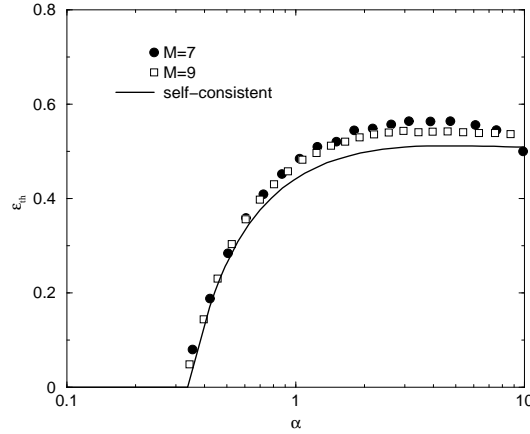


Figure 2.23:  $\epsilon$  versus  $\alpha = P/N$  ( $M = 7$  and  $M = 9$ ,  $S = 2$ ,  $300P$  iterations, average over 200 samples). The straight line is  $\bar{\epsilon}_{th}$ , the theoretical prediction of the self consistent theory.

$\bar{\epsilon}_{th}$  for  $\epsilon$  leads to qualitatively similar results, but underestimates  $\Delta\rho^2$  because  $\bar{\epsilon}_{th} < \epsilon$  (see figure 2.23).

The self-consistent replica calculation for the Minority Game of refs. [32, 34, 35] with the ansatz  $\rho = \tau/P$  and  $\tau$  given by the pdf (2.99) is discussed in the appendix. Fig 2.18 and 2.19 indicate that analytic predictions are well supported by numerical simulations.

In the asymmetric phase, which is arguably the most relevant and interesting in the MG [35], all quantities of MG change significantly if one replaces real histories with random uniform histories. A dependence on the frequencies  $\rho^\mu$  does not necessarily imply the relevance of the detailed dynamics of the histories. If the histories  $\mu$  were drawn randomly from the “correct” distribution  $\rho^\mu$ , the results would be the same (actually it suffices to know the pdf of  $\rho^\mu$ ). The problem is that the distribution  $\rho^\mu$  depends on the asymmetry  $\langle A^\mu \rangle$ , which in turn depends on the microscopic constitution of all agents [24]. In other words,  $\rho^\mu$  is a self-consistently determined quantity and hence it is only known *a posteriori*.

Therefore the dynamics of histories cannot be considered as irrelevant. Indeed, even for the canonical MG, it is relevant and cannot be replaced by randomly drawn histories. In addition, for many extensions and variations of the MG, the dynamics of histories is not only relevant, but crucial (see chapter 4).

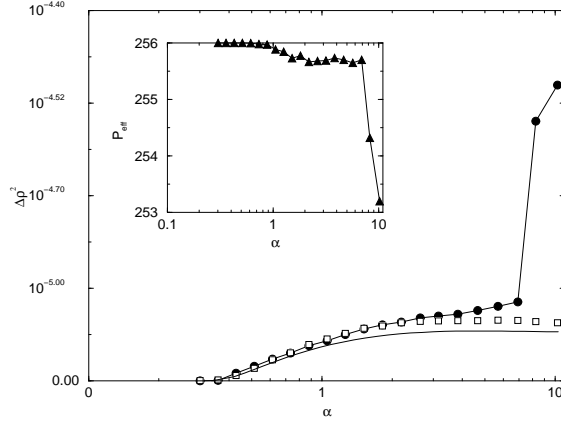


Figure 2.24: Inhomogeneity of the frequency of histories  $\Delta\rho^2$  versus  $\alpha = P/N$  from : numerical simulations (full circles), Eq (2.96) with  $\epsilon$  from numerical simulations (void squares) and Eq (2.96) with  $\bar{\epsilon}_{th}$  (continuous line) ; inset : average number of visited histories versus  $\alpha$  ; ( $M = 8$ ,  $S = 2$ ,  $300P$  iterations, average over 200 samples).

### Appendix : $(W_0^k)_{\mu,\nu}$

Let us prove by induction that

$$(W_0^k)_{\mu,\nu} = \frac{1}{2^k} \sum_{n=0}^{2^k-1} \delta_{[2^k \mu \% P] + n, \nu} \quad (2.101)$$

It is sufficient to calculate explicitly  $(W_0^k)_{\mu,\nu}$  from  $(W_0^{k-1})_{\mu,\nu}$

$$\begin{aligned} (W_0^k)_{\mu,\nu} &= \sum_{\tau=0}^{P-1} (W_0)_{\mu,\tau} (W_0^{k-1})_{\tau,\nu} \\ &= \sum_{n=0}^{2^{k-1}-1} \left\{ \delta_{[2^{k-1}([2\mu \% P]) \% P] + n, \nu} + \delta_{[2^{k-1}([2\mu \% P] + 1) \% P] + n, \nu} \right\} \\ &= \frac{1}{2^k} \sum_{n=0}^{2^k-1} \delta_{[2^k \mu \% P] + n, \nu} \end{aligned} \quad (2.102)$$

since  $A(B \% P) \% P = AB \% P$  and  $(2^k \mu + 2^{k-1}) \% P = [2^k \mu \% P] + 2^{k-1}$  if  $P = 2^M$  and  $k \leq M - 1$ .

**Appendix :**  $\langle ||\rho_{(k)}||^2 \rangle_\xi$ 

In order to simplify the notations, we define

$$(X^c)_{\mu,\nu} = \sum_{n=0}^{2^c-1} \delta_{[2^{c+1}\mu \% P] + n, \nu} - \delta_{[2^{c+1}\mu \% P] + n + 2^c, \nu} \quad (2.103)$$

This matrix is such that

$$(X^c)_{\mu,\nu} = \begin{cases} 1 & \text{if } 2^{c+1}\mu \% P \leq \nu < [2^{c+1}\mu \% P] + 2^c \\ -1 & \text{if } [2^{c+1}\mu \% P] + 2^c \leq \nu < 2^{c+1}(\mu + 1) \% P \\ 0 & \text{else} \end{cases} \quad (2.104)$$

With this formalism, one can write  $W_1 V$  as

$$(W_1 V)_{\mu,\nu} = \frac{\xi_\mu}{2} \sum_{c=0}^{M-1} \frac{1}{2^c} (X^c)_{\mu,\nu} \quad (2.105)$$

Let us calculate the perturbation at order 1 : one has to compute  $||\rho_{(1)}||^2$  in order to obtain an estimation of the typical value of a generic  $\rho_{(1)}^\nu$  : since the  $\xi$  are uncorrelated and  $\sum_{\nu=0}^{P-1} (X^c)_{\mu,\nu} (X^d)_{\mu,\nu} = 2^{c+1} \delta_{c,d}$ ,

$$\langle ||\rho_{(1)}||^2 \rangle_\xi = \frac{1}{4P^2} \sum_{\mu,\nu=0}^{P-1} \sum_{c=0}^{M-1} \frac{[(X^c)_{\mu,\nu}]^2}{2^{2c}} = \frac{(1 - 1/P)}{P} \quad (2.106)$$

The next orders of perturbation are much harder to handle. However, for large  $P$ , one can approximate them by supposing that

$$\langle ||\rho_{(k)}||^2 \rangle_\xi \sim (1 - 1/P) \langle ||\rho_{(k-1)}||^2 \rangle_\xi = \frac{(1 - 1/P)^k}{P}. \quad (2.107)$$

Consequently,  $\rho_{(k)}^\nu \sim (1 - 1/P)^{k/2} \frac{1}{P} \simeq \frac{1}{P}$  at leading order

**Appendix : replica calculus for real histories**

The generalization of the calculus of previous subsection to  $\rho^\mu = \tau^\mu / P$  drawn from the pdf given by Eq (2.99) and (2.98) is straightforward ; the free energy reads in the thermodynamic limit

$$\begin{aligned} f(\beta, Q, q, R, r) &= \left\langle \frac{\alpha}{2\beta} \log[1 + \chi\tau] \right\rangle_\tau + \frac{1+q}{2} \left\langle \frac{1}{\frac{1}{\tau} + \chi} \right\rangle_\tau \\ &+ \frac{\alpha\beta}{2} (RQ - rq) - \frac{1}{\beta} \left\langle \log \int_{-1}^1 ds e^{-\beta(\zeta s^2 - \sqrt{\alpha r} z s)} \right\rangle_z \end{aligned} \quad (2.108)$$

where  $\chi = \beta(Q - q)/\alpha$  and  $\zeta = -\sqrt{\alpha/r} \beta(R - r)$ . Next, the  $\beta \rightarrow \infty$  limit is taken while keeping finite  $\chi$  and  $\zeta$ . One obtains

$$H_c = \frac{1+Q}{2} \left[ \left\langle \frac{1}{\frac{1}{\tau} + \chi} \right\rangle_\tau - \chi \left\langle \frac{1}{[\frac{1}{\tau} + \chi]^2} \right\rangle_\tau \right] \quad (2.109)$$

and as previously,

$$\sigma_c^2 = H_c + \frac{1-Q}{2} \quad (2.110)$$

where  $Q$  and  $\chi$  take their saddle point values, given by the solution of

$$Q(\zeta) = 1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} - \left(1 - \frac{1}{\zeta^2}\right) \operatorname{erf}\left(\frac{\zeta}{\sqrt{2}}\right) \quad (2.111)$$

$$Q(\zeta) = \frac{1}{\alpha} \left[ \frac{\operatorname{erf}(\zeta/\sqrt{2})}{\chi \zeta} \right]^2 \frac{1}{\left\langle \frac{1}{[1/\tau + \chi]^2} \right\rangle_\tau} - 1 \quad (2.112)$$

$$\chi \left\langle \frac{1}{\frac{1}{\tau} + \chi} \right\rangle_\tau = \frac{\operatorname{erf}(\zeta/\sqrt{2})}{\alpha} \quad (2.113)$$

Eqs (2.112) and (2.113), together with Eq (2.98), form a closed set of equations that has to be solved numerically. Note that as in the random histories case,  $\chi$  becomes infinite at the critical point, where  $\alpha_c = \operatorname{erf}(\zeta/\sqrt{2})$ .

# Chapter 3

## Variations of MG

After having defined an extremely simple model, it is always tempting to experiment variations of the model. There are two main goals : to test which properties of the MG are robust under modifications, and to obtain new results.

### 3.1 Payoff

The first modification we did (in first chapter) concerned the payoff. Originally, it was defined as  $G(x) = \text{sgn}x$ , but it turned out that a linear payoff simplifies a lot all mathematical computations. Two questions are still open at this point of the discussion :

- How do the properties of the system vary when the payoff is changed ?
- How far the mathematical formalism can be applied ? Is it possible to find the exact solution for various payoffs ?

The first question has been partially answered in [43, 24, 47] : there are no qualitative changes to the structure of the MG when the payoff is changed. However, in the symmetric phase, depending on the payoff function, the probability distribution function of  $A(t)$  is modified : for some payoff functions,  $A(t) \propto N \forall t$ , whereas  $A \propto \sqrt{N}$  or  $N$  otherwise [22]. In addition, in the asymmetric phase, the stationary state depends on the payoff function, implying that also the macroscopic quantities vary. What remain is that the structure of the game (phase transition) is conserved.

The fact that there is a phase transition implies that some quantity characterizing the asymmetry of the game is minimized. As already discussed,

the central equation in MGs is that of strategies score update. For a given payoff function,

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}^\mu G[A(t)] \quad (3.1)$$

The thermodynamic limit has then to be taken. The difficulty arise from the rescaling of the payoff function, so for the moment being, assume that it is properly rescaled and noted by  $G_N(x)$  such that the proper time derivate is defined

$$\dot{U}_{i,s}^c(\tau) = - \lim_{\delta\tau \rightarrow 0} \limth \frac{1}{\delta\tau} \sum_{t=\tau P}^{(\tau+\delta\tau)P-1} a_{i,s}^{\mu(t)} G_N[A(t)] \quad (3.2)$$

The condition on  $G_N$  is that  $\limth G_N[A(t)]$  has to be defined for  $A(t) \sim \sqrt{N}$ . There are several choices, depending on the choice of the payoff function. For instance, if  $G(x) \sim x^\gamma$ ,  $G_N(x) = x/N^{(1+\gamma)/2}$ . Up to now, no Lyapunov function could be found for a general payoff function. Good candidates are  $\overline{G(A)^2}$  and  $\overline{G(\langle A \rangle)^2}$ .

## 3.2 Evolution

In any competitive situation, there are winners and losers. In real life, it is impossible to indefinitely lose, so that in principle, the worst performers eventually disappear. As originally defined, the Minority Game does not include an evolutionary process on the strategies, since agents' strategies sets are fixed during the game, and renewal is not allowed, for the sake of extreme simplicity. A selection-evolution is a fundamental process in competitive situations, therefore it is mandatory to implement evolution in the MG.

But, again, one has to design selection processes in the simplicity spirit of MG, that is, without introducing several parameters.

### 3.2.1 Evolutionary strategies

Evolution in strategies is maybe the simplest evolutionary process. When an agent switches to another strategy, it is because she feels that the one she had been using is less rewarding than the one she will be using from now on. A natural reaction is to discard, with some probability  $p_{ev}$ , the strategy she has been using and to draw a completely new strategy. Note that with this setup, an agent that has a good strategy does not evolve. Fig 3.1 shows that this leads to a MG with very different properties. In particular, the symmetric phase disappears, and agents are able to achieve a much better degree of cooperation.

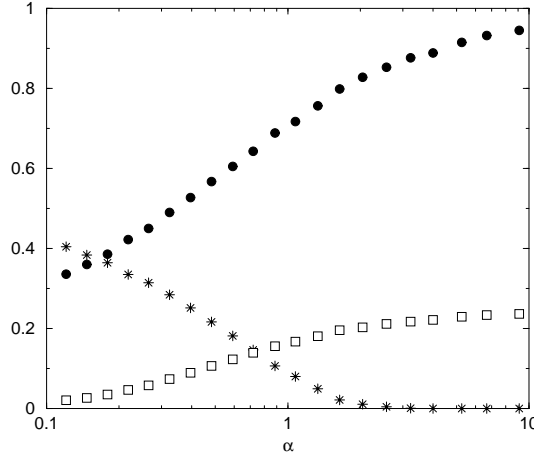


Figure 3.1:  $\sigma^2/N$  (circles),  $H/N$  (squares) and  $\phi$  (stars) versus  $\alpha$  for MG with evolutionary strategies ( $p_{\text{ev}} = 0.1$ ,  $M = 8$ ,  $S = 2$ ,  $400P$  iterations, average over 100 samples). There is neither a phase transition, nor herding effect.

### 3.2.2 Darwinism

The previous evolution permits agents to self-modify. However, in real life, worst agents do disappear. The following Darwinism process is studied in [20, 22] : every  $\tau$  time steps, the worst player is replaced by a clone of the best, except that one strategy of the clone is redrawn with a small probability  $p$  in order to allow regeneration, and that the virtual gains of the strategies are reset to zero, like a new born baby. If the new player is a pure clone of the best one, one says that both belong to the same species. A demonstration of the Darwinism's benefits can be seen in figure 3.2 : the latter shows a comparison between the variance of the attendance signal with and without Darwinism. The region where  $\sigma^2/N < 1$  is much greater when evolution takes place, in particular the  $\alpha < \alpha_c$  region is less affected by the overcrowding. Note that increasing  $p$  lowers  $\sigma^2/N$ , that is, the mutations are useful. The asymptotic behaviors of  $\sigma^2/N$  in the  $\alpha \rightarrow \infty$  limit depends on  $p$ , but the Darwinism is anyway harmful in this region. One also sees that the minimum of  $\sigma^2/N$  is not at the same  $\alpha$  than without Darwinism.

Since  $p < 1$ , the diversity (the number of different species) is reduced by the evolutionary process, and tends to a value that depends on  $p$ . When  $p$  decreases, the diversity decreases too, but stays over  $N/2$  even if  $p = 0$ . Even more, when one starts with only one species, but with random virtual gains, the system performs first very badly, slowly improves itself, and reaches an optimal diversity always greater than  $N/2$ . When the diversity is stable,

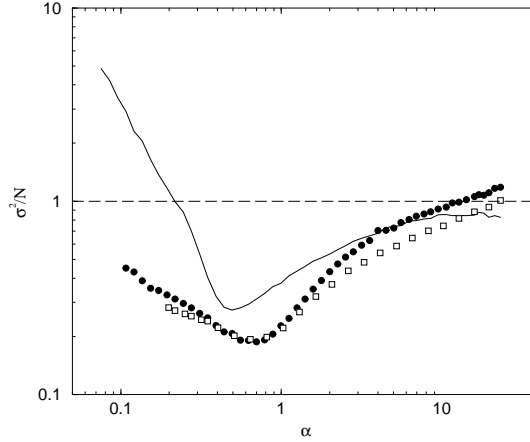


Figure 3.2: Dependence of  $\sigma^2/N$  in  $\rho$  with Darwinism for  $p = 0.01$  (stars) and  $p = 0.1$  (squares) ( $M = 5$ ,  $S = 2$ ,  $\tau = 10$ , average over 50 samples). The dashed line represents the random performance and the continuous line is  $\sigma^2/N$  without Darwinism.

the system is in a stationary state, and one can study several distributions. First, figure 3.3 shows the distribution of the species life time. It is a power law with exponent  $-2.02 \pm 0.02$ , which does not depend on the parameters ( $\tau$ ,  $p$ , ...). Fortunately, it is the same exponent as found in real life evolution [56]). Figure 3.4 helps to understand what happens. The average gain of several players during the game is plotted. One can see players remaining in the game, some other resisting for a while, then disappearing. The player that replaces a dead one is followed. This figure shows that the fluctuations of the average gain is very high when a player is young. Consequently such a player's death or reproduction are more likely than those of an old player ; this leads to punctuated equilibrium, which may be the origin of the power law distribution.

After sufficient time, it is interesting to rank the species according to the size of their population, and to Zipf plot the number of members composing each species against their rank (see figure 3.5). One finds a power law that depends at least on  $p$  ; indeed, if  $p = 1$ , the best performer will never be completely cloned and one obtains a flat line : every type of player has only one member.

In order to test the Darwinian process, we tried to apply the inverse process : a clone of the worse player replaces the best one. The figure 3.6 shows that the global performance suffers a lot.

One can wonder why the coordination is better in the crowded phase although there are multiple clones of a lot of players. This is not really



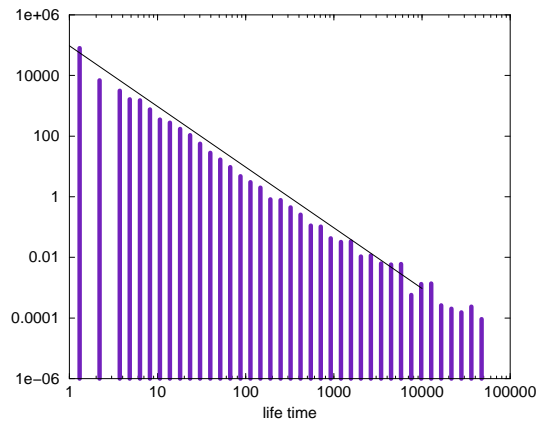


Figure 3.3: Histogram of species life time ( $N=101$ ,  $M=8$ ,  $S=2$ ,  $NIT=500000$ ,  $\tau=10$ ). The straight line has a -2 exponent.

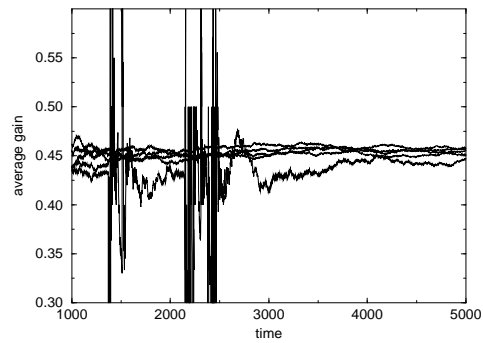


Figure 3.4: Temporal evolution of the average gain of several players ( $N = 51$ ,  $M = 10$ ,  $S = 2$ ,  $\tau=10$ ). Note the consequences of the death of a player at  $t=1450$  and  $t=2100$ .

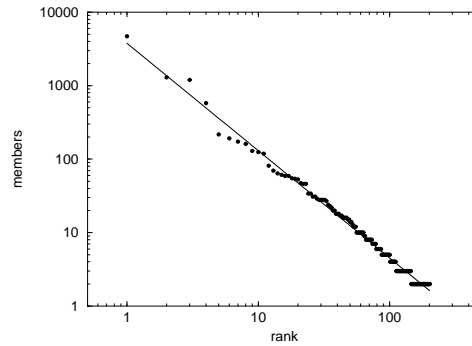


Figure 3.5: Rank of the members in a type of player ( $N = 20001$ ,  $S = 5$ ,  $M = 5$ ,  $\tau = 10$ ). It is a power-law, but the exponent depends on the system's parameters.

understood. In the asymmetric phase, the worst agent is very likely to be non frozen, and the best agent is either very young, or frozen. Cloning a frozen agent means that  $H$  is lowered, and accordingly  $\sigma^2$ . The fact that the fluctuations are minimal at a higher  $\alpha$  is consistent with this explanation : indeed, less agents are needed to cancel  $H$ . On the other hand, the fluctuations are also reduced in the symmetric phase, probably because the score of the freshers are set to zero : it is consequently more difficult for agents to coordinate, reducing accordingly the fluctuations.

Finally, note that the Darwinian processes described above is not the only one that has been studied in a MG (see ref. [57]), but they are thoses with the least parameters.

### 3.3 Local or personal information

In the standard MG, a common piece of history — random or real — is given to all agents. Some authors considered cases where agents do not share the same piece of information : they have rather their own one,  $\mu_i(t)$ . For instance,  $\mu_i(t)$  can encode the  $M$  last previous actions of agent  $i$  [48] ; agents still play a global MG ; the control parameter is  $\sqrt{P}/N$  in this case. In addition, some authors studied other variants where  $\mu_i(t)$  encodes the previous choice of  $M$  neighbours (local information) and agents play a global MG ([49] for random neighbours, [50] for agents on a circle). Finally, agents are placed on a regular lattice, play a local MG (there are as many MG as agents), and the local piece of information encode the last  $M$  minority choice of the local MG [51] ; in this case, there is no more frustration, that this, no

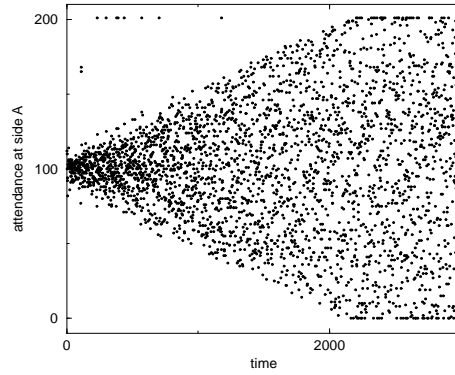


Figure 3.6: Anti Darwinism : a clone of the worst player replaces the best. The attendance at side A is plotted with dots.

more competition.

In ref. [49] it is claimed (but not showed) that the major cause of cooperation is the fact that all agents share the same piece of information. This happens to be wrong, as shown for instance by ref. [50]. The point is rather that agents cannot cooperate if the pieces of information agents receive are not correlated [48]. In all the cases reviewed above, the histories are — sometimes subtly — correlated. This is because they live in correlated environments. For instance, consider the following case :  $\mu_i(t)$  encodes the last  $M$  successes or failures of agents  $i$  ( $1 = \text{success}$ ,  $0 = \text{failure}$ ). Cooperation still arises, as illustrated by figure 3.7, but is less and less pronounced as  $M$  is increased, because the correlation is hard to attain for large  $M$ . Note that herding effects also take place (as in Refs. [50, 48]), showing that a phase transition occurs even in MGs with local or personal histories. This is a consequence of the very behavior of agents, and is a general feature of competition model with such agents [41].

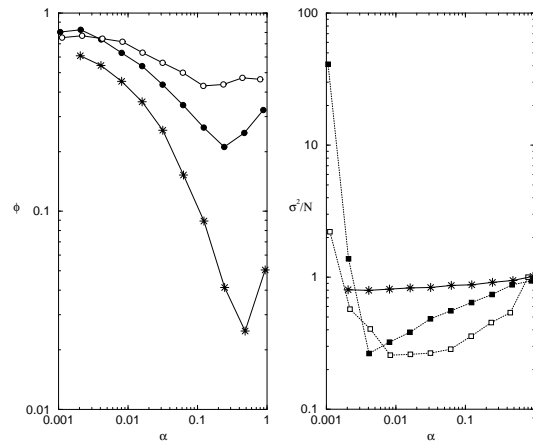


Figure 3.7:  $\phi$  and  $\sigma^2/N$  versus  $\alpha$  for agents playing a global MG and whose personal histories reflect their gain/losses, for  $M = 2$  (circles),  $M = 3$  (squares) and  $M = 4$  (stars). (300P iterations, average over 100 samples).

# Chapter 4

## From less naive to sophisticated agents

In this chapter, the perception that agents have of the success of their strategies is shown to play a crucial role on the Lyapunov function, thus all the properties of the system.

Agents as originally defined are naive in the sense that they do not account for their impact on the game. Indeed the strategies' scores are updated (Eq. (2.5)) as if the outcome  $A(t)$  did not depend on their action. In other words, agents behave as if they were facing an external process. This is justified if agents do not even know that they belong to the game, and that the latter is a minority game. On the contrary, if they are aware of the fact that they have an impact, they will be willing to modify their behavior, that is, the way they reward their strategies. The key observation is that they overestimate their non played strategies with respect to that they are playing. The question is : by how much.

### 4.1 Partial information

Suppose that agents only know that they have an impact<sup>1</sup>, but that they have no mean to estimate it. They can arbitrarily decide that their impact is  $\eta$  ; in other words, they can compensate the overestimation of the strategies they do not play by adding a supplementary reward  $\eta$  to the strategy there are playing. Eq (2.5) becomes [35]

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}^{\mu(t)} A^{\mu(t)}(t)/P + \eta \delta_{s,s_i(t)}/P. \quad (4.1)$$

---

<sup>1</sup>this is already a very important information

After some easy algebra, one obtains the new dynamics of the probabilities  $\pi_{i,s}$  :

$$\frac{d\pi_{i,s}}{d\tau} = -\Gamma_i \pi_{i,s} \left\{ \sum_j \left[ \overline{a_{i,s} \langle a_j \rangle} - \overline{\langle a_i \rangle \langle a_j \rangle} \right] - \eta (\pi_{i,s} - |\pi_i|^2) \right\} \quad (4.2)$$

and the Lyapunov function for this dynamics :

$$H_\eta = H - \eta \sum_i |\pi_i|^2 = H - \eta Q \quad (4.3)$$

where  $Q = \sum_i |\pi_i|^2$ . Indeed, observing that

$$\frac{\partial H_\eta}{\partial \pi_{i,s}} = -2 \frac{dU_{i,s}}{d\tau}$$

and using Eq. (2.34), one finds that

$$\frac{dH_\eta}{d\tau} = \sum_i \frac{\partial H_\eta}{\partial \pi_i} \cdot \frac{d\vec{\pi}_i}{d\tau} = -2 \sum_i \Gamma_i \sum_{s=1}^S \pi_{i,s} \left( \frac{dU_{i,s}}{d\tau} - \vec{\pi}_i \cdot \frac{d\vec{U}_i}{d\tau} \right)^2 < 0. \quad (4.4)$$

The dynamics converges therefore to the minima of  $H_\eta$ . It is very interesting to study the effects of  $\eta$  on the system's behavior. For  $\eta \ll 1$ , as long as  $H \gg \eta$ , that is, in the asymmetric phase, reasonably far from the critical point, the properties of the modified MG are the same as that of the standard MG. Near the critical point,  $H \sim (\alpha - \alpha_c)^2$ , therefore, for  $\alpha \sim \alpha_c(1 + \sqrt{\eta})$ , both terms in  $H_\eta$  are equally important. Consequently,  $H \neq 0$  for all  $\alpha$  : there is no phase transition for  $\eta > 0$ . The symmetric phase completely disappears, and  $H/N, \sigma^2/N \rightarrow 0$  in the  $\alpha \rightarrow 0$  limit : the system is arbitrage free (efficient) ( $H = 0$ ) and Pareto efficient ( $\sigma^2/N = 0$ ) in this limit.

Fig. 4.1 illustrates the spectacular effects of even a small  $\eta$  :  $\sigma^2/N$  does no more display a herding behavior, while  $H/N$  seems to be quite the same with and without  $\eta$  ; note that there is no phase transition, and  $H$  does not vanish, since the fraction of frozen agents tends to 1 when  $\alpha$  decreases.

The detailed agents' behavior is also changed. One can redo the discussion of section 2.14 in order to include the  $\eta$  effect. One can show that individual payoffs increase when  $\eta$  increases in  $[0, 1]$ , which is consistent with the fact that  $\sigma^2$  decreases. The case  $\eta < 1$  is treated first. In the long run the perceived performance of strategy  $s$  is

$$\begin{aligned} \langle \Delta U_s \rangle &= -\overline{a_s \langle A_{-i} \rangle} - \vec{\pi} \cdot \overline{\vec{a} a_s} + \eta \pi_s \\ &\cong -\overline{a_s \langle A_{-i} \rangle} - (1 - \eta) \pi_s \end{aligned} \quad (4.5)$$

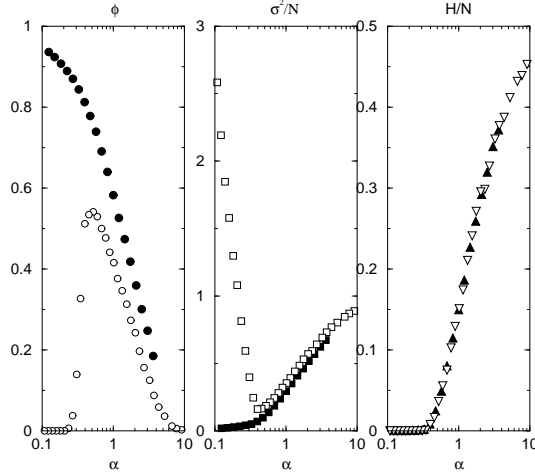


Figure 4.1:  $\phi$ ,  $\sigma^2/N$  and  $H/N$  for  $\eta = 0$  (void symbols) and  $\eta = 0.1$  (full symbols) ( $P = 100$ ,  $300P$  iterations, average over 100 samples).

where, again, the approximation in Eq. (4.5) holds for  $P \gg 1$ . The average score increase of the played strategies is now.

$$v = -\frac{1}{n} \sum_{s=1}^n \overline{a_s \langle A_{-i} \rangle} - \frac{1-\eta}{n}$$

Then Eq. (4.5) becomes

$$\pi_s = \frac{1}{n} + \frac{1}{1-\eta} \left( \frac{1}{n} \sum_{s'=1}^n \overline{a_{s'} \langle A_{-i} \rangle} - \overline{a_s \langle A_{-i} \rangle} \right).$$

The average payoff  $g = -\vec{\pi} \cdot \vec{a} \langle A_{-i} \rangle - 1$  delivered by a learning behavior with parameter  $\eta$  is

$$g = -\frac{1}{n} \sum_{s=1}^n \overline{a_s \langle A_{-i} \rangle} + \frac{1}{1-\eta} \sum_{s=1}^n \left( \overline{a_s \langle A_{-i} \rangle} - \frac{1}{n} \sum_{s'=1}^n \overline{a_{s'} \langle A_{-i} \rangle} \right)^2 - 1 \quad (4.6)$$

which is an increasing function of  $\eta$  for  $\eta < 1$ . Indeed at fixed  $n$ , this is trivially true. With some more algebra, it is easy to check that  $n$  is a non-increasing function of  $\eta$  and that  $g$  increases as  $n$  decreases. This means that for  $\eta < 1$  average payoffs are non-decreasing functions of  $\eta$  as claimed.

When  $\eta \rightarrow 1$  the only possible solution is that with  $n = 1$  which means that the agent plays her best response to  $A_{-i}$ . For  $\eta > 1$  the agent overweights the performance of her strategies. As a result she sticks to only one of her strategies, i.e.  $n = 1$ , but that need not be her best one. Without

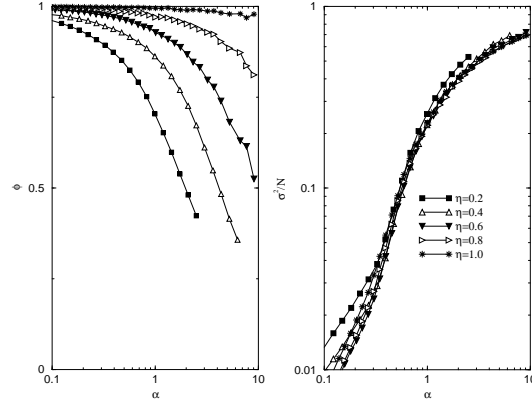


Figure 4.2: Fraction of frozen agents and normalized fluctuations versus  $\alpha$  for  $\eta$  ranging from 0.2 to 1 ( $P = 100$ ,  $S = 2$ ,  $300P$  iterations, average over 100 samples).

entering in too many details, let us only mention that for  $\eta > 1$  the agent plays always one strategy which is dynamically selected by initial conditions and stochastic fluctuations.

We expect that with  $\eta = 1$  agents behave almost optimally, in the sense that they converge to a stationary state which is close to a Nash equilibrium. Given the difference in the collective behavior of agents in the two cases – which may be appreciated comparing figure 2.1 for  $\eta = 0$  with figure 4.5 for  $\eta = 1$  – it is natural to ask what happens when  $\eta$  changes continuously from 0 to 1.

Fig. 4.2 illustrate the behavior of  $\phi$  and  $\sigma^2/N$  for various  $\eta$ . As  $\eta$  increases from 0 to 1,  $\phi$  increases and for  $\eta = 1$ , all agents are frozen ; this is consistent with the discussion above. Note that  $\sigma^2/N$  seems to first slightly decrease when  $\eta$  increases, and then to increase, specially for small  $\alpha$ . This is maybe due to very long time needed to reach equilibrium, thus, to too small simulation times. Figure 4.3 shows the analytical predictions for the dependence on  $\eta$  of  $\sigma^2/N$  for  $S = 2$ . These are based on the *replica symmetric ansatz* which is only valid for  $\alpha > \alpha_{\text{RSB}}(\eta)$ , where  $\alpha_{\text{RSB}}(\eta)$  marks a *replica symmetry breaking* phase transition, which will be discussed elsewhere in detail[61]. Here we just mention that  $\alpha_{\text{RSB}}(\eta) = 0$  for  $\eta < 0$ ,  $\alpha_{\text{RSB}}(0) = \alpha_c$  and  $\alpha_{\text{RSB}}(\eta) \geq 1 - 1/\sqrt{\pi\alpha}$  (for  $S = 2$  and)  $\eta > 0$ ). For  $\alpha < \alpha_{\text{RSB}}(\eta)$  the analytical results derived in the appendix 8.2 provides an approximate description of the behavior of the system which is however sufficient to appreciate the relevant features.

The most striking consequence of the result in fig. 4.3 is that the behavior of  $\sigma^2/N$  is quite different for  $\alpha > \alpha_c$  and for  $\alpha < \alpha_c$ . Indeed for large  $\alpha$ ,  $\sigma^2/N$  changes continuously with  $\eta$  whereas  $\sigma^2/N$  drops discontinuously to



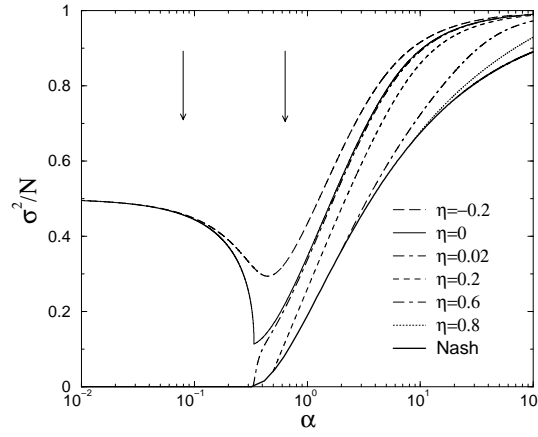


Figure 4.3: Theoretical estimate of global efficiency  $\sigma^2/N$  as a function of  $\alpha$  for  $S = 2$  and several values of  $\eta$  within the *replica symmetric ansatz*.

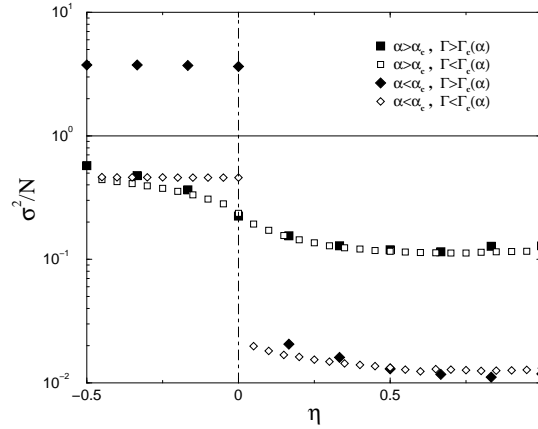


Figure 4.4:  $\sigma^2/N$  as a function of  $\eta$  for  $S = 2$  and  $\alpha \simeq 0.079 < \alpha_c \simeq 0.3374$  and  $\alpha \simeq 0.63 > \alpha_c$ . Results both of numerical simulations of the minority game and of the numerical minimization of  $H_\eta$  are shown.

zero as  $\eta \rightarrow 0$  for small  $\alpha$ . This feature is reproduced in figure 4.4 for two characteristic values of  $\alpha$  also shown as arrows in fig. 4.3. We show both the behavior derived from the numerical minimization of  $H_\eta$  and the behavior of the modified minority game with rewarding. Numerical results agree quite well in an intermediate range of values of  $\eta$  whereas for  $\eta > 0.5$  or  $\eta < -0.2$  some discrepancy – which we believe is due to finite size effects – is found. This effect can be even more spectacular when anti-persistence effects occur. Indeed the jump of  $\sigma^2/N$  at  $\eta = 0$  can be of several orders of magnitude !

The origin of this behavior lies in the dynamic degeneracy of the system for  $\alpha < \alpha_c$  and  $\eta = 0$ . Even an infinitesimal change in  $\eta$  can dramatically alter the nature of the minima of  $H_\eta$  : for negative  $\eta$  there is only one minimum which becomes shallower and shallower as  $\eta \rightarrow 0^-$ . At  $\eta = 0$  the minimum is always unique but it is no more point-like. Rather it is a connected set  $\mathcal{M}$ . An infinitesimal positive value of  $\eta$  is enough to lift this degeneracy and select only some extreme points of  $\mathcal{M}$  as the minima of  $H_\eta$ . The set of minima becomes suddenly disconnected. At fixed  $\alpha < \alpha_c$ , varying  $\eta$  across the transition  $H_\eta$  changes continuously – with a discontinuity in its first derivative – whereas  $G$  and hence  $\sigma^2/N$  change discontinuously with a jump.

The potential implications of this result are quite striking : *rewarding the strategy played more than those which have not been played by a small amount is always advantageous, both individually (see below) and globally. In particular, an infinitesimal reward is sufficient to avoid crowd effects when  $\alpha$  is small and to reduce the fluctuations by a finite amount.*

## 4.2 Full information

In the previous section, agents were able to account for their impact on the game, and did not know it exactly, because they only had a partial information.

The real question to be answered is “What would have been the outcome if I had played another strategy”. This requires the knowledge of  $A(t)$ , that is, agents need to have access to *full information*. In this case, agents can update their strategies score according to [35]

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}^{\mu(t)}(A(t) - a_{i,s_i(t)}^{\mu(t)} + a_{i,s}^{\mu(t)}) \quad (4.7)$$

or, with the notation  $A_{-i}(t) = \sum_{j \neq i} a_{j,s_j(t)}^{\mu(t)}$ ,

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}^{\mu(t)} A_{-i}^{\mu(t)}(t) - 1 \quad (4.8)$$

This equation suggests that, in first approximation we can regard the opponents of  $i$  as an external stationary stochastic process. It is known [42] that exponential learning with full information, for a single agent playing against a stationary stochastic process, converges to rational expectations, that is, he ends up using the strategy a deduction process would have selected. If this happens for all players the system converges to a Nash equilibrium. This is indeed what numerical experiments show (see figure 4.5).

The transformation to continuous time is exactly the same as previously, and Eq. (2.34) becomes

$$\frac{d\pi_{i,s}}{d\tau} = -\Gamma_i \pi_{i,s} \sum_{j,j \neq i} \left[ \overline{a_{i,s} \langle a_j \rangle} - \overline{\langle a_i \rangle \langle a_j \rangle} \right]. \quad (4.9)$$

Since the dynamics changed, the global quantity that agents minimize is different. In this case, the fluctuations  $\sigma^2 = \sum_{i,j \neq i} (\vec{\pi}_i \cdot \vec{a}_i)(\vec{\pi}_j \cdot \vec{a}_j) + N$  are a Lyapunov function under this dynamics. Indeed a little algebra leads to

$$\frac{d\sigma^2}{dt} = -2 \sum_i \sum_{s=1}^S \pi_{i,s} \left[ \frac{(a_{i,s} - \vec{\pi}_i \cdot \vec{a}_i) \sum_{j \neq i} \vec{\pi}_j \cdot \vec{a}_j}{\sum_{j \neq i} \vec{\pi}_j \cdot \vec{a}_j} \right]^2 \leq 0. \quad (4.10)$$

But the fluctuations are exactly equal to the total losses of agents. Therefore, Eq (4.10) implies that agents minimize their losses under this learning dynamics. In terms of Evolutionary Game theory, this means that they reach a Nash equilibrium. This is confirmed by the direct application of the multi-population standard replicator dynamics [17] (RD) which is known to lead to a Nash equilibrium and reads

$$\frac{d\pi_{i,s}}{dt} = -\pi_{i,s} \sum_{j,j \neq i} \left[ \overline{a_{i,s}(\vec{\pi}_j \cdot \vec{a}_j)} - \overline{(\vec{\pi}_i \cdot \vec{a}_i)(\vec{\pi}_j \cdot \vec{a}_j)} \right]. \quad (4.11)$$

Apart from the factor  $\Gamma_i$ , this coincides with the RD of Eq. (4.11). Again  $\sigma^2$  is minimized along the trajectories of Eq. (4.9) : it is easy to check that the time derivative of  $\sigma^2$  is given by Eq. (4.10) with an extra factor  $\Gamma_i$  inside the sum on  $i$ . We therefore conclude that *with exponential learning and full information agents coordinate on a Nash equilibrium*. Therefore Nash equilibria are local minima of  $\sigma^2$  in  $\Delta^N$ . Furthermore,  $\sigma^2$  is a linear function of  $\pi_{i,s}$  for any  $i, s$ , so that  $\frac{\partial^2 \sigma^2}{\partial \pi_{i,s}^2} = 0, \forall i, s$ . Therefore  $\sigma^2$  is an harmonic function in  $\Delta^N$  which implies that the minima are on the boundary of  $\Delta^N$ . This holds for any subset of variables  $\pi_{i,s}$  which therefore implies that minima are located in the corners of the simplex, i.e. Nash equilibria are in pure strategies ( $G = 1$ ). This, in its turn, implies that Nash equilibria have  $\sigma^2 \approx H$  by Eq. (2.33).

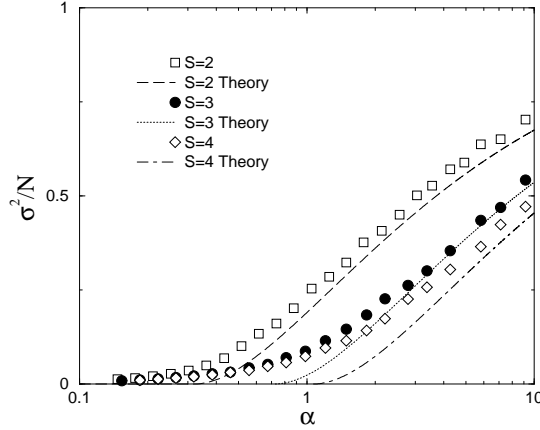


Figure 4.5: Global efficiency  $\sigma^2/N$  as a function of  $\alpha$  for  $S = 2, 3$ , and  $4$  from numerical simulations with  $P = 128$  averaged over 100 realizations of  $\vec{a}_i$  (symbols) and from the theoretical lower bound (lines).

Note that the Nash equilibrium to which agents converge depends on the initial conditions  $U_{i,s}(0)$ , i.e. on prior beliefs : Different initial conditions select different Nash equilibria

A detailed analytical characterization of these Nash equilibria is given elsewhere [61]. Here we briefly mention that also in this case Nash equilibria are exponentially many in  $N$ . This makes the analytic calculation a step more difficult than the one we shall present later. A simplified approximate calculation (see appendix) gives the following lower bound<sup>2</sup> :

$$\frac{\sigma^2}{N} \geq \begin{cases} \left[1 - \frac{z(S)}{\sqrt{\alpha}}\right]^2 & \text{for } \alpha > z(S)^2 \\ 0 & \text{for } \alpha \leq z(S)^2 \end{cases} \quad (4.12)$$

where  $\alpha = P/N$  and  $z(S) = \sqrt{2/\pi S} \int_{-\infty}^{\infty} dz e^{-z^2} z [1 - \text{erfc}(z)/2]^{S-1}$  is the expected value of the maximum among  $S$  standard random variable (for  $S \gg 1$ ,  $z(S) \simeq \sqrt{2 \ln S}$ ).

Figure 4.5 shows that the lower bound is already a good approximation to the typical value of  $\sigma^2$  in the Nash equilibrium, specially for small values of  $S$ . Eq. (4.12) implies that, for fixed  $S$ ,  $\sigma^2$  increases with  $\alpha$ , which is reasonable because the complexity of information increases and the resources of agents is limited by  $S$ . For fixed  $\alpha$ , Eq. (4.12) suggests that  $\sigma^2$  decreases with  $S$ . So if agents are given more resources (larger  $S$ ), they attain a better equilibrium. Both of these features are confirmed by numerical simulations (see fig. 4.5).

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<sup>2</sup>Strictly speaking, the meaning of this lower bound is that the probability to observe  $\sigma^2$  smaller than the lower bound decreases exponentially with  $N$

It is worth to point out that the game specified by the payoffs of Eq. (4.7), for  $N$  and  $P$  very large, implies a fantastic computational complexity. Deductive rational agents should be able to master a chain of logical deductions of formidable complexity in order to derive their best response. The efforts required by this strategic situation may well exceed the bounds of memory and computational capabilities of any realistic agent or more simply the resources she is likely to devote to the problem. Furthermore her assumption that everybody else behaves as a rational deductive player becomes more and more unrealistic as  $N$  grows large. Finally, even with deductive rational agents there would still be the problem of equilibrium selection which, in this case, involves a huge number of possible equilibria.

Note that the class of Nash equilibria discussed above are not the only Nash equilibria that exist in this kind of problem. Generally, Nash equilibria are defined with respect to all strategies of all agents. If the nature of strategies changes, then other kind of Nash equilibria appears (see refs [35, 36] for more details).



# Chapter 5

## Modeling market mechanisms

Although in principle the MG can be considered as a model for understanding competition between adaptative agents, its most popular interpretation concerns the financial markets. Most authors do not explicit the relationships between the MG and real markets, or only say that the MG is a metaphorical model which gives only access to stylized facts. Actually, even if the MG does not include some very important ingredients of financial markets, it is possible to extend it in order to include them. But the MG is really connected with real financial markets because of the minority mechanism. Indeed, it entails that it is not possible to make money by selling and buying at the same time, or in the MG context, that two agents earn a positive gain by forming a coalition.

The market interpretation of the MG consists of mapping the two possible actions to “buy one share” ( $a^\mu = +1$ ) and “sell one share” ( $a^\mu = -1$ ). Agents behave as price takers, that is, they buy or sell at the best available price. But at a given time  $t$ ,  $A(t)$  is always non zero, by definition of the MG. This is called excess demand/offer and changes the price. In first approximation, the agents buy or sell share to people that have placed sell or buy orders, that is, they “eat” some part of the distribution of bid/ask (see Fig. 1.1), which changes the price. For flat distributions and no spread, naively<sup>1</sup> the price variation is  $\delta p(t) \propto A(t)$ <sup>2</sup>, explaining how the MG in general and specially the linear payoff function are related to financial markets.

Consequently, even the basic minority game can be used for exploring the consequences of the minority mechanism. In particular, it allows to ask simple question and to obtain simple answers, some of them being supported by exact analytical results [34].

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<sup>1</sup>In reality, if  $A(t)$  is large, the market maker dilutes the order in time, leading to a smaller market impact [9].

<sup>2</sup>This approximation is found in [63, 4, 64] and holds for small  $A$ .

## 5.1 Speculators with diversified strategies

In the standard MG, it is assumed that the agents draw all their strategies randomly, and independently. One can argue that the agents can be less simple-minded so that they first draw a strategy, and then following their needs or what seems the best for them, draw the others strategies. For instance, if  $S = 2$ , an agent can believe that one strategy is enough and sticks to it (or takes two same strategies). Reversely, an agent might believe that it is better to have one strategy and the opposite one. More generally, we suppose that all the agents<sup>3</sup> draw their second strategy according to

$$P(a_{\uparrow}^{\mu} = a_{\downarrow}^{\mu}) = c \quad \forall \mu. \quad (5.1)$$

The parameter  $c$  counts the average fraction of histories for which the agents' choices are biased, that is, the average correlation between their two strategies. The standard MG corresponds to the independent case  $c = 1/2$ , while having only one strategy is obtained with  $c = 1$ . The other very special case is  $c = 0$  : all agents have two opposite strategies, thus there is no asymmetry in the outcome. As a result, the game is always in the symmetric phase : as  $\alpha$  is varied, no phase transition occurs (see section 2.13). Increasing  $c$  has two effects : on one hand it increases the bias of the outcome  $\Omega^{\mu} \sim \sqrt{cN}$ , on the other hand it reduces the ability of the agents of being adaptive, since they learn something about the game only when  $\xi_i^{\mu} \neq 0$ , which happens on average for  $(1 - c)P$  histories. The fact that the biases depend on  $c$  too implies that the second order phase transition also occurs when this parameter is varied. With the replica formalism (see appendix 5.7), one gets the phase diagram of the MG with parameter  $c$  (see figure 5.1). In the standard MG, one varies  $\alpha$  (dot-dashed vertical line). If one fixes  $\alpha$  and changes  $c$ , the symmetry is also broken (any horizontal line). Note that if  $c = 0$  and  $\alpha > 1$ , an infinitesimal  $c$  breaks the symmetry of the game.

## 5.2 Speculators and producers

Real markets are not *zero sum* games [9]. The fact that most participants are interested in playing is beyond doubt. In real markets the participants can be grossly divided into two groups : Speculators and Producers [44, 45, 9]. Producers can be characterized by those using the market for purposes other

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<sup>3</sup>This can be generalized to a  $c$  for each agent ; exact results also arise from the replica calculus



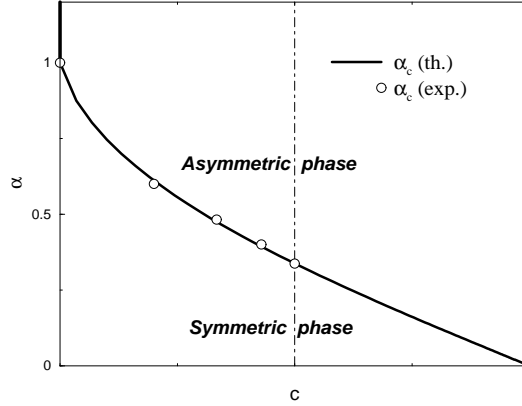


Figure 5.1: Phase diagram of the Minority Game with diversified strategies. The phase transition in the standard MG corresponds to the dash-dotted vertical line  $c = 1/2$ . The circles are numerical data

than speculation. They need market for hedging, financing, or any ordinary business, paying less or no attention to “timing the market”. Speculators, on the other hand, join the market with the aim of exploiting the marginal profit pockets. The two groups were shown to live in symbiosis [9] : the former inject information into the market prices, and the latter make a living carefully exploiting this information. One may wonder why do producers let themselves be taken advantage of. One answer is that they have other, probably more profitable business in mind. To conduct their business, they need the market, and their expertise and talents in other areas give them still better games to play. Speculators, being less capable in other areas, or by choice, make do exploiting the “meager margin” left in the competitive market.

In the MG, these general questions can be studied in detail. Producers will be limited in choice, their activities outside the game are not represented.

We define a **speculator** as an normal agent, and a **producer** as an agent limited to one strategy. The latter have a fixed pattern in their market behavior and put a measurable amount of information into the market, which is exploited by the speculators. We take a population of  $N$  speculators and always define  $\alpha = P/N$ . We add  $\rho N$  heterogeneous producers, so that  $\rho$  is the fraction of producers per speculator. The outcome is then

$$A^\mu = A_{\text{spec}}^\mu + A_{\text{prod}}^\mu. \quad (5.2)$$

The bias induced by the producers adds to the one caused by the speculators, so that the total bias is of order  $\sqrt{(c + \rho/2)N}$ . Therefore the phase

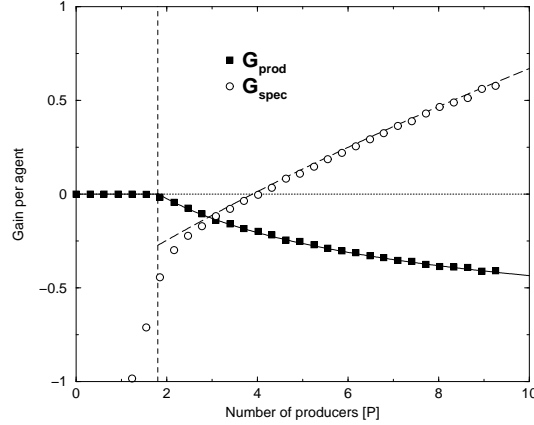


Figure 5.2: Gain of producers and speculators versus the number of producers (in  $P$  unit) ; the number of speculators is fixed at  $N = 641$  ( $c = 0$ ,  $M = 8$ ,  $S = 2$ ,  $\alpha = 0.4$ , average over 200 realizations). The lines are theoretical predictions.

transition can be obtained at fixed  $P$  by varying either  $N$ ,  $c$ , or the number of producers. Let us begin with the last possibility. We fix  $c = 0$ ,  $P = 2^8$ ,  $N = 641$  and plot the gains of the speculators and producers as a function of the number of producers (see figure 5.2). In the symmetric phase, the speculators wash out all available information, thus, by symmetry, the gain of the producers (squares) is zero. As the number of producers increases, the gain of the speculators (circles) stays negative but grows monotonically, while the gain of the producers remains zero as long as the symmetry of the outcome is not broken. When the number of producers reaches a critical value, the speculators are no more able to remove all available information, therefore the (second order) phase transition occurs (dashed line). Beyond this point, the producers lose more and more, while some (frozen) speculators gain more than zero on average (see 2.14.1). At one point, the gains of speculators and producers are the same. Finally, there are enough producers to make the gain of the speculators positive on average.

As illustrated by figures 5.3 and 5.4, if the number of speculators changes the behavior is qualitatively the inverse of the one of figure 5.2 : The gain of producers increases as the number of producers grows ; similarly, the gain of the speculators decreases when  $N$  increases for sufficiently large  $N$ . If there are not enough producers, the game is always negative sum for the speculators, and their gain has a maximum (see figure 5.3).

We now expose exact analytical results concerning the gain of the two types of agents. They rely on the generalization of the approach of refs.

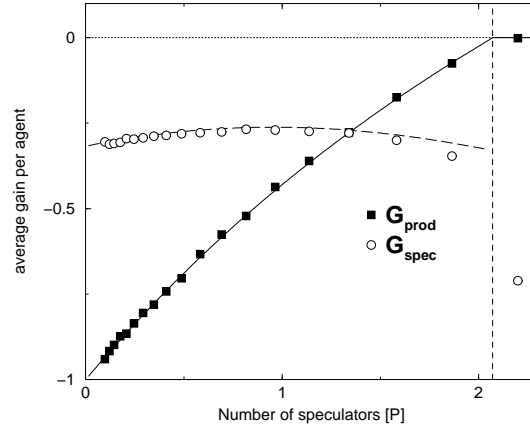


Figure 5.3: Gain of producers and speculators versus the number of speculators (in  $P$  unit) ; the number of producers is fixed at 64 ( $c = 0$ ,  $M = 8$ ,  $S = 2$ , average over 200 realizations). The lines are theoretical predictions.

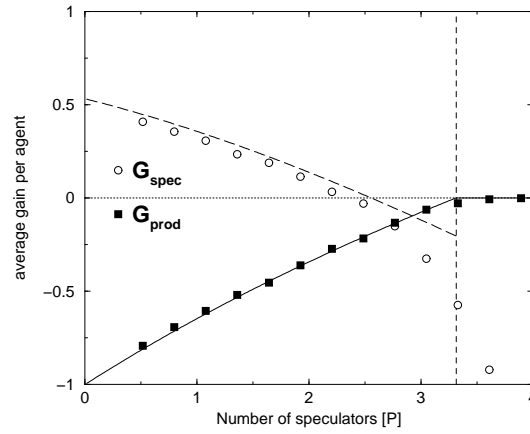


Figure 5.4: Gain of producers and speculators versus the number of speculators (in  $P$  unit) ; the number of producers is fixed at 256 ( $c = 0$ ,  $M = 6$ ,  $S = 2$ , average over 200 realizations). The lines are theoretical predictions.

[32, 35] : the calculus is carried out in detail in appendix 5.7. Let us introduce  $G_{\text{spec}}$ , the total gain of the speculators and  $G_{\text{prod}}$ , the one of the producers. Since  $\sigma^2$  is equal to the total losses,

$$G_{\text{spec}} + G_{\text{prod}} = -\sigma^2. \quad (5.3)$$

The results depend on the ratio  $\rho$  between the number of producers, on the number of speculators and on  $c$ , the parameter introduced in the previous section. We obtain

$$\frac{\sigma^2}{N} = \frac{c + \rho + (1 - c)Q}{(1 + \chi)^2} + (1 - c)(1 - Q) \quad (5.4)$$

where  $\chi$  and  $Q$  take their saddle point values.. These two quantities depend on  $\alpha$  and on  $(1 + \rho)/(1 - c)$  (see appendix 5.7). The average gain per producer is

$$\frac{G_{\text{prod}}}{\rho N} = -\frac{1}{1 + \chi} \quad (5.5)$$

and the average gain per speculator is

$$\frac{G_{\text{spec}}}{N} = -\frac{c + \rho + (1 - c)Q}{(1 + \chi)^2} - (1 - c)(1 - Q) + \frac{\rho}{1 + \chi}. \quad (5.6)$$

Figures 5.2, 5.3 and 5.4 completely agree with analytical results ; note that the small deviations are finite size effects.

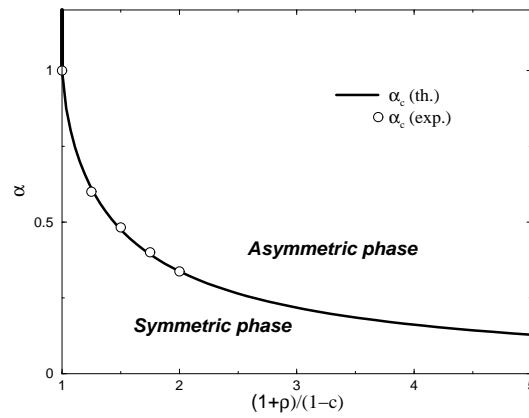
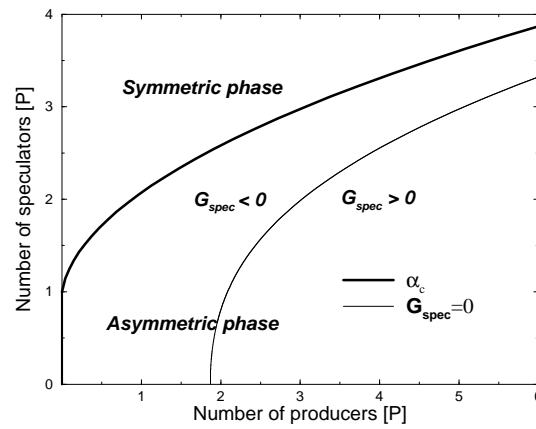
The fact that the gains of producers and speculators only depend on the ratio  $\rho$  and not on how many producers and speculators there are in the game explains why figures 5.3 and 5.4 look very much like the inverse of figure 5.2.

As it emerges for the replica calculus, the critical point  $\alpha_c$  only depends<sup>4</sup> on  $(1 + \rho)/(1 - c)$  (see figure 5.5), that is, on the distribution of the quenched disorder. Numerical data (circles) completely agree with our results. The vertical line corresponds to the standard MG ( $\rho = 0$  and  $c = 1/2$ ). A more intuitive version of this phase diagram is shown in figure 5.6 for  $c = 0$ .

The game becomes favorable, on average, for the speculators when their average gain is greater than zero. Using Eq (5.6), one can plot the curve of zero sum gain for the speculators (see figure 5.6). One can see that the number of producers must be greater than  $1.868 \dots P$  (this value depends on  $c$ ) in order to make the game positive sum for the speculators ; this is consistent with numerical simulations (figures 5.3 and 5.4).

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<sup>4</sup>This explain why evolutionary schemes that preserve the distribution of the quenched disorder have the same  $\alpha_c$  [57], while others that involve Darwinism, shift  $\alpha_c$  [20, 22].

Figure 5.5: Phase diagram  $\alpha_c[(1 + \rho)/(1 - c)]$ Figure 5.6: Phase diagram, and zero sum gain for speculators with  $c = 0$

The main message of these results is that producers always benefit from the presence of speculators, and reversely : both types of agents live in symbiosis. Indeed, the producers introduce systematic biases into the market, and without speculators, their losses would be proportional to these biases. The speculators precisely try to remove this kind of bias, reducing also systematic fluctuations in the market, thus reducing the losses of the producers and their own losses. Moreover, the efforts of speculators yield a positive gain only if the number of producers is sufficiently large. In this respect the symmetric phase, where producers do not lose and speculators lose a lot, is unrealistic : real speculators would rather withdraw from a market which is in this phase, thus increasing  $\alpha$ , and recovering the asymmetric phase<sup>5</sup>. This suggests that a grand-canonical MG is much more realistic<sup>6</sup>. Here we briefly present an over-simplified “grand-canonical” MG. An agent enters into the market only when she has a strategy with virtual points greater than zero. As a result, the game is always in the asymmetric phase, but almost at the transition point : the average losses of the producers are always extremely small (see figure 5.7). When the number of producers increases, the *a priori* asymmetry of the outcome increases, and more and more agents actually play the game (see figure 5.8), thus in this situation, the producers give incentives to play to the speculators. Accordingly, the average gain of the speculators, when positive, is higher in this grand-canonical MG than in the corresponding canonical MG.

### 5.3 Speculators, producers and noise traders

The debate about what the noise traders do to a competitive market is not closed [59]. In the economics literature a noise trader is not very precisely defined. Sometimes they are synonym with speculators. We define noise traders in the following way : they choose their actions without any basis. Compared with speculators, who analyze carefully the market information, noise traders take action in a purely random way (see appendix 5.7). Noise traders may be speculators who base their action on astrology, on “fengshui”, or on some “random number generators”. The model allows us to evaluate the influence of noise traders on the market. They increase the market volatility  $\sigma^2$ , as shown in Fig. 5.9 and in appendix 5.7. Therefore, in principle, they do harm to themselves as well to other participants. Actually in the linear-payoff version that we consider, the average gain of speculators and producers is not much affected by noise traders, since  $\langle A_{\text{noise}} \rangle = 0$ . However,

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<sup>5</sup>Or accounting for their impact on the game with a  $\eta$ -modified dynamics

<sup>6</sup>See also [58]. This kind of grand-canonical MG can be exactly solved [54]

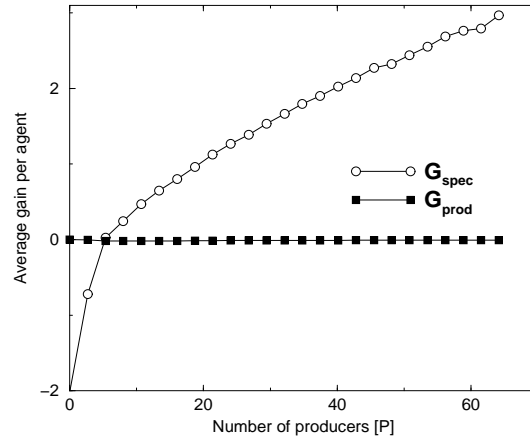


Figure 5.7: Average gain per agent versus the number of producers (in  $P$  units) in the grand canonical MG ( $N = 107$ ,  $M = 5$ ,  $\alpha = 0.3$ ,  $S = 2$ ,  $c = 1/2$ , average over 500 realizations)

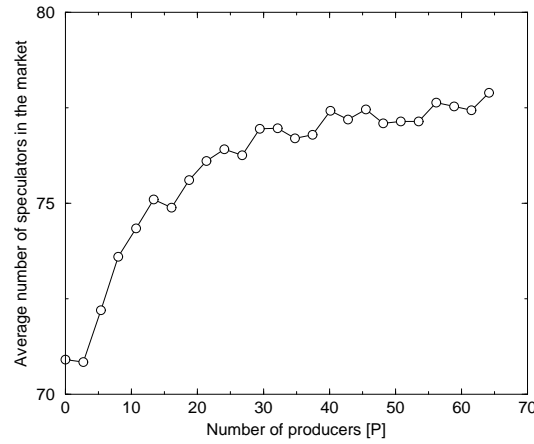


Figure 5.8: Average number of speculators versus the number of producers (in  $P$  units) in the grand canonical MG ( $N = 107$ ,  $M = 5$ ,  $\alpha = 0.3$ ,  $S = 2$ ,  $c = 1/2$ , average over 500 realizations, )

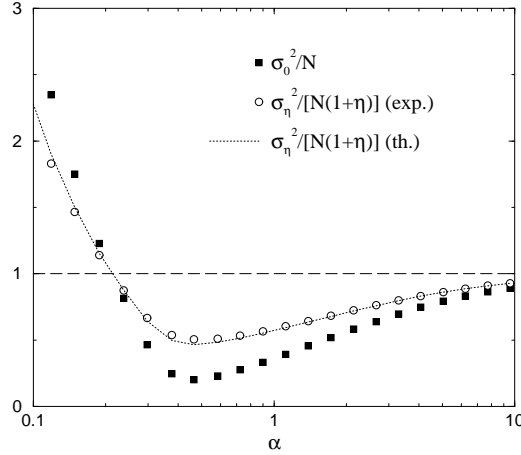


Figure 5.9: Normalized variance of the outcome with (opaque circles and without (black squares) noise traders ; the dotted line is the naive theoretical prediction ( $N = 101$  speculators, 50 noise traders, average over 1000 realizations).

it is easy to see that in the original version, where  $g_i(t) = -a_i(t)\text{sgn}A(t)$ , payoffs are reduced by the presence of noise traders [34].

Our numerical results of Fig. 5.9 also shows that deep in the symmetric phase, noise traders reduces the volatility per agent  $\sigma^2/(N+N_{\text{noise}})$ , when this becomes bigger than one. This is easy to understand assuming that the only effect of noise traders is to increase  $\sigma^2$  by a constant equal to  $N_{\text{noise}} \equiv \eta N$ . Let  $\sigma_0^2/N$  be the volatility per agent, without noise traders ( $\eta = 0$ ) and  $\sigma_\eta^2$  that with noise traders. The variation in the volatility per agent in the presence of noise traders is :

$$\frac{\sigma_\eta^2}{N(1+\eta)} - \frac{\sigma_0^2}{N} \simeq \frac{\sigma_0^2 + \eta N}{N(1+\eta)} - \frac{\sigma_0^2}{N} = \frac{1 - \sigma_0^2/N}{1 + 1/\eta}. \quad (5.7)$$

As illustrated by figure 5.9, numerical simulations globally confirm these conclusions, but also show that the effects of the noise traders are more pronounced than the theory predicts.

## 5.4 Market impact

In order to quantify the impact of an agent on the market let us first consider the case of an *external* agent with  $S$  strategies : This agent does not take part in the game but just observes it from the outside. From this position,



each of her strategies gives an average<sup>7</sup> *virtual* gain

$$u_s = -\overline{a_s \langle A \rangle}, \quad s = 1, \dots, S. \quad (5.8)$$

Given that the strategies  $a_s^\mu$  are drawn randomly,  $u_s$  are independent random variables. Since  $u_s$  is the sum of  $P \gg 1$  independent variables  $a_s^\mu \langle A^\mu \rangle / P$ , their distribution is Gaussian with zero mean and variance

$$\text{Var}(u_s) = \frac{1}{P^2} \sum_{\mu=1}^P \text{Var}(a_s^\mu) \langle A^\mu \rangle^2 = \frac{H}{P}.$$

Clearly, one of these strategies, that with  $u_{s^*} = \max_s u_s$ , is superior to all others<sup>8</sup>. It would be most reasonable for this agent to just stick to this strategy.

However, the same agent *inside* the game will typically use not only strategy  $s^*$ . This is because every strategy, when used, delivers a *real* gain which is reduced with respect to the virtual one by the “market impact”. Imagine the “experiment” of injecting the new agent in a MG. Then  $\langle A^\mu \rangle \rightarrow \langle A^\mu \rangle + a_s^\mu$ , where, in a first approximation, we neglect the reaction of other agents to the new-comer. Then the real gain of the newcomer is :

$$g_s \cong -\overline{a_s \langle A \rangle} - \langle a_s a_s \rangle = u_s - 1. \quad (5.9)$$

The agent will then update the scores  $U_s(t)$  with the real gain  $g_s$  for the strategy she uses and with the virtual one  $u_{s'} = g_{s'} + 1 - \overline{a_s a_{s'}}$ , for the strategies she does not use (in the following, we neglect the term  $\overline{a_s a_{s'}}$ ). Therefore inductive agents over-estimate the performance of the strategies they do not play. Then if strategy  $s$  is played with a frequency  $p_s$ , the virtual score increases *on average* by

$$\begin{aligned} \delta U_s &= U_s(t+1) - U_s(t) = p_s g_s + (1 - p_s)(g_s + 1) \\ &= g_s - p_s + 1 \end{aligned} \quad (5.10)$$

at each time step (on average). Instead of playing only his best strategy, this agent will end up playing  $n$  strategies. This is a consequence of the fact that she neglects her impact on the market.

So far we did not take into account the reaction of other agents to the new-comer. In order to quantify this effect, let us consider a MG in the asymmetric phase, and let us add a new agent with the *best* strategy  $a^\mu = -\text{sgn} \langle A^\mu \rangle$ .

<sup>7</sup>The average is meant over a long time here

<sup>8</sup>The distribution of  $u_{s^*}$  can be easily computed using extreme statistics. For  $S \gg 1$  typically  $u_{s^*} \simeq \sqrt{2H \log(S)/P}$ .

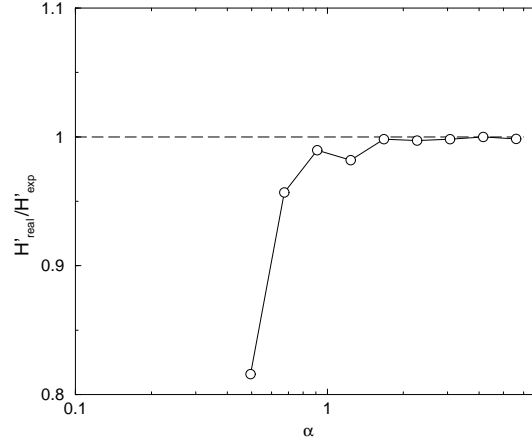


Figure 5.10: Ratio of real  $H'$  over approximated  $H' \simeq H - 2|\overline{\langle A \rangle}| + 1$  versus  $\alpha$  ( $N = 101$ , average over 100 realizations)

This gives us an idea of this effect in the extreme case and we expect that for a randomly drawn strategy the effect will be smaller. Neglecting the reaction of other agents, we find that available information with the new-comer should be  $H' \simeq H - 2|\overline{\langle A \rangle}| + 1$ . Figure 5.10 shows that the reaction of all agents is indeed negligible, excepted near the critical point, where  $H$  is of the order of 1.

## 5.5 Privileged agent or insider-trading

In this section we consider a MG where a particular agent has different characteristics. In particular we address the question of what additional resources would be advantageous for this agent and in which circumstances. In the first subsection, we consider an agent with  $S'$  strategies (with  $S' > S$ , where  $S$  is the number of strategies assigned to other agents). The last two subsections are devoted to the study of effects of asymmetric information, in which an agent has access to privileged information which the other cannot access. This can be achieved in several ways. First, we consider the case of a pure population with memory  $M$  and one agent with a longer memory  $M'$ . Then we consider the case of an agent who knows, in advance, how a subset of agents plays.

### 5.5.1 An agent with $S'$ strategies

In the symmetric phase, no matter how many strategies an agent has, there is no possibility of gaining. Therefore we focus in this section on the asymmetric phase.

As shown in Sect. 5.4, inductive agents over-estimate the performance of the strategies they do not play

Let us consider now the case where an agent with  $S'$  strategies enters into a MG. As shown in Sect. 5.4, to a good approximation, the value of  $H/P$  is the only relevant information we need to retain of the stationary state of the MG without the special agent. This quantity encodes all information such as the number of producers, the number of strategies played by the agents in the MG and the value of  $\alpha$ .

We carried out numerical simulations, and compared it to the analytical results derived in Sect. 5.4. These are shown in the figures 5.11, for  $H/P = 0.5$ , and 5.12, for  $H/P = 1$ . The virtual gain  $v$  is always larger than the actual gain  $g$ . Even though  $g$  is less than the gain agents would get playing only their best strategy  $E[g_{s*}]$  (maximal gain), it is not much smaller and has the same leading behavior  $g \propto \sqrt{\ln S}$ .

Numerical simulations agree well with analytical results, apart from finite size effects which become more pronounced if  $H/P$  is small<sup>9</sup>.

Figures 5.11 and 5.12 refer to values of  $H/P$  which are realistic of MG with producers. A moderately large  $S'$  suffices to obtain a positive gain  $g > 0$ . With  $S = 2$  and without producers  $H/P \sim 0.1$  at most. For these values the analytic approach suggests that, even playing only her best strategy an agent would need  $S' > 750$  strategies to have a positive gain, whereas inductive agents would need more than  $S' \simeq 2400$  strategies to obtain a positive gain. The same agent would find that her virtual gain becomes positive with only  $S' > 8$  strategies. These results for  $H/P = 0.1$  suffer from strong finite size effects (which indeed are of the order of  $P/H$ ). One would need system sizes  $N$  which are well beyond what our computational resources allow to confirm these conclusions.

It is also interesting to observe that the number of strategies actually used by the inductive agent increases with  $S$  (sub-linearly) and it decreases as  $H/P$  increases (see figures 5.11 and 5.12). That means that if there is more exploitable information in the system, agent's behavior becomes more peaked on the best strategy.

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<sup>9</sup>This is mostly due the term which we have neglected in the section 5.4 : it is typically of the order of  $P/H$

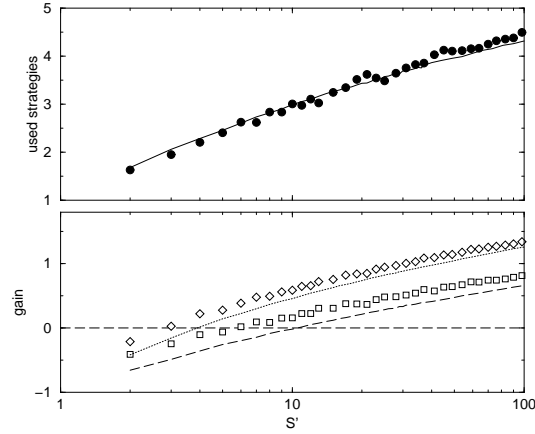


Figure 5.11: Upper graph : average number of played strategies (circles) versus  $S'$ . Below : average virtual (diamonds) and actual (squares) gains versus  $S'$  for  $H/P = 0.5$ , from top to below (averages over 500 realizations). The lines are theoretical predictions

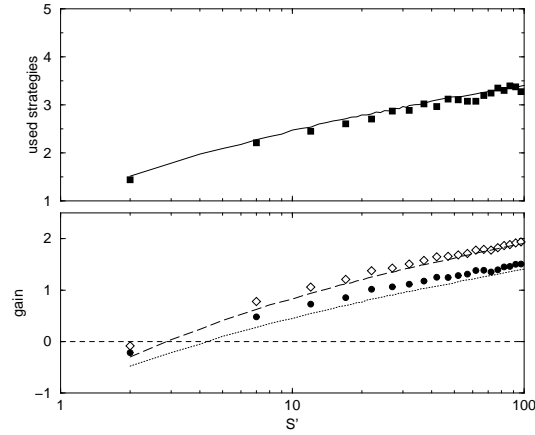


Figure 5.12: Upper graph : average number of played strategies (circles) versus  $S'$ . Below : average virtual (diamonds) and actual (squares) gains versus  $S'$  for  $H/P = 1$ , from top to below (averages over 500 realizations). The lines are theoretical predictions.

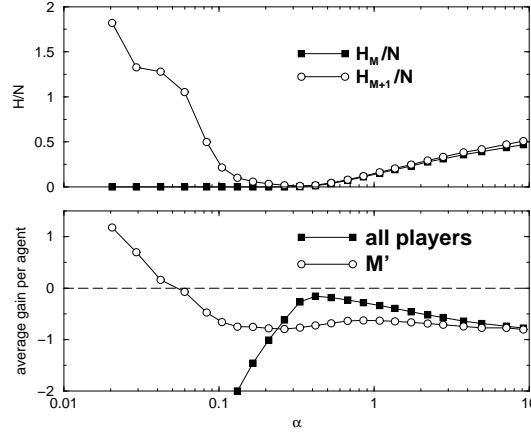


Figure 5.13: Upper graph : normalized available information for  $M$  and  $M + 1$ . Lower graph : Gain of an agent with  $M + 1$  within a pure population with  $M = 3$  ( $S = 2$ , average over 3000 realizations)

### 5.5.2 $M' > M$

Let us consider the case of a pure population with memory  $M$  and one agent with a longer memory<sup>10</sup>  $M'$ . Figure 5.13 plots the gain of such an agent with  $M' = M + 1$  as a function of  $\alpha$ . The average gain of all agents is also shown for comparison. In the asymmetric phase the special agent receives a lower payoff, which can be understood by observing that she has a number of histories  $P' = 2^{M'} = 2P$  bigger than that of the pure population. Thus her effective  $\alpha' = 2\alpha$  is larger, which is detrimental in the asymmetric phase.

The gain of the special agent is the same as that of normal agents at the point where there is neither persistence, nor anti-persistence ( $\alpha \simeq 0.25$  for  $M = 3$ , and  $\alpha_c$  in the thermodynamic limit).

By contrast, in the symmetric phase, the game is symmetric for normal agents but their anti-persistent behavior produces arbitrages who can be exploited by agents having a bigger memory. Indeed, as  $\alpha$  decreases, available information  $H_{M'}$  for the privileged agent grows<sup>11</sup>. As a result the gain of the privileged agent becomes larger than that of other agents and as  $\alpha$  becomes small enough, it becomes positive.

Can the anti-persistence be exploited even more if one increases  $M'$  ? The figure 5.14 answers clearly no. This is not surprising since again the effective  $\alpha$  is bigger and bigger as  $M'$  is increased. At the same time, available

<sup>10</sup>In this kind of numerical simulations, one has to keep the dynamics of histories

<sup>11</sup> $H_{M'}$  is defined as  $H = \overline{A^2}$ , but with an average over  $\mu' = 1, \dots, 2P$ .

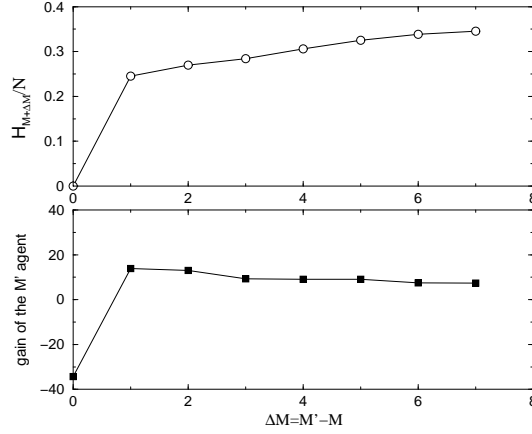


Figure 5.14: Gain of an agent with  $M' = M + \Delta M$  within a pure population with  $M = 3$  ( $\alpha = 0.1$ , average over 1000 realizations).

information increases, but too slowly.

### 5.5.3 Espionage

Some agents may have access to some information about other agents. This is the case of a stock broker who knows his clients' orders before execution, hence he has privileged information and should be barred from trading. When there is no available information, as in the symmetric phase, an agent who has access to asymmetric information can expect at least to lose much less than the others agents, or even to have a positive gain. Also, since having access to a little information is greatly preferable to no information at all, only a very limited amount of information is needed to get a considerable advantage. Suppose that agent  $b$  knows the sign  $s_B$  of the aggregate actions of a subset  $\mathcal{B}$  of other agents. Let  $B = |\mathcal{B}|$  be the number of agents in  $\mathcal{B}$ . Then  $s_B(t) = \text{sgn} \sum_{i \in \mathcal{B}} a_i(t)$ . She can exploit this supplementary information by having two virtual values  $U_{b,s}^+(t)$  and  $U_{b,s}^-(t)$  for each of her strategies. In other words, if agent  $b$  knows that  $s_B(t) = +1$  before having to choose, she takes her decision according to the scores  $U_{b,s}^+(t)$ , that is,

$$s_b(t) = \arg \max_{s=1, \dots, S} U_{b,s}^+(t) ; \quad (5.11)$$

she updates the scores of her strategies according to

$$U_{b,s}^+(t+1) = U_{b,s}^+(t) - a_{b,s}^{\mu(t)} A^{\mu(t)} \quad (5.12)$$

and analogously if  $s_B(t) = -1$ .

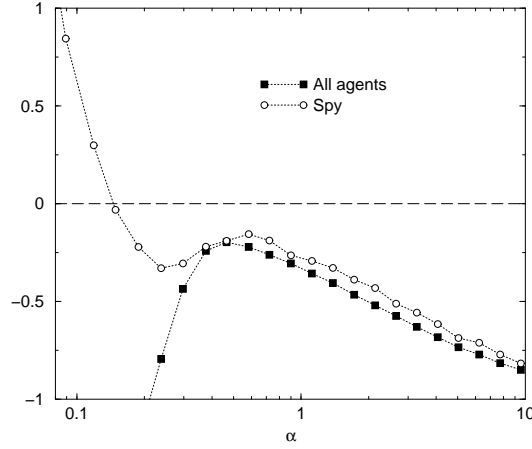


Figure 5.15: Gain of a spy and average gain of all agents versus  $\alpha$  ( $N=101$ ,  $N_B = 3$ ,  $100P$  iterations, average over 100 realizations)

What is the kind of the supplementary information this agent has access to ?

Since the outcome is anti-persistent in the symmetric phase and persistent in the asymmetric phase, only at the critical point there is no long term correlation in the outcome [24]. Accordingly, the spy always gain more than the average, except at the critical point where she gains the same (see figure 5.15). With this setting, the agent has access in particular to the anti-persistence of the symmetric phase, explaining why even if only one agent is spied, the gain of the broker is much bigger (figure 5.16).

Finally, the comparison between the two types of asymmetric information we have considered shows that it is much more interesting to spy than to have a larger memory : in the former case, one is sure to win more than the normal agents, except at the critical point.

## 5.6 From the MG to El Farol

Producers act as a symmetry breaking field on the game, that is, they induce an *a priori* asymmetry on the outcome. In other words, studying producers and speculators is equivalent to considering an asymmetric MG [46], that is, an El Farol's problem with binary histories and strategies. One is willing to obtain an mean attendance around the comfort level and small fluctuations. This is equivalent to small  $H$  and  $\sigma^2/N$ , which can be obtained in the MG for small  $\alpha$  if agents account for their market impact, as shown by Fig. 5.17

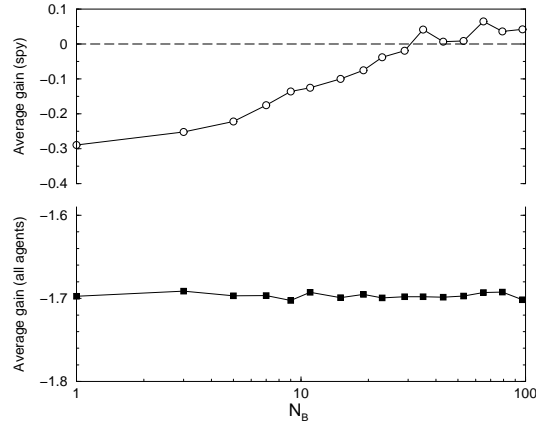


Figure 5.16: Gain of a spy versus the number of spied agents ( $N = 1001$ ,  $\alpha = 0.15$ , average over 1000 realizations)

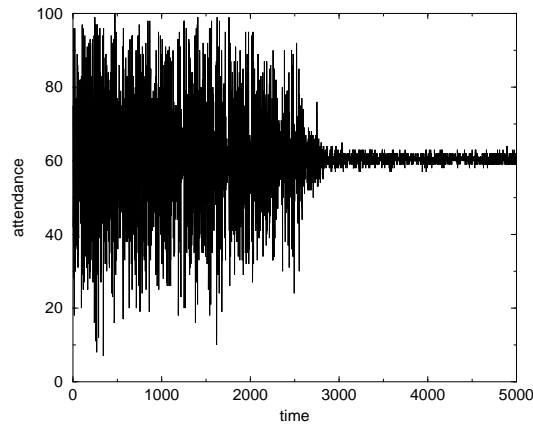


Figure 5.17: Attendance versus time in the revisited El Farol's bar problem ( $N = 100$ ,  $S = 2$ ,  $P = 10$ ,  $\eta = 0.1$ )



One knows in principle how to obtain exact analytical expressions of the mean attendance and of the fluctuations around the latter. However, some issues have still to be solved. Indeed, speculators and producers were studied with random histories. All the discussion can be extended to real histories. However, the bias on the histories is not of the same kind as in the original MG :  $\rho^\mu$  actually also depends on the number of 1 needed to encode  $\mu$ , which make the diffusion on De Bruijn graphs more difficult to study, but probably not impossible. Finally, one should decide if an El Farol's bar customer has  $c = 0$  or  $c = 1/2$ , that is, if (s)he also induces an asymmetry in the outcome or not.

Once these issues are overcome —an important work that has to be done in a near future— the general binary El Farol's bar problem will have an exact solution, except, of course, in the symmetric phase.

## 5.7 Replica method for the speculators and producers

### 5.7.1 Definition

There are three different population of agents :

1. the first population is composed of  $N$  **speculators**. These are adaptive agents and they have each two *speculative* strategies  $a_{\uparrow,i}^\mu$ ,  $a_{\downarrow,i}^\mu$  for  $i = 1, \dots, N$  and  $\mu = 1, \dots, P$ . These are drawn at random from the pool of all strategies, independently for each agent. We allow a correlation among the two strategies of the same agent :

$$\begin{aligned} P(a_{\uparrow}, a_{\downarrow}) &= \frac{c}{2} [\delta_{a_{\uparrow},+1} \delta_{a_{\downarrow},+1} + \delta_{a_{\uparrow},-1} \delta_{a_{\downarrow},-1}] \\ &+ \frac{1-c}{2} [\delta_{a_{\uparrow},-1} \delta_{a_{\downarrow},+1} + \delta_{a_{\uparrow},+1} \delta_{a_{\downarrow},-1}] \end{aligned} \quad (5.13)$$

Note that, for  $c = 0$  agents choose just one strategy  $a_{\uparrow}$  and fix  $a_{\downarrow} = -a_{\uparrow}$  as its opposite, whereas for  $c = 1$  they have one and the same strategy  $a_{\uparrow} = a_{\downarrow}$ . The original random case corresponds to  $c = 1/2$ . These agents assign scores  $U_{i,s}(t)$  to each of their strategies and play the strategy  $s_i(t)$  with the highest score, as discussed in the text. Therefore for speculators :

$$a_{\text{spec}}(t) = a_{s_i(t),i}^{\mu(t)}. \quad (5.14)$$

2. then we consider  $N_{\text{prod}}^{\text{indep}} = \rho N$  **producers** : They have only one randomly and independently drawn strategy  $b_i^\mu$  so

$$a_{\text{prod}}(t) = b_i^{\mu(t)}. \quad (5.15)$$

Producers have a predictable behavior in the market and they are not adaptive. Instead of  $\rho N$  *independent* producers one can also consider  $N_{\text{prod}}^{\text{dep}}$  correlated producers who all have the same predictable behavior  $b_{\text{prod}}^\mu$ .

It has been shown [32, 35] that the stationary state properties of the MG are described by the ground state of  $H$ . Note that this approach fails however to reproduce the anti-persistent behavior which is at the origin of crowd effects in the symmetric phase. In our case

$$A(t) = A_{\text{spec}}(t) + A_{\text{prod}}(t) \quad (5.16)$$

where

$$A_{\text{spec}}(t) = \sum_{j=1}^N a_{s_j(t),j}^{\mu(t)} \quad (5.17)$$

and

$$A_{\text{prod}}(t) = \sum_{j=1}^{\rho N} b_j^{\mu(t)} \equiv A_{\text{prod}}^{\mu(t)} \quad (5.18)$$

Let us define, for convenience,  $A^\mu = A_{\text{spec}}^\mu + \lambda A_{\text{prod}}^\mu$  where

$$A_{\text{spec}}^\mu = \sum_{i=1}^N \left[ a_{\uparrow,i}^\mu \frac{1+s_i}{2} + a_{\downarrow,i}^\mu \frac{1-s_i}{2} \right] \quad (5.19)$$

and  $A_{\text{prod}}^\mu$  is given in Eq. (5.18). Here  $s_i$  is the dynamical variable controlled by speculator  $i$ . We shall implicitly consider directly time averaged quantities so  $s_i$  is a real variable in  $[-1, 1]$  rather than a discrete one. The parameter  $\lambda$  is inserted so that, once we have computed the energy  $H = \overline{(A_{\text{spec}} + \lambda A_{\text{prod}})^2}$  we can compute the total gain  $G_{\text{prod}}$  of producers by

$$G_{\text{prod}} \equiv -\overline{A A_{\text{prod}}} = - \left. \frac{1}{2} \frac{\partial H}{\partial \lambda} \right|_{\lambda=1}.$$

The gain of speculators is obtained subtracting this contribution and that of noise traders from the total gain  $-\sigma^2$

$$G_{\text{spec}} = -\sigma^2 + \eta N - G_{\text{prod}}. \quad (5.20)$$

### 5.7.2 Replica calculation

In this case, it suffices to find the matrix  $\hat{T}$  and use the results of appendix 8.1. After having averaged the replicated partition function over the disorder, one obtains the matrix  $\hat{T}$

$$T_{a,b} = \delta_{a,b} + \frac{2\beta}{\alpha} [c + \rho + (1-c)Q_{a,b}].$$

For *correlated* producers we would have obtained the same result but with  $\rho \rightarrow \rho + \rho^2 N \epsilon^2$ , where  $\epsilon$  measures the bias of producers towards a particular action for a given  $\mu$ , or equivalently the correlation between the actions of two distinct producers. More precisely  $\epsilon^2$  is the average of  $b_i^\mu b_j^\mu$  for  $i \neq j$  and for all  $\mu$ . Therefore the limit  $\rho \rightarrow \infty$  also corresponds to a small share of producers  $\rho \ll 1$  with a small bias  $\epsilon \neq 0$ . Note that a bias  $\epsilon \sim \sqrt{N}$  corresponds indeed to  $\sim N$  independent producers. Equivalently  $\sim \sqrt{N}$  correlated producers, with  $\epsilon$  finite are equivalent to  $\sim N$  independent producers.

The free energy is given by

$$\begin{aligned} f(q, r) &= \frac{\alpha}{2\beta} \log \left[ 1 + \frac{2(1-c)\beta(Q-q)}{\alpha} \right] \\ &+ \frac{\alpha[c + \rho + (1-c)q]}{\alpha + 2(1-c)\beta(Q-q)} + \frac{\alpha\beta}{2}(RQ - rq) \\ &- \frac{1}{\beta} \langle \log \int_{-1}^1 ds e^{-\beta V_z(s)} \rangle \end{aligned} \quad (5.21)$$

Since  $\zeta$  is fixed as a function of  $\alpha$  by the equation

$$\sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} + \left(1 - \frac{1}{\zeta^2}\right) \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) + \frac{\alpha}{\zeta^2} = \frac{1+\rho}{1-c}, \quad (5.22)$$

thus  $\zeta$  depends on the combination  $(1+\rho)/(1-c)$  which runs from 1 – for  $\rho = c = 0$  i.e. no producers and “perfect” speculators – to  $\infty$ . The latter limit occurs either if  $c \rightarrow 1$ , i.e. when speculators become producers, or if  $\rho \rightarrow \infty$  (many producers).

Available information is given by

$$H_c = \frac{c + (1-c)Q + \rho}{(1+\chi)^2} \quad (5.23)$$

where  $Q$  and  $\chi$  take their saddle point values. The fluctuations are

$$\sigma_c^2 = H_c + (1-c)(1-Q) \quad (5.24)$$

The gain of producers, from Eq. (5.21), is

$$G_{\text{prod}}^c = -\frac{\rho}{1 + \chi} \quad (5.25)$$

and that of speculators is obtained from Eq. (5.20),

$$G_{\text{spec}}^c = \frac{\rho}{1 + \chi} - \frac{c + (1 - c)Q + \rho}{(1 + \chi)^2} - (1 - c)(1 - Q) \quad (5.26)$$

# Chapter 6

## Toward a realistic model of markets

The previous chapter showed how some very simple extensions to the standard MG allow to study market impact and the role of information in financial markets. In order to be more realistic, the MG needs to be modified without compromising its simplicity. In particular, exactly solvable extensions will be privileged.

The main issue is about the heterogeneity of agents. Indeed, if the basic MG captures heterogeneity in behaviors and beliefs, it does not include the effect of richness and poverty, that is, in the MG, all agents have the same weight. In this chapter, the MG is generalized to weighted agents, first with fixed weight, and then to dynamical weights (agents invest a fixed fraction of their variable capital). Finally, one considers another kind of heterogeneity in markets are the sensibility of agents to price changes, which explains why all the agents are not interested in playing in market at every time step.

### 6.1 Fixed weights

This modification can be easily handled : each agent has a fixed weight  $w_i$  in the market, meaning that he plays  $\pm w_i$  at each time step, where the weights are drawn from a given pdf.

Numerical simulations reveal that the structure of the resulting MG is exactly the same. In addition, the replica calculus can be adapted. Using the general replica calculus for stationary MG found in appendix 8.1, with  $f_0 = \rho \langle v^2 \rangle + \langle w^2 \rangle c$  and  $f_1 = \langle w^2 \rangle (1 - c)$ , one obtains

$$H_c = \frac{\langle w^2 \rangle (1 + Q)}{2(1 + \chi)^2} \quad (6.1)$$

and

$$\frac{\sigma_c^2}{N} = H_c + \frac{\langle w^2 \rangle_w (1 - Q)}{2} \quad (6.2)$$

where  $Q$  and  $\chi$ , as usual, take their saddle point value, which are slightly modified:

$$\langle w^2 \rangle_w Q = \langle w^2 \rangle_w - \langle (w^2 - 1/\zeta^2) \operatorname{erf} \left( \frac{\zeta w}{\sqrt{2}} \right) \rangle_w - \sqrt{\frac{2}{\pi}} \frac{1}{\zeta} \langle w e^{-\zeta^2 w^2/2} \rangle_w \quad (6.3)$$

The equation of  $\chi$  is now

$$\chi = \left[ \alpha / \langle \operatorname{erf} \left( \frac{\zeta w}{\sqrt{2}} \right) \rangle_w - 1 \right]^{-1} \quad (6.4)$$

Finally  $\zeta$  is fixed as a function of  $\alpha$  by the equation

$$\frac{\alpha}{\zeta^2} - \langle w^2 \rangle_w Q = 2 \langle w^2 \rangle_w, \quad (6.5)$$

that is,

Eq. (6.4) means that  $\chi$  diverges when  $\alpha \rightarrow \alpha_c(\rho, c)^+$ , which then implies that at the critical point

$$\langle \operatorname{erf} \left( \frac{\zeta_c w}{\sqrt{2}} \right) \rangle = \alpha_c. \quad (6.6)$$

These predictions hold as long as  $\langle w^2 \rangle$  is defined, that is, as long as the pdf has a finite variance : they are fully confirmed by Fig 6.2, where a Poisson distribution was used.

Another issue concerns the critical point location. How does it depend on the parameters of the distribution and of the distribution is a question difficult to answer analytically. However, Fig 6.1 reports the theoretical predictions for the pdf  $P(w) = \gamma w^{\gamma-1} \exp(-w^\gamma)$  and power laws pdf  $P_\gamma(w) = \gamma/(1+w)^{\gamma+1}$ . The pdfs's average and variance are respectively  $\Gamma(1/\gamma)/\gamma^2$  and  $\Gamma(1+2\gamma)/\gamma$ , and  $1/[(\gamma-1)(\gamma-2)]$  and  $1/[(\gamma-1)(\gamma-2)(\gamma-3)]$ . Interestingly, if the first moment is not defined, the MG is always in the asymmetric phase ( $\alpha_c = 0$ ), because the outcome  $A$  is dominated by the contribution of one agent. This happens for  $\gamma = 0^+$  for the first pdf, and  $\gamma \leq 2$  for the second pdf.

Note that for infinite variance, the replica calculus still can be done, but the control parameter is no more  $\alpha = P/N$  [55]

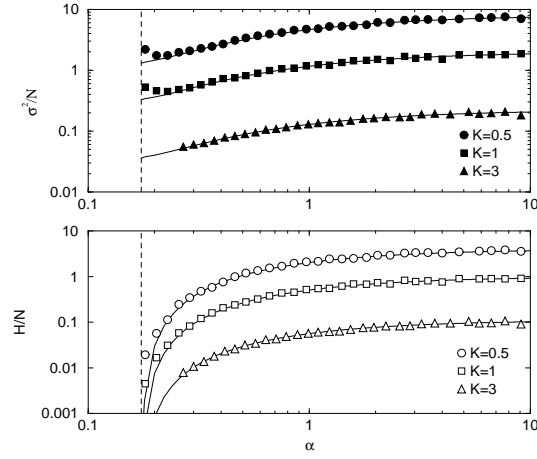


Figure 6.1: Normalized fluctuations versus  $\alpha$  for Poisson wealth and various parameters ( $P = 100$ ,  $S = 2$ ,  $300P$  iterations, average over 100 samples). The continuous lines are the corresponding theoretical predictions

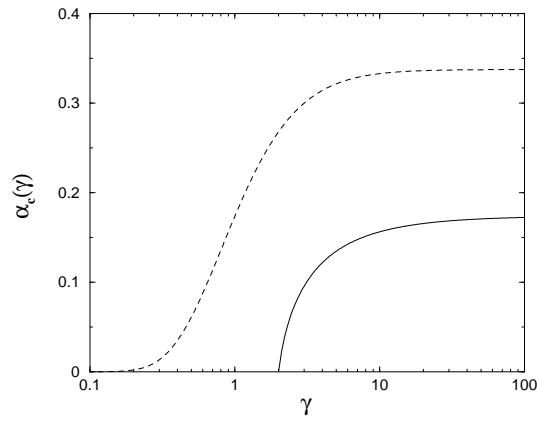


Figure 6.2: Location of the critical point versus the parameter  $\gamma$  of the weights' pdf for  $P_\gamma(w) = \gamma w^{\gamma-1} \exp(-w^\gamma)$  (dashed line) and  $P_\gamma(w) = \gamma/w^{\gamma+1}$  (continuous line)

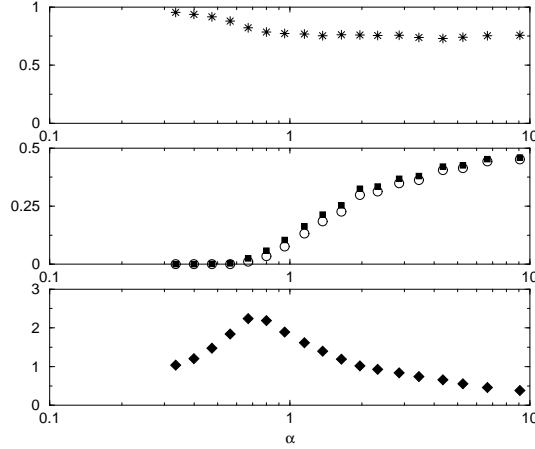


Figure 6.3: Fraction of frozen agents (stars), normalized fluctuations (circles), available information (squares) and average capital of agents (diamonds) ( $P = 100$ ,  $S = 2$ ,  $\epsilon = 0.1$ ,  $1500P$  iterations, average over 50 samples).

## 6.2 Evolving weights

In real markets, agents' capital is a dynamical quantity. The market and the decisions of agents decide of the evolution of their capital. Agents that become poorer and poorer cannot sustain constant investments, and richer agents can be tempted to invest more, since they are being successful. Agents here invest a constant fraction  $\epsilon$  of their capital [55]. Since the MG is a negative sum game without producers, one adds some of the latter in order to make the game more attractive for the agents. Interestingly, one can show that, again, the information is minimized by agents — a definitively universal feature of minority games —, and in the stationary state, the average gain of agents is zero, which is curiously the centennial assertion of Bachelier, and raises the question of how to interpret this finding. Fig 6.3 shows that there is also a phase transition. In this case, the average wealth is maximal at the critical point, whereas in the basic MG, the average gain is maximal at this point. Note that the fraction of frozen agents is always very high, and increase as the number of agents increase. This is similar to agents taking into account their impact on the game ; indeed, the fact that their capital reflects their own success, is a way of accountint for market impact.

The replica calculus, again, can be generalized and gives exact results, but is several folds more complicated. It is under investigation.



## 6.3 Heterogeneity in price tracking

Apart from the fixed weight hypothesis, the most unrealistic assumption is that agents have to play at each time step, implying that the volume is constant. This is not the case in financial markets.

In order to extend the MG, one has to answer the following question : why do agents not change their positions all at the same time ? For instance, market makers are forced to play every time step, day traders change their positions dozens of times per day. At the other end of scale, small investors update their positions few times a year. One explanation for this is their heterogeneity of goals and motivations [44, 45]. For a day trader, a small price change is enough for closing his position and taking his profits. A small investor does not react for so few, he waits (at least) for a change of several percents. Note that a heterogeneity in thresholds implies a heterogeneity if time scales.

The MG is modified as follows : each agent  $i$  has his own price change horizon  $r_i$  (market makers have  $r_i = 0$ ). The  $r_i$  are drawn from a given pdf. At times  $t_i(k)$ , agent  $i$  receives a signal, meaning that the price change ratio exceeds  $r_i$  since the last time he updated his position, that is,

$$r_i > \left| \frac{p[t_i(k)]}{p[t_i(k-1)]} - 1 \right| \quad (6.7)$$

His personal history  $\mu_i$  encodes the last  $M$  threshold signs and is also updated at time  $t_i(k)$ . The strategies' score are changed according to

$$U_{i,s}(t_i(k+1)) = U_{i,s} - [p[t_i(k)] - p[t_i(k-1)]]/p[t_i(k) - 1] \quad (6.8)$$

so that the game is a minority game in the short run and a majority game in the long run. At each time step,

$$N(t) = \sum_i \Theta \left( r_i - \left| \frac{p[t_i(k)]}{p[t_i(k-1)]} - 1 \right| \right) \quad (6.9)$$

agents are active and wish to change their position. They use their best strategy, as before, and put the order  $a_{i,s_i(t)}^{\mu_i(t)}$ , so that the excess demand/offer is

$$A(t) = \sum_{i : \text{active agent}} a_{i,s_i(t)}^{\mu_i(t)} \quad (6.10)$$

There is no consensus on the price formation equation given  $A(t)$  [63, 4, 9, 64]. In first approximation, one can write

$$x(t+1) = x(t) + \frac{A(t)}{\lambda} \quad (6.11)$$

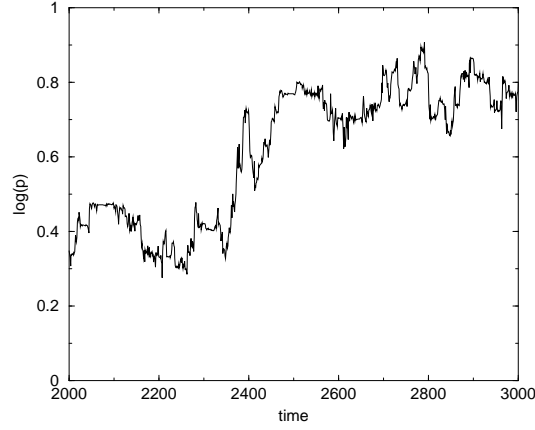
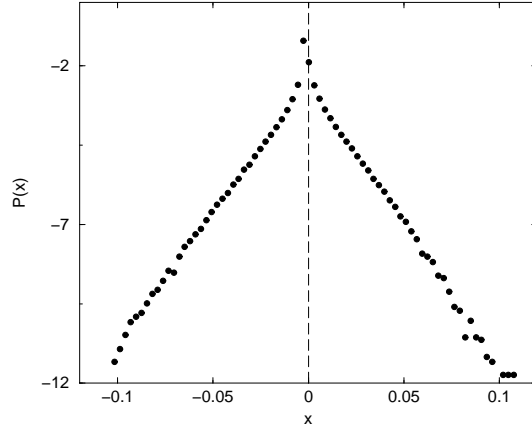
Figure 6.4: Temporal evolution of the price ( $M = 8$ ,  $N = 5001$ ,  $S = 2$ )

Figure 6.5: Histogram of returns of Fig. 6.4

where  $x(t) = \log p(t)$  and  $\lambda$  is the liquidity, or the depth of the market.

The resulting price is plotted in Fig. 6.4, where  $P(r) = \gamma/(1+r)^{\gamma+1}$ . The histogram of  $x(t)$  is found in Fig. 6.5. It shows that the price is strongly non Gaussian for small  $x$ , and has exponential tails. This model clearly show the way to create more realistic MG based models. It also begs for a more detailed study.

# Chapter 7

## Conclusions

By combining both simplicity and rich behavior, the Minority Game is becoming a paradigm of competition between adaptative agents. It also illustrates in a very simple way how physicists can use their tools for understanding problems outside their traditional area of expertise. It took more than one year for the Econophysics community to begin to understand it. Now the standard MG is reasonably understood, and one knows how to obtain exact analytical expressions for the whole range of parameters where the system reaches a stationary state, while the analytical description of the symmetric phase still remains a challenge.

Apart from completing the understanding of the standard MG, there are some very interesting extensions to be worked on :

- Obtaining the Lyapunov function for all kinds of reasonable payoffs. It is clear that the structure of the game is the same for all these payoffs, but that stationary state is not the same.
- Revisiting the El Farol's bar problem. Indeed, with correct inductive dynamics ( $\eta$ ), even agents with binary strategies are able to self-organize around the comfort level with very small fluctuations, which is considered as impossible by some economists.
- Studying thoroughly more realistic models of financial markets. They should include variable number of agents, evolving capital and minority-majority mechanisms. Initially, one should study and understand all of them separatly, and afterwards combine them in order to obtain a realistic model of market with as few parameters as possible, and hopefully exactly solvable.

## Outlook

The MG is relevant not only for the understanding of the El Farol's bar problem. It is fundamental since the minority mechanism is found ubiquitously in nature. It shows for instance that we are interacting and can cooperate with other people in numerous ways without being aware of it. Even more, selfishness is not an obstacle to cooperation, provided that all people have access to the same information. If someones have more relevant information than others, they can outperform the latter. Initially the El Farol's bar problem was thought as a provocation by Arthur, it shed eventually some light on real life situations and gave some very unexpected conclusions.

Even if the interest in the Minority Game has been more or less confined to the phycists' community, encouraging sign are coming from economists and biolgists.

Indeed, some economists are now working on MG-like models. They realize that even their own MG versions share the same characteristics as the MG. This suggests that the findings contained in this thesis are in some way "universal".

Biologists in turn use the Minority Game in order to understand why some birds species migrate in huge groups instead of very small ones, whereas they are more likely to be short on resources.

There are probably numerous other situations hiding an underlying minority mechanism. The challenge for physicists, economists, biologists and sociologists is now to find them.

# Chapter 8

## Appendix

### 8.1 General procedure of replica calculus for the MG :

In general, we obtain

$$\langle Z^n(\beta) \rangle = e^{-Nn\beta F(\hat{Q}, \hat{r})}$$

with

$$F(\hat{Q}, \hat{r}) = \frac{\alpha}{2n\beta} \text{Tr} \log \hat{T} + \frac{\alpha\beta}{2n} \sum_{a,b} r_{a,b} Q_{a,b} - \frac{1}{n\beta} \log \left[ \text{Tr}_s e^{\frac{\alpha\beta^2}{2} \sum_{a,b} r_{a,b} s_a s_b} \right]$$

where

$$T_{a,b} = \delta_{a,b} + \frac{2\beta}{\alpha} f(Q_{a,b}). \quad (8.1)$$

and  $f(x) = f_0 + f_1 x$ . With the replica symmetric ansatz

$$Q_{a,b} = q + (Q - q)\delta_{a,b}, \quad r_{a,b} = r + (R - r)\delta_{a,b}$$

the matrix  $\hat{T}$  is such that

$$T_{a,a} = 1 + \frac{2\beta}{\alpha} f(Q) \quad (8.2)$$

and

$$T_{a,b} = \frac{2\beta}{\alpha} f(q) \quad \text{for } a \neq b \quad (8.3)$$

Therefore,  $\hat{T}$  has  $n - 1$  degenerated eigenvalues  $\lambda_0 = 1 + f_1(Q - q)$  and one eigenvalue equal to  $\lambda_1 = (n - 1)f(q) + 1 + f(Q) = nf(q) + \lambda_0$  therefore, after

standard algebra,

$$F^{(RS)}(q, r) = \frac{\alpha}{2\beta} \log \left( 1 + \frac{2\beta}{\alpha} f_1(Q - q) \right) + \frac{f(q)}{1 + \frac{2\beta}{\alpha} f_1(Q - q)} + \frac{\alpha\beta}{2} (RQ - rq) - \frac{1}{\beta} \langle \log \int_{-1}^1 ds e^{-\beta V_z(s)} \rangle \quad (8.4)$$

where we found it convenient to define the “potential”

$$V_z(s) = -\frac{\alpha\beta(R - r)}{2} s^2 - \sqrt{\alpha r} z s \quad (8.5)$$

so that the last term of  $F^{(RS)}$  looks like the free energy of a particle in the interval  $[-1, 1]$  with potential  $V_z(s)$  where  $z$  plays the role of disorder.

The saddle point equations are given by :

$$\frac{\partial F^{(RS)}}{\partial q} = 0 \quad \Rightarrow \quad r = \frac{4f(q)f_1}{\alpha^2(1 + \chi)^2} \quad (8.6)$$

$$\frac{\partial F^{(RS)}}{\partial Q} = 0 \quad \Rightarrow \quad \beta(R - r) = -\frac{2f_1}{\alpha(1 + \chi)} \quad (8.7)$$

$$\frac{\partial F^{(RS)}}{\partial R} = 0 \quad \Rightarrow \quad Q = \langle \langle s^2 \rangle \rangle \quad (8.8)$$

$$\frac{\partial F^{(RS)}}{\partial r} = 0 \quad \Rightarrow \quad \beta(Q - q) = \frac{\langle \langle sz \rangle \rangle}{\sqrt{\alpha r}} \quad (8.9)$$

where  $\langle \langle \cdot \rangle \rangle$  stands for a thermal average over the above mentioned one particle system and  $\chi = \frac{2\beta f_1(Q - q)}{\alpha}$

In the limit  $\beta \rightarrow 0$  we can look for a solution with  $q \rightarrow Q$  and  $r \rightarrow R$ . It is convenient to define

$$\zeta = -\sqrt{\frac{\alpha}{r}} \beta(R - r) \quad (8.10)$$

and to require that  $\chi$  and  $\zeta$  stay finite in the limit  $\beta \rightarrow \infty$ . The averages are easily evaluated since, in this case, they are dominated by the minimum of the potential  $V_z(s) = \sqrt{\alpha r}(ys^2/2 - zs)$  for  $s \in [-1, 1]$ . The minimum is at  $s = -1$  for  $z \leq -\zeta$  and at  $s = +1$  for  $z \geq \zeta$ . For  $-\zeta < z < \zeta$  the minimum is at  $s = z/\zeta$ . With this we find

$$\langle \langle sz \rangle \rangle = \frac{1}{\zeta} \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) \quad (8.11)$$

and

$$\langle \langle s^2 \rangle \rangle = Q = 1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} - \left( 1 - \frac{1}{\zeta^2} \right) \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) \quad (8.12)$$

With some more algebra, one easily finds :

$$x = \left[ \alpha / \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) - 1 \right]^{-1} \quad (8.13)$$

Finally  $\zeta$  is fixed as a function of  $\alpha$  by the equation

$$1 - Q + \frac{\alpha}{\zeta^2} = \frac{f_0 + f_1}{f_1}, \quad (8.14)$$

that is,

$$\sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} + \left( 1 - \frac{1}{\zeta^2} \right) \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) + \frac{\alpha}{\zeta^2} = \frac{f_0 + f_1}{f_1}. \quad (8.15)$$

Eq. (8.13) means that  $\chi$  diverges when  $\alpha \rightarrow \alpha_c(\rho, c)^+$ , which then implies that at the critical point

$$\operatorname{erf} \left( \frac{\zeta_c}{\sqrt{2}} \right) = \alpha_c. \quad (8.16)$$

This back in the other saddle point equations, yields the following equation for  $\zeta = \zeta_c$  :

$$\sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{\zeta} + \operatorname{erf} \left( \frac{\zeta}{\sqrt{2}} \right) = \frac{f_0 + f_1}{f_1}. \quad (8.17)$$

The free energy, at the saddle point, for  $\beta \rightarrow \infty$ , is

$$\frac{H}{N} = \frac{2f(Q)}{(1 + \chi)^2} \quad (8.18)$$

where  $Q$  and  $x$  take their saddle point values Eqs. (8.12) and (8.13).

## 8.2 Replica calculus for any $S$

Our goal is to compute and characterize the minimum of  $H_\eta = \overline{\langle A \rangle^2} - \eta NG$ , with  $G$  given by Eq. (2.32), in  $\Delta^N = \{\vec{\pi}_i, i = 1, \dots, N\}$ . Considering  $H_\eta$  as an Hamiltonian of a statistical mechanic's system, this can be done analyzing the zero temperature limit. First we build the partition function

$$Z(\beta) = \operatorname{Tr}_\pi e^{-\beta H_\eta\{\pi\}}, \quad (8.19)$$

where  $\beta$  is the inverse temperature and  $\operatorname{Tr}_\pi$  stands for an integral on  $\Delta^N$  (we call simply  $\pi$  an element of  $\Delta^N$ ). The quantity of interest is then

$$\min_{\pi \in \Delta^N} H_\eta\{\pi\} = - \lim_{\beta \rightarrow \infty} \beta^{-1} \ln Z(\beta). \quad (8.20)$$

This in principle depends on the specific realization  $a_{i,s}^\mu$  of rules chosen by agents. In practice however, to leading order in  $N$ , all realizations of  $a_{i,s}^\mu$  yield the same limit, which then coincides with the average of  $\min_{\pi \in \Delta^N} H_\eta\{\pi\}$  over  $a_{i,s}^\mu$ . The average of  $\ln Z$  over the  $a$ 's, which we denote by  $\langle \dots \rangle_a$ , is reduced to that of moments of  $Z$  using the replica trick[52]:

$$\langle \ln Z \rangle_a = \lim_{n \rightarrow 0} \frac{1}{n} \ln \langle Z^n \rangle_a \quad (8.21)$$

With integer  $n$  the calculation of  $\langle Z^n \rangle_a$  amounts to study  $n$  replicas of the same system with the same realization of  $a_{i,s}^\mu$ . To do this we introduce a set of dynamical variables  $\pi_a \equiv \{\pi_{i,s,a}\}$  for each replica, which are labeled by the additional index  $a = 1, \dots, n$ . Each replica has its corresponding Hamiltonian, which we write as  $H_\eta^a\{\pi_a\} = \overline{A_a^2} - \eta N G_{a,a}$  where  $A_a^\mu = \sum_i \vec{\pi}_{i,a} \cdot \vec{a}_i$  and  $NG_{a,a} = \sum_i |\pi_{i,a}|^2$  (the reason for this notation shall become clear later). The set of all dynamical variables for all replicas is the direct product  $\Delta^{Nn}$  of  $n$  phase spaces  $\Delta^N$ . In order to compute the limit  $n \rightarrow 0$  in Eq. (8.21) one appeals to analytic continuation of  $\langle Z^n \rangle_a$  for real  $n$ . We give here the details of the calculation in our specific case. More details on the nature of the method can be found in ref. [52]. We write

$$\begin{aligned} \langle Z^n \rangle &= \text{Tr}_\pi \prod_{a=1}^n \prod_\mu \langle e^{-\beta \varrho^\mu [(A_a^\mu)^2 - \eta N G_{a,a}]} \rangle_a \\ &= \text{Tr}_\pi \prod_\mu \prod_{a=1}^n E_{z_a^\mu} \langle e^{-i\sqrt{2\beta\varrho^\mu} z_a^\mu A_a^\mu} \rangle_a e^{\beta\eta N G_{a,a}} \end{aligned} \quad (8.22)$$

where  $E_z[\dots]$  stands for the expectation over the Gaussian variable (unit variance and zero mean)  $z$  and we have introduced one such variable  $z_a^\mu$  for each  $a$  and  $\mu$ , using the identity  $E_z[e^{-ixz}] = e^{-x^2/2}$ . In addition we used the shorthand  $\text{Tr}_\pi$  for the integral over  $\Delta^{Nn}$ . The average over  $a_{i,s}^\mu$  now factorizes

$$\begin{aligned} \prod_{a=1}^n \langle e^{-i\sqrt{2\beta\varrho^\mu} z_a^\mu A_a^\mu} \rangle_a &= \prod_i \prod_{s=1}^S \langle e^{-i\sqrt{2\beta\varrho^\mu} (\sum_a z_a^\mu \pi_{i,s}^a) a_{i,s}^\mu} \rangle_a \\ &= \prod_i \prod_{s=1}^S \cos \left[ \sqrt{2\beta\varrho^\mu} \sum_{a=1}^n z_a^\mu \pi_{i,s}^a \right] \\ &\simeq \prod_i \exp \left[ -\beta\varrho^\mu \sum_{a,b=1}^n z_a^\mu z_b^\mu \sum_i \vec{\pi}_i^a \cdot \vec{\pi}_i^b \right]. \end{aligned} \quad (8.23)$$

In the last passage we used the relation  $\cos x \simeq e^{-x^2/2}$  which is correct to order  $x^2$  in a power expansion. This is justified as long as  $\varrho^\mu \rightarrow 0$  as  $P =$



$\alpha N \rightarrow \infty$  for each  $\mu = 1, \dots, P$ . Note that, for this reason, we would have got the same result for any generic distribution  $P_i(a)$  of  $a_{i,s}^\mu$  such that  $\langle a \rangle = 0$  and  $\langle a^2 \rangle = 1$ . This allows us to understand why models with continuum strategies  $a_{i,s}^\mu \in \mathbb{R}$ , such as the one proposed in ref. [29], yield the same results as the one with binary strategies, which we are discussing here. Before going back to Eq. (8.22), we introduce the matrices  $\hat{G} \equiv \{G_{a,b}, a, b = 1, \dots, n\}$  and  $\hat{r} \equiv \{r_{a,b}, a, b = 1, \dots, n\}$  through the identities

$$1 = \int dG_{a,b} \delta \left( G_{a,b} - \frac{1}{N} \sum_i \vec{\pi}_i^a \cdot \vec{\pi}_i^b \right) \propto \int dr_{a,b} dG_{a,b} e^{\frac{\alpha\beta^2 r_{a,b}}{2} (\sum_i \vec{\pi}_i^a \cdot \vec{\pi}_i^b - N G_{a,b})}$$

for all  $a \geq b$ , where  $\delta(x)$  is Dirac's delta function and we used its integral representation. The only part depending on the  $\pi_{i,s}^a$  in  $\langle Z^n \rangle$  is  $e^{\alpha\beta^2 \sum_{a \geq b} r_{a,b} \sum_i \vec{\pi}_i^a \cdot \vec{\pi}_i^b / 2}$ . This can be factorized in the agent's index  $i$  and so the integral  $\text{Tr}_\pi$  on  $\Delta^{N^n}$  can be factorized into  $N$  integrals over  $\Delta^n$  (=the direct product of the simplexes of the  $n$  replicas of the same agent's mixed strategies). With this we can write

$$\langle Z^n \rangle = \int dr_{a,b} dG_{a,b} e^{-\beta n N F_\beta(\hat{G}, \hat{r})} \quad (8.24)$$

where, specializing to the case  $\varrho^\mu = 1/P^1$ ,

$$\begin{aligned} F_\beta(\hat{G}, \hat{r}) &= \frac{\alpha}{2n\beta} \ln \det \left[ \hat{I} + \frac{2\beta}{\alpha} \hat{G} \right] + \frac{\alpha\beta}{2n} \sum_{a,b} r_{a,b} G_{a,b} \\ &\quad - \frac{1}{n\beta} \ln \text{Tr}_{\pi \in \Delta^n} \exp \left[ \frac{\alpha\beta^2}{2} \sum_{a,b} r_{a,b} \vec{\pi}^a \cdot \vec{\pi}^b \right] - \eta \sum_a G_{a,a}, \end{aligned} \quad (8.25)$$

where  $\hat{I}$  is the identity matrix. The first term arises from the expectation over  $z_a^\mu$ . This factorizes for each  $\mu$  and one is left with a Gaussian integral over  $\vec{z} \in \mathbb{R}^n$ . The second and the third terms arise from the integral representation of the delta functions<sup>2</sup>.

The key point is that, in the limit  $N \rightarrow \infty$  the integral over the matrices  $\hat{r}$  and  $\hat{G}$  in Eq. (8.24) are dominated by their saddle point value, i.e. by the values of  $r_{a,b}$  and  $G_{a,b}$  for which  $F$  attains its minimum value<sup>3</sup>. One should then study the first order conditions  $\partial F / \partial r_{a,b} = 0$  and  $\partial F / \partial G_{a,b} = 0$  for all  $a, b$ . Here we focus on the *replica symmetric* approximation where we assume that the matrices for which  $F$  attains its extreme have the form

$$G_{a,b} = g + (G - g)\delta_{a,b}, \quad r_{a,b} = r + (R - r)\delta_{a,b}. \quad (8.26)$$

<sup>1</sup>A generic distribution  $\varrho^\mu$  can also be handled, though with heavier notations.

<sup>2</sup>For simplicity we have also done the transformation  $r_{a,b} \rightarrow r_{a,b}/2$  for  $a \neq b$  so that  $\sum_{a \geq b} \rightarrow \sum_{a,b}$ .

<sup>3</sup>Note that, by Eq. (8.20), we shall also be interested in the limit of  $\beta \rightarrow \infty$  at the end!

This ansatz is correct for  $\eta \leq 0$  and for  $\eta > 0$  and  $\alpha$  large enough[61]. The reason for this is that  $H_\eta$  is a non-negative definite quadratic form in  $\Delta^N$ . Hence it has a very simple *energy landscape*, characterized by a single *valley*. Taking the limit  $n \rightarrow 0$ , Eq. (8.21) then gives

$$\begin{aligned} F_\beta^{(RS)}(Q, q, R, r) &= \frac{\alpha g}{\alpha + 2\beta(G - g)} + \frac{\alpha}{2\beta} \ln \left[ 1 + \frac{2\beta(G - g)}{\alpha} \right] - \eta G \\ &+ \frac{\alpha\beta}{2} (RG - rg) - \frac{1}{\beta} E_{\vec{z}} \{ \ln \text{Tr}_\pi \exp [-\beta V_{\vec{z}}(\vec{\pi})] \} \end{aligned} \quad (8.27)$$

where  $\text{Tr}_\pi$  is now the integral over the simplex  $\Delta$  of a single agent's mixed strategies and we defined, for convenience, the potential  $V_{\vec{z}}(\vec{\pi}) = \sqrt{\alpha r} \vec{z} \cdot \vec{\pi} - \frac{\alpha}{2} \beta (R - r) |\pi|^2$ . The parameters  $g, G, r$  and  $R$  are fixed by the first order conditions  $\partial F_\beta^{(RS)} / \partial g = 0$ ,  $\partial F_\beta^{(RS)} / \partial G = 0$ ,  $\partial F_\beta^{(RS)} / \partial r = 0$  and  $\partial F_\beta^{(RS)} / \partial R = 0$ . These equations, finally, have to be studied in the limit  $\beta \rightarrow \infty$ , where one recovers the minimum of  $H_\eta$  by Eq. (8.20), i.e.

$$\lim_{N \rightarrow \infty} \min_{\pi \in \Delta^N} \frac{H_\eta\{\pi\}}{N} = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta N} \langle \ln Z(\beta) \rangle = \lim_{\beta \rightarrow \infty} F_\beta^{(RS)}(Q, q, R, r) \Big|_{\text{sp}}$$

where the subscript sp means that we compute the function  $F_\beta^{(RS)}$  at the saddle point values of  $Q, q, R$  and  $r$ .

It is convenient to define the parameters

$$\chi = \frac{2\beta(G - g)}{\alpha}, \quad y = \frac{\sqrt{g/\alpha}}{1 + \eta(1 + \chi)} \quad (8.28)$$

In the limit  $\beta \rightarrow \infty$ , we first look for solutions where  $g \rightarrow G$  and  $\chi$ , which we call *susceptibility*, remains finite. This implies that two replicas of the same system converge in the long run to the same stationary state. Using the saddle point equations, and  $g = G$ , we can rewrite

$$V_{\vec{z}}(\vec{\pi}) = \frac{2y \vec{z} \cdot \vec{\pi} + \pi^2}{1 + \chi}, \quad \beta \rightarrow \infty \quad (8.29)$$

The last term in Eq. (8.27) is dominated by the mixed strategy  $\vec{\pi}^*(\vec{z})$  which is the solution of

$$\vec{\pi}^*(\vec{z}) = \arg \min_{\pi \in \Delta} V_{\vec{z}}(\vec{\pi}). \quad (8.30)$$

We find that  $G = g = E_{\vec{z}}[\vec{\pi}^*(\vec{z})]$ , which is then a function of  $y$  only  $G \equiv G(y)$ . Upon defining  $\zeta(y) = E_{\vec{z}}[\vec{z} \cdot \vec{\pi}^*(\vec{z})]$ , we find

$$\chi(y) = - \frac{\zeta(y)}{\sqrt{\alpha G(y) + \zeta(y)}}$$

The second of Eqs. (8.28) becomes an equation for  $y$  as a function of  $\alpha$  which has two implicit solutions. These can be expressed as explicit solutions for  $\alpha$  as a function of  $y$  and  $\eta$ :

$$\alpha = \frac{1}{G} \left[ \frac{G - y\zeta \pm \sqrt{(G + y\zeta)^2 - 4\eta y\zeta G}}{2(1 - \eta)y} \right]^2 \quad (8.31)$$

### 8.2.1 $\eta \leq 0$

The solutions of Eq. (8.31), for  $\eta < 0$  describe the two branches  $\alpha < \alpha_c$  and  $\alpha > \alpha_c$ . In particular for  $\eta \rightarrow 0^-$  these solutions become

$$\alpha y^2 = G(y), \quad \alpha G(y) = \zeta^2(y). \quad (8.32)$$

Let us discuss first the case  $\eta \rightarrow 0^-$ : The *free energy* per agent is

$$\lim_{N \rightarrow \infty} \frac{H}{N} = - \lim_{N \rightarrow \infty} \frac{\langle \ln Z(\beta) \rangle_a}{N} = \frac{G}{(1 + \chi)^2} \quad (8.33)$$

These equations are transcendental and we could not find an explicit solution for generic  $S$ . Nevertheless, they represent a great simplification with respect to the original problem. The main technical difficulty lies in the evaluation of the functions  $G(y) = E_{\vec{z}}[|\pi^*(\vec{z})|^2]$  and  $\zeta(y) = E_{\vec{z}}[\vec{z} \cdot \vec{\pi}^*(\vec{z})]$ , which can be computed numerically to any desired accuracy  $\forall S$ .

The first of Eqs. (8.32) gives the  $\alpha > \alpha_c$  phase. This solution has  $\chi > 0$  finite and  $H > 0$  non-zero. As  $\alpha$  decreases  $\chi$  increases and it diverges as  $|\alpha - \alpha_c|^{-1}$  when  $\alpha \rightarrow \alpha_c^+$ . In this limit Eq. (8.33) implies that  $H \sim |\alpha - \alpha_c|^2$  vanishes. The critical point  $\alpha_c = \alpha(y_c)$  is obtained imposing  $\chi = \infty$ , which gives  $G(y_c) = -y_c\zeta(y_c)$ . By the numerical evaluation of the functions  $G(y)$  and  $\zeta(y)$ , we find

$$\alpha_c(S) \cong \alpha_c(2) + \frac{S - 2}{2} \quad (8.34)$$

to a high degree of accuracy. It might be that this equation is exact but we could not prove it. An interesting relation for  $\alpha_c(S)$  can be derived by algebraic considerations: Note that for each  $\pi_{i,s} > 0$  the equation

$$\frac{\partial H}{\partial \pi_{i,s}} = 2 \sum_{j,s'} \overline{a_{i,s} a_{s',j}} \pi_{s',j} = 0 \quad (8.35)$$

must hold. This is a set of linear equations in the variables  $\pi_{i,s} > 0$ . The  $NS \times NS$  matrix  $\overline{a_{i,s} a_{j,s'}}$  is built with  $P$  dimensional vectors  $a_{i,s}^\mu$  and therefore has at most rank  $P$ . In other words there are only  $P$  independent equations

(8.35). In addition there are  $N$  normalization conditions on  $\pi_{i,s}$ . The system becomes dynamically degenerate when the number of free variables  $\pi_{i,s}$  becomes bigger than the number  $P + N$  of independent equations and, exactly at  $\alpha_c$  the two are equal. Dividing this condition by  $N$  gives the desired equation

$$\sum_{s=1}^S E_{\vec{z}}\{\theta[\pi_s^*(\vec{z})]\} = \alpha_c(S) + 1. \quad (8.36)$$

The left hand side is the average number of strategies used by agents,  $n$ . Note that this equation implies that  $\alpha_c(S)$  cannot grow faster than linear in  $S$ . Also  $\alpha_c(S) \propto S/2$  imply that agents use on average  $1/2$  of their strategies at  $\alpha_c$ .

The second of Eqs. (8.32) gives the  $\alpha < \alpha_c$  phase. Note indeed that with this choice  $\chi \simeq -1/\eta \rightarrow \infty$  and  $H \sim \eta^2 \rightarrow 0$  as  $\eta \rightarrow 0^-$ . At odds with the solution for  $\alpha > \alpha_c$ , this equation only arises if  $\eta < 0$  and in the limit  $\eta \rightarrow 0^-$ . With  $\eta = 0$  the saddle point equations have only a solution with  $G > g$  in the limit  $\beta \rightarrow \infty$ . This is because for  $\alpha < \alpha_c$  the set  $\mathcal{M}$  where  $H = 0$  is not a single point, but rather a connected set. The replica method with  $\eta = 0$  takes an average on all the set  $\mathcal{M}$  and so it gives results which are not representative of a particular system<sup>4</sup>. In order to select a single point in  $\mathcal{M}$  one may consider the limit  $\eta \rightarrow 0^-$ . Since the term  $-\eta N G$  in  $H_\eta$  breaks the degeneracy of equilibria for  $\eta = 0$ , the limit  $\eta \rightarrow 0^-$  selects the equilibrium which is closest to the random initial condition  $\pi_{i,s}(0) = 1/S$  for all  $i = 1, \dots, N$  and  $s = 1, \dots, S$ . This describes the stationary state of a system of agents with no prior beliefs ( $U_{i,s}(t=0) = 0, \forall i, s$ ).

In both phases, once the saddle point equations are solved, one can derive the full statistical characterization of the system. For instance the fraction of agents playing a strategies in a neighborhood  $d\vec{\pi}$  of  $\vec{\pi}$  is given by  $p(\vec{\pi})d\vec{\pi} = E_{\vec{z}}[\delta(\vec{\pi}^*(\vec{z}) - \vec{\pi})]d\vec{\pi}$ .

### 8.2.2 $\eta > 0$

Let us for simplicity consider the simpler case  $S = 2$ . The solution with  $G = g$  exists for  $\alpha \geq 1/\pi$ . For  $\alpha > [\pi(1 - \eta)^2]^{-1}$  this solution has  $G = g < 1$ , which means that agents do not all play pure strategies. When  $\alpha \rightarrow [\pi(1 - \eta)^2]^{-1}$ ,  $G \rightarrow 1$  and the solution becomes independent of  $\eta$ . In other words, the solution merges with the solution for  $\eta = 1$ . This solution breaks down too, with  $\chi \rightarrow \infty$  and  $H_1/N = \sigma^2/N \rightarrow 0$  when  $\alpha \rightarrow 1/\pi$ . Below this point, only solutions with  $G < g$  and  $H_1/N = 0$  exist. This behavior is

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<sup>4</sup>Note indeed that  $g$  has the interpretation of the overlap between two replicas of the same system, so that  $g < G$  means that the two replicas are not identical.

well documented in figure 4.4. However, for  $\eta > 0$ , one needs to go beyond the simple approximation for  $G_{a,b}$  and  $r_{a,b}$  in Eq. (8.26). Therefore we shall refrain from a more detailed discussion and rather refer the interested reader to a forthcoming publication [61].



# Bibliography

- [1] A. Dragulescu, V. M. Yakovenko, preprint cond-mat/0001432 ; A. Chakraborti, B. K. Chakrabarti, preprint cond-mat/0004256
- [2] V. Plerou, P. Gopikrishnan, L. A. N. Amaral, X. Gabaix, H. E. Stanley preprint cond-mat/9912051
- [3] W. B. Arthur, J. H. Holland, B. LeBaron, R. Palmer, P. Taylor, Physica D (1996)
- [4] J.-P. Bouchaud, R. Cont, preprint cond-mat/9801279
- [5] See the Minority Game's web page on <http://www.unifr.ch/econophysics>
- [6] L. Bachelier, "Théorie de la spéculation" Paris (1900), reprinted in P. Cootner, "Random Character of Stock Prices", MIT Press, Cambridge (1964)
- [7] D. Fama, J. of Finance, **25**, 383 (1965) Am. Econ. Assoc. Papers and Proc **84**, 406, 1994. ([http://www.santafe.edu/arthur/Papers/El\\_Farol.html](http://www.santafe.edu/arthur/Papers/El_Farol.html))
- [8] <http://www.proinvest.com/home/emh.htm>
- [9] Y.-C. Zhang, Physica A **269**, 30 (1999). (cond-mat/9901243)
- [10] M. Marsili, S. Maslov and Y.-C. Zhang, preprint cond-mat/9801239
- [11] S. Maslov and Yi-Cheng Zhang, Int. J. of Th. and Applied Fin, preprint cond-mat/9801240 (1999)
- [12] W. Brock, J. Lakonishok and B. LeBaron, J. of Finance (Dec. 1992)
- [13] H. Simon, *Models of Bounded Rationality*, MIT Press, Cambridge (1997)
- [14] Arthur W. B.,

- [15] T. R. Kaplan, preprint (1998).
- [16] A. Greenwald, preprints available at [http://cs.nyu.edu/phd\\_students/amygreen](http://cs.nyu.edu/phd_students/amygreen) (1997).
- [17] J. W. Weibull, “Evolutionary Game Theory”, MIT Press, Cambridge (1995)
- [18] J. L. Casti, Complexity, **5** 1, 7 (1996)
- [19] D. B. Fogel, K. Chellapilla and P. J. Angeline, IEEE Trans. Ev. Comp. Vol. 3 no 2 (1999).
- [20] Challet D. and Zhang Y.-C., Physica A **246**, 407 (1997) (adap-org/9708006).
- [21] Savit R., Manuca R., and Riolo R., PRL, **82**(10), 2203 (1999), preprint adap-org/9712006.
- [22] Challet D. and Zhang Y.-C., Physica A **256**, 514 (1998) (cond-mat/9805084) ;
- [23] Y.-C. Zhang, Europhys. News **29**, 51 (1998) (cond-mat/9803308)
- [24] D. Challet and M. Marsili, Phys. Rev. E **60**, 6271 (1999), preprint cond-mat/9904071
- [25] R. Mansilla, preprints cond-mat/9906017 and cond-mat/0002331
- [26] R. D’hulst, G. J. Rodgers, preprint cont-mat/9902001 (1999)
- [27] N. F. Johnson, M. Hart, P. M. Hui, preprints cond-mat/9811227 and cond-mat/0003486
- [28] N. Brunel and R. Zecchina, PRE **49** 3, R1823 (1994)
- [29] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999) preprint cond-mat/9903415 (1999).
- [30] D. Fudenberg and D. K. Levine, *The theory of learning in games* (MIT Press, 1998).
- [31] Luce R.D., *Individual Choice Behavior : A Theoretical Analysis*, New York, Wiley (1959).
- [32] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett **84**, 1824 (2000), preprint cond-mat/9904392



- [33] D. Challet, M. Marsili and R. Zecchina, Comment on “A thermal model for adaptative competition”, submitted to Phys. Rev. Lett. (2000)
- [34] Challet D., M. Marsili and Y.-C. Zhang, Physica A **276**, 284 (2000) preprint cond-mat/9909265
- [35] M. Marsili, D. Challet and R. Zecchina, to appear in Physica A (2000), preprint cond-mat/9908480
- [36] M. Marsili and D. Challet, submitted to Adv. Compl. Sys., preprint cond-mat/0004376
- [37] A. M. Sengupta, Mitra P. P. , Phys. Rev. E **60**, 3389 (1999).
- [38] A. Cavagna, Phys. Rev. E, **59**, R3783, preprint cond-mat/9812215, (1998) .
- [39] D. Challet and M. Marsili, submitted to Phys. Rev. E, preprint cond-mat/0004196 (2000)
- [40] See the web page <http://www.scs.carleton.ca/~quesnel/papers/debruijn/paper.html> for a nice introduction to De Bruijn graphs.
- [41] S. Berg, M. Marsili, A. Rustichini, R. Zecchina *et al.* forthcoming.
- [42] A. Rustichini, preprint (1999)
- [43] M. A. R. de Cara et al. preprint cond-mat/9811162 (1998)
- [44] R. B. Olsen, M. M. Dacorogna, U. A. Mller, O. V. Pictet, Olsen internal document RBO.1992-09-07 (1992) (available on [www.olsen.ch](http://www.olsen.ch))
- [45] U. A. Mller, M. M. Dacorogna, R. D. Dav, O. V. Pictet, R. B. Olsen and J.R. Ward, Olsen internal document UAM.1993-08-16 (1993) (available on [www.olsen.ch](http://www.olsen.ch))
- [46] N. F. Johnson et al., preprint cond-mat/9905039 (1999)
- [47] Y. Li, A. VanDeemen, R. Savit, preprint nlin/0002004
- [48] A. De Cara, P. Guinea, preprint cond-mat/9904187
- [49] M. Paczuski, K. E. Bassler and A. Corral, cond-mat/9905082 (1999)
- [50] T. Kalinowski, H.-J. Schulz and M. Briesse, Physica A **277** (3-4), 502-508 (2000)

- [51] P. De Los Rios, S. Moelbert unpublished ; M. A. R. de Cara, P. Guinea, unpublished.
- [52] M. Mezard, G. Parisi, M. A. Virasoro, *Spin glass theory and beyond* (World Scientific, 1987).
- [53] V. Dotsenko, *An introduction to the theory of spin glasses and neural networks* ; World Scientific Publishing (1995).
- [54] M. Marsili and R. Zecchina, unpublished
- [55] D. Challet, A. Chessa and M. Marsili, unpublished.
- [56] P. Bak *et al.* PNAS 92, 5202 (1995)
- [57] Y. Li et al., Physica A **276**, preprints cond-mat/9903415 and cond-mat/9906001
- [58] N. F. Johnson, M. Hart, P. M. Hui and D. Zheng, cond-mat/9910072
- [59] J. B. DeLong, A. Schleifer, L. H. Summers, R. J. Waldmann, *Noise Trader Risk in Financial Markets*, J. Political Econ. 98 (1990)703.
- [60] F. Ricci-Tersenghi *et al.*, unpublished
- [61] A. De Martino, M. Marsili, R. Zecchina, forthcoming
- [62] J. D. Farmer, Comp. Fin., Nov-Dec (1999), pp. 26-39
- [63] J. D. Farmer, Santa Fe working paper 98-12-117
- [64] T. S. Lo, P. M. Hui, N. F. Johnson, preprint cond-mat/0003379